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On a class of two-dimensional singular elliptic problems

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We consider Dirichlet problems of the form $-|x|^\alpha \Delta u = \lambda u + g(u)$ in Ω, $u = 0$ on $\partial\Omega$, where $\alpha, \lambda \in \mathbb{R}$, $g \in C(\mathbb{R})$ is a superlinear and subcritical function, and Ω is a domain in $\mathbb{R}^2$. We study the existence of positive solutions with respect to the values of the parameters α and λ, and according that $0 \in \Omega$ or $0 \in \partial\Omega$, and that Ω is an exterior domain or not.

1. Introduction

In this paper we deal with the equation

$$-|x|^\alpha \Delta u = f(u) \quad \text{in } \Omega, \tag{1.1}$$

where $f \in C(\mathbb{R})$, $\alpha \in \mathbb{R}$ and Ω is a domain in $\mathbb{R}^N$. Equation (1.1) is a model for a class of N-dimensional stationary Schrödinger equations with a singular potential [9, 15]. The case $\alpha > 0$ corresponds to a singularity at 0, while for $\alpha < 0$ the singularity is the point at infinity.

Singular elliptic problems somehow related to (1.1) have been studied in the case of dimensions $N = 1$ and $N \geqslant 3$ by several authors. We mention, for instance, [2–4, 6, 8, 10, 16]. In this paper we will mostly concentrate on the two-dimensional case, which exhibits some special features that do not appear in dimensions $N \neq 2$.

From the technical point of view, an important tool is given by the Kelvin transform $x \mapsto x/|x|^2$, which in two dimensions produces a sort of 'duality' between the cases 'Ω contains 0' and 'Ω is an exterior domain', without changing the structure of the equation.

The role played by the dimension of the domain is firstly underlined in § 1, where we prove some non-existence results in the spirit of the paper [4] by Brezis and Cabré. Here we first assume that $N \geqslant 2$ and that f is positive and superlinear at infinity, and we prove that no *very weak* positive solution to (1.1) exists when $0 \in \Omega$ and $\alpha \geqslant 2$ (see § 1 for the precise definitions).

Then we restrict our attention to the case $N = 2$ and we use the Kelvin transform to show that no *very weak* positive solution exists when Ω is an exterior domain

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and $\alpha \leqslant 2$. In particular, equation (1.1) on $\mathbb{R}^2$ has no very weak positive solution for any $\alpha \in \mathbb{R}$.

Notice that this last non-existence result is false in dimension $N \neq 2$. As an example, the equation

$$-|x|^{\alpha}\Delta u = |u|^{N^*_{\alpha}-1} \quad \text{on } \mathbb{R}^N, \tag{1.2}$$

where $N \geqslant 3$, $N^*_{\alpha} = 2(N-\alpha)/(N-2)$ and $\alpha \in [0,2)$ has a positive solution with finite energy (see [10]). This solution admits a variational characterization, since it solves the minimization problem

$$S_{\alpha}(\mathbb{R}^N) = \inf\biggl[\int_{\mathbb{R}^N} |\nabla u|^2 \bigg/ \biggl(\int_{\mathbb{R}^N} \frac{|u|^{N^*_{\alpha}}}{|x|^{\alpha}}\biggr)^{2/N^*_{\alpha}}\biggr] \tag{1.3}$$

on a suitable Sobolev space. Notice that for $N \geqslant 3$ the infimum $S_{\alpha}(\mathbb{R}^N)$ is positive by the Hardy–Sobolev inequality (see, for example, [5]). We remark also that for $N \geqslant 3$ the exponent N^*_{α} is the unique *critical exponent* associated to the singular operator $-|x|^{\alpha}\Delta u$, in the sense that it makes the equation (1.2) and the ratio in (1.3) invariant with respect to dilations $x \mapsto rx$ ($r > 0$).

In the two-dimensional case the situation is completely different. First we note that for every $q \geqslant 2$ and $\alpha \in \mathbb{R}$,

$$\inf\biggl\{\int_{\mathbb{R}^2} |\nabla u|^2 \bigg/ \biggl(\int_{\mathbb{R}^2} \frac{|u|^q}{|x|^{\alpha}}\biggr)^{2/q} : u \in C^{\infty}_c(\mathbb{R}^2 \setminus \{0\}),\ u \neq 0\biggr\} = 0,$$

which expresses the failure of the Hardy–Sobolev inequality in two dimensions (see [5]).

In addition, for $N = 2$ and $\alpha \neq 2$, no *critical exponent* associated to $-|x|^{\alpha}\Delta u$ exists, in the sense that (1.1) is never invariant with respect to dilations. On the other hand, for $\alpha = 2$, every nonlinearity f gives rise to a dilation-invariant equation, which, together with the invariance under Kelvin transform, reflects on phenomena of concentration at 0 or vanishing, and then on a possible lack of compactness for the corresponding variational problem.

The technical facts described above produce a variety of phenomena in the two-dimensional case which are missing for $N \neq 2$.

For this reason, in the second part of the paper we start to study more closely the two-dimensional case, by considering Dirichlet's problems of the form

$$\left.\begin{aligned} -\Delta u &= \frac{\lambda u + g(u)}{|x|^{\alpha}} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.4}$$

where $\lambda, \alpha \in \mathbb{R}$, $g \in C(\mathbb{R})$ is subcritical and superlinear (see § 2 for the precise assumptions).

First we consider the case $0 \notin \partial\Omega$, and we investigate some situations which do not fit into the non-existence results of § 1.

In particular, if Ω is a bounded smooth domain containing 0 and $\alpha < 2$, the standard Sobolev space $H^1_0(\Omega)$ turns out to be compactly embedded into the weighted spaces $L^q(\Omega, \mathrm{d}x/|x|^{\alpha})$ for any $q \geqslant 2$, and the weight $1/|x|^{\alpha}$ in fact gives rise to

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a removable singularity. Indeed, in this case, we find smooth positive solutions $u \in H_0^1(\Omega)$ with $u(0) > 0$, when λ lies in a suitable range. An analogous 'dual' result holds when Ω is an exterior domain with $0 \notin \bar{\Omega}$ and $\alpha > 2$ (see theorem 3.3).

A different situation occurs if Ω is a bounded smooth domain containing 0 and $\alpha \geqslant 2$, because in general the space $H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}x/|x|^\alpha)$ is not compactly embedded into $L^q(\Omega, \mathrm{d}x/|x|^\alpha)$ for $q \geqslant 2$. However, in the radially symmetric case, we can recover compactness, and when Ω is the unit ball, $\alpha > 2$ and $\lambda < 0$, we find a positive radial solution u of (1.4) with finite energy. The fact that

$$\int_\Omega \frac{u^2}{|x|^\alpha} < \infty$$

forces the solution to vanish at 0, and then in this case the singularity cannot be considered as removable. Moreover, thanks to the Kelvin transform, we state that a corresponding 'dual' result holds for $\Omega = \{x \in \mathbb{R}^2 : |x| > 1\}$, $\alpha < 2$ and $\lambda < 0$ (see theorem 3.4).

If $\alpha = 2$, concentration at 0 or vanishing may occur and indeed, in general, we have existence of a radially symmetric positive solution only on $\mathbb{R}^2$, provided that $\lambda < 0$ (theorem 3.5).

If Ω is an unbounded, non-exterior domain, with $0 \in \partial\Omega$, in general we cannot expect existence. For example, if Ω is a half-space, or more generally, a cone with vertex at the origin, a Pohozaev-type argument [14] prevents existence for $\alpha \neq 2$ (see lemma 4.1).

However, for these domains we can prove existence of positive and changing-sign solutions to (1.4) when $\alpha = 2$ (theorem 4.4). This is a consequence of the fact that the Hardy–Sobolev inequality (which fails on $\mathbb{R}^2$) actually holds if we take as domain $\mathbb{R}^2 \setminus \Gamma$, where $\Gamma = \{(x_1, 0) : x_1 \geqslant 0\}$. More precisely, we prove that *for every $q \geqslant 2$ there exists $C_q > 0$ such that*

$$\left(\int_{\mathbb{R}^2} \frac{|u|^q}{|x|^2}\right)^{2/q} \leqslant C_q \int_{\mathbb{R}^2} |\nabla u|^2 \quad \text{for any } u \in C_c^\infty(\mathbb{R}^2 \setminus \Gamma). \tag{1.5}$$

In a forthcoming paper [7] we shall complete the study of problem (1.4) for $\alpha = 2$ in the case of arbitrary domains Ω contained in $\mathbb{R}^2 \setminus (\Gamma_0 \cup \Gamma_\infty)$, where $\Gamma_0 = \{tx_0 : 0 \leqslant t \leqslant 1\}$ and $\Gamma_\infty = \{tx_\infty : t \geqslant 1\}$, with $x_0, x_\infty \in \mathbb{R}^2 \setminus \{0\}$.

The plan of the paper is as follows. In § 2 the non-existence of positive solutions on domains containing 0 and exterior domains is discussed. In § 3 we look at the Dirichlet problem on domains containing 0 and exterior domains, with the cases $\alpha \neq 2$ and $\alpha = 2$ studied in §§ 3.1 and 3.2, respectively. Finally, § 4 is devoted to the Dirichlet problem on cones.

2. Non-existence of positive solutions on domains containing 0 and exterior domains

In this section we consider the equation

$$-\Delta u = \frac{f(u)}{|x|^\alpha} \quad \text{in } \Omega, \tag{2.1}$$

where $\alpha \in \mathbb{R}$, and $f \in C(\mathbb{R})$ satisfies the following conditions.

(f1) There exists $p > 1$ such that

$$\liminf_{u \to +\infty} \frac{f(u)}{u^p} > 0.$$

(f2) $f(u) > 0$ for $u > 0$.

As regards the domain Ω, in this section we will assume that $0 \in \Omega$ or Ω is an exterior domain, i.e. there exists a compact set $K \subset \mathbb{R}^N$ such that $\Omega = \mathbb{R}^N \setminus K$.

Let us introduce a notion of supersolution to (2.1) in a very weak sense, in the spirit of analogous definitions stated in [4] by Brezis and Cabré.

Given a domain Ω in $\mathbb{R}^N$ and a Carathéodory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$, we say that a function $u : \Omega \to \mathbb{R}$ solves $-\Delta u \geqslant h(x, u)$ in $\mathcal{D}'(\Omega)$ if $u \in L^1_{\mathrm{loc}}(\Omega)$, $h(\cdot, u) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$-\int_\Omega u \Delta\varphi \geqslant \int_\Omega h(x, u)\varphi$$

for every $\varphi \in C_c^\infty(\Omega)$, $\varphi \geqslant 0$.

Definition 2.1. Let Ω be a domain in $\mathbb{R}^N$ and let $f \in C(\mathbb{R})$. A function $u : \Omega \to \mathbb{R}$ is a *weak supersolution* to (2.1) if u solves $-\Delta u \geqslant f(u)/|x|^\alpha$ in $\mathcal{D}'(\Omega)$. A function $u : \Omega \to \mathbb{R}$ is a *very weak supersolution* to (2.1) if u solves $-\Delta u \geqslant f(u)/|x|^\alpha$ in $\mathcal{D}'(\Omega \setminus \{0\})$. A weak or very weak supersolution u to (2.1) is positive if $u \geqslant 0$ a.e. in Ω and $u \neq 0$.

The main result of this section is the following.

Theorem 2.2. *Let $f \in C(\mathbb{R})$ satisfy (f1) and (f2).*

(i) *If Ω is a domain in $\mathbb{R}^N$ containing 0, $N \geqslant 2$ and $\alpha \geqslant 2$, then (2.1) admits no very weak positive supersolution.*

(ii) *If Ω is an exterior domain in $\mathbb{R}^2$ and $\alpha \leqslant 2$, then (2.1) admits no very weak positive supersolution.*

As a consequence of theorem 2.2, we have the following non-existence result concerning the case $\Omega = \mathbb{R}^2$.

Corollary 2.3. *Let $f \in C(\mathbb{R})$ satisfy (f1) and (f2). Then, for any $\alpha \in \mathbb{R}$, equation (2.1) on $\mathbb{R}^2$ admits no very weak positive supersolution.*

The proof of theorem 2.2 is based on the next crucial lemma.

Lemma 2.4. *Let Ω be a domain in $\mathbb{R}^N$ containing 0, $N \geqslant 2$, and let $f \in C(\mathbb{R})$ satisfy (f1) and (f2). If u is very weak non-negative supersolution to*

$$-\Delta u = \frac{f(u)}{|x|^2} \quad \text{in } \Omega, \tag{2.2}$$

then u is a weak supersolution to (2.2).

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Proof. Since u is a very weak supersolution to (2.2), it is enough to show that u solves $-\Delta u \geqslant f(u)/|x|^2$ in $\mathcal{D}'(B_r)$, where $B_r = \{x \in \mathbb{R}^N : |x| < r\} \subset \Omega$ for some $r \in (0, 1)$. To this end, we will adapt the techniques developed in [4], by using appropriate powers of testing functions (see also [1]).

STEP 1 ($f(u)/|x|^2 \in L^1(B_r)$). For every $n \in \mathbb{N}$ large enough, let $\chi_n \in C_c^\infty(B_r)$ satisfy

$$0 \leqslant \chi_n \leqslant 1 \text{ on } B_r,$$
$$\chi_n(x) = 0 \text{ for } |x| \leqslant 1/n \text{ and } |x| \geqslant \tfrac{2}{3}r,\ \chi_n(x) = 1 \text{ for } x \in B_{r/2} \setminus B_{2/n},$$
$$|\Delta\chi_n| \leqslant Cn^2 \text{ on } U_n = B_{2/n} \setminus \bar{B}_{1/n},\ |\Delta\chi_n| \leqslant C \text{ on } B_r \setminus B_{r/2}.$$

Taking a suitable $\beta \geqslant 2$, we have that $\chi_n^\beta \in C_c^2(\Omega)$ and, by standard density arguments (on any χ_n^β), since $u, f(u) \in L^1_{\mathrm{loc}}(\Omega \setminus \{0\})$, we have

$$\begin{aligned}\int_{B_r} \frac{f(u)}{|x|^2}\chi_n^\beta &\leqslant -\int_{B_r} u\Delta\chi_n^\beta \\ &= -\beta(\beta-1)\int_{B_r} u\chi_n^{\beta-2}|\nabla\chi_n|^2 - \beta\int_{B_r} u\chi_n^{\beta-1}\Delta\chi_n \\ &\leqslant \beta\int_{B_r} u\chi_n^{\beta-2}|\Delta\chi_n|,\end{aligned}$$

and then

$$\int_{B_r} \frac{f(u)}{|x|^2}\chi_n^\beta \leqslant Cn^2\int_{U_n} u\chi_n^{\beta-2} + C. \tag{2.3}$$

By (f1), there exist $\delta, k > 0$ such that

$$f(u) \geqslant ku^p \quad \text{for } u \geqslant \delta. \tag{2.4}$$

Letting $U_n^\delta = \{x \in U_n : u(x) \geqslant \delta\}$, we have

$$\begin{aligned}\int_{U_n} u\chi_n^{\beta-2} &\leqslant \delta|U_n \setminus U_n^\delta| + \int_{U_n^\delta} u\chi_n^{\beta-2} \\ &\leqslant \frac{C}{n^N} + \frac{C}{n^{2/p}}\int_{U_n^\delta} \frac{u}{|x|^{2/p}}\chi_n^{\beta-2} \\ &\quad \frac{C}{n^2} + \frac{C}{n^{2/p}}|U_n^\delta|^{1/p'}\biggl(\int_{U_n^\delta} \frac{u^p}{|x|^2}\chi_n^{(\beta-2)p}\biggr)^{1/p}\end{aligned}$$

where p' is the conjugate exponent to p. Choosing $\beta = 2p/(p-1)$ (note that $\beta > 2$), and using (2.4), we obtain

$$n^2\int_{U_n} u\chi_n^{\beta-2} \leqslant C + C\biggl(\int_{U_n} \frac{f(u)}{|x|^2}\chi_n^\beta\biggr)^{1/p}, \tag{2.5}$$

which, together with (2.3), implies

$$\int_{U_n} \frac{f(u)}{|x|^2}\chi_n^\beta \leqslant C + C\biggl(\int_{U_n} \frac{f(u)}{|x|^2}\chi_n^\beta\biggr)^{1/p}.$$

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Since $p > 1$, we infer that

$$\sup_n \int_{U_n} \frac{f(u)}{|x|^2}\chi_n^\beta < +\infty$$

and then, using (2.3) and (2.5),

$$\sup_n \int_{B_r} \frac{f(u)}{|x|^2}\chi_n^\beta < \infty.$$

Finally, by (f2) and by Fatou's lemma,

$$\int_{B_r} \frac{f(u)}{|x|^2}\,\mathrm{d}x < \infty.$$

Step 2 ($u \in L^1(B_r)$). This follows immediately from step 1, since we have that $0 \leqslant u \leqslant C(1 + f(u)/|x|^2)$.

Step 3.

$$-\int_{B_r} u\Delta\varphi \geqslant \int_{B_r} (f(u)/|x|^2)\varphi \quad \text{for every} \quad \varphi \in C_c^\infty(B_r), \quad \varphi \geqslant 0.$$

Proof. For every $n \in \mathbb{N}$, let $\phi_n \in C^\infty(\mathbb{R}^N)$ satisfy $0 \leqslant \phi_n \leqslant 1$, $\phi_n = 0$ on $B_{1/n}$, $\phi_n = 1$ on $\mathbb{R}^N \setminus B_{2/n}$, $|\nabla\phi_n| \leqslant Cn$ and $|\Delta\phi_n| \leqslant Cn^2$ on $U_n = B_{2/n} \setminus \bar{B}_{1/n}$. Let $\varphi \in C_c^\infty(B_r)$, $\varphi \geqslant 0$. Since u solves $-\Delta u \geqslant f(u)/|x|^2$ in $\mathcal{D}'(\Omega \setminus \{0\})$, for every $n \in \mathbb{N}$ we have

$$-\int_{B_r} u\Delta(\varphi\phi_n) \geqslant \int_{B_r} \tfrac{f(u)}{|x|^2}\varphi\phi_n. \tag{2.6}$$

By step 1,

$$\int_{B_r} \frac{f(u)}{|x|^2}\varphi\phi_n \to \int_{B_r} \frac{f(u)}{|x|^2}\varphi.$$

Moreover, $\Delta(\varphi\phi_n) = \phi_n\Delta\varphi + 2\nabla\phi_n \cdot \nabla\varphi + \varphi\Delta\phi_n$. By step 2,

$$\int_\Omega u\phi_n\Delta\varphi \to \int_\Omega u\Delta\varphi.$$

Furthermore,

$$\int_\Omega u|\nabla\phi_n \cdot \nabla\varphi| \leqslant Cn\int_{U_n} u \quad \text{and} \quad \int_\Omega u\varphi|\Delta\phi_n| \leqslant Cn^2\int_{U_n} u.$$

We claim that

$$n^2\int_{U_n} u \to 0 \quad \text{as } n \to \infty. \tag{2.7}$$

Indeed, by (f1) and (f2), for every $\epsilon > 0$ there exists $k_\epsilon > 0$ such that

$$f(u) \geqslant k_\epsilon u \quad \text{for } u \geqslant \epsilon. \tag{2.8}$$

Arguing as in step 2, and using (2.8), we have that

$$n^2\int_{U_n} u \leqslant n^2\epsilon|U_n \setminus U_n^\epsilon| + \frac{n^2}{k_\epsilon}\int_{U_n^\epsilon} f(u) \leqslant C\epsilon + \frac{C}{k_\epsilon}\int_{U_n} \frac{f(u)}{|x|^2},$$

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where $U_n^\epsilon = U_n \cap \{u \geqslant \epsilon\}$. By step 1,

$$\int_{U_n} \frac{f(u)}{|x|^2} \to 0,$$

and thus, by the arbitrariness of $\epsilon > 0$, equation (2.7) follows. Hence we can pass to the limit in (2.6) to get the thesis. □

Proof of theorem 2.2. Let us show the statement (i). By contradiction, suppose that (2.1) admits a very weak positive supersolution u. By lemma 2.4, since $\alpha \geqslant 2$, u solves $-\Delta u \geqslant f(u)/|x|^2$ in $\mathcal{D}'(\Omega \cap B_1)$. In particular, by (f2), $-\Delta u \geqslant 0$ in $\mathcal{D}'(\Omega \cap B_1)$ and, since $u \geqslant 0$, $u \neq 0$, by the maximum principle, $u \geqslant \epsilon$ in a neighbourhood $U \subset \Omega$ containing 0, for some $\epsilon > 0$. Then, since f is continuous and positive on $(0, +\infty)$, there exists $a > 0$ such that $f(u) \geqslant a$ on U. If $N = 2$, we have that

$$\int_U \frac{f(u)}{|x|^2} \geqslant \int_U \frac{a}{|x|^2} = +\infty,$$

contrary to the fact that $f(u)/|x|^2 \in L^1(U)$. If $N > 2$, we have that

$$-\Delta u \geqslant \frac{a}{|x|^2} = -\Delta\left(\frac{a}{N-2}\log\frac{1}{|x|}\right)$$

in $\mathcal{D}'(U)$ and then

$$u \geqslant \frac{a}{N-2}\log\frac{1}{|x|} - b \quad \text{on } U \tag{2.9}$$

for some $b > 0$ (see [4] for more details). Moreover, by (f1) and (f2) there exists $k > 0$ such that $f(u) \geqslant ku^p$ for $u \geqslant \epsilon$ and then on U. Let now $\chi \in C_c^\infty(\mathbb{R}^+)$ be such that $0 \leqslant \chi \leqslant 1$ on $\mathbb{R}^+$, $\chi = 1$ on $[0, 1]$ and $\chi = 0$ on $[2, +\infty)$. Setting $\chi_n(x) = \chi(n|x|)$, $\beta = 2p/(p-1)$, testing (2.2) with χ_n^β and arguing as in the proof of step 1 of lemma 2.4, we obtain that

$$\int_U \frac{f(u)}{|x|^2}\chi_n^\beta \leqslant \frac{C}{n^{N-2}} \tag{2.10}$$

for some $C > 0$. On the other hand, using (2.8) and (2.9), we have that

$$\begin{aligned}\int_U \frac{f(u)}{|x|^2}\chi_n^\beta &\geqslant k_\epsilon \int_{B_{1/n}} \frac{u}{|x|^2} \\ &\geqslant (A\log n - B)\int_{B_{1/n}} \frac{1}{|x|^2} \\ &= (A\log n - B)\frac{1}{n^{N-2}},\end{aligned}$$

which, for $n \in \mathbb{N}$ large enough, is in contradiction with (2.10).

Part (ii) turns out to be equivalent to (i) thanks to the Kelvin transform $x \mapsto x/|x|^2$. More precisely, the proof of (ii) is accomplished using the following lemma and the previous part (i).

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Lemma 2.5. *Let Ω be a domain in $\mathbb{R}^2$ and let $f \in C(\mathbb{R})$. Then u is a very weak supersolution to (2.1) if and only if $\tilde{u}$ is a very weak supersolution to*

$$-\Delta\tilde{u} = \frac{f(\tilde{u})}{|x|^{4-\alpha}} \quad in\ \tilde{\Omega}, \tag{2.11}$$

with $\tilde{\Omega} = \{x/|x|^2 : x \in \Omega,\ x \neq 0\}$ and $\tilde{u}(x) = u(x/|x|^2)$ for $x \in \tilde{\Omega}$.

Proof. Let u be a very weak supersolution to (2.1). One can easily check that $\tilde{u} \in L^1_{\mathrm{loc}}(\tilde{\Omega})$ because $0 \notin \tilde{\Omega}$ and, for $U \subset\subset \tilde{\Omega}$,

$$\int_U \tilde{u}^2 = \int_{\tilde{U}} \frac{u^2}{|x|^4},$$

where $\tilde{U} = \{x/|x|^2 : x \in U\}$. Similarly, one can see that $f(\tilde{u})/|x|^{4-\alpha} \in L^1_{\mathrm{loc}}(\tilde{\Omega})$. Furthermore, if $\varphi \in C^\infty_c(\tilde{\Omega})$ and $\tilde{\varphi}(x) = \varphi(x/|x|^2)$, then

$$\Delta\tilde{\varphi}(x) = \frac{1}{|x|^4}\Delta\varphi\Big(\frac{x}{|x|^2}\Big), \qquad \int_{\tilde{\Omega}} \tilde{u}\Delta\varphi = \int_\Omega u\Delta\tilde{\varphi}, \qquad \int_{\tilde{\Omega}} \frac{f(\tilde{u})}{|x|^{4-\alpha}}\varphi = \int_\Omega \frac{f(u)}{|x|^\alpha}\tilde{\varphi}.$$

Hence $\tilde{u}$ is a weak supersolution to (2.11). The other implication follows by the fact that the Kelvin transform is idempotent. □

Remark 2.6. In order to state an analogous result to lemma 2.5 in any dimension N, one has to consider a weighted Kelvin transform. More precisely, one can prove that, given a domain Ω in $\mathbb{R}^N$ and a Carathéodory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$, a function $u : \Omega \to \mathbb{R}$ is a weak supersolution to $-\Delta u = h(x, u)$ in $\Omega \setminus \{0\}$ if and only if $\tilde{u}$ is a weak supersolution to

$$-|x|^{N+2}\Delta\tilde{u} = h\bigg(\frac{x}{|x|^2}, |x|^{N-2}\tilde{u}\bigg) \quad \text{in } \tilde{\Omega}, \tag{2.12}$$

with $\tilde{\Omega} = \{x/|x|^2 : x \in \Omega,\ x \neq 0\}$ and $\tilde{u}(x) = |x|^{2-N}u(x/|x|^2)$ for $x \in \tilde{\Omega}$.

Consider now the case $h(x, u) = f(u)/|x|^\alpha$ with an arbitrary $f \in C(\mathbb{R})$. In order that (2.12) for $\tilde{u}$ has the same structure of the equation for u, namely, can be written in the form $-\Delta\tilde{u} = f(\tilde{u})/|x|^\beta$ for some $\beta \in \mathbb{R}$, a necessary and sufficient condition is that $N = 2$.

A special situation occurs if f is homogeneous. More precisely, if $h(x, u) = |u|^p/|x|^\alpha$, equation (2.12) becomes

$$-\Delta\tilde{u} = \frac{|\tilde{u}|^p}{|x|^{\tilde{\alpha}}} \quad \text{in } \tilde{\Omega},$$

where $\tilde{\alpha} = N + 2 - \alpha - p(N - 2)$. In particular, for $N \neq 2$, we have $\tilde{\alpha} = \alpha$ if and only if $p = N^*_\alpha - 1$, where $N^*_\alpha = 2(N - \alpha)/(N - 2)$. For $N = 2$, we have $\tilde{\alpha} = \alpha$ if and only if $\alpha = 2$, whatever p is. More generally, we note that in the two-dimensional case, for $\alpha = 2$, the equations (2.1) and (2.11) have the same form (invariance under Kelvin transform for $\alpha = 2$ and $N = 2$).

3. The Dirichlet problem on domains containing 0 and exterior domains

In this section and in the following one we study the existence of positive solutions for the Dirichlet problem

$$\left.\begin{aligned} -\Delta u &= \frac{f(u)}{|x|^\alpha} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}\right\} \tag{3.1}$$

in some cases that do not fit into the non-existence results of § 1. In particular, in this section we deal with smooth domains Ω in $\mathbb{R}^2$ which are bounded and contain 0, or which are exterior domains and $0 \notin \bar{\Omega}$. In § 3 we will consider the case in which Ω is a cone in $\mathbb{R}^2$ with vertex at 0.

As concerns the notion of solution to problem (3.1), we are interested in variational solutions, according to the following definition.

Definition 3.1. Let Ω be a smooth domain in $\mathbb{R}^2$ and let $f \in C(\mathbb{R})$. A *variational solution* to problem (3.1) is a function $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $\nabla u \in (L^2(\Omega))^2$, $u = 0$ on $\partial\Omega$ in the sense of the traces, $f(u)/|x|^\alpha \in L^1_{\mathrm{loc}}(\Omega \setminus \{0\})$ and

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega \frac{f(u)}{|x|^\alpha}\varphi$$

for every $\varphi \in C_c^\infty(\Omega \setminus \{0\})$.

Looking for variational solutions to problem (3.1) leads us to make suitable assumptions on the behaviour of f at 0 and at infinity. More precisely, we will restrict ourselves to the case $f(u) = \lambda u + g(u)$, where $\lambda \in \mathbb{R}$ and $g \in C(\mathbb{R})$ satisfies the following conditions.

(g1) $\log|g(u)| = o(u^2)$ as $|u| \to \infty$.

(g2) $g(u) = o(u)$ as $u \to 0$.

(g3) There exists $\mu > 2$ such that $0 < \mu G(u) \leqslant g(u)u$ for any $u \neq 0$, with

$$G(u) = \int_0^u g(t)\,\mathrm{d}t.$$

Consider now the problem

$$\left.\begin{aligned} -\Delta u &= \frac{\lambda u + g(u)}{|x|^\alpha} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.2}$$

As we will see in the following theorems, the existence of positive variational solutions depends also on the range in which λ lies, as one expects. To this end, it is useful to introduce the value

$$\lambda_{1,\alpha}(\Omega) = \inf\left\{\int_\Omega |\nabla u|^2 : u \in C_c^\infty(\Omega),\ \int_\Omega \frac{u^2}{|x|^\alpha} = 1\right\}$$

defined for an arbitrary domain Ω in $\mathbb{R}^2$ and for any $\alpha \in \mathbb{R}$.

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Note that for a bounded domain Ω containing 0, $\lambda_{1,\alpha}(\Omega)=0$ if $\alpha \geqslant 2$ (see [5]), while $\lambda_{1,\alpha}(\Omega)>0$ and is attained by a positive function in $H_0^1(\Omega)$ if $\alpha<2$, since, in this case, $H_0^1(\Omega)$ is compactly imbedded in $L^2(\Omega, \mathrm{d}x/|x|^\alpha)$.

Moreover, by the Kelvin transform, one can easily see that if Ω is a domain in $\mathbb{R}^2$ and $\tilde{\Omega}=\{x\in\mathbb{R}^2\setminus\{0\} : x/|x|^2\in\Omega\}$, then $\lambda_{1,\alpha}(\tilde{\Omega})=\lambda_{1,4-\alpha}(\Omega)$.

Remark 3.2 (Evaluation of $\lambda_{1,\alpha}(B_1)$). Let $\Omega=B_1$ be the unit ball in $\mathbb{R}^2$ centred at 0. Let us prove that for every $\alpha<2$,

$$\lambda_{1,\alpha}(B_1)=(1-\tfrac{1}{2}\alpha)^2\lambda_1(B_1).$$

Indeed, fixing $\alpha<2$, there exists $u_\alpha\in H_0^1(B_1)$, $u_\alpha\geqslant 0$, such that

$$\int_{B_1}\frac{u_\alpha^2}{|x|^\alpha}=1 \quad\text{and}\quad \int_{B_1}|\nabla u_\alpha|^2=\lambda_{1,\alpha}(B_1).$$

Then, up to a multiplicative constant, u_α is the unique (classical) positive solution to the Dirichlet problem

$$\left.\begin{aligned} -\Delta u_\alpha &= \lambda\frac{u_\alpha}{|x|^\alpha} && \text{in } B_1,\\ u_\alpha &= 0 && \text{on } \partial B_1,\end{aligned}\right\} \tag{3.3}$$

with $\lambda=\lambda_{1,\alpha}(B_1)$. Considering in particular the case $\alpha=0$, it is known that the problem

$$\left.\begin{aligned} -\Delta u &= \lambda_1(B_1)u && \text{in } B_1,\\ u &= 0 && \text{on } \partial B_1\end{aligned}\right\}$$

admits a positive radially symmetric solution $u_0\in H_0^1(B_1)$. Then the function $u_\alpha(x)=u_0(x/|x|^{\alpha/2})$ solves (3.3) with $\lambda=(1-\frac{1}{2}\alpha)^2\lambda_1(B_1)$.

3.1. The case $\alpha\neq 2$

The next theorem describes a case in which compactness holds.

Theorem 3.3. *Let $g\in C(\mathbb{R})$ satisfy (g1)–(g3).*

(i) *If Ω is a bounded smooth domain and $\alpha<2$, then (3.2) admits a variational positive solution $u\in H_0^1(\Omega)$ if and only if $\lambda<\lambda_{1,\alpha}(\Omega)$. In this case, $u\in C^{1,\gamma}(\bar{\Omega})$ for $\gamma<1$, and $u>0$ in Ω. In particular, $u(0)>0$ when $0\in\Omega$.*

(ii) *If Ω is a domain with $0\notin\bar{\Omega}$ and $\alpha>2$, then (3.2) admits a variational positive solution u if and only if $\lambda<\lambda_{1,4-\alpha}(\Omega)$. In this case, $u\in C^{1,\gamma}(\bar{\Omega})$ for $\gamma<1$, $u>0$ in Ω. In particular, if Ω is an exterior domain, then $u(x)\to u_\infty>0$ as $|x|\to\infty$.*

Proof. (i) For every $u\in H_0^1(\Omega)$ let us set

$$J(u)=\int_\Omega\frac{G(u)}{|x|^\alpha}.$$

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Since $\alpha < 2$ and Ω is bounded, there exists $C > 0$ and $\sigma > 1$ such that

$$J(u)^{\sigma} \leqslant C \int_{\Omega} G(u)^{\sigma}.$$

Using (g1)–(g3), one easily checks that $G(u) = o(u^2)$ as $u \to 0$ and $\log G(u) = o(u^2)$ as $|u| \to \infty$. Therefore, thanks to the above estimate on J and to the Trudinger–Moser inequality [13], following a standard procedure, one infers that the functional J is well defined and continuous on $H_0^1(\Omega)$. In fact, using analogous estimates and standard arguments, one can see that J is Fréchet differentiable on $H_0^1(\Omega)$ and its gradient is completely continuous (see [12, § II.4]). This allows us to treat (3.2) with classical variational techniques. In particular, the necessity of $\lambda < \lambda_{1,\alpha}(\Omega)$ is obtained as usual by multiplying by the first eigenfunction. Instead, if $\lambda < \lambda_{1,\alpha}(\Omega)$, by assumptions (g1)–(g3), a variational positive solution $u \in H_0^1(\Omega)$ to (3.2) can be obtained as a mountain-pass critical point of the energy functional associated to (3.2). Moreover, classical regularity arguments apply to show that $u \in C^{1,\gamma}(\bar{\Omega})$ and $u > 0$ in Ω.

(ii) It is enough to observe that a function $u : \Omega \to \mathbb{R}$ is a variational positive solution to (3.2) if and only if, setting $\tilde{\Omega} = \{x \in \mathbb{R}^2 \setminus \{0\} : x/|x|^2 \in \Omega\}$, the function $\tilde{u} : \tilde{\Omega} \to \mathbb{R}$ defined by $\tilde{u}(x) = u(x/|x|^2)$ for $x \in \tilde{\Omega}$ is a variational positive solution to

$$\left.\begin{aligned} -\Delta \tilde{u} &= \frac{\lambda \tilde{u} + g(\tilde{u})}{|x|^{4-\alpha}} && \text{in } \tilde{\Omega}, \\ \tilde{u} &= 0 && \text{on } \partial\tilde{\Omega}. \end{aligned}\right\}$$

Then part (ii) follows by (i) and by the fact that $\lambda_{1,\alpha}(\tilde{\Omega}) = \lambda_{1,4-\alpha}(\Omega)$. □

For $\alpha \geqslant 2$ and Ω a bounded domain containing 0, the space $H_0^1(\Omega)$ is not contained in $L^2(\Omega, \mathrm{d}x/|x|^{\alpha})$ and in general a lack of compactness occurs even considering the space $H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}x/|x|^{\alpha})$. For instance, if $\alpha = 4$, then, using the Kelvin transform, problem (3.2) on a bounded domain containing 0 turns out to be equivalent to the non-compact problem

$$\left.\begin{aligned} -\Delta u &= \lambda u + g(u) && \text{in } \tilde{\Omega}, \\ u &= 0 && \text{on } \partial\tilde{\Omega}, \end{aligned}\right\}$$

where $\tilde{\Omega}$ is an exterior domain.

However, considering radially symmetric situations, we recover compactness and, because of theorem 2.2, we need a changing-sign nonlinearity, that is, we have to assume $\lambda < 0$.

THEOREM 3.4. *Let $g \in C(\mathbb{R})$ satisfy (g2) and (g3), and let $\lambda < 0$.*

(i) *If $\alpha < 2$ and $\Omega = \{x : |x| > 1\}$, then (3.2) admits a variational positive radial solution $u \in C^2(\Omega) \cap C_0^0(\bar{\Omega})$, $u > 0$ in Ω, $u(\infty) = 0$.*

(ii) *If $\alpha > 2$ and $\Omega = \{x : |x| < 1\}$, then (3.2) admits a variational positive radial solution $u \in C^2(\Omega \setminus \{0\}) \cap C_0^0(\bar{\Omega})$, $u > 0$ in $\Omega \setminus \{0\}$, $u(0) = 0$.*

Proof. (i) Setting $u(x) = v(|\lambda|^{1/2} \log |x|)$ and $t = |\lambda|^{1/2} \log |x|$, we can see that u is a variational radial solution to (3.2) in $\Omega = \{x : |x| > 1\}$ if $v : \mathbb{R}^+ \to \mathbb{R}$ satisfies

$$-\ddot{v} + \mathrm{e}^{\beta t} v = \mathrm{e}^{\beta t} \bar{g}(v) \quad \text{for} \quad t > 0, \qquad v \in H_0^1(\mathbb{R}^+), \tag{3.4}$$

where $\beta = (2 - \alpha)/|\lambda|^{1/2}$ and $\bar{g} = |\lambda|^{1/2} g$. In particular, let us point out that

$$\int_\Omega |\nabla u|^2 = 2\pi |\lambda|^{1/2} \int_0^\infty \dot{v}^2.$$

To show the existence of a positive solution to (3.4), let us introduce the following variational setting. Let

$$X = \left\{ v \in H_0^1(\mathbb{R}^+) : \int_0^\infty \mathrm{e}^{\beta t} v^2 < \infty \right\},$$

endowed with the inner product

$$\langle v_1, v_2 \rangle = \int_0^\infty (\dot{v}_1 \dot{v}_2 + \mathrm{e}^{\beta t} v_1 v_2)$$

and with the corresponding norm $\|v\| = \langle v, v \rangle^{1/2}$. One can check that X is a Hilbert space, continuously imbedded in $H_0^1(\mathbb{R}^+)$, and then in $L^q(\mathbb{R}^+)$ for every $q \in [2, \infty]$. We claim that X is compactly imbedded into $L^\infty(\mathbb{R}^+)$. Indeed, let $(v_n) \subset X$ be such that $v_n \to 0$ weakly in X, and suppose, by contradiction, that $\limsup \|v_n\|_{L^\infty} \geqslant 2\delta > 0$. Then, passing to a subsequence, there exist $(s_n), (t_n) \subset \mathbb{R}^+$ such that $s_n < t_n$, $|v_n(s_n)| = \delta$, $|v_n(t_n)| = 2\delta$ and $\delta < |v_n(t)| \leqslant 2\delta$ for $t \in (s_n, t_n)$. Let $\tau_n = t_n - s_n$. On one hand, we have that

$$\delta \leqslant \int_{s_n}^{t_n} |\dot{v}_n| \leqslant \tau_n^{1/2} \biggl(\int_{s_n}^{t_n} \dot{v}_n^2 \biggr)^{1/2} \leqslant \tau_n^{1/2} \|v_n\|$$

and then, since (v_n) is bounded, $\inf \tau_n > 0$. On the other hand,

$$\|v_n\|^2 \geqslant \int_{s_n}^{t_n} \mathrm{e}^{\beta t} v_n^2 \geqslant \frac{\delta^2}{\beta} (\mathrm{e}^{\beta t_n} - \mathrm{e}^{\beta s_n}) = \delta^2 \mathrm{e}^{\beta \sigma_n} \tau_n$$

for some $\sigma_n \in (s_n, t_n)$. Since $\inf \tau_n > 0$, we obtain that (σ_n) is bounded and then (s_n) too. Therefore, $\inf_n |v_n(s_n)| = \delta > 0$, in contrast to the fact that $v_n \to 0$ uniformly on $[0, t]$ for any $t > 0$.

Now the existence of a positive solution to (3.4) is obtained with standard variational techniques. Indeed, let

$$\bar{G}(u) = \int_0^u \bar{g}(s) \, \mathrm{d}s$$

and, for every $v \in X$, let

$$I(v) = \tfrac{1}{2} \|v\|^2 - \int_0^\infty \mathrm{e}^{\beta t} \bar{G}(v).$$

By (g2), $I \in C^1(X)$ and the critical points of I are solutions to (3.4), of class C^2 on $\mathbb{R}^+$. Then, using (g2) and (g3), one can check that the functional I has a

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mountain-pass geometry. Moreover, thanks to the compact embedding of X into $L^\infty(\mathbb{R}^+)$, the Palais Smale condition holds. Hence I admits a critical point $v \in X$, with $I(v) > 0$. Finally, with standard arguments, changing $\bar{g}$ by 0 on $\mathbb{R}^-$, we can conclude that $v > 0$ in $\mathbb{R}^+$. Therefore, part (i) is proved.

(ii)Arguing as in the proof of theorem 3.3, one has that the statement (ii) is in fact equivalent, by the Kelvin transform, to part (i). □

3.2. The case $\alpha = 2$

In this subsection we investigate some special features occurring when $\alpha = 2$.

We remark that, in addition to the invariance under Kelvin transform (see lemma 2.5), the case $\alpha = 2$ exhibits also invariance under dilation.

Let us consider first the problem on $\mathbb{R}^2$ associated to the equation

$$-\Delta u = \frac{\lambda u + g(u)}{|x|^2} \quad \text{in } \mathbb{R}^2. \tag{3.5}$$

By corollary 2.3, we know that (3.5) has no weak positive supersolution for $\lambda \geqslant 0$. On the other hand, we can state the following existence result.

Theorem 3.5. *Let $g \in C(\mathbb{R})$ satisfy (g2) and (g3), and let $\lambda < 0$. Then (3.5) admits a variational positive radial solution $u \in C^2(\mathbb{R}^2 \setminus \{0\}) \cap C^0(\mathbb{R}^2)$, $u > 0$ in $\mathbb{R}^2 \setminus \{0\}$, $u(0) = u(\infty) = 0$.*

Proof. Arguing as in part (i) of the proof of theorem 3.4, we have that u is a variational radial solution to (3.2) in $\mathbb{R}^2$ if and only if $u(x) = v(|\lambda|^{1/2} \log |x|)$ and v satisfies

$$-\ddot{v} + v = \bar{g}(v) \quad \text{in } \mathbb{R}, \quad v \in H^1(\mathbb{R}), \tag{3.6}$$

where $\bar{g} = |\lambda|^{1/2} g$. Now standard ODE or variational arguments show that (3.6) admits a positive solution $v \in C^2(\mathbb{R})$ and $v(\pm\infty) = 0$. Thus the proof is complete. □

The possibility of also finding a variational positive solution to

$$\left.\begin{aligned} -\Delta u &= \frac{\lambda u + g(u)}{|x|^2} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.7}$$

when $\lambda < 0$ and Ω is a bounded domain containing 0 (as it happens if $\alpha \neq 2$) is prevented, at least when Ω is star shaped, by the following Pohozaev-type result that, in fact, holds without any sign assumption on u.

Lemma 3.6. *Let Ω be a bounded smooth domain containing* 0*, and let $f \in C(\mathbb{R})$. If u is a variational solution to (3.1) with $\alpha = 2$, and if $F(u)/|x|^2 \in L^1(\Omega)$, where*

$$F(u) = \int_0^u f(s)\,\mathrm{d}s,$$

then

$$\int_{\partial\Omega} |\nabla u|^2 x \cdot \nu = 0$$

(here ν is the outward normal at $\partial\Omega$). If, in particular, Ω is star shaped, then $u = 0$.

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Proof. Let $\chi_n \in C^\infty(\mathbb{R}^2)$ be a non-negative radial function such that $\chi_n(x) = 0$ for $|x| \leqslant 1/n$, $\chi_n(x) = 1$ for $|x| \geqslant 2/n$, and $|\nabla\chi_n| \leqslant Cn$ on $\mathbb{R}^2$. By standard arguments, u is regular enough far from 0, and then

$$\int_\Omega \nabla u \cdot \nabla(\chi_n x \cdot \nabla u) = \int_\Omega \frac{f(u)}{|x|^2}\chi_n x \cdot \nabla u. \tag{3.8}$$

On one hand, we have that

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla(\chi_n x \cdot \nabla u) &= \int_\Omega (\nabla u \cdot \nabla\chi_n)(x \cdot \nabla u) \\ &\quad - \frac{1}{2}\int_\Omega (\nabla\chi_n \cdot x)|\nabla u|^2 + \frac{1}{2}\int_\Omega \operatorname{div}(\chi_n x|\nabla u|^2). \end{aligned} \tag{3.9}$$

By the divergence theorem,

$$\int_\Omega \operatorname{div}(\chi_n x|\nabla u|^2) = \int_{\partial\Omega} |\nabla u|^2 x \cdot \nu.$$

Moreover, since

$$\int_\Omega |x||\nabla\chi_n||\nabla u|^2 \leqslant C\int_{1/n<|x|<2/n} |\nabla u|^2$$

and $|\nabla u| \in L^2(\Omega)$, equation (3.9) gives

$$\int_\Omega \nabla u \cdot \nabla(\chi_n x \cdot \nabla u) = \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 x \cdot \nu + o(1), \tag{3.10}$$

where $o(1) \to 0$ as $n \to \infty$. On the other hand, since $F(0) = 0$ and $u = 0$ on $\partial\Omega$, we have that

$$\begin{aligned}\int_\Omega \frac{f(u)}{|x|^2}\chi_n x \cdot \nabla u &= \int_{\partial\Omega} \frac{F(u)}{|x|^2}\chi_n x \cdot \nu - \int_\Omega F(u)\operatorname{div}\left(\chi_n \frac{x}{|x|^2}\right) \\ &= -\int_\Omega \frac{F(u)}{|x|^2}\nabla\chi_n \cdot x.\end{aligned}$$

Noting that

$$\int_\Omega \frac{|F(u)|}{|x|^2}|\nabla\chi_n \cdot x| \leqslant C\int_{1/n<|x|<2/n} \frac{|F(u)|}{|x|^2},$$

and, by assumption, $F(u)/|x|^2 \in L^1(\Omega)$, we conclude that

$$\int_\Omega \frac{f(u)}{|x|^2}\chi_n x \cdot \nabla u \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

Finally, equations (3.8), (3.10) and (3.11) imply the thesis. □

Using lemma 3.6 we obtain the following non-existence result. Note that also in this case we do not require any condition on the sign of u.

Corollary 3.7. *Let $g \in C(\mathbb{R})$ satisfy (g1) and (g2), $\lambda \in \mathbb{R}$ and let Ω be a smooth bounded star-shaped domain in $\mathbb{R}^2$ containing 0. If u is a variational solution to (3.7) such that $u^2/|x|^2 \in L^1(\Omega)$, then $u = 0$.*

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Proof. Let X be the closure of $C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^2} (|\nabla u|^2 + \frac{u^2}{|x|^2}) \right)^{1/2}.$$

If $u \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ and $v(s,\theta) = u(\mathrm{e}^s \cos\theta, \mathrm{e}^s \sin\theta)$, then

$$\int_{\mathbb{R}^2} |\nabla u|^2 = \int_{\Sigma} |\nabla v|^2 \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} = \int_{\Sigma} v^2,$$

where $\Sigma = \mathbb{R} \times (0, 2\pi)$. This shows that X is isomorphic to $H^1(\Sigma)$. Under the assumptions (g1) and (g2), it holds that the functional

$$v \mapsto \int_{\Sigma} G(v)$$

is well defined and continuous on $H^1(\Sigma)$. Thus, since

$$\int_{\mathbb{R}^2} \frac{G(u)}{|x|^2} = \int_{\Sigma} G(v),$$

the functional

$$u \mapsto \int_{\Sigma} \frac{G(u)}{|x|^2}$$

is well defined and continuous on X. In particular, if u satisfies the assumptions of the corollary, then $G(u)/|x|^2 \in L^1(\Omega)$ and thus, since $u^2/|x|^2 \in L^1(\Omega)$, $F(u)/|x|^2 \in L^1(\Omega)$. Finally, the corollary follows by applying lemma 3.6. □

4. The Dirichlet problem on cones

In this section we study the Dirichlet problem (3.2) when Ω is a proper cone with vertex at 0, including the case of a half-plane, or $\mathbb{R}^2 \setminus \Gamma$, where $\Gamma = \{(x_1, 0) : x_1 \geqslant 0\}$.

The special role played by the exponent $\alpha = 2$ in the study of (2.1) is in fact emphasized in a still more striking way, in the case of a domain like a cone, as stated in the next result. Note that the cones with vertex at 0 are all the domains in $\mathbb{R}^2$, which are invariant under dilation and Kelvin transform.

LEMMA 4.1. *Let Ω be a cone in $\mathbb{R}^2$ with vertex at 0 and let $f \in C(\mathbb{R})$ satisfy (f1). If u is a variational solution to (3.1) such that $F(u)/|x|^\alpha \in L^1(\Omega)$ (where $F(u) = \int_0^u f(s)\,\mathrm{d}s$) and*

$$\int_{\Omega} \frac{F(u)}{|x|^\alpha} \neq 0,$$

then $\alpha = 2$.

Proof. Let $\chi_n \in C^\infty(\mathbb{R}^2)$ be a non-negative radial function such that $\chi_n(x) = 0$ for $|x| \leqslant 1/n$ and $|x| > 2n$, $\chi_n(x) = 1$ for $2/n \leqslant |x| \leqslant n$, $|\nabla \chi_n| \leqslant Cn$ for $1/n \leqslant |x| 2/n$, and $|\nabla \chi_n| \leqslant C/n$ for $n \leqslant |x| \leqslant 2n$. Arguing as in the proof of lemma 3.6, we can test the equation $-\Delta u = f(u)/|x|^\alpha$ with $\chi_n x \cdot \nabla u$, to get

$$\int_{\Omega} \nabla u \cdot \nabla(\chi_n x \cdot \nabla u) = \int_{\Omega} \frac{f(u)}{|x|^\alpha} \chi_n x \cdot \nabla u. \tag{4.1}$$

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The left-hand side of (4.1) can be estimated with similar calculations as in (3.9), (3.10), obtaining that

$$\int_\Omega \nabla u \cdot \nabla(\chi_n x \cdot \nabla u) = \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \chi_n x \cdot \nu + o(1), \tag{4.2}$$

where $o(1) \to 0$ as $n \to \infty$. As regards the right-hand side of (4.1), we have

$$\begin{aligned}\int_\Omega \frac{f(u)}{|x|^\alpha}\chi_n x \cdot \nabla u &= \int_\Omega F(u)\,\mathrm{div}\biggl(\chi_n \frac{x}{|x|^\alpha}\biggr) \\ &= (2-\alpha)\int_\Omega \frac{F(u)}{|x|^\alpha}\chi_n - \int_\Omega \frac{F(u)}{|x|^\alpha} x \cdot \nabla\chi_n. \end{aligned}\tag{4.3}$$

Using the fact that $F(u)/|x|^\alpha \in L^1(\Omega)$, equation (4.3) implies

$$\int_\Omega \frac{f(u)}{|x|^\alpha}\chi_n x \cdot \nabla u = (2-\alpha)\int_\Omega \frac{F(u)}{|x|^\alpha} + o(1). \tag{4.4}$$

Finally, by (4.1), (4.2), (4.4), and since $x \cdot \nu = 0$ on the boundary of a cone, we infer that u satisfies

$$(2-\alpha)\int_\Omega \frac{F(u)}{|x|^\alpha} = 0$$

that implies the thesis. □

According to lemma 4.1, when Ω is a cone, we are lead to study just the case $\alpha = 2$. Moreover, by the rotational invariance of (4.5), we can reduce ourselves to consider cones of the form

$$\Omega_\theta = \{(\rho\cos\tau, \rho\sin\tau) : \rho > 0,\ 0 < \tau < \theta\},$$

where $\theta \in (0, 2\pi]$. As we will see in the next proposition, it is convenient to introduce the space

$$D_0^1(\Omega_\theta) = \text{closure of } C_c^\infty(\Omega_\theta) \text{ with respect to the norm } \|u\| = \|\nabla u\|_{L^2}$$

and the values

$$S_q(\Omega_\theta) = \inf\biggl\{\int_{\Omega_\theta} |\nabla u|^2\,\mathrm{d}x : u \in C_c^\infty(\Omega_\theta),\ \int_{\Omega_\theta} \frac{|u|^q}{|x|^2} = 1\biggr\},$$

where $q \geqslant 2$. Note that $S_2(\Omega_\theta) = \lambda_{1,2}(\Omega_\theta)$, according to the definition given in § 2.

Proposition 4.2. *For every $\theta \in (0, 2\pi]$, $(D_0^1(\Omega_\theta), \|\cdot\|)$ is a Hilbert space isomorphic to $H_0^1(\Sigma_\theta)$, where $\Sigma_\theta = \mathbb{R} \times (0, \theta)$. Moreover,*

(i) $\lambda_{1,2}(\Omega_\theta) = \pi^2/\theta^2$;

(ii) *for any $q > 2$, the value $S_q(\Omega_\theta)$ is positive and is attained by a positive function $u \in D_0^1(\Omega_\theta)$.*

Proof. The mapping $\phi : \Sigma_\theta \to \Omega_\theta$, defined by $\phi(s,\tau) = (\mathrm{e}^s \cos\tau, \mathrm{e}^s \sin\tau)$, is a diffeomorphism from Σ_θ onto Ω_θ with Jacobian given by $|J_\phi(s,\tau)| = \mathrm{e}^{2s}$. Furthermore, if $u \in C_c^\infty(\Omega_\theta)$ and $v = u \circ \phi$, then

$$\int_{\Omega_\theta} |\nabla u|^2 = \int_{\Sigma_\theta} |\nabla v|^2 \quad \text{and} \quad \int_{\Omega_\theta} \frac{|u|^q}{|x|^2} = \int_{\Sigma_\theta} |v|^q.$$

Since the Poincaré inequality holds in $D_0^1(\Sigma_\theta)$, we infer that $D_0^1(\Sigma_\theta) = H_0^1(\Sigma_\theta)$ and $\lambda_{1,2}(\Omega_\theta) = \lambda_1(\Sigma_\theta)$, that is, the 'first eigenvalue' for $-\Delta$ in $H_0^1(\Sigma_\theta)$. Hence $D_0^1(\Omega_\theta)$ turns out to be a Hilbert space endowed with the norm $\|\nabla u\|_{L^2}$ and the mapping $\Phi : D_0^1(\Omega_\theta) \to D_0^1(\Sigma_\theta)$, defined by $\Phi(u) = u \circ \phi$, is an isomorphism of Hilbert spaces. Then, thanks to the isomorphism Φ, parts (ii) and (iii) follow by well-known results. Finally, the equality $\lambda_{1,2}(\Omega_\theta) = \pi^2/\theta^2$ is obtained by using the known fact that $\lambda_1(\Sigma_\theta) = \lambda_1(I_\theta) = \pi^2/\theta^2$, where $I_\theta = (0,\theta)$. □

Remark 4.3.

(i) We point out the difference between the case $\Omega = \mathbb{R}^2$, for which the Hardy inequality fails, and $\Omega = \Omega_\theta$ (including the case $\Omega = \mathbb{R}^2 \setminus \Gamma$, with $\Gamma = \{x \in \mathbb{R}^2 : x_1 \geqslant 0\}$), for which the Hardy inequality holds, since $\lambda_{1,2}(\Omega_\theta) > 0$. In fact, as it is clear from the proof of proposition 4.2, the Hardy inequality on Ω_θ is equivalent to the Poincaré inequality on Σ_θ.

(ii) By proposition 4.2, for any $\lambda < \lambda_{1,2}(\Omega_\theta)$, the functionals

$$u \mapsto \left(\int_{\Omega_\theta} |\nabla u|^2 - \lambda \frac{u^2}{|x|^2} \right)^{1/2}$$

define equivalent Hilbertian norms in $D_0^1(\Omega_\theta)$.

Proposition 4.2 allows us to tackle the Dirichlet problem

$$\left.\begin{aligned} -\Delta u &= \frac{\lambda u + g(u)}{|x|^2} && \text{in } \Omega_\theta, \\ u &= 0 && \text{on } \partial\Omega_\theta \end{aligned}\right\} \tag{4.5}$$

by using variational methods.

Theorem 4.4. *Let $g \in C(\mathbb{R})$ satisfy (g1)–(g3).*

(i) *Given $\theta \in (0, 2\pi]$, problem (4.5) on Ω_θ admits a positive solution if $\lambda < \lambda_{1,2}(\Omega_\theta)$.*

(ii) *For every $\lambda \in \mathbb{R}$, there exists $\theta_\lambda \in (0, 2\pi]$ such that, for every $\theta \in (0, \theta_\lambda)$, problem (4.5) on Ω_θ admits a positive solution.*

(iii) *Given $\theta \in (0, 2\pi]$, for every $\lambda \in \mathbb{R}$, problem (4.5) on Ω_θ admits infinitely many solutions changing sign.*

Proof. Let $\theta \in (0, 2\pi]$ be fixed. Using the isomorphism $\Phi : D_0^1(\Omega_\theta) \to H_0^1(\Sigma_\theta)$ defined in the proof of proposition 4.2, one can easily check that a function $u \in D_0^1(\Omega_\theta)$ is a solution to (4.5) if and only if $v = \Phi(u)$ is a solution to

$$-\Delta v = \lambda v + g(v) \quad \text{in } \Sigma_\theta, \quad v \in H_0^1(\Sigma_\theta), \tag{4.6}$$

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where $\Sigma_\theta = \mathbb{R} \times (0, \theta)$. Under the assumptions (g1)–(g3), standard variational arguments apply to prove the existence of a positive solution to problem (4.6) if $\lambda < \lambda_1(\Sigma_\theta) = \lambda_{1,2}(\Omega_\theta)$.

(ii) By proposition 4.2 (ii), for every $\lambda \in \mathbb{R}$, there exists $\theta_\lambda \in (0, 2\pi]$ such that $\lambda_{1,2}(\Omega_\theta) > \lambda$ if $\theta \in (0, \theta_\lambda)$. Then, by (i), problem (4.5) admits a positive solution.

(iii) Given $\theta \in (0, 2\pi]$ and $\lambda \in \mathbb{R}$, by proposition 4.2 (ii), there exists $k_0 \in \mathbb{N} \cup \{0\}$ such that $\lambda_{1,2}(\Omega_{\theta_k}) > \lambda$ for any $k \geqslant k_0$, where $\theta_k = \theta/2^k$. Fix $k \geqslant k_0$ and, setting $\Sigma_k = \Sigma_{\theta_k}$, consider the positive solution v of (4.6) on Σ_k obtained in part (i). Note that, by regularity theory, $v \in C^{1,\gamma}(\bar{\Sigma}_k)$ and $v = 0$ on $\partial \Sigma_k$. We want to construct a solution to (4.6) on Σ_θ, and then, by the isomorphism Φ, a solution to (4.5) on Ω_θ, by gluing together suitable translations of $\pm v$, with an 'accordion' procedure. Precisely, let $\theta_i = \theta 2^{i-k}$, for $i = 0, \dots, k$, $v_0 = v$ and, for $i = 1, \dots, k$,

$$v_i(s, \tau) = \begin{cases} v_{i-1}(s, \tau) & \text{as } \tau \in (0, \theta_{i-1}], \quad s \in \mathbb{R}, \\ -v_{i-1}(s, \tau - \theta_{i-1}) & \text{as } \tau \in (\theta_{i-1}, \theta_i), \ \ s \in \mathbb{R}. \end{cases}$$

By construction, v_k turns out to be a solution to (4.6) on Σ_θ having 2^{k-1} positive components. Since this procedure can be repeated for any $k \geqslant k_0$, part (iii) is proved. □

In the special case in which $\theta = 2\pi$, we have the following multiplicity result on $\mathbb{R}^2$, to compare with corollary 2.3 and theorem 3.5.

COROLLARY 4.5. *Let $g \in C(\mathbb{R})$ satisfy (g1)–(g3). Then, for every $\lambda \in \mathbb{R}$, equation (3.5) admits infinitely many solutions changing sign in $D_0^1(\mathbb{R}^2 \setminus \Gamma)$.*

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