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ADDITIVITY OF THE P^n-INTEGRAL (2)

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1. Introduction. The problem of additivity of the P^n-integral on abutting intervals was considered in [**2**] and in [**5**]. It was noted in [**2**] that the necessary and sufficient conditions for additivity for the P^2-integral obtained by Skvorcov in [**5**] could be completely generalized to the P^n-integral, $n > 2$, if a key lemma (corresponding to Skvorcov's Lemma 3 [**6**]) could be proved. We provide a proof of that lemma in this paper and hence obtain the general additivity result.

The definitions and notation of [**2**] are used in the following, except that we shall take the following as the definition of P^n-major and minor functions:

Definition 1.1. Let $f(x)$ be a function defined in $[a, b]$ and let a_1, $i = 1, 2, \ldots, n$, be fixed points such that $a = a_1 < a_2 < \ldots < a_n = b$. The functions $Q(x)$ and $q(x)$ are called *P^n-major and minor functions* respectively of $f(x)$ over $(a_i) = (a_1, a_2, \ldots, a_n)$ if

(1.4) $Q(x)$ and $q(x)$ satisfy condition $A_n{}^*$ in $[a, b]$;

(1.5 $Q(a_i) = q(a_i) = 0$, $i = 1, 2, \ldots, n$;

(1.6) $\partial^n Q(x) \geqq f(x) \geqq \Delta^n q(x)$, $x \in (a, b) - E$, $|E| = 0$;

(1.7) $\partial^n Q(x) \neq -\infty$, $\Delta^n q(x) \neq +\infty$, $x \in (a, b) - S$, S a scattered set;

(1.8) Q and q are n-smooth in S.

(Condition (1.8) is stronger than the corresponding condition in [**3**] and [**2**] but seems more natural. Compare the corresponding smoothness conditions in [**4**] and [**6**].)

2. Main results. The property of additivity of the P^n-integral may be stated as follows:

THEOREM 2.1. *Let $f(x)$ be P^n-integrable over $(a_i; x)$, where*

$$A_1 \equiv \{a_i\} = (a, d_1, c_2, d_2, c_3, \ldots, d_{(n/2)-1}, c_{n/2}, d_{n/2}),$$

$(d_{n/2} = c)$, with associated integral $F_1(x)$ and over $(b_i; x)$, where

$$A_2 \equiv \{b_i\} = (d_{n/2}, c_{(n/2)+1}, d_{(n/2)+1}, c_{(n/2)+2}, \ldots, c_{n-1}, d_{n-1}, b)$$

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with associated integral $F_2(x)$. *Then* $f(x)$ *is* P^n*-integrable on* $[a, b]$ *if and only if there exist constants* $\{\theta_j\}$, $j = 1, 2, \ldots, n - 1$, *such that the function*

$$F(x) = \begin{cases} F_1(x) + \sum_{j=1}^{n/2} \lambda(A_1; x, d_j)\theta_j, & a \leqq x \leqq c. \\ F_2(x) + \sum_{j=n/2}^{n-1} \lambda(A_2; x, d_j)\theta_j, & c \leqq x \leqq b, \end{cases}$$

(*where for a set* $A = \{x_0, x_1, \ldots, x_n\}$ *of distinct numbers*,

$$\lambda(A; x, x_r) \equiv \lambda(x, x_r) \equiv \prod_{i \neq r} \left(\frac{x - x_i}{x_r - x_i}\right), \quad r = 0, 1, \ldots, n),$$

is n*-smooth and possesses Peano unsymmetric derivatives up to order* $(n - 2)$ *at* $x = c$. *If such numbers exist then the function* $F(x)$ *is the associated* P^n*-integral of* $f(x)$ *over* $(a, c_2, c_3, \ldots, c_{n-1}, b)$.

The following result is crucial to our construction in the proof of Lemma 2.2.

THEOREM 2.2. *If* $G(x)$ *is* n*-convex on* $[a, b]$ *and* $G(a_i) = 0$, $i = 1, 2, \ldots, n$, *where* $a = a_1 < a_2 < \ldots < a_n = b$, *then*

$$G_{(n-1),-}(b) \geqq 0 \text{ and } G_{(n-1),+}(a) \leqq 0.$$

Proof. Since $G(x)$ is n-convex and has zeros at $a_1, a_2, \ldots, a_{n-1}$, and a_n, the graph of G lies alternately above and below the x-axis, lying below if $a_{n-1} \leqq x \leqq a_n = b$ (Theorem 5, [**1**]). We may choose points x_i, $a_i < x_i < a_{i+1}$, $i = 1, 2, \ldots, n - 1$, and $x_{n-1} < x_n < b$, so that

$$G(x_k)/w'(x_k) > 0, \; k = 1, 2, \ldots, n - 1,$$

where

$$w(x) = \prod_{k=1}^{n} (x - x_k),$$

and x_n is close enough to b so that

$$\sum_{k=1}^{n} \frac{G(x_k)}{w'(x_k)} \geqq 0.$$

It then follows from Theorem 7 [**1**] that

$$G_{(n-1),-}(x) \geqq 0 \text{ for } x_n \leqq x < b,$$

and consequently

$$G_{(n-1),-}(b) \geqq 0.$$

Similarly it may be proved that

$$G_{(n-1),+}(a) \leqq 0.$$

LEMMA 2.1. *If $f(x)$ is P^n-integrable with respect to the basis (a_i) on $[a, b]$ then it is P^n-integrable on each interior interval $[c, d]$, $a < c < d < b$. Furthermore, given $\epsilon > 0$, there exists a major function $Q(x)$ and a minor function $q(x)$ for $f(x)$ on $[c, d]$, such that, if $F(x)$ denotes a P^n-integral of $f(x)$ on $[c, d]$ (with respect to some basis (b_i)), $R(x) = Q(x) - F(x)$ and $r(x) = F(x) - q(x)$, then*

$$|R(x)| < \epsilon,\ |r(x)| < \epsilon,\ |R_{(k),+}(c)| < \epsilon,\ |r_{(k),+}(c)| < \epsilon,$$
$$|R_{(k),-}(d)| < \epsilon, \text{ and } |r_{(k),-}(d)| < \epsilon, \text{ for } 1 \leqq k \leqq n-1.$$

Proof. Let

$$B = \sup_i \sup_{0 \leqq k \leqq n-1} \{\lambda^{(k)}(x; b_i)|_{x=c}\},$$

and

$$C = \sup_{1 \leqq k \leqq n-1} \sup \left\{\frac{1}{(b-c)^k}, \frac{1}{(c-a)^k}\right\}.$$

Choose K such that $\epsilon/2 > \sup(K, KAC, BnK)$ where A is the constant determined in Corollary 8(b), [**1**]. Then pick a P^n-major function $Q_1(x)$ for $f(x)$ on $[a, b]$ with respect to the basis $\{a_i\}$ such that if $F_1(x)$ is the P^n-integral of $f(x)$ on $[a, b]$ with respect to the basis $\{a_i\}$, then we have

$$|R_1(x)| < K < \epsilon/2$$

where

$$R_1(x) = Q_1(x) - F_1(x).$$

Define the function R on $[c, d]$ by

$$R(x) = R_1(x) - \sum_{i=1}^{n} \lambda(x; b_i) R_1(b_i).$$

Because of the choice of K, $R(x)$ is seen to satisfy the required inequalities and thus the function Q defined by

$$Q(x) = Q_1(x) - \sum_{i=1}^{n} \lambda(x; b_i) Q_1(b_i)$$

is the major function required.

In a similar way a minor function with the required properties may be shown to exist.

It follows incidentally that the P^n-integral of $f(x)$ on $[c, d]$ with respect to the basis (b_i) is the function F defined by

$$F(x) = F_1(x) - \sum_{i=1}^{n} \lambda(x; b_i) F_1(b_i).$$

LEMMA 2.2. *Suppose $f(x)$ is P^n-integrable with respect to the basis $\{a_i\}$ on $[a, b]$, and let $F(x)$ be the associated P^n-integral with respect to the basis*

$\{a_i\}$ on $[a, b]$. Then corresponding to $\epsilon > 0$ there is a P^n-major function $Q(x)$ and a P^n-minor function $q(x)$ such that if $R(x) = Q(x) - F(x)$ and $r(x) = F(x) - q(x)$, we have

$$|R(x)| < \epsilon,\ |r(x)| < \epsilon,\ |R_{(k),+}(a)| < \epsilon,$$
$$|R_{(k),-}(b)| < \epsilon,\ |r_{(k),+}(a)| < \epsilon,\ |r_{(k),-}(b)| < \epsilon,$$
$$1 \leqq k \leqq n - 1.$$

Proof. Let

$$K_1 = \max_{1\leqq i\leqq n} \sup_{x\in[a,b]} \lambda(x; a_i),$$
$$K_2 = \sup_{1\leqq k\leqq n-1} \lambda_{(k),+}(a; a_i),$$
$$K_3 = \sup_{1\leqq k\leqq n-2} (b - a)^k/k!.$$

Suppose $\{\alpha_k\}_{k=1}^{\infty}$, $\{\beta_k\}_{k=1}^{\infty}$ are two sequences of points in the interval $[a, b]$ such that

$$\alpha_1 < \beta_1 < \beta_2 < \ldots < \beta_k < \ldots,$$
$$\alpha_1 > \alpha_2 > \alpha_2 < \ldots > \alpha_k > \ldots,$$

and $\lim_{k\to+\infty} \alpha_k = a$, $\lim_{k\to+\infty} \beta_k = b$.

Let $\{\epsilon_k\}$ be a sequence of positive numbers such that

$$\lim_{k\to+\infty} \frac{\epsilon_k}{(b - \beta_k)^j} = 0 \quad \text{and}$$

$$\lim_{k\to+\infty} \frac{\epsilon_k}{(\alpha_k - a)^j} = 0, \quad j = 1, 2, \ldots, (n - 1),$$

$$\sum_{k=1}^{\infty} \epsilon_k < \min\left\{\frac{(\epsilon)(n-2)!}{8(b-a)^{n-1}(1 + nK_1)}, \frac{\epsilon(n-2)!}{16(b-a)^{n-1}nK_2}, \frac{\epsilon}{16K_3}\right\},$$

$$\epsilon_k < \min\left(\epsilon/4, \frac{\epsilon}{2nK_2}, \frac{\epsilon}{4nK_1}\right), \quad k = 1, 2, \ldots.$$

For the closed interval $[\alpha_1, \beta_1]$ and $\epsilon = \epsilon_1$, construct a function $R_1(x)$ corresponding to the function $R(x)$ of Lemma 2.1. Similarly for the closed intervals $[\alpha_k, \alpha_{k-1}]$ and $[\beta_{k-1}, \beta_k]$ and $\epsilon = \epsilon_k$, $k \geqq 2$, construct functions $\bar{R}_k(x)$ and $R_k(x)$ corresponding to the function $R(x)$ of Lemma 2.1. Then define the function $\bar{R}^0(x)$ on $[a, b]$ by

$$\bar{R}^0(x) = \begin{cases} \bar{R}_k(x), x \in [\alpha_k, \alpha_{k-1}], & k = 2, 3, \ldots \\ R_k(x), x \in [\beta_{k-1}, \beta_k], & k = 2, 3, \ldots \\ R_1(x), x \in [\alpha_1, \beta_1] \\ 0, x = a, x = b. \end{cases}$$

It is easy to verify that

$$\bar{R}^0_{(j),+}(a) = \bar{R}^0_{(j),-}(b) = 0,\ 1 \leqq j \leqq n - 1.$$

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Now construct a function $p(x)$, constant on the intervals (α_1, β_1), (α_k, α_{k-1}), and (β_{k-1}, β_k), $k \geqq 2$, such that $p(a) = 0$ and its jump at a point of discontinuity d is equal to $\bar{R}^0_{(n-1),-}(d) - \bar{R}^0_{(n-1),+}(d)$. Since the functions $R_1(x)$, $\bar{R}_k(x)$, and $R_k(x)$ are n-convex on their respective intervals of definition, it follows from Theorem 2.2 that

$$\bar{R}^0_{(n-1),-}(d) \geqq \bar{R}^0_{(n-1),+}(d),$$

and $p(x)$ is monotonic increasing on $[a, b]$.

By construction we have that the jump in the function $p(x)$ at α_k and β_k is not more than $\epsilon_k + \epsilon_{k+1}$. Moreover

$$0 \leqq p(x) \leqq p(b) < 4 \sum_{k=1}^{\infty} \epsilon_k.$$

Now define $G(x)$ as the $(n-1)^{\text{th}}$ indefinite integral of $p(x)$ on the interval $[a, b]$:

$$G(x) = \frac{1}{(n-2)!} \int_a^x (x-t)^{n-2} p(t) dt,$$

and let the function L be defined on $[a, b]$ by

$$L(x) = G(x) - \sum_{i=1}^{n} \lambda(x; a_i) G(a_i).$$

Then $L(a_i) = 0$, $i = 1, 2, \ldots, n$,

$$|L(x)| \leqq \frac{4(b-a)^{n-1}}{(n-2)!} \sum_{k=1}^{\infty} \epsilon_k + \frac{4nK_1}{(n-2)!} (b-a)^{n-1} \sum_{k=1}^{\infty} \epsilon_k$$

$$= (1 + nK_1) \left[\frac{4(b-a)^{n-1}}{(n-2)!} \right] \sum_{k=1}^{\infty} \epsilon_k < \epsilon/2, \quad a \leqq x \leqq b,$$

$$|L_{(k),+}(a)| \leqq |G_{(k),+}(a)| + \sum_{i=1}^{n} |\lambda_{(k),+}(a; a_i)| \, |G(a_i)|$$

$$\leqq 4K_3 \sum_{k=1}^{\infty} \epsilon_k + (nK_2) \frac{4(b-a)^{n-1}}{(n-2)!} \sum_{k=1}^{\infty} \epsilon_k$$

$$< \epsilon/4 + \epsilon/4 = \epsilon/2, \quad 1 \leqq k \leqq n-1.$$

Similarly $|L_{(k),-}(b)| < \epsilon$, for $1 \leqq k \leqq n-1$. Now define the functions R^0 and R on $[a, b]$ by

$$R^0(x) = \bar{R}^0(x) - \sum_{i=1}^{n} \lambda(x; a_i) \bar{R}^0(a_i),$$

and

$$R(x) = L(x) + R^0(x).$$

Then if d is a point of discontinuity of $\bar{R}^0(x)$, we have

$$\frac{1}{h^{n-1}}\left[\frac{G(d+h)+G(d-h)}{2} - \sum_{k=0}^{(n/2)-1}\frac{h^{2k}}{(2k)!}D^{2k}G(d)\right.$$
$$\left. + \frac{\bar{R}^0(d+h)+\bar{R}^0(d-h)}{2} - \sum_{k=0}^{(n/2)-1}\frac{h^{2k}}{(2k)!}D^{2k}\bar{R}^0(d)\right]$$
$$= \frac{1}{2h^{n-1}}\left[G(d+h) - G(d) - \sum_{k=1}^{n-2}\frac{h^k}{k!}G_{(k),+}(d)\right]$$
$$- \frac{1}{2(-h)^{n-1}}\left[G(d-h) - G(d) - \sum_{k=1}^{n-2}\frac{(-h)^k}{k!}G_{(k),-}(d)\right]$$
$$+ \frac{1}{2h^{n-1}}\left[\bar{R}^0(d+h) - \bar{R}^0(d) - \sum_{k=1}^{n-2}\frac{h^k}{k!}\bar{R}^0_{(k),+}(d)\right]$$
$$- \frac{1}{2(-h)^{n-1}}\left[\bar{R}^0(d-h) - \bar{R}^0(d) - \sum_{k=1}^{n-2}\frac{(-h)^k}{k!}\bar{R}^0_{(k),-}(d)\right]$$
$$\to \pm\tfrac{1}{2}[G_{(n-1),+}(d) - G_{(n-1),-}(d) + \bar{R}^0_{(n-1),+}(d) - \bar{R}^0_{(n-1),-}(d), \quad \text{as } h \to \pm 0,$$
$$= \pm\tfrac{1}{2}[p(d+) - p(d-) - \bar{R}^0_{(n-1),+}(d) - \bar{R}^0_{(n-1),-}(d)] = 0,$$

by definition of $p(x)$. Since each $\lambda(x; a_i)$ is n-smooth it follows that $R(x)$ is n-smooth at each point of discontinuity of $\bar{R}^0(x)$. Since $R(x)$ is n-convex on each of the intervals determined by the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ it follows that $R(x)$ is n-convex on $[a, b]$.

We have then $|R(x)| < \epsilon$, $|R_{(k),+}(a)| < \epsilon$ and $|R_{(k),-}(b)| < \epsilon$, $1 \leqq k \leqq n-1$

Now let Q be defined on $[a, b]$ by

$$Q(x) = F(x) + R(x),$$

where $F(x)$ is defined in the statement of the lemma. Then

$$Q(x) = F(x) + L(x) + R^0(x),$$

and on a typical interval, say $[\beta_{k-1}, \beta_k]$,

$$Q(x) = F(x) + Q_k(x) - F_k(x) + L(x)$$

where $Q_k(x)$ is a major function for $f(x)$ on $[\beta_{k-1}, \beta_k]$ and $F_k(x)$ is its P^n-integral. Since $F(x) - F_k(x)$ and $L(x)$ are polynomials of degree at most $(n-1)$, it is easy to see that $Q(x)$ has the required properties of a P^n-major function on $[\beta_{k-1}, \beta_k]$.

$Q(x)$ is obviously continuous and the existence of $Q_{(k)}(x)$, $1 \leqq k \leqq n-2$, follows from the n-convexity of $R(x)$ and the existence of $F_{(k)}(x)$, $1 \leqq k \leqq n-2$ [**2**]. $F(x)$ is n-smooth in (a, b) and since $R(x)$ is n-convex then $R^{(n-1)}(x)$ exists except at a countable number of points in (a, b)

[1] and so $R(x)$ is n-smooth except at a countable number of points in (a, b). This shows that condition (1.4) in the definition of a major function is satisfied.

Conditions (1.5) and (1.6) are satisfied by $Q(x)$ since, clearly, $Q(a_i) = 0$, $i = 1, 2, \ldots, n$, and

$$\partial^n Q(x) \geqq \partial^n Q_k(x) \geqq f(x), \text{ a.e. in } [\beta_{k-1}, \beta_k].$$

We have moreover that $\partial^n Q(x) \geqq \partial^n Q_k(x) > -\infty$ except on a scattered set in (β_{k-1}, β_k) where $Q_k(x)$, and hence $Q(x)$, is n-smooth. But the set which is the union of all the scattered sets from the intervals (α_1, β_1), (α_k, α_{k-1}), (β_k, β_{k+1}), $k = 1, 2, \ldots$, is scattered, as is its union with the set of end points $T = \{\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots\}$. Since $F(x)$ is n-smooth everywhere and $R(x)$ is n-smooth at the points of T, condition (1.7) is verified for $Q(x)$.

In a similar way we can construct a minor function with the required properties.

Now Theorem 2.1 follows because of the results of [3] (in particular, Remark, page 796).

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