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Multi-components chemotactic system in the absence of conflicts

G. WOLANSKY

Department of Mathematics, Technion, Haifa 32000, Israel
email: gershonw@math.technion.ac.il

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A generalization of the Keller–Segel model for chemotactic systems is studied. In this model there are several populations interacting via several sensitivity agents in a two-dimensional domain. The dynamics of the population is determined by a Fokker–Planck system of equations, coupled with a system of diffusion equations for the chemical agents. Conditions for global existence of solutions and equilibria are discussed, as well as the possible existence of time-periodic attractors. The analysis is based on a variational functional associated with the system.

1 Introduction

1.1 General description

We consider a model equation for the time-dependent distribution $\rho(x,t)$ of organisms (bacteria, cells, humans, etc.), sensitive to the gradient of a self-produced chemical agent (sensitivity) whose distribution is given by $S(x,t)$. Both the organisms and the sensitivity agents are subjected to independent diffusive fluctuations. In addition, any individual organism 'smells' the local sensitivity substance S, and tends to climb up its gradient if it 'likes' the smell or down the gradient if it 'dislikes' it.

The system (1.1, 1.2) below was suggested by Keller & Segel [10] as a model for chemotactic aggregation:

$$v\frac{\partial\rho}{\partial t} + \theta\nabla\cdot(\rho\nabla S) = \Delta\rho, \tag{1.1}$$

$$\sigma\frac{\partial S}{\partial t} = \Delta S - \alpha S + \gamma\rho + f. \tag{1.2}$$

Equations (1.1) and (1.2) are satisfied in a habitat domain $\Omega \subset \mathbb{R}^2$. The boundary conditions for (1.1) are chosen to guarantee the conservation of the number of individuals $N = \int_\Omega \rho(x,t)dx$ for all times:

$$[\theta\rho\nabla S - \nabla\rho]\cdot\vec{n} = 0, \;\; x \in \partial\Omega, \tag{1.3}$$

where $\vec{n}$ is the normal to the boundary of Ω. The mobility parameter θ determines the tendency of the population to climb up the sensitivity gradient ($\theta > 0$) or down ($\theta < 0$). Likewise, γ is the rate of production (if positive) or consumption (if negative) of the agent S by the population. The population-independent source $f(x)$ is a prescribed function,

$\alpha \geqslant 0$ is the disintegration rate of S. The positive parameters $\sigma > 0$ and $v > 0$ stand for the (possibly different) time scales of the substance concentration and population dynamics, respectively.

In this paper, we consider a generalization of (1.1, 1.2) to a system of n populations $\{\rho_1, \dots \rho_n\}$ and k sensitivity agents $\{s_1, \dots s_k\}$ living together in a common habitat domain Ω:

$$v_i \frac{\partial \rho_i}{\partial t} + \sum_{j=1}^{k} \theta_{i,j} \nabla \cdot \left(\rho_i \nabla s_j\right) = \Delta \rho_i, \quad i \in [1, \dots n], \tag{1.4}$$

$$\sigma_j \frac{\partial s_j}{\partial t} = \Delta s_j - \alpha s_j + \sum_{i=1}^{n} \gamma_{i,j} \rho_i + f_j, \quad j \in [1, \dots k]. \tag{1.5}$$

Here $\gamma_{i,j}$, $\theta_{i,j}$ define a pair of $n \times k$ matrices for the production/consumption rate and the mobilities, respectively.

We show that the tendency of a population i_1 towards a population i_2 due to the action of all the agents can be quantified by the parameter $\lambda_{i_1,i_2} = \sum_{j=1}^{k} \theta_{i_1,j} \gamma_{i_2,j}$. The condition $\lambda_{i_1,i_2} > 0$ means that i_1 is attracted to i_2, while $\lambda_{i_1,i_2} < 0$ means that i_1 is repelled by i_2. In particular, $\lambda_{i,i} > 0$ (< 0) is the condition of self-attraction (self-repulsion) of the population i.

The situation where λ_{i_1,i_2} and λ_{i_2,i_1} are opposite in sign is of a particular interest. We denote this case as a 'conflict of interests' between the i_1 and i_2 populations. We shall, however, concentrate on the conflict-free case in this paper, and defer the discussion on conflicts to a separate publication.

The paper is organized as follows. In § 2.1–2.2 we set the foundation of our variational approach: let $\mathbf{Y}$ be the state space of the populations density vectors $\vec{\rho} \equiv \{\rho_1, \dots \rho_n\}$ subjected to the integral constraint

$$\int_\Omega \rho_i dx = N_i, i \in [1, \dots n], \tag{1.6}$$

where $N_i > 0$ is the prescribed population size (determined by the initial data). In addition, denote the state space of vector-valued sensitivity agents $\vec{s} \equiv \{s_1, \dots s_k\}$ by $\mathbf{X}$.

It is shown that, under some additional assumptions, a variational functional $\Psi = \Psi(\vec{\rho}, \vec{s})$ can be introduced on $\mathbf{Y} \otimes \mathbf{X}$. The state space $\mathbf{Y}$ is composed of two disjoint components $\mathbf{Y}_+$ and $\mathbf{Y}_-$, where Ψ is convex on $\mathbf{Y}_+$ and concave on $\mathbf{Y}_-$. The conflict-free case is characterized by the condition $\mathbf{Y}_- = \emptyset$, i.e. Ψ is convex with respect to $\vec{\rho}$. Likewise, the space $\mathbf{X}$ is composed into a direct sum $\mathbf{X} = \mathbf{X}^+ \oplus \mathbf{X}^-$, where Ψ is convex on $\mathbf{X}^+$ and concave on $\mathbf{X}^-$. Any stationary solution of (1.4, 1.5) is determined by a critical point of the functional Ψ on the combined space. In general, Ψ is not monotone along solutions of (1.4, 1.5). § 2.3 and 2.4 deal with singular limits of the system, where the population dynamics (1.4) is much faster (resp. slower) than the sensitivity dynamics (1.5).

In § 2.3 we consider the fast population dynamics, i.e. $\sigma_j = 0$, for all $1 \leqslant j \leqslant k$. In this case, we extremize the free energy Ψ with respect to the sensitivities $\vec{s}$ to obtain

$$\mathscr{F} \equiv \mathscr{F}(\vec{\rho}) := \inf_{\vec{s}_1 \in \mathbf{X}^+} \sup_{\vec{s}_2 \in \mathbf{X}^-} \Psi(\vec{\rho}, \vec{s}_1 + \vec{s}_2).$$

We show that $\mathscr{F}$ *is* a monotone functional for (1.4) in this singular limit, if and only if the system is conflict-free.

In § 2.4 we consider the analogous limit of slow population dynamics, i.e. $v_i = 0$ for all $1 \leqslant i \leqslant n$. Define a functional

$$\mathscr{D}_{N_1,...N_n} \equiv \mathscr{D}_{N_1,...N_n}(\vec{s}) := \inf_{\vec{\rho}_1 \in \mathbf{Y}_+} \sup_{\vec{\rho}_2 \in \mathbf{Y}_-} \Psi(\vec{\rho}_1 + \vec{\rho}_2, \vec{s}),$$

where the extrema on $\vec{\rho} \in \mathbf{Y}$ is subject to (1.6). The condition for monotonicity of $\mathscr{D}_{N_1,...N_n}$ for the system (1.5) in this singular limit is obtained as well. It follows that the relative values of the diffusion rates $\sigma_1, \dots \sigma_k$ play an important rule in this limit.

In § 3 we study the existence and stability of local minimizers to the functionals Ψ, $\mathscr{F}$ and $\mathscr{D}_{\vec{N}}$. The global solvability of (1.4, 1.5) is obtained by joining together two independent results: the first concerns the global solvability of the single population system (1.1, 1.2) [1], and the second concerns the properties of Liouville systems and its relation to the functional $\mathscr{F}$, studied in Chipot *et al.* [3].

In § 4 we study periodic solutions, using the Andronov–Hopf bifurcation theorem [6].

Before turning to the main parts of the paper, we introduce a short review of known results for the single component system (1.1, 1.2) and an informal discussion on its multi-component analog. This discussion provides motivation for the investigation of periodic solutions in § 4.

1.2 Background and motivation

The system (1.1, 1.2) has been extensively investigated, in particular, for the singular limit $\sigma = 0$. In this case with $f = 0$ and $v = 1$, equations (1.1) and (1.2) take the form

$$\frac{\partial \rho}{\partial t} + \theta \nabla \cdot (\rho \nabla S) = \Delta \rho, \quad S(x,t) = \gamma \int_\Omega G_\alpha(x,y)\rho(y,t)dy, \tag{1.7}$$

where G_α is the Green's function for the operator $-\Delta + \alpha$ subject to the prescribed boundary conditions associated with (1.2) (say, Dirichlet). Particular attention has been given to the self-attractive ('gravitational') case $\theta\gamma > 0$. In this case, it is known that time-dependent solutions of (1.7) may blow up in a finite time [5, 7, 8, 10, 12]. For two dimensional, starlike domains Ω there exist blow-up solutions provided the conserved, total population number $N \equiv \int \rho$ exceeds a certain critical number

$$N^c := \frac{8\pi}{\theta\gamma}. \tag{1.8}$$

It is also known that stable equilibria of (1.7) *do* exist, and that it is globally solvable for two-dimensional domains provided $N < N^c$ if a Dirichlet boundary condition is assumed for (1.2), and for $N < 4\pi/\theta\gamma$ in the case of a Neumann boundary condition [1, 13].

This type of result is a manifestation of the critical role played by the spatial dimension two [11] in the context of the system (1.1, 1.2). If the total population number N is above a critical number, then the self-attraction dominates the smoothing effect of the diffusion in (1.1), resulting in a finite time blow up of the solution. If, however, $N < N^c$, then the smoothing effect of the diffusion dominates the self-attraction, preventing the blow-up and implying the asymptotic convergence of solution toward an equilibrium.

An essential tool for the study of (1.7) is the 'Free-Energy' functional

$$F := F(\rho) = \int_\Omega \rho \ln \rho dx - \frac{\theta\gamma}{2} \int_\Omega \int_\Omega \rho(x) G_\alpha(x,y) \rho(y) dxdy.$$

We chose the name Free Energy after the classical definition

$$F = -T \cdot Entropy + Energy$$

of Free energy in thermodynamics. Recall that $-\int \rho \ln \rho$ is the statistical-mechanics definition for entropy, while the second component resembles the energy of self-interaction via gravitational or electrostatic interaction. The 'temperature' T is normalized to one (in accordance with our choice of the diffusion coefficient in (1.1)).

It is known that the Free Energy is monotone non-increasing (Lyapunov) functional for the singular limit $\sigma = 0$ in (1.7) for both the self-attractive $\theta\gamma > 0$ and self-repelling ('electrostatic') $\theta\gamma < 0$ cases (cf. Wolansky [14]). It yields useful *a priori* estimates for the time-dependent solution ρ in this singular limit, as well as a variational formulation for stable steady solutions via its local minimizers. It was investigated in other contexts as well (in particular, statistical mechanics and fluid-dynamics: cf. Wolansky [14] and Caglioti *et al.* [4]) and its properties are well understood.

A basic feature of this functional is the convexity of the first (negative entropy) term and the positivity of the Green's function $(-\Delta + \alpha)^{-1}$. This implies, in particular, that F is strictly convex and bounded from below in the self-repelling case, which yields the existence and uniqueness of a minimizer.

Deeper mathematical questions arise for the self-attracting case. Here the second (energy) term is concave and the boundedness of F is conditional on the total mass (or population number) $N = \int \rho$. The fundamental inequality of Moser and Trudinger (e.g. see McLeod & McLeod [11]) implies that F is bounded if and only if $N \leqslant N^c$ (1.8).

Turning back to the application of F to the system (1.1, 1.2), we note that F is *not* a Lyapunov functional unless the singular limit $\sigma = 0$ is assumed. However, for $\sigma > 0$ we can still define an extension of F into a Lyapunov functional of (1.1, 1.2), namely

$$\psi(\rho, S) = \int_\Omega \rho \ln \rho + \frac{\theta}{2\gamma} \int_\Omega \left[|\nabla S|^2 + \alpha S^2 \right] - \theta \int_\Omega \rho S$$

provided $\theta\gamma > 0$. F is related to ψ by

$$F(\rho) = \inf_S \psi(\rho, S). \tag{1.9}$$

It is remarkable that this extension is not a Lyapunov functional in the repelling case (cf. Wolansky [15]).

Obviously, the existence of a time-monotone (Lyapunov) functional excludes the possibility of time-periodic solutions. This is the case for self-attraction in general ($\sigma \geqslant 0$), and self-repulsion in the singular limit ($\sigma = 0$). The natural question which we now pose is: can time-periodic solutions to (1.1, 1.2) exist in the non-singular limit $\sigma > 0$ in the self-repelling case?

We conjecture that the answer to this question is negative. In fact, we cannot point to any mechanism which will drive a periodic solution for the self-repelling case for $\sigma > 0$, but exclude it in the singular limit. Still, this conjecture is awaiting for a proof and intuition can be a bad consultant, as we shall see below.

The limit $\sigma = 0$ is not the only one possible. The second singular limit $\nu = 0$ is also of interest. Note that, in that case, equation (1.1) takes the form

$$\nabla \cdot (\nabla \rho - \theta \rho \nabla S) = 0$$

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which, together with the boundary condition (1.3) and the prescribed mass condition $\int \rho = N$, yields

$$\rho = N \frac{e^{\theta S}}{\int e^{\theta S}}. \tag{1.10}$$

When (1.10) is substituted in (1.2), we obtain the singular limit $v = 0$ as the non-local diffusion equation

$$\frac{\partial S}{\partial t} = \Delta S - \alpha S + \gamma N \frac{e^{\theta S}}{\int e^{\theta S}}. \tag{1.11}$$

Unlike the singular limit $\sigma = 0$ (1.7), equation (1.11) has been studied very little. Global (in time) existence of solutions is proved for $N < N^c$ in the attractive case $\gamma\theta > 0$ [15]. We are not aware of any finite time blow-up result in the super-critical case $N \geqslant N^c$.

It is interesting to note that (1.11) has a variational formulation analogous to the Free Energy F, namely

$$D_N(S) = \inf_{\rho} \psi(\rho, S),$$

where the infimum is taken over the set $\rho \geqslant 0$; $\int \rho = N$. Explicitly,

$$D_N(S) = \frac{\theta}{2\gamma} \int_\Omega \left[|\nabla S|^2 + \alpha S^2 \right] - N \ln \left[\int e^{\theta S} \right].$$

The functional D_N is, again, a Lyapunov functional for the singular limit (1.11). It is monotone non-increasing in the self-attractive case, and monotone non-decreasing in the self-repelling case. In particular, its critical points are equilibria (steady states) of (1.11).

The existence of monotone functionals in both singular limits excludes the possibility of limit cycles for the single-component system (1.1, 1.2), at least in the singular limit $\sigma = 0$ or $v = 0$. In the multi-component analogue (1.4, 1.5), on the other hand, we do expect periodic solutions (and more general attracting sets) due to the possibility of conflicting pairs. In this case there is a population i_1 which is attracted to a population i_2, while the population i_2 is repelled from i_1. This dynamics points to a mechanism by which i_1 is chasing i_2 like a dog chasing its tail, namely, a limit cycle.

The reason we concentrate on the *conflicts free* multi-component system (1.4, 1.5) is that, in fact, periodic solutions do exist, under certain assumptions, also in the absence of conflicts. This is shown in §4, using Andronov–Hopf bifurcation. We find this result quite surprising since, in the absence of conflicts, all populations have a mutual interest to settle in a local minimizer of the Free Energy functional $\mathscr{F}$, which is the generalization of F in the multi-component case. This is indeed the case for the singular limit $\sigma_j = 0$, but, as we prove in §4, the diffusion rate σ_j play an essential rule in the dynamics.

The multi-component system introduces an additional challenge: what is the analogue of the critical mass N^c obtained for the self-attracting, single component system? Recall that the condition $N < N^c$ is sufficient (and, in some cases, necessary) for the global solvability of the single component system (1.1, 1.2) in the self-attractive case.

A version of the Free Energy functional $\mathscr{F}$ for a multi-component system was introduced in Chipot *et al.* [3]. The condition (1.8) is generalized (for n components) into $2^n - 1$ conditions:

$$\sum_{i,l \in I} a_i \lambda_{i,l} N_i N_l - 8\pi \left(\sum_{i \in I} N_i \right) < 0 \quad \forall I \subset \{1, \dots n\}, \quad I \neq \emptyset \tag{1.12}$$

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where $N_i = \int \rho_i$, a_i and $\lambda_{i,j}$ are certain constants defined in § 2. Note that in the case of a single component ($n = 1$), $a_1 = 1$, $\lambda_{1,1} = \theta\gamma$ and (1.12) is reduced to (1.8).

In § 3 we prove that condition (1.12) is sufficient for the global solvability of (1.4, 1.5) in the singular limit $\sigma_j = 0$, and for the asymptotic convergence of its solutions into a regular steady state. This result is also generalized to the case of finite diffusion rate $\sigma_j > 0$, provided some conditions on σ_j are assumed. It is also optimal in the case where all pairs are mutually attracting (i.e. $\lambda_{i,l} > 0$ for all $1 \leqslant i, l \leqslant n$).

2 Preliminaries

2.1 Basic assumptions

We consider a boundary condition of zero flux for (1.4), that is

$$(\nabla\rho_i - \rho_i\nabla S_i) \cdot \hat{n}(x) = 0; \quad \text{on} \quad \partial\Omega \times R^+; \quad 1 \leqslant i \leqslant n$$

where $S_i = \sum_{j=1}^{k} \theta_{i,j} s_j$ and $\hat{n}(x)$ is the normal vector to the boundary at $x \in \partial\Omega$. As for the boundary condition for (1.5) we adopt either Dirichlet ($s_j = 0$) or Neumann ($\nabla s_j \cdot \hat{n} = 0$) for any $x \in \partial\Omega$. In the first case, the boundary condition for (1.4, 1.5) is nonlinear, given by

$$0 = s_j = (\nabla\rho_i - \rho_i\nabla S_i) \cdot \hat{n} = 0 \quad \text{on} \quad \partial\Omega \times R^+ \quad 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant k \tag{2.1}$$

while in the second case, it is a linear set of boundary conditions:

$$\nabla\rho_i \cdot \hat{n} = \nabla s_j \cdot \hat{n} \equiv 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+ \quad 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant k. \tag{2.2}$$

Both (2.1, 2.2) lead to the conservation of the population's size for each population separately:

$$\int_\Omega \rho_i(x,t)dx = \int_\Omega \rho_i(x,0)dx \equiv N_i > 0, \forall t \geqslant 0, 1 \leqslant i \leqslant n. \tag{2.3}$$

Let $\vec{\theta}_i \equiv \{\theta_{i,1}, \ldots \theta_{i,k}\} \in \mathbb{R}^k$ be the n- $\mathbb{R}^k$ valued *mobility* vectors. Likewise, let $\vec{\gamma}_i \equiv \{\gamma_{i,1}, \ldots \gamma_{i,k}\} \in \mathbb{R}^k$ be the n-$\mathbb{R}^k$ valued *production/consumption* vectors. Let us define the $n \times n$-matrix $\Lambda \equiv \{\lambda_{i,l}\}$ as:

$$\lambda_{i,l} \equiv \sum_{j=1}^{k} \theta_{i,j}\gamma_{l,j} = \vec{\theta}_i \cdot \vec{\gamma}_l. \tag{2.4}$$

The following definition is self-explanatory.

Definition 1 (i) *A population i_1 is attracted (resp. rejected) to (resp. from) a population i_2 if $\lambda_{i_1,i_2} > 0$ (resp. $\lambda_{i_1,i_2} < 0$). In particular, a population i is self-attracting (self-repelling) if $\lambda_{i,i} > 0$ (resp. $\lambda_{i,i} < 0$).*
(ii) *A pair of populations i_1, $i_2 \in \{1, \ldots n\}$ is said to be in a conflict if $\lambda_{i_1,i_2} \times \lambda_{i_2,i_1} < 0$.*

In general, the matrix Λ is not a symmetric one. In this paper, however, we shall assume that there exists n constants $a_1, \ldots a_n$, all different from zero, for which

$$a_i\lambda_{i,l} = a_l\lambda_{l,i}, \tag{2.5}$$

i.e. the matrix $\mathbf{D}_{\vec{a}}\Lambda$ is a symmetric matrix, where $\mathbf{D}_{\vec{a}} \equiv \mathbf{Diag}\{a_1, \ldots a_n\}$.

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If we assume that Λ is non-singular then there exists a $k \times k$ matrix $\mathbf{B}$ which transforms $\vec{\gamma}_i$ into $a_i\vec{\theta}_i$ for all $1 \leqslant i \leqslant n$. Moreover, (2.5) implies that $\mathbf{B}$ can be chosen as a symmetric matrix. This leads us to our fundamental hypothesis:

Hypothesis A *There exists a vector $\vec{a} \equiv \{a_1, \dots a_n\}$ whose components are all different from zero, and a nonsingular, symmetric matrix $\mathbf{B} \equiv \{b_{j,l}\}$ which satisfy*

$$\mathbf{B}\vec{\gamma}_i = a_i\vec{\theta}_i \quad \text{for all} \quad 1 \leqslant i \leqslant n. \tag{2.6}$$

Remark 1 *Hypothesis A is equivalent to* (2.5) *if Λ is non-singular. We shall use the convention by which there exists at least one population i for which $a_i > 0$. Evidently, we may always achieve this by switching form $\{\mathbf{B}, \vec{a}\}$ to $\{-\mathbf{B}, -\vec{a}\}$, if necessary. In particular, the conflict-free case is characterized, via Definition* 1 *and* (2.5)*, by $a_i > 0$ for all $i \in \{1, \dots n\}$.*

A special case of Hypothesis A is $n = 2$ and $k \geqslant 1$ arbitrary. In this case, the matrix $\mathbf{B}$ can always be found, provided $\lambda_{1,2} \times \lambda_{2,1} \neq 0$, where

$$\frac{a_1}{a_2} = \frac{\lambda_{2,1}}{\lambda_{1,2}}. \tag{2.7}$$

Another special case is obtained if $k = 1$ and $n \geqslant 1$ is arbitrary. Then we may choose the scalar 1×1 matrix $\mathbf{B} \equiv \{1\}$ and $a_i = \gamma_i/\theta_i$. In particular, it follows that a_i are all positive if all the populations are attracted to the sensitivity s (and hence attracted to each other). In the opposite situation where $\theta_i < 0$ for all populations, we may define $a_i = -\gamma_i/\theta_i$ and $\mathbf{B} = \{-1\}$, using the convention defined below (2.6). In the third case we may have mobility coefficients of different signs. This is interpreted as a 'conflict of interests', where one of the populations is attracted to a second, while the second is rejected from the first one. In particular, the coefficients a_i are *not* all of the same sign.

Remark 2 *In the remainder of the paper, we shall always assume Hypothesis A. The results are valid for either boundary conditions* (2.1) *or* (2.2)*, unless a specific reference is made to one of the boundary conditions.*

2.2 Variational formulation

Set

$$F_j = \sum_{l=1}^{k} f_l b_{l,j}.$$

Define

$$\Psi\,(\rho_1, \dots \rho_n, s_1, \dots s_k) \equiv$$

$$\sum_{i=1}^{n} a_i \int_\Omega \rho_i \ln \rho_i dx + \frac{1}{2} \sum_{j=1}^{k} \sum_{l=1}^{k} b_{l,j} \int_\Omega \left[\nabla s_j \cdot \nabla s_l + \alpha s_j s_l\right] dx$$

$$- \sum_{i=1}^{n} \sum_{j=1}^{k} a_i \theta_{i,j} \int_\Omega \rho_i s_j dx - \sum_{j=1}^{k} \int_\Omega F_j s_j dx. \tag{2.8}$$

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Let $\vec{\rho} = \{\rho_1, \ldots \rho_n\}$, $\vec{s} = \{s_1, \ldots s_k\}$. Set $\mathbf{D}_q \equiv \mathbf{Diag}\{q_1, \ldots q_l\}$ be the diagonal $l \times l$ matrix where $q = \{q_1, \ldots q_l\}$. A direct application of the definition of Ψ yields the following lemma.

Lemma 1 *The system* (1.4, 1.5) *can be written as*

$$\mathbf{D}_v \frac{\partial \vec{\rho}}{\partial t} = \mathbf{D}_a^{-1} \nabla_x \cdot \left[\vec{\rho} \cdot \nabla_x \frac{\delta \Psi}{\delta \vec{\rho}} \right], \quad \frac{\partial \vec{s}}{\partial t} = \mathbf{D}_\sigma^{-1} \mathbf{B}^{-1} \frac{\delta \Psi}{\delta \vec{s}}. \tag{2.9}$$

In particular, it follows that

$$\frac{d}{dt} \Psi\left(\vec{\rho}(\cdot, t), \vec{s}(\cdot, t)\right) = -\sum_{i=1}^{n} (a_i v_i)^{-1} \int_\Omega \rho_i \left| \nabla \frac{\delta \Psi}{\delta \rho_i} \right|^2 dx - \int_\Omega \left[\frac{\partial \vec{s}}{\partial t} \right]^T \mathbf{B} \mathbf{D}_\sigma \frac{\partial \vec{s}}{\partial t} dx \tag{2.10}$$

if $\{\vec{\rho}(\cdot, t), \vec{s}(\cdot, t)\}$ *is a classical solution of* (1.4, 1.5).

Recall that the conflict-free case is characterized by $a_i > 0$ for all $i \in \{1, \ldots n\}$. From this and Lemma 1, we obtain the following corollary.

Corollary 1 $\Psi(\vec{\rho}, \vec{s})$ *is monotone along classical solutions of* (1.4, 1.5) *if*

(a) *There are no conflicts;*
(b) $\mathbf{B}\mathbf{D}_\sigma + (\mathbf{B}\mathbf{D}_\sigma)^T$ *is positive definite.*

In general, the evolution system (1.4, 1.5) is not a gradient flow of Ψ and the right-hand side of (2.10) is rarely definite. Still, we may characterize *all* stationary solutions by Theorem 1.

Definition 2 *If the boundary condition* (2.1) *is assumed, let*

$$\{s_1, \ldots s_k\} \in \mathbf{X} := \mathbb{H}_0^1(\Omega, \mathbb{R}^k).$$

If, on the other hand, (2.2) *is assumed, set* $\mathbf{X} = \mathbb{H}^1(\Omega, \mathbb{R}^k)$. *Also,*

$$\{\rho_1, \ldots \rho_n\} \in \mathbf{Y}_{\vec{N}} := \left\{ \vec{\rho} : \Omega \to \mathbb{R}^+ \; \middle| \int_\Omega \rho_i = N_i; \; \int_\Omega \rho_i \log \rho_i dx < \infty \right\} \tag{2.11}$$

where $\vec{N} = \{N_1, \ldots N_n\}$.

Theorem 1 $\left\{ \rho_1^{(0)}, \ldots \rho_n^{(0)} \; ; \; s_1^{(0)}, \ldots s_k^{(0)} \right\} \in \mathbf{Y}_{\vec{N}} \otimes \mathbf{X}$ *is a stationary solution to* (1.4, 1.5) *if and only if it is a critical point of* Ψ *in the above domain. Moreover,* $\rho_i^{(0)}$ *are strictly positive on* $\overline{\Omega}$.

Proof of Theorem 1 The representation (2.9) immediately implies that a critical point of Ψ is an equilibrium of (1.4, 1.5). Conversely, let $S_i = \sum_{j=1}^k \theta_{i,j} s_j$. Then $\delta_{\rho_i} \Psi = \ln\left(e^{-S_i} \rho_i^{(0)}\right)$ and (2.9) implies

$$\nabla \cdot \left[\rho_i^{(0)} \nabla \left(e^{-S_i} \rho_i^{(0)} \right) \right] = 0.$$

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The boundary condition (2.1) implies

$$\nabla\left(e^{-S_i}\rho_i^{(0)}\right)\cdot\hat{n}=0 \quad \forall x\in\partial\Omega.$$

The maximum principle now implies that $e^{-S_i}\rho_i^{(0)}$ must be a constant throughout Ω, hence $\delta_{\rho_i}\Psi=\mu_i$, where μ_i is a constant, corresponding to the Lagrange multiplier of the constraint (2.11). In particular, we obtain that $\rho_i^{(0)}=\mu_i e^{S_i}$ is strictly positive on $\overline{\Omega}$. □

2.3 The singular limit $\vec{\sigma}=0$

Define the subspaces $\mathbf{X}^{\pm}$ of $\mathbf{X}$ to be the positive/negative decomposition

$$\vec{\phi}\equiv\{\phi_1,\dots\phi_k\}\in\mathbf{X}^+\Rightarrow\sum_{j=1}^{k}\sum_{l=1}^{k}b_{j,l}\int_{\Omega}\nabla_x\phi_l\cdot\nabla_x\phi_j dx\geqslant 0$$

and

$$\vec{\phi}\in\mathbf{X}^-\Rightarrow\sum_{j=1}^{k}\sum_{l=1}^{k}b_{j,l}\int_{\Omega}\nabla_x\phi_l\cdot\nabla_x\phi_j dx\leqslant 0.$$

Since $\mathbf{B}$ is non-singular

$$\mathbf{X}=\mathbf{X}^+\oplus\mathbf{X}^-. \tag{2.12}$$

Define

$$\mathscr{F}(\rho_1,\dots\rho_n)\equiv\inf_{\vec{s}_1\in\mathbf{X}^+}\sup_{\vec{s}_2\in\mathbf{X}^-}\Psi(\rho,\vec{s}_1+\vec{s}_2)\equiv\sup_{\vec{s}_2\in\mathbf{X}^-}\inf_{\vec{s}_1\in\mathbf{X}^+}\Psi(\rho,\vec{s}_1+\vec{s}_2). \tag{2.13}$$

It is easy to find an explicit expression for $\mathscr{F}$ as follows: let $\mathbf{U}=u_{l,j}$ be the orthogonal matrix diagonalizing $\mathbf{B}$ and set $S_j=\sum_{m=1}^{k}u_{m,j}s_m$. Then

$$\frac{1}{2}\sum_{j=1}^{k}\sum_{l=1}^{k}b_{l,j}\int_{\Omega}\left[\nabla s_j\cdot\nabla s_l+\alpha s_j s_l\right]dx=\frac{1}{2}\sum_{j=1}^{k}b_j\int_{\Omega}\left[|\nabla S_j|^2+\alpha S_j^2\right]dx,$$

where b_j is the jth eigenvalue of $\mathbf{B}$ (as in Definition 1). Similarly,

$$\sum_{i=1}^{n}\sum_{j=1}^{k}a_i\theta_{i,j}\int_{\Omega}\rho_i s_j dx+\sum_{j=1}^{k}\int_{\Omega}F_j s_j dx=\sum_{i=1}^{n}\sum_{j=1}^{k}a_i q_{i,j}\int_{\Omega}\rho_i S_j dx+\sum_{j=1}^{k}\int_{\Omega}\Xi_j S_j dx,$$

where

$$q_{i,j}=\sum_{l=1}^{k}\theta_{i,l}u_{l,j};\quad \Xi_j\equiv\sum_{l=1}^{k}F_l u_{l,j}.$$

If $b_j>0$, let S_j be the minimizer of

$$\frac{b_j}{2}\int_{\Omega}\left(|\nabla S|^2+\alpha|S|^2-\frac{2}{b_j}\Xi_j S\right)dx-\sum_{i=1}^{n}a_i\int_{\Omega}q_{i,j}S\rho_i dx \tag{2.14}$$

over $H_0^1(\Omega)$ (resp. $H^1(\Omega)$). The minimum of (2.14) is obtained at $S=S_j$, which solves the elliptic boundary value problem

$$-\Delta S_j+\alpha S_j-\frac{1}{b_j}\sum_{i=1}^{n}a_i q_{i,j}\rho_i-\frac{\Xi_j}{b_j}=0. \tag{2.15}$$

That is,

$$S_j(x) = \frac{1}{b_j} \sum_{i=1}^{n} a_i q_{i,j} \int_\Omega G_\alpha(x,y)\rho_i(y)dy + \phi_j \tag{2.16}$$

where

$$\phi_j \equiv \frac{1}{b_j} \int_\Omega G_\alpha(x,y)\Xi_j(y)dy$$

and $G_\alpha(\cdot,\cdot)$ is the Green kernel corresponding to the Dirichlet problem

$$(-\Delta_x + \alpha)\, G_\alpha(x,y) = \delta_{(|x-y|)}.$$

Multiply (2.15) by S_j and integrate by parts over Ω to obtain

$$b_j \int_\Omega \left(|\nabla S_j|^2 + \alpha |S_j|^2 - \frac{\Xi_j}{b_j} S_j \right) dx - \sum_{i=1}^{n} a_i q_{i,j} \int_\Omega \rho_i S_j dx = 0. \tag{2.17}$$

Insert (2.17) in (2.14) to obtain the minimum of (2.14):

$$\frac{1}{2} \int_\Omega \Xi_j S_j dx - \frac{1}{2} \sum_{l=1}^{n} a_i q_{l,j} \int_\Omega \rho_l S_j dx \tag{2.18}$$

where for S_j, we use (2.16). Apply the same argument for the analogous problem (2.14) corresponding to $b_j < 0$, where this time we *maximize* over $S \in H^1$ ($S \in H_0^1$) and sum over all j and i to obtain

$$\mathscr{F}(\rho_1, \dots \rho_n) = \sum_{i=1}^{n} a_i \int_\Omega \rho_i \ln \rho_i dx$$

$$-\frac{1}{2} \sum_{i=1}^{n} \sum_{l=1}^{n} a_i \lambda_{i,l} \int_\Omega \int_\Omega \rho_i(x) G_\alpha(x,y) \rho_l(y) dx dy + \sum_{i=1}^{n} \int_\Omega \rho_i \phi_i dx, \tag{2.19}$$

where we have used the definition of $q_{i,l}$ and (2.5, 2.6) to obtain

$$a_l \sum_{j=1}^{k} \frac{q_{i,j} q_{l,j}}{b_j} = \lambda_{i,l}.$$

The singular limit $\vec{\sigma} = 0$ for the system (1.4, 1.5) is now *defined* as

$$v_i \frac{\partial \rho_i}{\partial t} = a_i^{-1} \nabla \cdot \left[\rho_i \nabla \left(\delta_{\rho_i} \mathscr{F} \right) \right]. \tag{2.20}$$

Equation (2.20) can be written explicitly as

$$v_i \frac{\partial \rho_i}{\partial t} + \nabla \cdot \left(\rho_i \nabla \left[\sum_{l=1}^{n} \lambda_{i,l} \nabla \mathscr{S}_l + \phi_i \right] \right) = \Delta \rho_i, \tag{2.21}$$

where $\mathscr{S}_i$ are the 'virtual' agents determined by

$$\mathscr{S}_i(x) = \int_\Omega G_\alpha(x,y) \rho_i(y) dy.$$

The following is an immediate conclusion of Lemma 1.

Lemma 2 *Let $\vec{\rho}(\cdot, t)$ be a classical solution of* (2.20). *Then*

$$\frac{d}{dt}\mathscr{F}(\vec{\rho}(\cdot,t)) = -\sum_{i=1}^{n}(v_i a_i)^{-1}\int_\Omega \rho_i \left|\nabla\frac{\delta\mathscr{F}}{\delta\rho_i}\right|^2 dx$$

$$\equiv -\sum_{i=1}^{n}\frac{a_i}{v_i}\int_\Omega \rho_i\left|\nabla\left[\ln\rho_i - \sum_{j=1}^{k}\theta_{i,j}s_j\right]\right|^2 dx. \quad (2.22)$$

Analogously to Corollary 1, an immediate Corollary of Lemma 1 is:

Corollary 2 *$\mathscr{F}$ is monotone non-increasing along classical solutions of* (2.20) *if and only if there are no conflicts.*

It is evident that the set of stationary solutions of (1.4, 1.5) are independent of v_i and σ_j. In the case $\vec{\sigma} = 0$, one may eliminate $s_1, \ldots, s_k$ from the equations, using the definition of $\mathscr{F}$. The proof of Theorem 1 yields

Theorem 2 *Let the conditions of Theorem 1 be satisfied. Then $\{\vec{\rho}^{(0)}, \vec{s}^{(0)}\}$ is a stationary solution of* (1.4, 1.5) *subjected to* (2.1) *if and only if $\vec{\rho}^{(0)}$ is a critical point of $\mathscr{F}$, subject to the constraint* (2.11). *If this is the case, then $\vec{s}^{(0)}$ is obtained from $\vec{\rho}^{(0)}$ via* (2.16).

2.4 The singular limit $\vec{v} = 0$

Define

$$I_\pm = \{i \in \{1, \ldots n\}\ ;\ \pm a_i > 0\},$$

$$\vec{N}_\pm := \{\mathscr{N}_1, \ldots \mathscr{N}_n\} \text{ where } \mathscr{N}_i = N_i \ \ \textit{if}\ \ i \in I_\pm,\ \ \mathscr{N}_i = 0 \ \text{ otherwise.} \quad (2.23)$$

Let $\mathbf{Y}^{\pm}_{\vec{N}} := \mathbf{Y}_{\vec{N}_\pm}$ so

$$\mathbf{Y}_{\vec{N}} = \mathbf{Y}^{+}_{\vec{N}} \oplus \mathbf{Y}^{-}_{\vec{N}} \quad (2.24)$$

and define

$$\mathscr{D}_{\vec{N}} \equiv \sup_{\vec{\rho}_1\in\mathbf{Y}^{-}_{\vec{N}}}\ \inf_{\vec{\rho}_2\in\mathbf{Y}^{+}_{\vec{N}}} \Psi(\vec{s}, \vec{\rho}_1 + \vec{\rho}_2) \equiv \inf_{\vec{\rho}_1\in\mathbf{Y}^{+}_{\vec{N}}}\ \sup_{\vec{\rho}_2\in\mathbf{Y}^{-}_{\vec{N}}} \Psi(\vec{s}, \vec{\rho}_1 + \vec{\rho}_2). \quad (2.25)$$

For an explicit expression for $\mathscr{D}_{\vec{N}}$, observe that the infimum (supremum) in (2.25) is attained at

$$\rho_i = \mu_i e^{\sum_{j=1}^{k}\theta_{i,j}s_j} \quad \text{where} \quad \mu_i = \frac{N_i}{\int_\Omega e^{\sum_{j=1}^{k}\theta_{i,j}s_j}} \quad (2.26)$$

provided the integral in (2.26) makes sense. Substitute ρ_i from (2.26) in the definition (2.8) of Ψ to obtain

$$\mathscr{D}_{\vec{N}}(\vec{s}) = -\sum_{i=1}^{n} a_i N_i \ln\left[\int_\Omega e^{\sum_{j=1}^{k}\theta_{i,j}s_j}dx\right]$$

$$+\frac{1}{2}\sum_{j=1}^{k}\sum_{l=1}^{k} b_{l,j}\int_\Omega\left[\nabla s_j\cdot\nabla s_l + \alpha s_j s_l\right]dx - \sum_{j=1}^{k}\int_\Omega F_j s_j dx. \quad (2.27)$$

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The singular limit $\vec{v}=0$ under the initial data $\rho(\cdot,0)$ satisfying $\int_\Omega \rho(x,0)dx = N_i$, $1\leqslant i\leqslant n$, is now *defined*, analogously to (2.20), as

$$\frac{\partial \vec{s}}{\partial \tau} = \mathbf{D}_\sigma^{-1}\mathbf{B}^{-1}\frac{\delta \mathscr{D}_{\vec{N}}}{\delta \vec{s}}. \tag{2.28}$$

The explicit form of (2.28) is given by

$$\sigma_j \frac{\partial s_j}{\partial t} = (\Delta - \alpha)s_j + \sum_{i=1}^{n} \gamma_{i,j} \frac{N_i e^{\sum_{l=1}^{k}\theta_{i,l}s_j}}{\int_\Omega e^{\sum_{l=1}^{k}\theta_{i,l}s_j}} + f_j. \tag{2.29}$$

The analogue of Lemma 2, Corollary 2 and Theorem 2 evidently holds for the singular limit (2.29). We summarize it in Theorem 3.

Theorem 3 *Let* $\vec{s}(\cdot,t)$ *be a classical solution of* (2.29). *Then*

$$\frac{d}{dt}\mathscr{D}_{\vec{N}}(\vec{s}(\cdot,t)) = -\int_\Omega \left[\frac{\partial \vec{s}}{\partial t}\right]^T \mathbf{BD}_\sigma \frac{\partial \vec{s}}{\partial t} dx \tag{2.30}$$

In particular, $\mathscr{D}_{\vec{N}}(\vec{s}(\cdot,t))$ *is monotone non-increasing along time-dependent solution of* (2.28) *if* $\mathbf{BD}_\sigma + (\mathbf{BD}_\sigma)^T$ *is positive definite.* $\{\vec{\rho}^{(0)}, \vec{s}^{(0)}\}$ *is a stationary solution of* (1.4, 1.5) *subjected to* (2.11) *iff* $\{\vec{s}^{(0)}\}$ *is a critical point of* $\mathscr{D}_{\vec{N}}$ *and* $\vec{\rho}^{(0)}$ *given by* (2.26).

3 Equilibria in the conflict-free case

Our interest in this section is the case in which the functionals Ψ, $\mathscr{F}$ are monotone for the corresponding systems (1.4, 1.5) and (2.20), respectively. It is evident from Corollary 1 and Corollary 2 that a conflict-free condition is necessary for Ψ to be monotone for (1.4, 1.5), and sufficient for $\mathscr{F}$ to be monotone for (2.20). Hence, in this section we shall impose the standing assumption of a conflict-free system. This is equivalent to $a_i > 0$ for $1\leqslant i \leqslant n$ which, in turn, is equivalent to $\mathbf{Y}_{\vec{N}} \equiv \mathbf{Y}_{\vec{N}}^+$ (cf. (2.24)). In addition, we shall concentrate on two-dimensional domains Ω and Dirichlet b.c. (2.1). To elaborate, we summarize our standing assumption below:

H0

(i) $\Omega \subset \mathbb{R}^2$ is a bounded domain with a C^1 smooth boundary.

(ii) The source terms f_j are in $\mathbf{L}^2(\Omega)$ for any $1\leqslant j\leqslant k$.

(iii) The system is conflict-free.

(iv) The Dirichlet boundary conditions (2.1) hold for the sensitivity agents s_j.

In addition to the standing assumption (H0), we shall refer to some of the following assumptions:

H1 All populations are self-attracting, i.e. $\lambda_{i,i} > 0$ for all $1\leqslant i\leqslant n$.

H1.1 All populations are mutually attractive, i.e. $\lambda_{i,l} > 0$ for all $1\leqslant i,l\leqslant n$,

H1.2 The matrix $\mathbf{B}$ is positive definite, i.e. $\mathbf{X}^+ = \mathbf{X}$.

The fact that Assumption (H1.1) implies (H1) is trivial. In addition (H1.2), together with the conflict-free assumption ($a_i > 0$), implies assumption (H1), since

$$(\vec{\gamma}_i)^T \mathbf{B}\vec{\gamma}_i = a_i\lambda_{i,i} > 0.$$

In certain cases, we shall replace (H1.2) by the still stronger assumption:

H1.2.1 The matrix $\mathbf{D}_\sigma\mathbf{B} + \mathbf{B}\mathbf{D}_\sigma$ is positive definite.

Remark 3 *Assumption* (H1.2.1) *is indeed stronger than* (H1.2). *In fact, the following is an elementary exercise: If* $\mathbf{B}$ *and* $\mathbf{D}$ *are symmetric matrices,* $\mathbf{D}$ *and* $\mathbf{DB} + \mathbf{BD}$ *are positive definite, then* $\mathbf{B}$ *is positive definite as well.*

The purpose of this section is to prove the stability of minimizers under either (1.4, 1.5) or (2.20), respectively. For this, we need first to prove the global (in time) solvability of the above systems, as well as the actual existence of such equilibria. To answer the last question, we introduce the following theorem.

Theorem 4 *Assume* (H0) *and either*

(i) (H1) *and* Ω *is the disk* $|x| < R \subset \mathbb{R}^2$, *or*

(ii) (H1.1).

Assume, in addition,

(iii) *For any* $I \subset \{1, 2 \ldots n\}$, $I \neq \emptyset$, *the inequality*

$$8\pi\left(\sum_{i\in I} a_i N_i\right) - \sum_{i\in I}\sum_{l\in I} a_i\lambda_{i,l}N_iN_l > 0$$

holds. Then

(a) *If alternative* (ii) *holds, then* $\mathscr{F}$ *is bounded from below on* $\mathbf{Y}_{\vec{N}}$, *and there exists a minimizer* $\{\vec{\rho}^{(0)}\} \in \mathbf{Y}_{\vec{N}}$ *of* $\mathscr{F}$. *If alternative* (i) *holds, then the same is true for the set of all* radial *functions* $\vec{\rho} = \vec{\rho}(|x|)$ *in* $\mathbf{Y}_{\vec{N}}$.

(b) *If condition* (H1.2) *replaces* (H1.1), *then* Ψ *and* $\mathscr{D}_{\vec{N}}$ *are bounded from below on* $\mathbf{X} \times \mathbf{Y}_{\vec{N}}$ *and* $\mathbf{X}$, *respectively. Moreover, a minimizer* $\{\vec{\rho}^{(0)}, \vec{s}^{(0)}\}$ *(resp.* $\{\vec{s}^{(0)}\}$*) exists for* Ψ *and* $\mathscr{D}_{\vec{N}}$ *on the underlying spaces.*

Proof of Theorem 4 The first part of the theorem follows from Theorem 1.1 in Chipot *et al.* [3]. To show the second part, we introduce:

Lemma 3 *Under assumptions* (H0) *and* (H1.2), *the following are equivalent (Here* $\vec{\rho}^{(0)}$ *and* $\vec{s}^{(0)}$ *are related by* (2.26) *and* (2.16), *respectively):*

(i) $\vec{\rho}^{(0)}$ *is a local (global) minimizer of* $\mathscr{F}$ *on* $\mathbf{Y}_{\vec{N}}$.

(ii) $\vec{s}^{(0)}$ *is a local (global) minimizer of* $\mathscr{D}_{\vec{N}}$ *on* $\mathbf{X}$.

(iii) $\{\vec{\rho}^{(0)}\ ;\ \vec{s}^{(0)}\}$ *is a local (global) minimizer of* Ψ *on* $\mathbf{Y}_{\vec{N}} \otimes \mathbf{X}$.

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Before turning to the proof of Lemma 3, we need

Definition 3 *Let $\vec{S}_{\vec{\rho}}$ be the extremizer of $\Psi(\vec{\rho},\vec{s})$ on $\mathbf{X}$, where $\vec{\rho}$ is prescribed. Equivalently, $\vec{S}_{\vec{\rho}}$ is the solution vector of* (2.15)*, where $\vec{\rho}$ is the source term. Likewise, let $\vec{\rho}_{\vec{s}} \in \mathbf{Y}_{\vec{N}}$ be the extremizer of $\Psi(\vec{\rho},\vec{s})$ on $\mathbf{Y}_{\vec{N}}$ where $\vec{s}$ is prescribed. Equivalently, $\vec{\rho}_{\vec{s}}$ is given by* (2.26).

Claim Under assumption (H0), $\vec{S}_{\vec{\rho}}$ is a continuous mapping from $\mathbf{Y}_{\vec{N}}$ to $\mathbf{X}$, and $\vec{\rho}_{\vec{s}}$ is a continuous mapping from $\mathbf{X}$ to $\mathbf{Y}_{\vec{N}}$.

Proof of Lemma 3 Let $\vec{\rho}^{(0)}$ be a local minimizer of $\mathscr{F}$. To show (i) $\Longrightarrow$ (iii),

$$\Psi(\vec{\rho},\vec{s}) \geqslant \Psi\left(\vec{\rho},\vec{S}_{\vec{\rho}}\right) \equiv \mathscr{F}(\vec{\rho}) > \mathscr{F}\left(\vec{\rho}^{(0)}\right)$$

for any $\vec{\rho}$ in a $\mathbf{Y}_{\vec{N}}$ neighbourhood of $\vec{\rho}^{(0)}$, and for any $\vec{s} \in \mathbf{X}$. The argument for (i) $\Longrightarrow$ (iii) is analogous.

Assume now (iii). Then

$$\Psi(\vec{\rho},\vec{s}) \geqslant \Psi\left(\vec{\rho}^{(0)},\vec{s}^{(0)}\right) \tag{3.1}$$

for any $\{\vec{\rho};\vec{s}\}$ in a $\mathbf{Y}_{\vec{N}} \oplus \mathbf{X}$-neighbourhood of $\left\{\vec{\rho}^{(0)};\vec{s}^{(0)}\right\}$. By our claim, $\vec{S}_{\vec{\rho}}$ is also in a $\mathbf{X}$-neighbourhood of $\vec{s}^{(0)}$, hence

$$\mathscr{F}(\vec{\rho}) \equiv \Psi\left(\vec{\rho},\vec{S}_{\vec{\rho}}\right) \geqslant \Psi\left(\vec{\rho}^{(0)},\vec{s}^{(0)}\right) \equiv \mathscr{F}\left(\vec{\rho}^{(0)}\right).$$

for any $\vec{\rho}$ in a $\mathbf{Y}_{\vec{N}}$ neighbourhood of $\vec{\rho}^{(0)}$, hence (iii) $\Longrightarrow$ (i). Analogously, (iii) $\Longrightarrow$ (ii). This implies the proof of Lemma 3 and Theorem 4. □

□

By Theorem 4 we obtain a non-trivial set of critical points of $\mathscr{F}$ in $\mathbf{Y}_{\vec{N}}$, hence a non-trivial set of critical points of $\mathscr{D}_{\vec{N}}$ on $\mathbf{X}$ and of Ψ on $\mathbf{X} \times \mathbf{Y}_{\vec{N}}$. To prove the stability of such minimizers we need, in particular, the global solvability of the corresponding systems:

Theorem 5 *Assume the conditions of Theorem 4. Then, for any $\vec{\rho}(\cdot,0) \in \mathbf{Y}_{\vec{N}}$ ($\vec{\rho}(\cdot,0) = \vec{\rho}(|x|,0)$ if alternative* (i) *holds) there exists a global (in time) classical solution of* (2.20) *with $\vec{\rho}(\cdot,0)$ as an initial data. If we replace conditions* (H1) *in* (i) *and* (H1.2) *in* (ii) *by* (H1.2.1)*, then the same holds for the systems* (1.4, 1.5) *and* (2.29)*, where $\{\vec{\rho}(\cdot,0),\vec{s}(\cdot,0)\} \in \mathbf{Y}_{\vec{N}} \times \mathbf{X}$ and $\{\vec{s}(\cdot,0)\}$, respectively, are initial data. Moreover, the limit*

$$\lim_{t\to\infty} \vec{\rho}(\cdot,t) = \vec{\rho}^{(0)}; \quad \lim_{t\to\infty} \vec{s}(\cdot,t) = \vec{s}^{(0)} \tag{3.2}$$

holds in $\mathbf{L}^\infty$, where $\vec{\rho}^{(0)} \in \mathbf{Y}_{\vec{N}}$ and $\vec{s}^{(0)} \in \mathbf{X}$ are critical points of $\mathscr{F}$ and $\mathscr{D}_{\vec{N}}$, respectively.

Corollary 3 *Under the assumptions of Theorem 4, $\{\vec{\rho}^{(0)}\}$ is a stable equilibrium of* (2.20) *iff it is a local minimizer of $\mathscr{F}$ on $\mathbf{Y}_{\vec{N}}$. If this is the case, then the corresponding $\vec{s}^{(0)}$ is a local minimizer of $\mathscr{D}_{\vec{N}}$ on $\mathbf{X}$, $\{\vec{\rho}^{(0)},\vec{s}^{(0)}\}$ is a local minimizer of Ψ in $\mathbf{Y}_{\vec{N}} \oplus \mathbf{X}$ and are stable equilibria of* (2.29) *and* (1.4, 1.5)*, respectively.*

Proof of Theorem 5 The global solvability and regularity of solutions to the system (2.20) and (1.4, 1.5) is a slight generalization of known results for the case of single component system [1]. The local solvability and regularity of (2.28) is evident by elementary arguments.

To obtain the global solvability of (2.28) (which is the easier part of the proof) we only need to derive an *a priori* $\mathbf{L}^p(\Omega)$ estimate on the right-hand side of (2.29) for some $p > 1$. We know by Theorem 4 that $\mathscr{D}_{\vec{N}}$ is bounded from below. Since $\mathscr{D}_{\vec{N}}$ is non-increasing by Theorem 3 and assumption (H1.2.1), it follows that $\mathscr{D}_{\vec{N}}(\vec{s}(\cdot,t))$ is bounded from below for any $t \geqslant 0$ for which the local solution exists. Denote $\mathscr{D}^{\varepsilon}_{\vec{N}}$ the functional $\mathscr{D}_{\vec{N}}$ where $\mathbf{B}$ is replaced by $\mathbf{B} - \varepsilon\mathbf{I}$, where $\mathbf{I}$ is the unit $k \times k$ matrix and $\varepsilon > 0$. Evidently, we may choose ε small enough for which the conditions of Theorem 4 are still satisfied for $\mathscr{D}^{\varepsilon}_{\vec{N}}$. Then

$$\varepsilon\,||\vec{s}(\cdot,t)||_{\mathbf{X}} = \mathscr{D}_{\vec{N}}(\vec{s}(\cdot,t)) - \mathscr{D}^{\varepsilon}_{\vec{N}}(\vec{s}(\cdot,t)) \leqslant \mathscr{D}_{\vec{N}}(\vec{s}(\cdot,0)) + C$$

where $\mathscr{D}^{\varepsilon}_{\vec{N}} > -C$ on $\mathbf{X}$. Hence $||\vec{s}(\cdot,t)||_{\mathbf{X}}$ is a priori estimated for $t > 0$. In particular, $\vec{\theta}_i \cdot \vec{s}(\cdot,t)$ is uniformly bounded in $\mathbf{H}^1_0$ for any $t \geqslant 0$. Using the Trudinger inequality [11], we obtain that $\exp(\vec{\theta}_i \cdot \vec{s}(\cdot,t))$ is uniformly bounded in $\mathbf{L}^p(\Omega)$ for *any* $1 \leqslant p < \infty$ and any $1 \leqslant i \leqslant n$. In addition, we observe that $\int \exp(\vec{\theta}_i \cdot \vec{s}(\cdot,t))$ is uniformly bounded from *below*, for otherwise $\mathscr{D}_{\vec{N}}(\vec{s}(\cdot,t))$ will be unbounded from above (cf. (2.27)). This implies that the source terms in (2.29) are uniformly bounded in $\mathbf{L}^p$ for some $p > 1$.

We turn now to the proof of global solvability in the case of the system (1.4, 1.5). The corresponding problem for a single-component system and Neumann b.c. was proved in Biler [1]. For this reason, we shall only sketch the proof, indicating the points of difference between our case and Biler's.

We shall use standard notation $||\cdot||_{k,p}$ for the Sobolev norm in $W^{k,p}$ and shall abbreviate $||\cdot||_{0,p} \equiv |\cdot|_p$.

We first show a uniform (in time) estimate on $|\rho_i \ln \rho_i|_1$ for $1 \leqslant i \leqslant n$. Set $i = 1$ and $a^{\varepsilon}_1 = a_1 - \varepsilon$, $a^{\varepsilon}_i = a_i$ for $n \geqslant i > 1$, $\theta^{\varepsilon}_{1,j} = \left(1 + \varepsilon(a_1 - \varepsilon)^{-1}\right)\theta_{1,j}$ and $\theta^{\varepsilon}_{i,j} = \theta_{i,j}$ for $n \geqslant i > 1$, $1 \leqslant j \leqslant k$. Note that $a_i\theta_{i,j} = a^{\varepsilon}_i\theta^{\varepsilon}_{i,j}$ for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k$. Let us denote Ψ^{ε} the functional (2.8), where $\{a_i\}$ and $\{\theta_{i,j}\}$ are replaced by $\{a^{\varepsilon}_i\}$ and $\{\theta^{\varepsilon}_{i,j}\}$. Again, we choose ε sufficiently small for which the conditions of Theorem 4 still hold for Ψ^{ε}. It then follows that

$$\varepsilon\,|\rho_1(\cdot,t)\ln\rho_1(\cdot,t)|_1 = \Psi(\vec{\rho}(\cdot,t),\vec{s}(\cdot,t)) - \Psi^{\varepsilon}(\vec{\rho}(\cdot,t),\vec{s}(\cdot,t)) \leqslant C + \Psi(\vec{\rho}(\cdot,0),\vec{s}(\cdot,0)),$$

where $\Psi^{\varepsilon} > -C$. By symmetry $|\rho_i \ln \rho_i|_1$ is uniformly bounded for all $1 \leqslant i \leqslant n$.

Next we show *a priori* estimate for $||\vec{s}||_{\mathbf{X}}$ (equivalent to $||s||_{1,2}$) independent of t. For this, choose some $\vec{\zeta} \in \mathbb{R}^k$ and set $\mathbf{B}^{\varepsilon} = \mathbf{B} - \varepsilon\vec{\zeta} \otimes \vec{\zeta}^T$. If we replace $\vec{\gamma}_i$ by

$$\vec{\gamma}^{\varepsilon}_i := \left[\mathbf{I} + \left(\mathbf{B} - \varepsilon\vec{\zeta} \otimes \vec{\zeta}^T\right)^{-1} \vec{\zeta} \otimes \vec{\zeta}^T\right]\vec{\gamma}_i,$$

then $\mathbf{B}\vec{\gamma}_i = \mathbf{B}^{\varepsilon}\vec{\gamma}^{\varepsilon}_i := a_i\vec{\theta}_j$. Set Ψ^{ε} as before, where $\mathbf{B}$ replaced by $\mathbf{B}^{\varepsilon}$, then

$$\varepsilon\left|\vec{\zeta} \cdot \nabla\vec{s}(\cdot,t)\right|^2_2 = \Psi(\vec{\rho}(\cdot,t),\vec{s}(\cdot,t)) - \Psi^{\varepsilon}(\vec{\rho}(\cdot,t),\vec{s}(\cdot,t)) \leqslant C + \Psi(\vec{\rho}(\cdot,0),\vec{s}(\cdot,0)).$$

Since $\vec{\zeta}$ is arbitrary we obtain a priori estimate for $||\vec{s}||_{1,2}$.

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Next we multiply the ith equation of (1.4) by ρ_i and integrate by parts, using the boundary condition (2.1), to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}|\rho_i|_2^2 + v_i|\nabla\rho_i|_2^2 = \sum_{j=1}^k \theta_{i,j}\int_\Omega \rho_i\nabla\rho_i\cdot\nabla s_j \leqslant C|\rho_i|_4|\nabla\rho_i|_2\,|\nabla\vec{s}|_4\,. \tag{3.3}$$

To estimate the right-hand side of (3.3) we use the logarithmic inequality

$$|\rho|_4^4 \leqslant \delta||\rho||_{1,2}^3|\rho\ln\rho|_1 + C_\delta|\rho|_1^4$$

for arbitrary small $\delta > 0$ (cf. [2, eq. (32)], and the Sobolev inequality

$$|\nabla s|_4 \leqslant C|\nabla^3 s|_2^{1/4}|\nabla s|_2^{3/4}.$$

In addition,

$$|\rho|_2^2 \leqslant C||\rho||_{1,2}|\rho|_1$$

implies that $|\rho_i|_2$ can be controlled by $||\rho_i||_{1,2}$, using $|\rho_i|_1 \equiv N_i$. Finally, the estimate

$$\int_0^t ||s_j(\cdot,t)||_{3,2}^2 \leqslant C\left(\max_{1\leqslant i\leqslant n}\int_0^t ||\rho_i(\cdot,s)||_{1,2}^2 ds + 1\right) \tag{3.4}$$

follows from the diffusion equations (1.5). The same estimate was given in Biler [1] for Neumann b.c., where integration by parts of the corresponding diffusion equation is possible. In our case (2.1), such integration by parts is impossible, so we justify (3.4) in the appendix.

Collecting all the above estimates, we end up with an *a priori* bound on $|\vec{\rho}|_2$ which is sufficient for extending the local solution into a global one and to (3.2) [2]. The case of the system (2.20) requires only minor modifications. □

4 Limit cycles

From Lemma 2 we obtain that, in the case of a conflict-free system ($a_i > 0$), the functional $\mathscr{F}$ is monotone for the singular system (2.20), hence no limit cycles can exist. The same result holds for the system (1.4, 1.5) under the additional condition that $\mathbf{D}_\sigma\mathbf{B} + \mathbf{B}\mathbf{D}_\sigma$ is positive definite (see (2.10)). We shall now demonstrate, via an explicit example, that if the condition $\mathbf{D}_\sigma\mathbf{B} + \mathbf{B}\mathbf{D}_\sigma > 0$ is relaxed into the condition that $\mathbf{B}$ is positive definite (cf. the remark below condition H1.2.1, § 3) then non-stationary attractors may exist even in the case of a conflict-free system.

For our result we use the Neumann b.c. (2.2) and $f_j = 0$. In this case there always exists a constant steady state, determined only by the populations N_i, namely

$$\rho_i^{(0)} = \frac{N_i}{|\Omega|}; \quad i = 1,\dots n \tag{4.1}$$

where $|\Omega|$ is the volume (area) of the domain Ω. The corresponding sensitivities are (c.f (1.5))

$$s_j^{(0)} = \frac{\sum_i \gamma_{i,j}N_i}{\alpha|\Omega|}; \quad j = 1,\dots k. \tag{4.2}$$

We shall consider the singular limit (2.29). Our result is based on the Andronov–Hopf

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bifurcation theorem. To start with, let us compute the linearization of (2.29) at the steady state. The second derivative $\delta^2_{\vec{s}}\mathscr{D}_{\vec{N}}$ at a stationary $\vec{s}^{(0)}$ induces the operator $\mathscr{L}$ via

$$\left\langle \vec{\psi}, \delta^2_{\vec{s}^{(0)}}\mathscr{D}_{\vec{N}}\vec{\phi} \right\rangle = \left\langle \vec{\psi}, \mathscr{L}\vec{\phi} \right\rangle \ ,$$

where

$$\mathscr{L}\vec{\phi} = -\mathbf{B}\,(\Delta - \alpha)\,\vec{\phi} - \mathbf{G}\vec{\phi} + \vec{\mathbf{P}}(\vec{\phi}).$$

Here $\mathbf{G}$ is a $k \times k$ matrix-valued function given by

$$\mathbf{G}_{j,l} = \sum_{i=1}^{k} a_i \theta_{i,j} \theta_{i,l} \rho_i^{(0)}, \tag{4.3}$$

while $\vec{\mathbf{P}}$ is a projection-matrix operator given by

$$\vec{\mathbf{P}} = \{\mathbf{P}_{j,l}\}\ ; \quad \mathbf{P}_{j,l}(\phi) \equiv \sum_{i=1}^{k} a_i \theta_{i,j} \theta_{i,l} \mathbf{P}_{\rho_i^{(0)}}(\phi)$$

and

$$\mathbf{P}_{\rho_i^{(0)}}(\phi) \equiv \frac{\rho_i^{(0)}}{N_i} \int_\Omega \rho_i^{(0)} \phi\, dx.$$

Using (4.1) we set

$$\mathbf{G}_{j,l} = \sum_{i=1}^{k} a_i \theta_{i,j} \theta_{i,l} \frac{N_i}{|\Omega|}. \tag{4.4}$$

We note that $\mathscr{L}$ is a self-adjoint operator. In addition, it is positive definite if and only if the operator

$$\mathbf{B}^{-1/2}\mathscr{L}\mathbf{B}^{-1/2} = -\,(\Delta - \alpha) - \mathbf{B}^{-1/2}\mathbf{G}\mathbf{B}^{-1/2} + \mathbf{B}^{-1/2}\vec{\mathbf{P}}\mathbf{B}^{-1/2}$$

is positive definite.

Given a domain $\Omega \in \mathbb{R}^m$, let

$$0 = \lambda_0 < \lambda_1 \leqslant \ldots \leqslant \lambda_j \to \infty \tag{4.5}$$

be the eigenvalues of the negative Laplacian $-\Delta$ under Neumann b.c. in Ω.

Lemma 4 *The eigenvalues and eigenfunctions of $\mathbf{B}^{-1/2}\mathscr{L}\mathbf{B}^{-1/2}$ are given by*

$$\alpha; \quad \psi_0^{(j)}; \ j = 1, 2 \ldots k \quad \textit{is a basis of} \quad \mathbb{R}^k$$

$$\lambda_i + \alpha - \mu_j; \quad \psi_i^{(j)} = \vec{v}_j \phi_i \quad i = 1, 2 \ldots, j = 1, 2 \ldots k \tag{4.6}$$

where λ_i are given by (4.5), ϕ_i the corresponding eigenfunctions of the Laplacian, while μ_j and $\vec{v}_j \in \mathbb{R}^k$ are the eigenvalues of the eigenvectors of the matrix $\mathbf{B}^{-1/2}\mathbf{G}\mathbf{B}^{-1/2}$. In particular, $\mathbf{B}^{-1/2}\mathscr{L}\mathbf{B}^{-1/2}$ and $\mathscr{L}$ are positive iff

$$\lambda_1 + \alpha - \mu > 0 \tag{4.7}$$

where $\mu = \max \mu_j$.

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Proof The non-constant eigenstates are obtained by separation of variables. For the constant states ψ_0, note that $\left(\vec{\mathbf{P}}-\mathbf{G}\right)\psi^j_{(0)}=0$ hence $\mathbf{B}^{-1/2}\mathscr{L}\mathbf{B}^{-1/2}\psi^j_{(0)}=\alpha\psi^{(0)}$ as claimed. □

Corollary 4 *Under condition* (4.7)*, the constant steady state* (4.2) *is stable if* $\sigma_j=\sigma_l>0$ *for all* $1\leqslant l,j\leqslant k$.

Proof By Theorem 4 we obtain that $\mathscr{D}_N$ is monotone if $\mathbf{D}_\sigma\mathbf{B}+\mathbf{B}\mathbf{D}_\sigma$ is positive definite. Since $\mathbf{D}$ is a multiple of the identity and $\mathbf{B}$ is positive definite by assumption, then we have the monotonicity of $\mathscr{D}_N$ in our case. By Lemma 1 we have that the constant state $\vec{s}^{(0)}$ is a local minimizer of $\mathscr{D}_N$. This implies Lyapunov-stability. □

Let us now write the linearization of (2.28) at the steady solution $\vec{s}^{(0)}$:

$$\frac{\partial\vec{\phi}}{\partial t}=-\mathbf{D}^{-1}_{\vec{\sigma}}\mathbf{B}^{-1}\mathscr{L}\vec{\phi}. \tag{4.8}$$

Let $\vec{\phi}(0)\in\mathbf{Z}:=\mathbb{H}^1(\Omega,\mathbb{R}^k)$. Then (4.8) induces a semigroup flow in $\mathbf{Z}$. We may split $\mathbf{Z}$ into a direct sum $\mathbb{R}^k\oplus\mathbf{Z}_0$ (i.e. $\mathbf{Z}_0$ is the orthogonal complement to the constants in $\mathbf{Z}$). Since $(\vec{\mathbf{P}}-\mathbf{G})\vec{\phi}=0$ on constants and $\mathbf{P}\vec{\phi}=0$ on the orthogonal complement, we may reduce the linearized equation to the orthogonal complement of constants, and omit the action of $\mathbf{P}$ from $\mathscr{L}$. Thus

$$\frac{\partial\vec{\phi}}{\partial t}=\mathbf{D}^{-1}_{\vec{\sigma}}\left\{(\Delta-\alpha)\mathbf{I}+\mathbf{B}^{-1}\mathbf{G}\right\}\vec{\phi};\quad \phi(0)\in\mathbf{Z}_0 \tag{4.9}$$

defines a semigroup flow on $\mathbf{Z}_0$.

Let

$$\mathbf{G}^*=\mathbf{B}^{-1/2}\mathbf{G}\mathbf{B}^{-1/2}.$$

With this definition we may rewrite (4.9) as

$$\frac{\partial\vec{\phi}}{\partial t}=\mathbf{D}^{-1}_{\vec{\sigma}}\mathbf{B}^{-1/2}\left\{(\Delta-\alpha)\mathbf{I}+\mathbf{G}^*\right\}\mathbf{B}^{1/2}\vec{\phi}. \tag{4.10}$$

Assume that λ_1 is a simple (non-degenerate) eigenvalue of the Laplacian and let ϕ_1 be the corresponding eigenstate. Then the space $\mathbf{Z}_1=Sp\{\vec{v}\phi_1\ ;\ \vec{v}\in\mathbb{R}^k\}$ is a k-dimensional invariant space of $\mathbf{Z}_0$ for (4.10). With

$$\mathbf{Q}^*=\mathbf{G}^*-(\lambda_1+\alpha)\mathbf{I};\quad \mathbf{Q}=\mathbf{B}^{-1/2}\mathbf{Q}^*\mathbf{B}^{1/2},$$

we may represent (4.10) on $\mathbf{Z}_1$ simply as

$$\frac{\partial\vec{v}}{\partial t}=\mathbf{D}^{-1}_{\vec{\sigma}}\mathbf{Q}\vec{v}. \tag{4.11}$$

We shall now show that a set of mobility and production vectors can be found such that there exists $\vec{\sigma}=\{\sigma_1,\dots\sigma_k\}\in\left(\mathbb{R}^+\right)^k$ and positive population numbers $\{N_1,\dots N_n\}$ for which the system (2.28) satisfies the condition of the Hopf bifurcation at the critical solution (4.2). Let $\mathbf{Q}^*$ be some symmetric, negative definite matrix, and $\mathbf{B}$ a positive definite matrix so that $\mathbf{Q}=\mathbf{B}^{-1/2}\mathbf{Q}^*\mathbf{B}^{1/2}$ has a positive diagonal element. As an example, we take $k = 2$ and define

$$\mathbf{Q}^* = \begin{pmatrix} q_1 & p \\ p & q_2 \end{pmatrix}, \qquad \mathbf{B}^{1/2} = \begin{pmatrix} B & b \\ b & B \end{pmatrix},$$

where $q_1 q_2 - p^2 > 0$, $B^2 - b^2 > 0$, and $B > 0$, $q_1 + q_2 < 0$. The diagonal elements of $\mathbf{Q}$ are given by $q_1 B^2 - q_2 b^2$ and $q_2 B^2 - q_1 b^2$, respectively. If $-q_1$ is sufficiently large then we can choose a pair of positive $\{\sigma_1, \sigma_2\}$ for which

$$Trace\left[\mathbf{D}_\sigma^{-1}\mathbf{Q}\right] := \frac{q_1 B^2 - q_2 b^2}{\sigma_1} + \frac{q_2 B^2 - q_1 b^2}{\sigma_2} = 0.$$

This implies that the spectrum of (4.11) is composed of a pair of purely imaginary eigenvalues and that $\vec{\sigma}$ is a Hopf bifurcation point for the original system (2.28) at the critical solution (4.2), as required.

We now show how the corresponding production and mobility vectors as well as the population densities can be obtained, such that the resulting system is conflict-free. Let us first reconstruct the matrix $\mathbf{G}$ from the matrices $\mathbf{Q}$, $\mathbf{Q}^*$ and $\mathbf{B}$ given above. Note that

$$\mathbf{G} = \mathbf{B}^{1/2}\mathbf{G}^*\mathbf{B}^{1/2} = \mathbf{B}\left[\mathbf{Q} + (\lambda_1 + \alpha)\mathbf{I}\right]$$

is a symmetric $k \times k$ matrix. Moreover, we can shrink the domain Ω so that the second eigenvalue λ_1 of the Laplacian in Ω is sufficiently large, to guarantee that $\mathbf{G}$ is positive definite. Thus, if we chose $n = k$ and the mobility vectors $\vec{\theta}_j = \{\theta_{1,j}, \ldots \theta_{k,j}\} \in \mathbb{R}^k$ to be the eigenvectors of $\mathbf{G}$, then

$$\mathbf{G}_{j,l} = \sum_1^n \beta_i \theta_{i,j} \theta_{i,l} \tag{4.12}$$

where $\beta_i > 0$ are the eigenvalues of $\mathbf{G}$. Comparing (4.12) with (4.4) we may set a_i and the population densities N_i such that

$$a_i N_i = \beta_i |\Omega|.$$

The production vectors $\vec{\gamma}_i \in \mathbb{R}^n$ are now determined, via (2.6), by

$$\vec{\gamma}_i = a_i \mathbf{B}^{-1} \vec{\theta}_i.$$

Since a_i are all positive by the positivity of N_i and β_i, the system we obtained is consistent with the conflict-free assumption.

Conclusion 1

There exists a conflict-free system which admits a periodic solution in the singular limit $\vec{v} = 0$.

5 Conclusion

The Keller–Segel model of chemotaxis can be extended to a system of populations and sensitivity agents. For such a system, there is a structure which exhibit the interaction between different populations via the agents. In particular, the notion of ‘conflict’ can be introduced. A conflict free system admits, under certain additional assumption, a Lyapunov functional. Using this formulation, the steady states of the systems can be investigated via variational methods. In addition, this structure is useful to investigate time-periodic solutions in a parameter range where a Lyapunov functional does not exist.

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6 Appendix

Consider the equation

$$\frac{\partial u}{\partial t} = \Delta u + \rho \quad ; \quad \{x, t\} \in \Omega \times \mathbb{R}^+ \ ,$$

where $\Omega \in \mathbb{R}^2$ is a bounded domain and $u \equiv 0$ on $\partial\Omega \times \mathbb{R}^+$ and on $\Omega \times \{0\}$.

We will show the estimate:

$$\int_0^T |\nabla^3 u(\cdot, t)|^2 dt \leqslant \int_0^T ||\rho(\cdot, t)||_{1,2}^2 dt$$

Let $u_i(t)$, $r_i(t)$ be the Fourier coefficients of $u(\cdot, t)$, $\rho(\cdot, t)$ with respect to the eigenstates of the Dirichlet Laplacian. If λ_i are the eigenvalues of $-\Delta$, then $||\rho(\cdot, t)||_{1,2}^2$ is equivalent to $\sum(\lambda_i + 1)|r_i(t)|^2$ while $|\nabla^3 u(\cdot, t)|_2^2$ is equivalent to $\sum \lambda_i^3 |u_i(t)|^2$. In addition

$$u_i(t) = \int_0^t e^{\lambda_i(s-t)} r_i(s) ds$$

Let $\overline{r}_i = \sqrt{\lambda_i} r_i$. Then

$$\sum \lambda_i^3 |u_i|^2(t) = \sum \lambda_i^3 \left(\int_0^t e^{\lambda_i(s-t)} \hat{r}_i(s) ds \right)^2 \leqslant \sum \lambda_i \int_0^t e^{\lambda_i(s-t)} \overline{r}_i^2(s) ds$$

We now integrate over t from 0 to T to obtain

$$\int_0^T |\nabla^3 u(\cdot, t)|_2^2 dt \leqslant C \sum \lambda_i \int_0^T \int_0^t e^{\lambda_i(s-t)} \overline{r}_i^2(s) ds dt \leqslant \sum \int_0^T \overline{r}_i^2(t) dt \leqslant \int_0^T ||\rho(\cdot, t)||_{1,2}^2 dt.$$

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