The Compton Effect in Wave Mechanics. By Dr P. A. M. Dirac, St John's College.

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§ 1. *Introduction.*

The problem of the scattering of radiation by a free electron has been treated by the author* on the basis of Heisenberg's matrix mechanics, which was first modified to be in agreement with the principle of relativity. The main point of this modification is that, whereas in the non-relativity theory one deals with matrices whose elements vary with the time according to the law $e^{i\omega t}$, in the relativity theory the elements of the matrices must vary according to the law $e^{i\omega t'}$ where $t' = t - (l_1 x_1 + l_2 x_2 + l_3 x_3)/c$ if they are to determine correctly the radiation emitted in the direction specified by the direction cosines (l_1, l_2, l_3), x_1, x_2 and x_3 being the coordinates of the electron at the time t. These matrices were obtained by writing the Hamiltonian equation of the system in the form

$$H' - W' = 0 \qquad \text{.......................(1)},$$

where W' is a variable canonically conjugate to t' and H' commutes with t', and then using H' as an ordinary Hamiltonian function of a dynamical system that has W' for its energy and t' for its time variable.

This method is rather artificial; particularly so since on the quantum theory there is no unique way of writing the Hamiltonian equation in the form (1) [owing to ambiguity in the order of factors in H'], and it has to be proved that all reasonable ways of doing so lead to the same results. A more natural and more easily understood method of obtaining the matrices is provided by Schrödinger's wave mechanics†.

In this method we do not need to write the Hamiltonian equation in any special form, such as (1), in order to obtain Schrödinger's wave equation from it, by applying the rules which consist of letting the momentum symbols denote certain operators. When we have obtained the solutions ψ_α of the wave equation, then we can get a matrix representation of the dynamical variables by obtaining expansions of the type

$$A\psi_\alpha = \Sigma_{\alpha'} A_{\alpha'\alpha} \psi_{\alpha'},$$

* *Roy. Soc. Proc.* A, vol. 111, p. 405 (1926). This paper is referred to later by *loc. cit.*

† See Schrödinger's papers in the *Ann. d. Phys.*, vols. 79—81 (1926). For the general method of obtaining matrices from the wave mechanics, see section 2 of the author's paper, *Roy. Soc. Proc.* A, vol. 112, p. 661 (1926).

where A is a dynamical variable, and the coefficients $A_{\alpha'\alpha}$ are functions of any steadily increasing variable such as t or t'. These coefficients are then the elements of the matrix that represents A. If we do choose just t' to be the variable of which the $A_{\alpha'\alpha}$'s are functions, then we get the matrix representation that is suitable for determining the radiation emitted in the direction (l_1, l_2, l_3). The matrix elements would now be of the form $ae^{2\pi i\omega t'}$, where the a's and ω's are constants, and we would then take the ω's to be the frequencies and the a's (if A is a component of total polarisation) to determine, according to classical formulae, the intensities of the components of radiation emitted in the direction (l_1, l_2, l_3).

§ 2. *Solution of the Wave Equation.*

We consider a free electron exposed to plane polarised incident radiation of wave number $\nu/2\pi$ moving in the direction of the x_1 axis with its electric vector in the direction of the x_2 axis. The Hamiltonian equation of the system is*

$$m^2c^2 = W^2/c^2 - p_1^2 - [p_2 + a' \cos \nu (ct - x_1)]^2 - p_3^2,$$

where p_1, p_2, p_3 and $-W$ are the variables canonically conjugate to x_1, x_2, x_3 and t, and a' is a constant, which may be assumed to be small. [p_2 differs from the ordinary component of momentum in the x_2 direction by the term $a' \cos \nu (ct - x_1)$.]

Schrödinger's wave equation is now

$$\{m^2c^2 - W^2/c^2 + p_1^2 + [p_2 + a' \cos \nu (ct - x_1)]^2 + p_3^2\} \psi = 0 \quad \text{...(2)},$$

in which the symbols p_1, p_2, p_3 and W mean the operators

$$-ih\frac{\partial}{\partial x_1}, \quad -ih\frac{\partial}{\partial x_2}, \quad -ih\frac{\partial}{\partial x_3} \text{ and } ih\frac{\partial}{\partial t}.$$

We must find the solutions ψ_α of this equation, and we can then determine the matrix representing a component X of the polarisation by obtaining an expansion for $X\psi_\alpha$ of the form

$$X\psi_\alpha = \Sigma_{\alpha'} X_{\alpha'\alpha} \psi_{\alpha'},$$

where the coefficients $X_{\alpha'\alpha}$ are functions of the single variable t'.

We now, as in the previous treatment of the problem, apply the linear canonical transformation

$$\left.\begin{aligned} x_1' &= ct - x_1 & p_1 &= -p_1' + l_1 W'/c \\ x_2' &= x_2 & p_2 &= p_2' + l_2 W'/c \\ x_3' &= x_3 & p_3 &= p_3' + l_3 W'/c \\ t' &= t - (l_1 x_1 + l_2 x_2 + l_3 x_3)/c & W &= W' - cp_1' \end{aligned}\right\} \quad \text{...(3)}.$$

The purpose of this transformation is two-fold; first it introduces explicitly the variable t' in terms of which we want the matrix

* *Loc. cit.*, equation (21).

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elements, and secondly we shall be able to solve the wave equation in the new co-ordinates directly by separation of the variables. By substituting for p_1, p_2, p_3 and W in (2) their values given by the second set of equations (3), we find for the wave equation in the new co-ordinates, with neglect of terms involving a'^2,

$$\{m^2c^2 + 2W'/c \,.\, [(1-l_1)p_1' + l_2p_2' + l_3p_3' + l_2a'\cos\nu x_1'] \\ + p_2'^2 + p_3'^2 + 2a'p_2'\cos\nu x_1'\}\,\psi = 0 \quad \ldots\ldots(4).$$

The symbols p_1', p_2', p_3' and W' here mean the operators

$$-ih\frac{\partial}{\partial x_1'},\quad -ih\frac{\partial}{\partial x_2'},\quad -ih\frac{\partial}{\partial x_3'}\text{ and } ih\frac{\partial}{\partial t'}.$$

Since the wave equation (4) does not involve x_2', x_3' or t' explicitly, it will have solutions of the form

$$\psi_\alpha = e^{i\alpha_2 x_2'/h}\,.\,e^{i\alpha_3 x_3'/h}\,.\,e^{-i\alpha_4 t'/h}\,.\,\chi(x_1'),$$

where α_2, α_3 and α_4 are arbitrary constants, and $\chi(x_1')$ is a function of the single variable x_1'. With this form for ψ_α we have

$$p_2'\psi_\alpha = \alpha_2\psi_\alpha,\quad p_3'\psi_\alpha = \alpha_3\psi_\alpha,\quad W'\psi_\alpha = \alpha_4\psi_\alpha,$$

and, more generally, if $f(p_2', p_3', W')$ is any function of p_2', p_3' and W', which may also involve x_1' and p_1' (f is thus an operator) we have

$$f(p_2', p_3', W')\,\psi_\alpha = f(\alpha_2, \alpha_3, \alpha_4)\,\psi_\alpha.$$

This rule applied to the left-hand side of equation (4) gives

$$\{2\alpha_4/c\,.\,[(1-l_1)p_1' + l_2\alpha_2 + l_3\alpha_3 + l_2a'\cos\nu x_1'] \\ + b + 2a'\alpha_2\cos\nu x_1'\}\,e^{i\alpha_2x_2'/h}\,.\,e^{i\alpha_3x_3'/h}\,.\,e^{-i\alpha_4t'/h}\,.\,\chi(x_1') = 0,$$

where $$b = m^2c^2 + \alpha_2^2 + \alpha_3^2.$$

The factors $e^{i\alpha_2x_2'/h}$, $e^{i\alpha_3x_3'/h}$ and $e^{-i\alpha_4t'/h}$ may be cancelled out, and a simple differential equation for the function $\chi(x_1')$ will then be left, whose solution is

$$\chi(x_1') = \exp - i\left\{\left(l_2\alpha_2 + l_3\alpha_3 + \frac{cb}{2\alpha_4}\right)x_1' \\ + \frac{a'}{\nu}\left(l_2 + \frac{c\alpha_2}{\alpha_4}\right)\sin\nu x'\right\}\Big/(1-l_1)\,h.$$

The solution of equation (4) is thus

$$\psi_\alpha = \psi_{\alpha_2,\alpha_3,\alpha_4} = \exp i\left\{(1-l_1)(\alpha_2x_2' + \alpha_3x_3' - \alpha_4t') \\ - \left(l_2\alpha_2 + l_3\alpha_3 + \frac{cb}{2\alpha_4}\right)x_1' - \frac{a'}{\nu}\left(l_2 + \frac{c\alpha_2}{\alpha_4}\right)\sin\nu x_1'\right\}\Big/(1-l_1)\,h \\ \ldots\ldots\ldots\ldots(5),$$

and since it contains the three arbitrary parameters α_2, α_3, α_4, and may also be multiplied by a fourth arbitrary parameter, it is a general solution.

§ 3. *The Integrals of the Equations of Motion.*

The first integrals of the equations of motion are obviously p_2', p_3' and W'. We must now determine the second integrals. We can do this in a way that is completely analogous to the way followed on the classical theory when one is solving a problem by the Hamilton-Jacobi method. Put

$$\psi_\alpha = \exp iS_\alpha/h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Then S_α corresponds to the principal function in the Hamilton-Jacobi method, and $\partial S/\partial\alpha_r$ is a constant of integration of the system, where α_r is any one of the parameters occurring in ψ_α.

To prove this theorem on the quantum theory, differentiate (6) partially with respect to α_r. This gives

$$\frac{\partial\psi_\alpha}{\partial\alpha_r} = \frac{i}{h}e^{iS_\alpha/h}\frac{\partial S_\alpha}{\partial\alpha_r} = \frac{i}{h}\frac{\partial S_\alpha}{\partial\alpha_r}\psi_\alpha,$$

which shows that $\partial S_\alpha/\partial\alpha_r \,.\, \psi_\alpha$ is of the form

$$-ih\iiint c(\alpha', \alpha)\,\psi_{\alpha'}\,d\alpha_2'\,d\alpha_3'\,d\alpha_4',$$

where $c(\alpha', \alpha)$ is a certain very discontinuous "function" of α_2, α_3, α_4 and α_2', α_3', α_4', which is equal to zero except when $|\alpha_2 - \alpha_2'|$, $|\alpha_3 - \alpha_3'|$ and $|\alpha_4 - \alpha_4'|$ are very small, and is very large and positive when α_r' is a little greater than α_r, and very large and negative when α_r' is a little less than α_r. Now $-ihc(\alpha', \alpha)$ gives the elements of the matrix that represents $\partial S_\alpha/\partial\alpha_r$, and since these elements are all constants (*i.e.* independent of t'), $\partial S_\alpha/\partial\alpha_r$ must be a constant of integration of the system.

To apply this theorem in the present example, we must differentiate the index of the exponential in (5) partially with respect to α_2, α_3 and α_4. This gives

$$\left.\begin{aligned}(1-l_1)\,x_2' - \left(l_2 + \frac{c\alpha_2}{\alpha_4}\right)x_1' - \frac{ca'}{\nu\alpha_4}\sin\nu x_1' &= c_2\\ (1-l_1)\,x_3' - \left(l_3 + \frac{c\alpha_3}{\alpha_4}\right)x_1' &= c_3\\ -(1-l_1)\,t' + \frac{cb}{2\alpha_4^2}x_1' + \frac{ca'\alpha_2}{\nu\alpha_4^2}\sin\nu x_1' &= c_4\end{aligned}\right\} \quad \ldots\ldots(7),$$

where c_2, c_3 and c_4 are the second integrals. These formulae correspond to equations (35), (34) and (30) respectively in *loc. cit.*, which were there obtained by more laborious methods.

§ 4. *The Matrix Elements.*

We must now find the matrices that represent the components of the polarisation of the system. We need do this only for two components in directions that are perpendicular to the direction of emission (l_1, l_2, l_3) and mutually perpendicular. We can take these components to be the quantities

$$X = l_3 x_1 - \frac{l_2 l_3}{1 - l_1} x_2 + \left(\frac{l_2^2}{1 - l_1} - l_1\right) x_3$$

and
$$Y = l_2 x_1 + \left(\frac{l_3^2}{1 - l_1} - l_1\right) x_2 - \frac{l_2 l_3}{1 - l_1} x_3$$

multiplied by the electronic charge, as in the previous treatment of the problem*. Expressed in terms of the dashed variables, X and Y are given by

$$(1 - l_1) X = l_3 ct' - l_3 x_1' + (1 - l_1) x_3'$$
$$(1 - l_1) Y = l_2 ct' - l_2 x_1' + (1 - l_1) x_2'.$$

With the help of equations (7) we find

$$(1 - l_1) X = c_3 + \frac{2\alpha_3 \alpha_4}{b} c_4 + l_3 ct' + \frac{2(1 - l_1) \alpha_3 \alpha_4}{b} t' - \frac{2ca' \alpha_2 \alpha_3}{\nu b \alpha_4} \sin \nu x_1'$$

$$(1 - l_1) Y = c_2 + \frac{2\alpha_2 \alpha_4}{b} c_4 + \frac{2(1 - l_1) \alpha_2 \alpha_4}{b} t' + \left(\frac{ca'}{\nu \alpha_4} - \frac{2ca' \alpha_2^2}{\nu b \alpha_4}\right) \sin \nu x_1' + l_2 ct' \qquad \ldots\ldots\ldots\ldots(8).$$

In these expressions for X and Y we know the matrices that represent all the terms except those involving $\sin \nu x_1'$. [The matrices representing the c's have already been discussed in the preceding section, while the matrix that represents t' is simply the diagonal matrix with diagonal elements t'.] Thus, in order to complete the matrix determination of X and Y, we have only to find the matrix that represents $\sin \nu x_1'$ and multiply it by the (constant) coefficients of $\sin \nu x_1'$ in the expressions (8). Since we are working only to the first order in a', and an a' occurs already in the coefficients of $\sin \nu x_1'$, we may neglect a' altogether when finding the matrix that represents $\sin \nu x_1'$.

With this approximation, we have from (5)

$$e^{i\nu x_1'} \psi_{\alpha_2, \alpha_3, \alpha_4} = \exp i \{(1 - l_1)(\alpha_2 x_2' + \alpha_3 x_3' - \alpha_4 t') - [l_2 \alpha_2 + l_3 \alpha_3 + cb/2\alpha_4 + (1 - l_1) h\nu] x_1'\}/(1 - l_1) h$$
$$= \exp - i\nu' ct' . \exp i \{(1 - l_1)(\alpha_2 x_2' + \alpha_3 x_3' - \alpha_4' t') - [l_2 \alpha_2 + l_3 \alpha_3 + cb/2\alpha_4'] x_1'\}/(1 - l_1) h,$$

* *Loc. cit.*, equations (33).

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where ν' and α_4' are two new constants satisfying

$$\alpha_4 = hc\nu' + \alpha_4' \quad \text{.........................(9)}$$

and
$$\frac{cb}{2\alpha_4} + (1 - l_1)\, h\nu = \frac{cb}{2\alpha_4'} \quad \text{.................(10).}$$

We now have the relation

$$e^{i\nu x_1'}\,\psi_{\alpha_2, \alpha_3, \alpha_4} = e^{-i\nu' ct'}\,\psi_{\alpha_2, \alpha_3, \alpha_4'},$$

which gives an expansion for the left-hand side of the required type. It shows that the general element of the matrix representing $e^{i\nu x_1'}$, namely $e^{i\nu x_1'}(\alpha_2, \alpha_3, \alpha_4;\ \alpha_2', \alpha_3', \alpha_4')$, vanishes except when

$$\alpha_2' = \alpha_2, \quad \alpha_3' = \alpha_3 \quad \text{....................(11),}$$

and α_4 and α_4' satisfy the condition (10), and is equal to $e^{-i\nu' ct'}$ when these conditions are fulfilled. In the same way it may be shown that the general element of the matrix representing $e^{-i\nu x_1'}$, which we write for convenience $e^{-i\nu x_1'}(\alpha_2', \alpha_3', \alpha_4';\ \alpha_2, \alpha_3, \alpha_4)$, vanishes except when the same conditions (10) and (11) are fulfilled, in which case it equals $e^{i\nu' ct'}$. We can now write down without further trouble all the elements of the matrices representing X and Y.

§ 5. *Physical Interpretation of the Matrix Elements.*

The constant terms and the terms proportional to t' in the expressions (8) for X and Y will contribute nothing to the emitted radiation, and may be ignored. We are thus left with only the terms proportional to $\sin \nu x_1'$ to investigate. We see that their matrices contain only elements referring to transitions in which p_2', p_3' and W' change from a set of values α_2, α_3, α_4 to a set α_2', α_3', α_4' given in terms of α_2, α_3, α_4 by equations (10) and (11) (or from the α''s to the α's), and further that the wave number of the radiation emitted during such a transition is $\nu'/2\pi$, where ν' is given by (9). Using the symbol Δ to denote the increase in a constant of integration of the system during a transition, we have from (11), (9) and (10)

$$\Delta p_2' = 0, \quad \Delta p_3' = 0, \quad \Delta W' = -hc\nu' \quad \text{...........(12)}$$

and
$$\Delta\,(cb/2W') = (1 - l_1)\, h\nu \quad \text{..................(13).}$$

These relations are sufficient to determine the recoil momentum of the electron and the frequency of the scattered radiation. Equation (13) or its equivalent equation (10) can be most conveniently used if we observe that, with neglect of a', p_1' is represented by a diagonal matrix whose diagonal elements α_1 are given

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by $-ih$ times the coefficient of x_1' in the index of the exponential in (5), *i.e.*

$$\alpha_1 = -(l_2\alpha_2 + l_3\alpha_3 + cb/2\alpha_4)/(1 - l_1)$$

so that

$$\Delta p_1' = \left(\frac{cb}{2\alpha_4} - \frac{cb}{2\alpha_4'}\right)\Big/(1 - l_1) = -h\nu \qquad (14).$$

The fact that we have to neglect a' in order to be able to give a meaning to $\Delta p_1'$ corresponds physically to the fact that in order to give a meaning to the recoil momentum of the electron we must neglect the oscillations of the electron due to the incident radiation. Equations (12) and (14) give, when written in terms of the undashed variables,

$$\Delta p_1 = h\nu - l_1 h\nu'$$
$$\Delta p_2 = -l_2 h\nu'$$
$$\Delta p_3 = -l_3 h\nu'$$
$$\Delta W/c = h\nu - h\nu',$$

which are just the equations that express the conservation of momentum and energy on Compton's light-quantum theory of scattering. (We are neglecting a' again when we count p_2 as an ordinary momentum.)

The elements of the matrix that represents the periodically varying part of X are

$$X(\alpha_2, \alpha_3, \alpha_4;\ \alpha_2', \alpha_3', \alpha_4') = \frac{ica'\alpha_2\alpha_3}{\nu(m^2c^2 + \alpha_2^2 + \alpha_3^2)\,\alpha_4(1 - l_1)}\, e^{-i\nu'ct'}$$

and

$$X(\alpha_2', \alpha_3', \alpha_4';\ \alpha_2, \alpha_3, \alpha_4) = \frac{-ica'\alpha_2'\alpha_3'}{\nu(m^2c^2 + \alpha_2'^2 + \alpha_3'^2)\,\alpha_4'(1 - l_1)}\, e^{i\nu'ct'},$$

where the α''s are given in terms of the α's by (10) and (11). The product of these two matrix elements gives a quarter of the square of the amplitude, which, by insertion in the formula of the classical theory, determines the intensity of the emitted radiation. The amplitude squared is thus

$$C_X^2 = 4\,\frac{c^2a'^2\alpha_2^2\alpha_3^2}{\nu^2(m^2c^2 + \alpha_2^2 + \alpha_3^2)^2\,\alpha_4\alpha_4'(1 - l_1)^2} \qquad (15).$$

Instead of actually evaluating the intensity from this formula, we can obtain results more simply by comparing C_X^2 with its value according to the classical theory. This classical value is given by one putting $h = 0$ in (15), which comes to writing α_4 for the α_4' in the denominator. Hence the quantum value of C_X^2 is just α_4/α_4' times its classical value. This result may easily be verified to hold also for C_Y^2, the amplitude squared of Y.

From (9) and (10) we have

$$\alpha_4 - \alpha_4' = hc\nu'$$

and

$$\frac{1}{\alpha_4'} - \frac{1}{\alpha_4} = \frac{2(1-l_1)}{cb} h\nu,$$

which give, by division,

$$\alpha_4 \alpha_4' = \frac{c^3 b}{2(1-l_1)} \frac{\nu'}{\nu} \quad \text{.....................(16).}$$

If the electron is initially at rest, we have initially

$$p_1 = p_2 = p_3 = 0, \quad W = mc^2,$$

from which, using the transformation equations (3), we find for the initial values of p_2', p_3' and W',

$$\alpha_2 = -\frac{l_2 mc}{1-l_1}, \quad \alpha_3 = -\frac{l_3 mc}{1-l_1}, \quad \alpha_4 = \frac{mc^2}{1-l_1}.$$

This gives

$$b = m^2c^2 + \alpha_2^2 + \alpha_3^2 = \frac{2m^2c^2}{1-l_1},$$

so that from (16)

$$\frac{\alpha_4}{\alpha_4'} = \frac{2(1-l_1)\alpha_4^2}{c^3 b} \frac{\nu}{\nu'} = \frac{\nu}{\nu'}.$$

The amplitude squared is thus greater than its value according to the classical theory in the ratio ν/ν'. Since the intensity is proportional to the amplitude squared multiplied by the fourth power of the frequency, and the frequency of the scattered radiation is less than its value on the classical theory in the ratio ν'/ν, it follows that the intensity is less than its value on the classical theory in the ratio $(\nu'/\nu)^3$. This result is the same as that obtained in *loc. cit.*, and is in good agreement with experiment*.

* *Note added in proof*:—A paper by W. Gordon dealing with the same subject has recently appeared [*Zeits. f. Phys.*, vol. 40, p. 117 (1926)]. Gordon's method makes use of an expression for the electric density for determining the field produced by the scattering electron, and differs from the method of the present paper, in which the wave equation is used merely as a mathematical help for the calculation of the matrix elements, which are then interpreted in accordance with the assumptions of matrix mechanics.