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WAVE FRONT HOLONOMICITY OF C^exp-CLASS DISTRIBUTIONS ON NON-ARCHIMEDEAN LOCAL FIELDS

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Abstract

Many phenomena in geometry and analysis can be explained via the theory of D-modules, but this theory explains close to nothing in the non-archimedean case, by the absence of integration by parts. Hence there is a need to look for alternatives. A central example of a notion based on the theory of D-modules is the notion of holonomic distributions. We study two recent alternatives of this notion in the context of distributions on non-archimedean local fields, namely $\mathscr{C}^{\exp}$-class distributions from Cluckers *et al.* ['Distributions and wave front sets in the uniform nonarchimedean setting', *Trans. Lond. Math. Soc.* **5**(1) (2018), 97–131] and WF-holonomicity from Aizenbud and Drinfeld ['The wave front set of the Fourier transform of algebraic measures', *Israel J. Math.* **207**(2) (2015), 527–580 (English)]. We answer a question from Aizenbud and Drinfeld ['The wave front set of the Fourier transform of algebraic measures', *Israel J. Math.* **207**(2) (2015), 527–580 (English)] by showing that each distribution of the $\mathscr{C}^{\exp}$-class is WF-holonomic and thus provides a framework of WF-holonomic distributions, which is stable under taking Fourier transforms. This is interesting because the $\mathscr{C}^{\exp}$-class contains many natural distributions, in particular, the distributions studied by Aizenbud and Drinfeld ['The wave front set of the Fourier transform of algebraic measures', *Israel J. Math.* **207**(2) (2015), 527–580 (English)]. We show also another stability result of this class, namely, one can regularize distributions without leaving the $\mathscr{C}^{\exp}$-class. We strengthen a link from Cluckers *et al.* ['Distributions and wave front sets in the uniform nonarchimedean setting', *Trans. Lond. Math. Soc.* **5**(1) (2018), 97–131] between zero loci and smooth loci for functions and distributions of the $\mathscr{C}^{\exp}$-class. A key ingredient is a new resolution result for subanalytic functions (by alterations), based on embedded resolution for analytic functions and model theory.

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1.   Introduction

1.1.   A key missing tool in the combination of p-adic geometry and analysis is the interplay between differentiation and integration. Techniques related to Bernstein–Sato polynomials and D-modules, which in the reals give a plenitude of results, seem to break down when one wants to use them in a p-adic context. In the real and complex setting, one thinks for example of the strong link between eigenvalues of monodromy with zeros of Bernstein–Sato polynomials, and with poles of certain real integrals, a link that remains elusive in the p-adic setting; see for example, the monodromy conjecture [**4**, Ch. 1, Section 3.4], [**17**, **27**].

Owing to this deficiency, there is a decades-old challenge for developing alternative theories, and this is a major driver behind motivic integration and non-archimedean geometry; see for example, [**19**, **28**]. This paper is also driven by this challenge. In a word, we study holonomicity in the p-adic setting. First, one needs an alternative notion for holonomicity, not for D-modules but for distributions on p-adic analytic manifolds. Such an alternative has been provided recently in [**1**] by the notion of WF-holonomicity, based on wave front sets [**24**, **26**]. As is evident from [**1**], it is not easy to show WF-holonomicity of a given distribution; indeed, one works hard in [**1**] to show it for certain forms of distributions. As a good holonomicity notion needs to do, it brings in geometry and a control of certain dimensions. However, it is shown in [**1**] that WF-holonomicity is, in general, not preserved under Fourier transformation, and the question was raised in [**1**] to find a class of distributions with better behavior. Here we bring in the extra geometry and control from a broad class of distributions, which contain the ones from [**1**]: the notion of distributions of the $\mathscr{C}^{\exp}$-class from [**11**]. Our main result provides a framework of WF-holonomic distributions on $\mathbb{Q}_p^n$, which is moreover stable under Fourier transform: the $\mathscr{C}^{\exp}$-class distributions. This is a flexible class of distributions, which is, for example, stable under regularization as we show in Theorem 2.8. The stability under Fourier transform is [**11**, Theorem 3.3.5]; the holonomicity is shown by Theorem 2.7, the main result of this paper. Let us go into some more details.

1.2.   The notion of WF-holonomicity of a distribution ξ, introduced in [**1**], is based on the wave front set WF(ξ) of ξ. So to say, the nicest distributions are the ones that come from integrating the product of a test function with a smooth (that is, C^∞, that is, locally constant) density function. The wave front set of ξ on a manifold X is a subset of the cotangent bundle of X, and it sees how far away ξ is from a smooth density function, roughly by looking at decay when nearing infinity and after Fourier transformation. If the wave front set WF(ξ) is small enough, then ξ is called WF-holonomic. In more detail, if WF(ξ) is contained in a finite

union of conormal bundles of submanifolds of X, then ξ is called WF-holonomic; see Definition 2.2.

1.3. In [11], the notion of distributions of $\mathscr{C}^{\exp}$-class on p-adic manifolds is introduced. The stability of this class is shown under operations like Fourier transforms, pullbacks, and pushforwards. (The definition of Heifetz [24] for pullbacks is made precise in [11] by specifying topologies on distributions in relation to their wave front sets, similar to the real case in [26]. This specification also applies to [1, Proposition 2.3.10].) In this paper, we show stability in a new sense, namely under regularization: any distribution of $\mathscr{C}^{\exp}$-class on U can be regularized to a distribution on X, which is still of $\mathscr{C}^{\exp}$-class, where U is open in the p-adic manifold X; see Theorem 2.8. Not only is the $\mathscr{C}^{\exp}$-class of distributions stable under all these operations but also this class contains many natural distributions, like the ones studied in [1]. Distributions of $\mathscr{C}^{\exp}$-class have a geometric flavor as reflected by the main result of this paper on WF-holonomicity, and by their definition based on model theory. Moreover, our results about $\mathscr{C}^{\exp}$-class distributions hold uniformly throughout all p-adic fields, as we explain in Section 8.

1.4. The notion of distributions of $\mathscr{C}^{\exp}$-class is tightly linked to the notion of $\mathscr{C}^{\exp}$-class functions, *grosso modo* via the continuous wavelet transform. For functions of $\mathscr{C}^{\exp}$-class, the zero loci have played important roles in transfer principles (to change the characteristic of the local field) and in the description of geometric and analytic objects; see [7–9, 11]. In [11], it is shown that the smooth locus (and even the microlocally smooth locus) of a distribution of $\mathscr{C}^{\exp}$-class equals the zero locus of a function of $\mathscr{C}^{\exp}$-class. We show the following converse: for any zero locus $Z(g)$ of a $\mathscr{C}^{\exp}$-class function g on a manifold X such that $Z(g)$ is moreover open and dense in X, there is a distribution of $\mathscr{C}^{\exp}$-class whose smooth locus equals $Z(g)$; see Theorem 2.9. Together with the results [11, Theorems 3.4.1, 4.1.2], this exhibits yet another complete role played by the zero loci of $\mathscr{C}^{\exp}$-class functions. An analogous converse for the microlocally smooth locus remains to be discovered in the future.

1.5. A key ingredient in the proofs consists of a (new variant of a) resolution result for definable functions, which creates monomials times units, but which allows finite fibers; see Theorem 2.11. However, this variant does not directly allow a reduction to a Cartesian product situation when proving the WF-holonomicity from Theorem 2.7. Indeed, an additive character evaluated in a unit times a quotient of monomials is not at all a Cartesian product situation.

However, via general properties of distributions and their wave front sets, we manage to proceed by induction on the dimension. By the finite fibers, our resolution maps are similar to alterations and remind us of smoothing of real subanalytic sets as in [**2**].

1.6. Typically, the results of this paper and of [**1**, **11**] hold uniformly throughout all p-adic fields and in (definable) family settings. For simplicity of notation, a large part of the paper will be formulated for a fixed non-archimedean local field F, which is often assumed to be of characteristic zero, and a fixed nontrivial additive character ψ on F. We will also state uniformity in the local field (including local fields of positive but large characteristic) and family aspects of most of our results. See for example Theorem 8.3 for a family variant of the regularization result. Very recently, some results of [**24**] and [**11**] have been presented in a motivic framework instead of (uniform) p-adic; see [**31**].

1.7. The motivation for this line of research lies in part in the search for p-adic analogues for the strong interplay between real analysis and real geometry. Another challenge came from more global geometrical aspects than usually dealt with in model theory. Indeed, wave front sets for distributions on a manifold X cannot be seen by working piecewise on X (unless the pieces are clopen), while most results on definable sets and functions are piecewise in nature. This represented the challenge to this project to combine a global geometric with a definable viewpoint.

1.8. Structure of the paper. In Section 2, we recall the relevant terminology and formulate the main result of the paper when working over a fixed local field of characteristic 0. In Section 3, we explain the main ingredients of all the proofs in the paper, with all the essential parts and omitting technical details. In Sections 4–7, we prove all the results for a fixed local field of characteristic 0. In Section 8, we explain how to deduce uniform versions of those results when we vary the local field over all p-adic fields and over all local fields of positive (but high enough) characteristic. Only local fields with a small positive characteristic fall outside our scope.

2. Wave front sets and holonomicity: definitions and main results

2.1. Let F be any non-archimedean local field, namely, a finite field extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$ for some prime p. Let $\mathcal{O}_F$ denote the valuation ring of F with maximal ideal $\mathcal{M}_F$ and residue field k_F with q_F elements and characteristic p_F.

We identify the value group of F with $\mathbb{Z}$. We write ord for the valuation map, and $|x|$ for the norm of $x \in F$, defined as $q_F^{-\operatorname{ord} x}$ for nonzero x and with $|0| = 0$.

Let X be an analytic submanifold of F^n of dimension m for some $n \geqslant m \geqslant 0$; see [3], where this is called F-analytic instead of analytic, and see [11, Section 2.1] for the more general notion of strict C^1 submanifolds. As in [11, Section 2.1], we will always assume that our analytic manifolds are smooth, nonempty, and of pure dimension, meaning of constant local dimension m for some integer $m \geqslant 0$. Define the tangent bundle TX and the cotangent T^*X bundle of X as usual (see [11, Section 2.1]). That is, T^*X is the cotangent bundle, which at $x \in X$ is the dual of the tangent space to X at x. By the wave front set WF(ξ) of a distribution ξ on X, we mean the $F^\times$-wave front set in $T^*X \smallsetminus X \times \{0\}$ in the sense of [11, 2.8.6] (based on Heifetz [24]). (Note that Definition A.0.1 of [1] of wave front sets is slightly different since it includes the zero section on the support of ξ; this is a harmless difference.) Let us recall these definitions. By a Schwartz–Bruhat function is meant a $\mathbb{C}$-valued locally constant function with compact support. The $\mathbb{C}$-vector space of Schwartz–Bruhat functions on X is denoted by $\mathcal{S}(X)$. A distribution on X is nothing other than a linear function $\mathcal{S}(X) \to \mathbb{C}$ (as usual in the p-adic case, there are no topological requirements). We write $\mathcal{S}^*(X)$ for the $\mathbb{C}$-vector space of distributions on X. The support of a distribution is meant as usual; see for example, [11, Definition 2.4.3].

DEFINITION 2.1 (Wave front sets). Let $U \subset F^n$ be open and let ξ be a distribution on U. Let (x_0, y_0) be in $T^*U \smallsetminus U \times \{0\} = U \times (F^n \smallsetminus \{0\})$. Say that ξ is microlocally smooth at (x_0, y_0) if there are open neighborhoods U_0 of x_0 and V_0 of y_0 such that for any Schwartz–Bruhat function φ with support contained in U_0, there is an integer N such that for all $\lambda \in F^\times$ with $|\lambda| > N$, one has

$$\mathcal{F}(\varphi\xi)(\lambda \cdot y) = 0 \quad \text{for all } y \text{ in } V_0. \tag{2.1.1}$$

Here, the Fourier transform $\mathcal{F}(\varphi\xi)$ of the product of φ with ξ is a continuous function (see [11, Theorem 2.5.2]) and $\lambda \cdot y$ stands for $(\lambda y_1, \ldots, \lambda y_n)$. The wave front set of ξ is defined as the complement in $T^*U \smallsetminus U \times \{0\}$ of the set of microlocally smooth points of ξ and is denoted by WF(ξ). More generally, for ξ a distribution on an analytic submanifold X of F^n of dimension m, the wave front set of ξ is defined by using analytic charts on X (see [11, 2.8.6] with $\Lambda = F^\times$).

Define the conormal bundle

$$CN_Y^X \subset T^*(X)$$

of an analytic submanifold $Y \subset X$ as usual; see [11, Section 2.1]. That is, the conormal bundle CN_Y^X is the dual bundle of the normal bundle N_Y^X, where N_Y^X at $y \in Y$ is the quotient of the tangent space to X at y by the tangent space

to Y at y. Note that the set CN_Y^X is an isotropic and even Lagrangian analytic submanifold of (the symplectic manifold) T^*X.

The following notions of WF-holonomicity come from [1, Section 3.2], with a slight generalization of [1, Section 3.2] and [11, Definition 4.1.1] (by relaxing the smoothness conditions in the algebraic case).

DEFINITION 2.2 (WF-holonomicity). Let X be an analytic submanifold of F^n and let ξ be a distribution on X. Say that ξ is analytically WF-holonomic if the wave front set of ξ is contained in a finite union of conormal bundles $CN_{Y_i}^X$ where each $Y_i \subset X$ is an analytic submanifold of X. Call ξ algebraically WF-holonomic if moreover one has $\dim X = \dim \overline{X}^{\mathrm{Zar}}$ and $\dim Y_i = \dim \overline{Y_i}^{\mathrm{Zar}}$, with $\overline{X}^{\mathrm{Zar}}$ and the $\overline{Y_i}^{\mathrm{Zar}}$ the Zariski closure of X and the Y_i.

If X and the Y_i are merely strict C^1 submanifolds (instead of analytic), then we say that ξ is strict C^1 WF-holonomic.

REMARK 2.3. Note that we do not assume smoothness of $\overline{X}^{\mathrm{Zar}}$ (neither of the $\overline{Y_i}^{\mathrm{Zar}}$), and that F is allowed to have positive characteristic. In these ways, the above definition of algebraic WF-holonomicity generalizes the notion of algebraic WF-holonomicity of [1, Definition 3.2.1] and [11, Definition 4.1.1]. In the case that $\overline{X}^{\mathrm{Zar}}$ is smooth and that F has zero characteristic, all these definitions of algebraic WF-holonomicity coincide (note that the smoothness condition is forgotten in the final part of [11, Proposition 4.3.1]; see Remark 6.6).

2.2. Definitions and results for a fixed local field F.

From now on and until the end of Section 7, we fix a local field F of characteristic zero and an additive character $\psi : F \to \mathbb{C}^\times$, which is trivial on $\mathcal{M}_F$ and nontrivial on $\mathcal{O}_F$. (Thus, F is a finite field extension of $\mathbb{Q}_p$ for some prime p.) (An additive character is a continuous group homomorphism from the additive group on F to $\mathbb{C}^\times$.)

The advantage of working with fixed F is the ease of presentation both for definable sets and for the rings of complex valued functions that we integrate (called functions of $\mathscr{C}^{\exp}$-class). Uniformity in F will come at the end of the paper, in Section 8.

2.3. Languages on F: subanalytic and semialgebraic.

For each integer $n \geqslant 0$, let $\mathcal{O}_F \langle x_1, \ldots, x_n \rangle$ be the p-adic completion of $\mathcal{O}_F[x_1, \ldots, x_n]$ inside $\mathcal{O}_F[[x_1, \ldots, x_n]]$, for the Gauss norm. Note that $\mathcal{O}_F \langle x_1, \ldots, x_n \rangle$ consists of power series $\sum_{i \in \mathbb{N}^n} a_i x^i$ in multi-index notation and with $a_i \in \mathcal{O}_F$ such that $|a_i|$ goes to zero when $|i| := i_1 + \cdots + i_n$ goes to infinity. For f in $\mathcal{O}_F \langle x_1, \ldots, x_n \rangle$, write $\bar{f}$ for the restricted analytic function associated to f, namely, the function $\bar{f} : F^n \to F$

sending $z \in \mathcal{O}_F^n$ to the evaluation $f(z)$ of f at z (that is, the p-adic limit of the partial sums) and sending all other z to 0.

We use the following two languages (in the first-order sense of model theory). Let $\mathcal{L}^F$ be the ring language (namely, having symbols $+, -, \cdot, 0, 1$), together with constant symbols from $\mathcal{O}_F$. Let $\mathcal{L}_{\text{an}}^F$ be $\mathcal{L}^F$ together with for each $f \in \mathcal{O}_F\langle x_1, \ldots, x_n\rangle$ a function symbol for the restricted analytic function $\bar{f}$ associated to f. A set $X \subset F^n$ is called $\mathcal{L}^F$-definable, respectively $\mathcal{L}_{\text{an}}^F$-definable, if there is an $\mathcal{L}^F$-formula, respectively an $\mathcal{L}_{\text{an}}^F$-formula $\varphi(x)$ with free variables $x_1, \ldots, x_n$, such that X consists of the values for $x = (x_1, \ldots, x_n)$ that make φ valid in F. A function between definable sets is called definable if its graph is a definable set. (All this is standard in model theory and first-order logic.) These definable sets and functions are called semialgebraic, respectively subanalytic, for $\mathcal{L}^F$, respectively $\mathcal{L}_{\text{an}}^F$, and have many geometric properties, enabled in the first place by quantifier elimination results in closely related (slightly bigger) languages; see for example, [29], [30, Theorem 5.6], and [16, 18, 22].

The dimension of a nonempty $\mathcal{L}_{\text{an}}^F$-definable set $C \subset F^n$ is defined as the maximum of the dimensions of analytic submanifolds of F^n contained in C (see [18, Section 3.15] and Lemma 5.4 for the dimension theory of $\mathcal{L}_{\text{an}}^F$-definable sets). If moreover C is $\mathcal{L}^F$-definable, then it is equal to the dimension of the Zariski closure of C; see [21] or [23].

2.4. $\mathscr{C}_{\mathcal{L}}^{\text{exp}}$-class and $\mathcal{L}$-WF-holonomicity.

From now on (and until the end of Section 7), we fix $\mathcal{L}$ to be either $\mathcal{L}^F$ or $\mathcal{L}_{\text{an}}^F$. By an $\mathcal{L}$-manifold we mean an $\mathcal{L}$-definable set $X \subset F^n$ for some $n \geqslant 0$ such that X is moreover an analytic submanifold of F^n. By an $\mathcal{L}$-analytic map we mean an $\mathcal{L}$-definable, analytic map between $\mathcal{L}$-manifolds. By an analytic isomorphism we mean an analytic bijection between analytic manifolds whose inverse is also analytic. We now come to our key definitions.

DEFINITION 2.4 (Functions of $\mathscr{C}_{\mathcal{L}}^{\text{exp}}$-class). Let $X \subset F^n$ be an $\mathcal{L}$-definable set. The $\mathbb{C}$-algebra $\mathscr{C}_{\mathcal{L}}^{\text{exp}}(X)$ is defined as the sub-$\mathbb{C}$-algebra of all complex valued functions on X generated by functions $X \to \mathbb{C}$ of the following forms:

(1) $x \mapsto |f(x)|$;

(2) $x \mapsto \operatorname{ord} g(x)$;

(3) $x \mapsto \psi(h(x))$;

where $f : X \to F, h : X \to F$, and $g : X \to F^\times$ are $\mathcal{L}$-definable functions and where ψ is the additive character fixed above. A function in $\mathscr{C}_{\mathcal{L}}^{\text{exp}}(X)$ is called a function of $\mathscr{C}_{\mathcal{L}}^{\text{exp}}$-class.

These algebras of Definition 2.4 are versatile because of their stability under integration (and thus under Fourier transforms) (see [**15**, Section 8.6] and [**7**, Theorem 3.2.1]), and they inherit geometrical properties from their definable building blocks; see for example, [**9**].

For $x \in F^n$ and $r \in \mathbb{Z}$, write $B_r(x)$ for the ball $\{y \in F^n \mid \operatorname{ord}(y - x) \geqslant r\}$, where the order of a tuple is the minimum of the orders of the entries. We call q_F^{-r} the (normative) radius of the ball $B_r(x)$. Write $\mathbf{1}_A$ for the characteristic function of a subset $A \subset S$ (where the superset S is usually implicitly clear).

DEFINITION 2.5 (Distributions of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class). We say that a distribution ξ on an $\mathcal{L}$-submanifold $X \subset F^n$ is of class $\mathscr{C}_{\mathcal{L}}^{\exp}$ if the following condition on the continuous wavelet transform of ξ is satisfied. The function

$$D_\xi : X \times F^\times \to \mathbb{C}$$

is a $\mathscr{C}^{\exp}$-class function, where

$$D_\xi(x, \rho) = \begin{cases} \xi(\mathbf{1}_{B_{\operatorname{ord}\rho}(x) \cap X}) & \text{if } B_{\operatorname{ord}\rho}(x) \cap X \text{ is compact,} \\ 0 & \text{otherwise.} \end{cases}$$

We call D_ξ the B-function of ξ (where the letter B comes from ball).

Proposition 4.2 implies that the condition for a distribution ξ on X to be of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class is independent of the embedding of X into F^n. Note that D_ξ is a continuous wavelet transform of ξ.

DEFINITION 2.6 ($\mathcal{L}$-WF-holonomicity). Let ξ be a distribution of class $\mathscr{C}_{\mathcal{L}}^{\exp}$ on an $\mathcal{L}$-manifold X. Then ξ is called $\mathcal{L}$-WF-holonomic if the wave front set of ξ is contained in a finite union of conormal bundles $CN_{Y_i}^X$, where each $Y_i \subset X$ is an $\mathcal{L}$-submanifold of X.

2.5. Main results for fixed F. Now we can formulate our main results. The holonomicity result is in fact the key goal of this paper. (Recall that F has characteristic zero from Section 2.2 until the end of Section 7.)

THEOREM 2.7 (Holonomicity). *Let X be an $\mathcal{L}$-manifold and let ξ be a distribution on X of class $\mathscr{C}_{\mathcal{L}}^{\exp}$. Then ξ is $\mathcal{L}$-WF-holonomic.*

In particular, ξ is analytically WF-holonomic, and if $\mathcal{L}$ is $\mathcal{L}^F$, then ξ is algebraically WF-holonomic.

The following regularization result, allowing us to extend distributions to larger domains without leaving the $\mathscr{C}_{\mathcal{L}}^{\exp}$-class, will be useful to show Theorem 2.7.

THEOREM 2.8 (Regularization). *Consider an $\mathcal{L}$-manifold X. Let U be a nonempty $\mathcal{L}$-definable open subset of X. Then the restriction map*

$$\mathcal{S}^*(X) \to \mathcal{S}^*(U)$$

admits a linear section $\mathcal{S}^*(U) \to \mathcal{S}^*(X)$ that maps distributions of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class to distributions of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class.

With U and X as in the theorem, for any ξ in $\mathcal{S}^*(U)$ and any linear section $\kappa : \mathcal{S}^*(U) \to \mathcal{S}^*(X)$, $\kappa(\xi)$ is usually called a regularization of ξ.

In [11], Theorem 3.4.1 (respectively Remark 4.3.3), it is shown that the wave front set of a $\mathscr{C}_{\mathcal{L}}^{\exp}$-class distribution equals the complement of the zero locus of a function of the same class. It would be interesting to find a precise criterion for such zero loci so that they are the complement of a wave front of some $\mathscr{C}_{\mathcal{L}}^{\exp}$-class distribution; we give a partial answer to this question in Theorem 2.9. The smooth locus of a distribution ξ on an analytic submanifold $X \subset F^n$ is defined as the set of those $x \in X$ that allow an open neighborhood U such that the restriction of ξ to U is a smooth measure. We know that the smooth locus of a $\mathscr{C}_{\mathcal{L}}^{\exp}$-class distribution ξ on $X \subset F^n$ is dense open in X by [11, Theorem 4.1.2]; see Theorem 6.1. The following result gives the converse to these properties.

THEOREM 2.9 (Correspondence of loci). *Let X be an $\mathcal{L}$-manifold and let g be in $\mathscr{C}_{\mathcal{L}}^{\exp}(X)$. If the zero locus $Z(g)$ of g is dense open in X, then there exists a distribution ξ on X which is of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class and such that the smooth locus of ξ equals $Z(g)$.*

The result shows that zero loci of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class functions are the right objects to describe smooth loci of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class. (For other objects described precisely by loci of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class functions, see [7, 9].)

The following is our resolution result for $\mathcal{L}_{an}^F$-definable functions on $\mathcal{L}_{an}^F$-definable sets, refining [18, Theorems 2.2 and 2.4] (see also Proposition 5.1). Note that the resolving maps φ_i have restrictions to $U = (\mathcal{O}_F \smallsetminus \{0\})^m$, which are only locally isomorphisms.

DEFINITION 2.10. An analytic function $u : \mathcal{O}_F^m \to \mathcal{O}_F$ is called an analytic unit on $\mathcal{O}_F^m$ if it is given by a power series that is an invertible element in the ring $\mathcal{O}_F\langle x_1, \ldots, x_m \rangle$. By a monomial on $\mathcal{O}_F^m$ we mean a function $M : \mathcal{O}_F^m \to \mathcal{O}_F$ sending x to $d \cdot \prod_{i=1}^m x_i^{\mu_i}$ with exponents μ_i, which are natural numbers and with $d \in \mathcal{O}_F$. By an $\mathcal{L}$-basic differential m-form on an $\mathcal{L}$-manifold $O \subset \mathcal{O}_F^n$ of dimension m we mean an analytic differential form of the form $\omega = g \bigwedge_{i_j \in I} dx_{i_1}$,

with $g : O \to \mathcal{O}_F$ analytic and $\mathcal{L}$-definable and with $I \subset \{1, \dots, n\}$ having m elements.

THEOREM 2.11 (Resolution result for $\mathcal{L}_{an}^F$-definable sets and functions). *Let $X \subset \mathcal{O}_F^n$ be a closed $\mathcal{L}_{an}^F$-definable set of pure dimension m and let $f : X \to \mathcal{O}_F^k$ be an $\mathcal{L}_{an}^F$-definable function for some k and $n \geqslant m \geqslant 0$. Consider a dense open $O \subset X$, which is an $\mathcal{L}_{an}^F$-manifold, and let ω be an $\mathcal{L}_{an}^F$-basic differential m-form on O. Write U for $(\mathcal{O}_F \smallsetminus \{0\})^m$. Then there exist finitely many $\mathcal{L}_{an}^F$-definable functions*

$$\phi_i : \mathcal{O}_F^m \to X$$

such that each ϕ_i is continuous, proper, and the following properties hold for some positive integers d_i.

(1) *The set $U_i := \phi_i(U)$ is an open subset of O and thus an $\mathcal{L}_{an}^F$-manifold.*

(2) *The restriction $\phi_i|_U$ is a local analytic isomorphism to U_i with finite fibers of fixed size d_i.*

(3) *There are analytic units u_{ij} on $\mathcal{O}_F^m$ and monomials M_{ij} on $\mathcal{O}_F^m$ such that for each component f_j of f, one has*

$$f_j(\phi_i(x)) = u_{ij}(x) M_{ij}(x) \quad \text{for each } x \text{ in } U \text{ and each } i, j. \qquad (2.5.1)$$

(4) *The U_i are pairwise disjoint, and the union of the U_i is dense open in X.*

(5) *There are analytic units u_i on $\mathcal{O}_F^m$ and monomials M_i on $\mathcal{O}_F^m$ such that*

$$(\phi_i|_U)^*(\omega|_{U_i}) = u_i M_i \bigwedge_{j=1}^m dx_j \quad \text{for each } i, \qquad (2.5.2)$$

where $\bigwedge_{j=1}^m dx_j$ is the standard differential form on U.

(6) *For each i, one has either $u_{i1} = 1$ or $M_{i1} = d$ for some constant $d \in \mathcal{O}_F$.*

The resolution theorem can of course also be applied to other situations, for example, to F-valued functions $f_j : X \to F$ instead of $\mathcal{O}_F$-valued, by working on pieces where $|f_j| \leqslant 1$, respectively where $|f_j| > 1$ and replacing f_j by $1/f_j$ on the latter. Note that a dense open O of X as in the theorem always exists by [**18**, Theorem 3.14] and Lemma 5.4.

We will give uniform versions of Theorems 2.7–2.9 below in Section 8.

3. Sketch of the proofs

We start with proving regularization (Theorem 2.8). By partition of unity, the question is local, so it is enough to extend a $\mathscr{C}_{\mathcal{L}}^{\exp}$-class distribution for an open definable subset U to a larger open definable subset X in F^n. We can stratify the complement $Z := X \smallsetminus U$ in definable manifolds. Proceeding by induction on the strata, we can assume that Z is smooth. Again using locality of the question and a suitable version of the implicit function theorem, we can assume that Z is a graph. In this case, we can extend our distribution using a chosen definable tubular neighborhood of Z in X.

The proof of holonomicity (Theorem 2.7) is more work and is based on a theorem from [11] stating that any $\mathscr{C}_{\mathcal{L}}^{\exp}$-class distribution is smooth on a definable open dense subset. We use a partition of unity and regularization to reduce to proving WF-holonomicity of distributions ξ on $X = \mathcal{O}_F^n$. The next ingredient in the proof is a key lemma (Lemma 6.2) stating that any smooth (that is, locally constant) $\mathscr{C}_{\mathcal{L}}^{\exp}$ function f on a open dense definable set $U \subset X$ can be extended to a $\mathscr{C}_{\mathcal{L}}^{\exp}$-class WF-holonomic distribution on X (maybe after restricting it first to a smaller open dense subset). Note that just extending the function f by zero outside U does not work in general, since that may not correspond to a distribution on X as it may not be locally L^1 on X. The key lemma and the result from [11] allow us to replace ξ with a distribution ξ' whose support is of smaller dimension. We resolve (using Theorem 2.11) the support of ξ'. We then use regularization in order to construct a distribution on the resolution, which coincides with ξ' on an open dense set. Using the pushforward of that distribution (along the resolving maps) and the induction assumption, we can replace ξ' with another distribution whose support has even smaller dimension. We continue by induction until we kill ξ completely.

The proof of the key lemma is based on resolution of singularities for $\mathcal{L}_{\mathrm{an}}^F$-definable functions (Theorem 2.11). Using it, we can reduce to the case that $U = (\mathcal{O}_F \smallsetminus \{0\})^n$ and f has an explicit form containing (quotients of) monomials, units, the absolute value, the valuation, and the additive character ψ. In this case, we explicitly construct a $\mathscr{C}_{\mathcal{L}}^{\exp}$-class distribution on X that extends f and prove that it is WF-holonomic.

REMARK 3.1.

- Both the holonomicity theorem and the key lemma are proven by induction. However, the key lemma is less suitable for induction since it only claims **existence** of an extension with certain properties. Therefore when we prove the key lemma, it is more convenient to use the holonomicity theorem for smaller dimension rather than the key lemma itself. Because of that, we prove both results together by induction.

- The proof of the regularization result works uniformly for the algebraic language and in the analytic one. However, this is not the case for the holonomicity theorem. The proof of the holonomicity theorem relies on resolution of singularities for definable functions, which does not (yet) work so well in the algebraic language. Though resolution of definable sets in the algebraic language is not a problem (modulo Hironaka's theorem), resolution of definable functions seems to be hard to deduce from the existing literature. Therefore we chose to prove the holonomicity for the analytic language first and then deduce it for the algebraic language based on [11].

- The resolution result is, in fact, an alteration rather than a modification since it is only a **local** isomorphism on an open dense set. An actual modification cannot resolve a root function to a monomial function. However, it is not a problem since our use of resolution is for pushing forward distributions, so we just have to divide by the size of the fiber (which we ensured to be constant) to get the desired result.

- Although the resolution result is crucial for the key lemma, it does not resolve it compliantly, and some additional analysis is required. The reason is that even after the resolution, the explicit form of the function f is not a product of functions each of which depends on only one coordinate. This is because composition of an additive character with a quotient of monomials is not such a function.

The proof of the resolution of singularities of definable sets and functions (Theorem 2.11) is based on Hironaka's theorem and results from [22] on decompositions of definable functions to simpler functions called terms (in the sense of model theory), of a suitable, slightly bigger language. We first use Hironaka's theorem to resolve terms on $\mathcal{O}_F^n$, similar to that in [18]. We then use this in order to resolve closed definable subsets of full dimension in $\mathcal{O}_F^n$. This is possible since such subsets can be defined (quantifier free) by terms. This is the first place where we actually need alterations and not just modifications since definable sets can be of the form $\{x | \exists y$ such that $x = y^n\}$, which cannot be resolved just by modification. We next resolve a general definable function f on a closed definable set $X \subset \mathcal{O}_F^n$ of full dimension. For this, we use the results from [22] and alternate the following two procedures that are possible because of the previous steps:

- Decompose X to into definable subsets (of full dimension) and deal with (the closure of) each one separately.

- Resolve any term that we need in order to resolve f.

Finally, we do the general case by decomposing any definable set to graphs over definable sets of full dimension. Note that two kinds of terms are used: one without root functions (which suffices for quantifier elimination) and one with root functions (in which definable functions become piecewise terms). Some extra work is done to reduce the resolution of terms in the richer language to terms in the smaller language (without roots). Also, in this reduction we need alterations and not just modifications.

In order to prove that any zero locus of a $\mathscr{C}^{\mathrm{exp}}$ function can be a smooth locus of a $\mathscr{C}^{\mathrm{exp}}$ distribution (Theorem 2.9), we first show that a zero locus of a $\mathscr{C}^{\mathrm{exp}}$ function is also a zero locus of a bounded $\mathscr{C}^{\mathrm{exp}}$ function. Then we prove the theorem using the following statements:

- For any $\mathscr{C}^{\mathrm{exp}}$ function g on F^n, there is a stratification of F^n by manifolds such that g is smooth on each strata.

- Any submanifold X of F^n has a canonical measure with full support on X.

The uniform versions of Theorems 2.8 and 2.9 are proved in the same way as the theorems themselves. We deduce the uniform version of Theorem 2.7 from results of [11] about the uniform nature of the wave front of a $\mathscr{C}^{\mathrm{exp}}$ distribution and Theorem 2.7 itself.

4. Proof of the regularization

To prove Theorem 2.8, we will need to work with $\mathscr{C}_{\mathcal{L}}^{\mathrm{exp}}$-families of Schwartz–Bruhat functions, which we now define and which combine well with distributions of $\mathscr{C}_{\mathcal{L}}^{\mathrm{exp}}$-class by Proposition 4.2.

DEFINITION 4.1. Consider an $\mathcal{L}$-manifold X and let Y be an $\mathcal{L}$-definable set. A family $(\varphi_y)_{y \in Y}$ of functions φ_y in $\mathcal{S}(X)$ is called a $\mathscr{C}_{\mathcal{L}}^{\mathrm{exp}}$-family when the function

$$(x, y) \in X \times Y \mapsto \varphi_y(x)$$

lies in $\mathscr{C}_{\mathcal{L}}^{\mathrm{exp}}(X \times Y)$.

Theorem 2.8 will follow from the following results.

PROPOSITION 4.2 [11]. *Consider an $\mathcal{L}$-manifold X and let Y be an $\mathcal{L}$-definable set. Let ξ be a distribution on X of $\mathscr{C}_{\mathcal{L}}^{\mathrm{exp}}$-class and let $(\varphi_y)_{y \in Y}$ be a $\mathscr{C}_{\mathcal{L}}^{\mathrm{exp}}$-family of functions φ_y in $\mathcal{S}(X)$ for some definable set Y. Then the function sending $y \in Y$ to $\xi(\varphi_y)$ is of $\mathscr{C}_{\mathcal{L}}^{\mathrm{exp}}$-class.*

Proof. This follows from [**11**, Proposition 3.3.4 and Remark 4.3.3]. □

LEMMA 4.3 (Definable Urysohn's Lemma). *Let $X \subset F^n$ be an $\mathcal{L}$-manifold and consider $\mathcal{L}$-definable sets $Z \subset U \subset X$ such that U is open in X and Z is closed in X. Then there exists an $\mathcal{L}$-definable clopen $C \subset X$ such that $Z \subset C \subset U$. (A clopen set is a set that is open and closed.)*

Proof. For any $x \in Z$, let B_x be the maximal ball around x satisfying

- B_x is of (normative) radius $\leqslant 1$;

- $B_x \cap X$ is compact;

- $B_x \cap X \subset U$.

Note that such a ball B_x exists for every $x \in Z$. Put

$$C = \bigcup_{x \in Z} B_x \cap X.$$

We obviously have $Z \subset C \subset U$ and that C is an $\mathcal{L}$-definable open subset of U. It remains to prove that C is closed in X. Let $\alpha_i \in C$ be such that $(\alpha_i)_{i \in \mathbb{N}}$ is a converging sequence with limit α in X. It is sufficient to show that α lies inside C. To this end, it is enough to find a converging subsequence with limit in C. Let $z_i \in Z$ such that $\alpha_i \in B_{z_i}$ for each i. There are two cases to consider.

Case 1: The B_{z_i} become identical to each other when i is large enough.

In this case, we can assume that all α_i are in one ball $B_{z_{i_0}} \cap X$ and the statement follows from the compactness of $B_{z_{i_0}} \cap X$ and the fact that $B_{z_{i_0}} \cap X \subset C$.

Case 2: Up to passing to a subsequence, the B_{z_i} are pairwise different.

Up to replacing by the subsequence and by the ultrametric, the B_{z_i} are pairwise disjoint. On the other hand, α_i forms a Cauchy sequence. Thus, the (normative) radius of B_{z_i} converges to 0 when i grows. This implies that $\lim z_i = \alpha$. Since Z is closed in X, this implies that $\alpha \in Z$. □

PROPOSITION 4.4 (Partition of unity). *Let $X \subset F^n$ be an $\mathcal{L}$-manifold and let $X = \bigcup_{i=1}^{N} U_i$ be a finite cover with $\mathcal{L}$-definable open subsets U_i of X. Then there exists a finite cover $X = \bigcup_{j=1}^{N'} U_j'$ with disjoint $\mathcal{L}$-definable clopen sets refining the cover $X = \bigcup_{j=1}^{N} U_i$. (Refining means that for any j, the set U_j' is contained in some U_i.)*

Proof. For each $i = 1, \ldots, N$, define U_i' recursively to be an $\mathcal{L}$-definable clopen set given by the previous lemma (Lemma 4.3) such that

$$\left(X \setminus \bigcup_{j<i} U_j' \right) \setminus \bigcup_{j>i} U_j \subset U_i' \subset U_i.$$

This gives the desired refinement with $N' = N$. $\qquad\square$

The following refines the results [23, Theorem 1.1], [18, Theorem 3.14], and [6, Proposition 1.5.3] in the sense that our covers are moreover open.

LEMMA 4.5. *Let $Y \subset F^n$ be an $\mathcal{L}$-submanifold. Then one can find a finite cover $Y = \bigcup U_i$ by $\mathcal{L}$-definable sets U_i, which are open in Y and such that each U_i is a graph of an analytic $\mathcal{L}$-definable function from an open subset $V_i \subset L_i$ of a linear subspace of F^n of dimension $\dim Y$ to a linear complement of this subspace.*

Proof. Since the case $\dim Y = n$ is obvious, we will assume that $\dim Y < n$. Since Y is an analytic submanifold of F^n, for each $y \in Y$ there exists a coordinate subspace L such that the differential of the projection p from Y to L at the point y is an analytic isomorphism and hence $p : Y \to L$ is a local analytic isomorphism around y by the analytic inverse function theorem. Without loss of generality, we can pass to an open cover and assume that the same subspace L can be used for all points $y \in Y$. By the existence of $\mathcal{L}$-definable sections and since the cardinality of the fibers of $p : Y \to L$ is bounded (both statements follow, for example, from the cell decomposition theorems from [5, 16]; indeed, any finite partition into cells yields in particular definable sections, and the number of occurring cells is an upper bound on the cardinality of the fibers), we can find finitely many definable (not necessarily continuous) sections $s_i : p(Y) \to Y$ such that the images of the s_i cover Y.

By [23, Theorem 1.1] and [18, Theorem 3.14] (or [6, Proposition 1.5.3]), we can partition $p(Y) = \bigcup_j S_j$ into finitely many $\mathcal{L}$-manifolds such that for each i, j, the restriction $s_i|_{S_j}$ is analytic and such that S_j is the graph of an $\mathcal{L}$-analytic function from an open $W_j \subset L_j$ to L_j', where $L = L_j \oplus L_j'$ as linear spaces. It suffices to show that, up to refining the partition $\bigcup_j S_j$, one can extend $s_i|_{S_j}$ analytically to an open $\mathcal{L}$-definable neighborhood V_{ij} of S_j in L.

To this end, fix i and j, and for any $x \in S_j$, let $B_{x,i}$ be the maximal ball in F^n around $s_i(x)$ and of (normative) radius at most 1 such that $p|_{B_{x,i} \cap Y}$ is injective. Let $\nu_{x,i}$ be the inverse of $p|_{B_{x,i} \cap Y} : B_{x,i} \cap Y \to p(B_{x,i} \cap Y)$ and put $V_{ijx} := p(B_{x,i} \cap Y) \cap (x + L_j')$. We now obtain an extension of $s_i|_{S_j}$ to the open $\mathcal{L}$-definable set $V_{ij} := \bigcup_{x \in S_j} V_{ijx}$ as needed, namely, sending $x + y$ in V_{ij} with $x \in S_j$ and $y \in L_j'$ to $\nu_{x,i}(x + y)$. $\qquad\square$

PROPOSITION 4.6. *Consider an $\mathcal{L}$-manifold X. Let U be a nonempty $\mathcal{L}$-definable open subset of X and write Z for the complement of U inside X. Suppose that Z is an $\mathcal{L}$-submanifold of X. Then the restriction map*

$$\mathcal{S}(X) \to \mathcal{S}(Z)$$

admits a linear section

$$\nu : \mathcal{S}(Z) \to \mathcal{S}(X)$$

that maps $\mathscr{C}_{\mathcal{L}}^{\exp}$-class families to $\mathscr{C}_{\mathcal{L}}^{\exp}$-class families. (Namely, if $(\varphi_y)_{y \in Y}$ is a $\mathscr{C}_{\mathcal{L}}^{\exp}$-class family of functions φ_y in $\mathcal{S}(Z)$, then $(\nu(\varphi_y))_{y \in Y}$ is a $\mathscr{C}_{\mathcal{L}}^{\exp}$-class family of functions φ_y in $\mathcal{S}(X)$.)

Proof. Let n be such that $X \subset F^n$.

Case 1: X is open in F^n and Z is a graph of an analytic map ϕ from an open $V \subset F^k$ to F^{n-k}.

Let $p : F^n \to F^k$ be the coordinate projection so that $p(Z) = V$. For any $z \in V$, let B_z be the maximal ball in F^n of (normative) radius at most 1 around $(z, \phi(z))$ that is contained in X and such that $p(B_z)$ is contained in V. For a function $f \in \mathcal{S}(Z)$ and $x \in X$, define

$$\nu(f)(x) := \begin{cases} f(p(x), \phi(p(x))) & \text{if } p(x) \in V \text{ and } x \in B_{p(x)}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that ν is a section as desired. (The fact that $\nu(f)$ has compact support follows from the continuity of ϕ and thus that the projection $Z \to V$ is a homeomorphism.)

Case 2: X is open in F^n.

By Lemma 4.5, we can find a finite $\mathcal{L}$-definable open cover $Z = \bigcup U_i$ such that each U_i is a graph of an analytic map from an open subset of L_i to L_i', where $F^n = L_i \oplus L_i'$ is a decomposition to vector spaces. We can find $\mathcal{L}$-definable open sets $V_i \subset X$ such that $U_i = V_i \cap Z$. Adding the set $X \smallsetminus Z$, we obtain a finite open cover of X. Applying partition of unity (Proposition 4.4) to this cover, we reduce to the previous step.

Case 3: General case.

By Lemma 4.5 again, we can cover X by finitely many open $\mathcal{L}$-sets, each of which is isomorphic to an open $\mathcal{L}$-definable subset of F^k with k the dimension of X. Again applying partition of unity (Proposition 4.4) to this cover, we reduce to the previous step. □

Proof of Theorem 2.8. Let Z be the complement of U in X.

Case 1. Z is a closed $\mathcal{L}$-submanifold of X.

Let ν be the section obtained from Proposition 4.6. To φ in $\mathcal{S}(X)$, we associate $\widetilde{\varphi}$ in $\mathcal{S}(U)$ by defining $\widetilde{\varphi}$ as the restriction of $\varphi - \nu(\varphi|_Z)$ to U. Now define the section κ of $\mathcal{S}^*(X) \to \mathcal{S}^*(U)$ by sending ξ in $\mathcal{S}^*(U)$ to the distribution $\kappa(\xi) = \xi_X$, where $\xi_X(\varphi)$ for any φ in $\mathcal{S}(X)$ is defined as $\xi(\widetilde{\varphi})$. That ξ_X is a distribution of class $\mathscr{C}_{\mathcal{L}}^{\exp}$ now follows from Proposition 4.2 and the fact that ν maps $\mathscr{C}_{\mathcal{L}}^{\exp}$-class families to $\mathscr{C}_{\mathcal{L}}^{\exp}$-class families. Linearity is clear by construction.

Case 2. General case.

By [6, Theorem 4.2.5] (more concisely, by the combination of the frontier condition for stratifications and [23, Theorem 1.1], [18, Theorem 3.14] as shown in particular in [6]), there exist definable sets Z_i for $i = 0, \ldots, k$ for some $k > 0$ such that $Z = Z_0 \supset Z_1 \supset \cdots \supset Z_n = \emptyset$ and such that $Z_i \setminus Z_{i+1}$ is a closed $\mathcal{L}$-submanifold of Z_i. Put $U_i := X \setminus Z_i$. By composition, it is enough to prove that $\mathcal{S}^*(U_{i+1}) \to \mathcal{S}^*(U_i)$ admits a section κ_i as desired, but this follows from the previous case. □

5. Proof of resolution for definable sets and functions

We will prove our variant of resolution of singularities for $\mathcal{L}_{\mathrm{an}}^F$-definable sets and functions. This is a resolution by alterations rather then by modifications in the sense that finite fibers are allowed. For $\mathcal{L}_{\mathrm{an}}^F$-definable functions, which are moreover compositions of converging power series and restricted division, similar resolution results have been obtained in [18], using Hironaka's embedded resolution of singularities from [25]. Similar to Hironaka's result, one of our main purposes is to make the pullback of a definable function a product of a monomial with a unit, while also controlling the pullback of a top degree differential form. Since in the p-adic case, fragments of rth root functions are definable for integers $r > 0$, we will need to combine power maps and monomialization to get our resolution result. For us, after monomialization, the unit will still have an important role since it can in general not be neglected inside the argument of the additive character ψ. Again because of the argument inside ψ, to prove our main theorem (Theorem 2.7) will require additional work, even after resolving singularities, and will not reduce directly to a Cartesian product situation.

Write $\mathcal{L}_{\mathrm{an,qe}}^F$ for $\mathcal{L}_{\mathrm{an}}^F$ together with a function symbol for field division sending nonzero x to x^{-1} and 0 to 0, and relation symbols P_n for each $n > 0$ for the set of nonzero nth powers in F. Similarly, write $\mathcal{L}_{\mathrm{an,D}}^F$ for the language $\mathcal{L}_{\mathrm{an}}^F$ together with a function symbol for restricted division D sending $(x, y) \in \mathcal{O}_F^2$ to x/y when $|x| \leqslant |y| \neq 0$ and to zero otherwise, and relation symbols P_n for

each $n > 0$ for the set of nonzero nth powers in $\mathcal{O}_F$. By the variant from [22] of the quantifier elimination result from [18], the structure F, respectively $\mathcal{O}_F$, has quantifier elimination in the language $\mathcal{L}_{\mathrm{an,qe}}^F$, respectively $\mathcal{L}_{\mathrm{an,D}}^F$.

We will derive Theorem 2.11 from the resolution results (2.2) and (2.4) from [18], and the piecewise description of definable functions by terms in a slightly larger language. Recall that a term in a language is a finite composition of function symbols from the language. Similarly as for quantifier elimination, one needs an adapted language to ensure that definable functions are piecewise equal to terms (and to ensure that definable sets are given by quantifier free formulas). See Definition 2.10 for the notions of analytic units and monomials on $\mathcal{O}_F^m$.

First we give a combination of Theorems (2.2) and (2.4) from [18].

PROPOSITION 5.1 [18]. *Let $f : \mathcal{O}_F^m \to \mathcal{O}_F^k$ be a map whose component functions are $\mathcal{L}_{\mathrm{an,D}}^F$-terms. Then there exist a compact analytic manifold C and an analytic map $h : C \to \mathcal{O}_F^m$ such that h is an analytic isomorphism on the preimage of a dense open subset of $\mathcal{O}_F^m$ and such that for each $c \in C$, there are an open neighborhood O_c of c in C and an analytic isomorphism $\varphi_c : \mathcal{O}_F^m \to O_c$ such that for each $i = 1, \ldots, k$,*

$$f_i \circ h \circ \varphi_c = u_{ic} M_{ic}$$

and

$$(h \circ \varphi_c)^*(dy_1 \wedge \cdots \wedge dy_m) = u_c M_c \cdot dx_1 \wedge \cdots \wedge dx_m,$$

where the u_c and u_{ic} are analytic units on $\mathcal{O}_F^m$ and the M_c and M_{ic} are monomials on $\mathcal{O}_F^m$, and where the y_i and the x_i are coordinate functions on $\mathcal{O}_F^m$.

Proof. In the special case that f is moreover analytic, the proof goes as the proof of Theorem 2.2 from [18], where h is even a composition of suitable blowing up maps, using the excellence result (Theorem 1.2) of [20] and Hironaka's results from [25]. The statement about the differential forms follows from the description of h in [18] in terms of blowing up maps. For general f, in the proof of Theorem (2.4) of [18], one constructs an analytic map $h_0 : C_0 \to \mathcal{O}_F^m$ for some compact analytic manifold C_0 such that h_0 is an analytic isomorphism on the preimage of a dense open subset of $\mathcal{O}_F^m$, such that h_0 is obtained as a finite composition of blowing up maps with respect to closed analytic submanifolds in codimension at least 2 and such that $f \circ h_0$ is analytic. Now apply the previous case to $f \circ h_0 \circ \varphi_c$. (In [18, Theorems 2.2 and 2.4], only $F = \mathbb{Q}_p$ is considered but the proof applies for general F as well.) □

The following result relates $\mathcal{L}_{\mathrm{an}}^F$-definable functions to terms, at the cost of taking powers.

PROPOSITION 5.2 [**22**]. *Let $f_i : X \subset F^n \to F$ be finitely many $\mathcal{L}_{\mathrm{an}}^F$-definable functions. Then there is an integer $M > 0$ and there are finitely many disjoint definable subsets X_s of X and finitely many definable functions $f_{is\ell}$ on X_s such that*

$$f_{is\ell}^M$$

is given by an $\mathcal{L}_{\mathrm{an,qe}}^F$-term for each i, s, ℓ and such that

$$\sum_\ell f_{is\ell} = f_i \text{ on } X_s.$$

Proof. This follows from [**22**, Theorem 5.5] and its proof. □

We will use Proposition 5.2 only once, namely in Case 3 of the proof of Theorem 2.11. One can also treat that case in an alternative way and avoid the use of Proposition 5.2 by proceeding instead by induction on the complexity of terms based on the weaker result (than Proposition 5.2) that $\mathcal{L}_{\mathrm{an}}^F$-definable functions are piecewise equal to terms in an expansion of $\mathcal{L}_{\mathrm{an,qe}}^F$-terms with root functions; see [**14**, Theorem 7.5].

The following (basic) lemma states that $\mathcal{L}_{\mathrm{an,qe}}^F$-terms correspond piecewise to $\mathcal{L}_{\mathrm{an,D}}^F$-terms. The lemma after this one gives a similar result for definable sets.

LEMMA 5.3. *Consider a tuple $t(x)$ of $\mathcal{L}_{\mathrm{an,qe}}^F$-terms in n variables x_i. Let $X \subset \mathcal{O}_F^n$ be an $\mathcal{L}_{\mathrm{an,qe}}^F$-definable set. Assume that $t(x) \in \mathcal{O}_F^s$ for each $x \in X$. Then there exist finitely many tuples t_j of $\mathcal{L}_{\mathrm{an,D}}^F$-terms such that for each $x \in X$, one has*

$$t(x) = t_j(x)$$

for some j.

Proof. Since a term by definition is a composition of function symbols in the language, any $\mathcal{L}_{\mathrm{an,qe}}^F$-term can be written as a composition of functions $f_i : F^l \to F^k$ such that each of the f_i satisfies one of the following:

(1) All the components of f_i are rational functions of the form $p \cdot (q)^{-1}$ for some polynomials p and q.

(2) All the components of f_i are restricted analytic functions.

Since a composition of rational functions as in (1) is piecewise equal to a rational function as in (1) (with definable pieces), we may suppose that

$$t = a_1 \circ b_1 \circ \cdots \circ a_N \circ b_N \circ a_{N+1},$$

where the a_i satisfy (2) and the b_i satisfy (1). For any rational function r of the form $p \cdot (q)^{-1}$ for some polynomials p and q, write r_D for the function $D(p, q)$, where D stands for restricted division. Likewise, for a tuple r of such rational functions r_i, write r_D for the corresponding tuple of the $r_{i,D}$. Note that $a_1 \circ b_1(z)$ equals $a_1 \circ b_{1,D}(z)$ for any z in $\mathcal{O}_F^\beta$ with β the arity of b_1. The lemma now follows easily by induction on N, by focusing on the remaining part $a_2 \circ b_2 \circ \cdots \circ a_N \circ b_N \circ a_{N+1}$. $\qquad\square$

LEMMA 5.4. *Any $\mathcal{L}_{\mathrm{an,qe}}^F$-definable set $X \subset \mathcal{O}_F^n$ is $\mathcal{L}_{\mathrm{an,D}}^F$-definable and vice versa.*

Proof. Clearly, any $\mathcal{L}_{\mathrm{an,D}}^F$-definable set in $\mathcal{O}_F^n$ is also $\mathcal{L}_{\mathrm{an,qe}}^F$-definable. For the other direction, consider the $\mathcal{L}_{\mathrm{an,qe}}^F$-definable embedding $i : F \to \mathcal{O}_F^2$ given by

$$
i(x) = \begin{cases} (x, 0) & \text{if } x \in \mathcal{O}_F, \\ (x^{-1}, 1) & \text{otherwise.} \end{cases}
$$

The assertion now follows from the following simple observations:

- There is an $\mathcal{L}_{\mathrm{an,D}}^F$-definable function $a : \mathcal{O}_F^4 \to \mathcal{O}_F^4$ such that for any $x, y \in F$, we have $a(i(x), i(y)) = i(x + y)$.

- There is an $\mathcal{L}_{\mathrm{an,D}}^F$-definable function $m : \mathcal{O}_F^4 \to \mathcal{O}_F^4$ such that for any $x, y \in F$, we have $m(i(x), i(y)) = i(xy)$.

- For any restricted analytic f function on F, there exists an $\mathcal{L}_{\mathrm{an,D}}^F$-definable function $A_f : \mathcal{O}_F^2 \to \mathcal{O}_F^2$ such that for any $x \in F$, we have $A_f(i(x)) = i(f(x))$.

- There is an $\mathcal{L}_{\mathrm{an,D}}^F$-definable function $inv : \mathcal{O}_F^2 \to \mathcal{O}_F^2$ such that for any $x \in F$, we have $inv(i(x)) = i(x^{-1})$ (recall that x^{-1} is interpreted as 0 if $x = 0$).

- For any integer n, there is a $\mathcal{L}_{\mathrm{an,D}}^F$-definable set $\Pi_n \subset \mathcal{O}_F^2$ such that for any $x \in F$, we have $i(x) \in \Pi_n$ if and only if $x \in P_n$. $\qquad\square$

Finally, in order to prove Part (6) of the resolution theorem, we will use the following standard result.

LEMMA 5.5. *Let u be an analytic unit on $\mathcal{O}_F^n$ and M be a monomial on $\mathcal{O}_F^n$. Consider the function $f = uM$ on $\mathcal{O}_F^n$. Then there exist a finite disjoint $\mathcal{L}$-definable open cover $\mathcal{O}_F^n = \bigcup_{i=1}^N U_i$ and $\mathcal{L}$-definable analytic isomorphisms $\phi_i : \mathcal{O}_F^n \to U_i$ such that*

- $\phi_i^*(f)$ is either a monomial on $\mathcal{O}_F^n$ or a constant times an analytic unit on $\mathcal{O}_F^n$;

- for any analytic unit u' on $\mathcal{O}_F^n$ and any monomial M' on $\mathcal{O}_F^n$, the function $\phi_i^*(u'M')$ equals the product of a monomial with an analytic unit on $\mathcal{O}_F^n$.

Proof. Let us first treat the special case that $M(x_1, \ldots, x_n) = x_1^{k_1} \cdots x_n^{k_n}$ for some natural numbers k_i, that $k_1 \geqslant 1$, and that there is an analytic unit v on $\mathcal{O}_F^n$ such that $v^{k_1} = u$ and such that $x_1 v(x)$ is a special restricted power series in the sense of [27, Section 2.2]. Consider the map $\psi : \mathcal{O}_F^n \to \mathcal{O}_F^n$ given by

$$\psi(x_1, \ldots, x_n) = (x_1 v(x_1, \ldots, x_n), x_2, \ldots, x_n).$$

By [27, Corollary 2.2.1], the map ψ is an analytic isomorphism, say, with inverse ϕ. Then $\phi_1 = \phi$ is as desired (with $N = 1$); in particular, one has $\phi^*(f) = M$.

Let us now reduce to the conditions of the special case. Fix $a = (a_1, \ldots, a_n) \in \mathcal{O}_F^n$. Since any two balls in $\mathcal{O}_F^n$ are either disjoint or contained in one another and by compactness of $\mathcal{O}_F^n$, it is enough to find a ball B_a around a and an $\mathcal{L}$-definable analytic isomorphism $\phi_a : \mathcal{O}_F^n \to B_a$ satisfying both conditions of the lemma. We may assume that there is i with $a_i = 0$ and that the monomial M depends nontrivially on the coordinate x_i. Indeed, otherwise we can take a small enough ball B_a around a and ϕ to be a homothety after a translation to make $\phi^*(f)$ an analytic unit times a constant. Without loss of generality, we may assume that $i = 1$. Also, we can assume that $a_1 = \cdots a_k = 0$ for some $k \geqslant 1$ and $a_{k+1}, \ldots, a_n \neq 0$. Write

$$M(x_1, \ldots, x_n) = x_1^{k_1} \cdots x_n^{k_n}.$$

Let $L := (x_1, \ldots, x_n) := x_1^{k_1} \cdots x_m^{k_m}$ and $K(x_1, \ldots, x_n) := x_{m+1}^{k_{m+1}} \cdots x_n^{k_n}$. We have $M = KL$. Note that K is a unit in a small ball B' around a. Consider the affine transformation t, which is a homothety after a translation and which maps 0 to a and $\mathcal{O}_F^n$ onto B'. Up to choosing B' small enough, the pullback $t^*(f)$ is of the form as in our special case, and the lemma now follows from this special case applied to $t^*(f)$. □

We can now give the complete proof of Theorem 2.11.

Proof of Theorem 2.11. It is enough to prove only statements (1)–(5) since statement (6) can be deduced from them by Lemma 5.5 (and refining the cover U_i and changing the maps ϕ_i). Write $\omega = g \bigwedge_{i_j \in I} dx_{i_j}$ for some g and some I. Without harm, we may suppose that g appears among the functions f_i.

Case 1: $X = O = \mathcal{O}_F^n$ and the f_i are $\mathcal{L}_{an,D}^F$-terms.

This case follows from Proposition 5.1. Indeed, the disjointness of the $\phi_i(U)$ is easily obtained on top of the conclusions of Proposition 5.1 by the total disconnectedness and compactness.

Case 2: X and O are of dimension $m = n$, and the f_i are $\mathcal{L}^F_{\mathrm{an,D}}$-terms.

By Lemma 5.4 and quantifier elimination in $\mathcal{L}^F_{\mathrm{an,D}}$, each of the sets X and O is defined by a finite Boolean combination of condition of the form $t_i(x) \in P_{n_i}$ for some $\mathcal{L}^F_{\mathrm{an,D}}$-terms t_i and some $n_i \geqslant 1$. (Note that a condition $t = 0$ corresponds to t not being in P_1.) Let N be the product of all occurring n_i. By the previous case, applied to the terms t_i and f_i, we can assume that the terms t_i and the terms f_i are analytic units times monomials on $\mathcal{O}^n_F$. Without loss of generality, we can suppose that the occurring units have constant coset in $F^\times$ modulo P_N. For any $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathcal{O}_F \setminus \{0\})^n$, define $\phi_\lambda : \mathcal{O}^n_F \to \mathcal{O}^n_F$ by $\phi_\lambda(x_1, \ldots, x_n) = (\lambda_1 x_1^N, \ldots, \lambda_n x_n^N)$. We can find finitely many $\lambda^1, \ldots \lambda^K \in \mathcal{O}^n_F$ such that the images of ϕ_{λ^j} cover the entire $\mathcal{O}^n_F$ and such that the sets $\phi_{\lambda^j}(U)$ are disjoint. Thus $\dim \mathcal{O}^n_F \setminus \bigcup_j \phi_{\lambda^j}(U) < n$. This implies that $\dim X \setminus \bigcup_j \phi_{\lambda^j}(U) < n = \dim X$ and thus (since X has pure dimension) $\bigcup_j \phi_{\lambda^j}(U) \cap X$ is dense in X. Note that for any j, either $\phi_{\lambda^j}(U) \subset O$ or $\phi_{\lambda^j}(U) \cap O = \emptyset$. We obtained that the collection

$$\{\phi_{\lambda^j} | \phi_{\lambda^j}(U) \subset O\}$$

meets the requirements.

Case 3: X and O are of dimension $m = n$ (and the f_i are general).

By Lemma 5.3 and Proposition 5.2, each f_i is piecewise equal to a sum of definable roots of $\mathcal{L}^F_{\mathrm{an,D}}$-terms t_{ij}. By working piecewise, we may suppose that O equals the interior of one such piece and that X is its closure. By Case 2, we can suppose that $X = \mathcal{O}^n_F$ and that $O = U$, and that the t_{ij} are analytic units times monomials on $\mathcal{O}^n_F$. By composing with Nth power maps (as the ϕ_λ of Case 2) for some highly divisible N, we reduce to the case that also the occurring roots of the t_{ij} are $\mathcal{L}^F_{\mathrm{an,D}}$-terms (since we have composed with power maps), and hence also that f is given by a tuple of $\mathcal{L}^F_{\mathrm{an,D}}$-terms. Now we end by Case 2 for these $\mathcal{L}^F_{\mathrm{an,D}}$-terms.

Case 4: General case.

Write $\dim X = m$ as in the theorem. By Proposition 4.4 and Lemma 4.5, and up to working piecewise, we may suppose that O equals the graph of an $\mathcal{L}^F_{\mathrm{an}}$-definable function $\tau : V \to \mathcal{O}^{n-m}_F$ for some $\mathcal{L}^F_{\mathrm{an}}$-definable open V of $\mathcal{O}^m_F$ and for some coordinates on affine space. Extend τ by 0 to a function $\bar\tau$ on $\bar V$. Using Case 3 for $\bar\tau$ on $\bar V$, we can assume that $\bar V = \mathcal{O}^m_F$ and that each component of $\bar\tau|_U$ equals the restriction to U of a monomial times an analytic unit on $\mathcal{O}^m_F$. This means that τ can be extended continuously to a function $\tilde\tau$ on $\mathcal{O}^m_F$ and thus that X equals the graph of $\tilde\tau$. This gives us an $\mathcal{L}$-definable analytic isomorphism of $\mathcal{L}$-manifolds $i : \mathcal{O}^m_F \to X$. The assertion follows now from Case 3 applied to $\mathcal{O}^m_F$, U, $f \circ i$, and $(i|_U)^*\omega$ in the roles of X, O, f, ω. $\qquad\square$

It may be interesting to look for a definable, strict C^1 variant of Theorem 2.11 that can be shown without using analyticity and without using Hironaka's resolution results; see Remark 8.7 about a possible axiomatic approach in which piecewise analyticity may not hold.

6. Proof of holonomicity

6.1. Proof of the analytic case. In this section, we prove Theorem 2.7 for $\mathcal{L} = \mathcal{L}_{\text{an}}^F$.

Consider for any $n > 0$ the Haar measure $|dx|$ on F^n normalized so that $\mathcal{O}_F^n$ has measure 1. By a smooth measure on X we mean a distribution on X that is locally (at any point $x \in X$) either zero or given by integration against the measure associated to an analytic volume form on X. Note that the wave front set in a way describes the nonsmooth aspect of a distribution and, in particular, the wave front set of a smooth measure is empty. We will deduce Theorem 2.7 from the following theorem from [**11**] and our Key Lemma 6.2 by using regularization and induction on the dimension of X.

THEOREM 6.1 [**11**]. *Let X be an $\mathcal{L}$-submanifold of F^n of dimension m and let ξ be a distribution on X. If ξ is of class $\mathscr{C}_{\mathcal{L}}^{\exp}$, then there exists an $\mathcal{L}$-definable set $C \subset X$ of dimension less than m such that ξ is smooth when restricted to $X \smallsetminus C$. Moreover, the locally constant density function on $X \smallsetminus C$ is of class $\mathscr{C}_{\mathcal{L}}^{\exp}$ when taken against the measure on X induced from the inclusion in F^n as in [**11**], Section 2.3.*

Proof of Theorem 6.1. If ξ is of class $\mathscr{C}^{\exp}$, then this is a special case of [**11**, Theorem 4.1.2]. For the $\mathscr{C}_{\text{an}}^{\exp}$-class, one moreover uses [**11**, Remark 4.3.3]. □

As mentioned in Section 3, naively extending μ by zero outside U does not give a distribution on X in general as this extension may not be locally L^1 on X. In fact, the proof of the following key lemma is quite involved and intertwined with the one for Theorem 2.7.

LEMMA 6.2 (Key Lemma). *Suppose that $\mathcal{L} = \mathcal{L}_{\text{an}}^F$. Let U be an $\mathcal{L}$-definable dense open subset of a compact $\mathcal{L}$-manifold X. Let μ be a smooth measure of class $\mathscr{C}_{\mathcal{L}}^{\exp}$ on U. Then there is an $\mathcal{L}$-definable dense open $V \subset U$ such that $\mu|_V$ can be extended to an $\mathcal{L}$-WF-holonomic distribution on X of class $\mathscr{C}_{\mathcal{L}}^{\exp}$.*

Based on the key lemma for $\dim X \leqslant n$ and assuming Theorem 2.7 for $\dim X < n$ and with $\mathcal{L} = \mathcal{L}_{\text{an}}^F$, we can now prove Theorem 2.7 for X with

$\dim X = n$ and $\mathcal{L} = \mathcal{L}_{\mathrm{an}}^F$. In Section 6.1.1, we will prove the key lemma for $\dim X = n$ assuming Theorem 2.7 for X with $\dim X < n$. This will complete the proof of Theorem 2.7 for $\mathcal{L} = \mathcal{L}_{\mathrm{an}}^F$. In Section 6.2, we will deduce Theorem 2.7 in the general case.

Proof of Theorem 2.7 assuming $\mathcal{L} = \mathcal{L}_{\mathrm{an}}^F$. Denote by $\mathrm{dimSupp}(\xi)$ the minimal dimension of an $\mathcal{L}$-definable set $Y \subset X$ such that $\mathrm{Supp}(\xi) \subset Y$. The proof is by induction on $\mathrm{dimSupp}(\xi)$. The base case is trivial.

By Lemma 4.5, we find a finite, open $\mathcal{L}$-definable cover $X = \bigcup_i U_i$ together with open $\mathcal{L}$-definable embeddings $U_i \to \mathcal{O}_F^n$ of $\mathcal{L}$-manifolds. (Recall that $\dim X = n$.) Using partition of unity (Proposition 4.4), we can reduce to the case that X is an open $\mathcal{L}$-definable set in $\mathcal{O}_F^n$. Using the regularization result (Theorem 2.8), we can reduce to the case $X = \mathcal{O}_F^n$. We proceed by analyzing the following cases.

Case 1: $\mathrm{dimSupp}(\xi) = \dim(X)$

By Theorem 6.1, there is a definable open dense subset $V \subset X$ such that the restriction $\xi|_V$ is smooth. By our Key Lemma 6.2, and up to making V smaller if necessary, we can extend $\xi|_V$ to an $\mathcal{L}$-WF-holonomic distribution ξ' on X of class $\mathscr{C}_{\mathcal{L}}^{\exp}$. Decompose ξ as the sum $\xi = \xi' + (\xi - \xi')$. By the induction assumption (on dimSupp), $\xi - \xi'$ is $\mathcal{L}$-WF-holonomic (indeed, its support lies in $X \setminus V$, which is of dimension less than $\mathrm{dimSupp}(\xi)$ by properties of dimensions of definable sets). Since the sum of $\mathcal{L}$-WF-holonomic distributions is $\mathcal{L}$-WF-holonomic, the theorem follows.

Case 2: $\mathrm{dimSupp}(\xi) < \dim(X)$

Let $Y \subset X$ be a closed $\mathcal{L}$-definable set such that $\mathrm{Supp}(\xi) \subset Y$ and $\dim(Y) = \mathrm{dimSupp}(\xi)$.

We apply resolution of singularities (Theorem 2.11) for the definable set Y and the constant function 1 on it. Let U, ϕ_i, U_i and d_i be as in Theorem 2.11. Let $V = \bigcup U_i$. Put

$$\xi' := \xi|_{X \setminus (Y \setminus V)}.$$

The distribution ξ' is supported on V and thus can be thought of as a distribution on V. It is easy to see that as such it is also of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class. Put $\xi_i = \xi'|_{U_i}$. Using regularization (Theorem 2.8), we can extend $\phi_i^*(\xi_i)$ to a $\mathscr{C}_{\mathcal{L}}^{\exp}$-class distribution ξ_i' on $\mathcal{O}_F^{\dim Y}$. Each ξ_i' is $\mathcal{L}$-WF-holonomic by the inductive hypothesis (indeed, $\dim Y < n$). Let

$$\xi'' := \sum (\phi_i)_*(\xi_i')/d_i.$$

Note that $\xi''|_V = \xi|_V$.

By [**1**, Proposition 3.2.7(2)] (see also [**24**, Theorem 2.9] and [**11**, Theorem 2.9.6]) on pushforwards, ξ'' is also $\mathcal{L}$-WF-holonomic. Furthermore, ξ'' is of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class by the stability under pushforward from [**11**, Theorem 3.4.5 and Remark 4.3.3]. Decompose

$$\xi = \xi'' + (\xi - \xi'').$$

Again we are done by induction on $\dim \mathrm{Supp}(\xi)$. Indeed, $(\xi - \xi'')$ has a lower dimensional support than ξ. □

6.1.1. Proof of the key lemma. Define a regular triple to be a triple (μ, U, X) that satisfies the assumptions of the key lemma, namely, μ is a $\mathscr{C}_{\mathcal{L}}^{\exp}$-class smooth measure on an $\mathcal{L}$-definable dense open U of the compact $\mathcal{L}$-manifold X and where $\mathcal{L} = \mathcal{L}_{\mathrm{an}}^F$. Call such a regular triple good if the key lemma holds for it. Precisely, a regular triple (μ, U, X) is called good if there is an $\mathcal{L}$-definable dense open V of U such that $\mu|_V$ has an extension to an $\mathcal{L}$-WF-holonomic distribution on X, which is of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class.

The key lemma will follow from the resolution theorem for definable functions, the following straightforward proposition and lemma, and an inductive procedure in tandem with the proof of our main holonomicity result.

PROPOSITION 6.3. *Suppose that $\mathcal{L} = \mathcal{L}_{\mathrm{an}}^F$. Consider a regular triple (μ, U, X). Then the following properties hold.*

(1) *If (μ_1, U, X) and (μ_2, U, X) are good, then so is $(\mu_1 + \mu_2, U, X)$.*

(2) *If (μ_1, U_1, X_1) (μ_2, U_2, X_2) are good, then so is $(\mu_1 \boxtimes \mu_2, U_1 \times U_2, X_1 \times X_2)$, with $\mu_1 \boxtimes \mu_2$ the product measure.*

(3) *If (μ, U, X) is good and f is a smooth (namely, locally constant) $\mathscr{C}_{\mathcal{L}}^{\exp}$-function on X, then $(\mu f|_U, U, X)$ is good.*

(4) *Assume that (μ_1, U_1, X_1) and (μ_2, U_2, X_2) are regular triples and that $\varphi : X_1 \to X_2$ is a proper $\mathcal{L}$-analytic map such that*

- *$\varphi(U_1)$ is open in U_2;*

- *$\varphi|_{U_1}$ is a local isomorphism onto $\varphi(U_1)$;*

- *$(\varphi|_{U_1})_*(\mu_1) = \mu_2$, that is, the pushforward along $\varphi|_{U_1}$ of the distribution μ_1 on U_1 equals μ_2 (as distributions on U_2).*

Then, if (μ_1, U_1, X_1) is good, then so is (μ_2, U_2, X_2).

Proof. Only (4) needs a proof (the other properties follow more easily from the corresponding properties of wave front sets and analytic manifolds).

For Property (4), take an $\mathcal{L}$-definable dense open V_1 of U_1 and a good extension ξ_1 on X_1 of $\mu_1|_{V_1}$ (namely, an extension that is of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class and $\mathcal{L}$-WF-holonomic). The pushforward $\varphi_*(\xi_1)$ is of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class and $\mathcal{L}$-WF-holonomic. Indeed, the pushforward of an analytically WF-holonomic distribution under a proper analytic map is again analytically WF-holonomic by [1, Proposition 3.2.7(2)] (see also [24, Theorem 2.9] and [11, Theorem 2.9.6]), and the pushforward of a $\mathscr{C}_{\mathcal{L}}^{\exp}$-class distribution under a proper $\mathcal{L}$-analytic map is again of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class (see [11, Theorem 3.4.5 and Remark 4.3.3]). Let V_2 be the union of $\varphi(V_1)$ with $U_2 \smallsetminus \overline{\varphi(U_1)}$, where $\overline{\varphi(U_1)}$ is the closure of $\varphi(U_1)$. Now, by construction and definability properties, V_2 is an $\mathcal{L}_{\mathrm{an}}^F$-definable dense open in U_2, and $\varphi_*(\xi_1)$ extends $\mu_2|_{V_2}$. Hence, we are done. $\square$

LEMMA 6.4. *Consider an $\mathcal{L}$-submanifold $X \subset F^n$. Let $U \subset X$ have a finite complement in X, and let ξ be a distribution of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class on U such that ξ is $\mathcal{L}$-WF-holonomic. Then there is a distribution ξ_X on X, which is of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class, whose restriction to U equals ξ, and such that ξ_X is $\mathcal{L}$-WF-holonomic.*

Proof. Regularize ξ to a distribution ξ_X on X using a section as given by Theorem 2.8. Then ξ_X is as desired. Indeed, ξ_X is $\mathcal{L}$-WF-holonomic since U has a finite complement in X. $\square$

We show the following lemma assuming Theorem 2.7 for X of dimension less than n.

LEMMA 6.5. *Suppose that $\mathcal{L} = \mathcal{L}_{\mathrm{an}}^F$. Let a good triple (μ, U, X) be given with $\dim X \leqslant n$. Let ξ be any $\mathscr{C}_{\mathcal{L}}^{\exp}$-class distribution on X, which coincides with μ on a dense open V of U. Then ξ is $\mathcal{L}$-WF-holonomic.*

Proof. Let a distribution ξ_X on X be given by the goodness of the triple. Write $\xi = \xi_X + (\xi - \xi_X)$. Then, by the argument of Case 2 of the proof of Theorem 2.7 and by our assumption that Theorem 2.7 holds when $\dim X < n$, we find that $\xi - \xi_X$ and hence also that ξ are $\mathcal{L}$-WF-holonomic. $\square$

We can now prove our key lemma for $\dim X = n$ assuming the main holonomicity theorem (Theorem 2.7) for X with $\dim X < n$ and $\mathcal{L} = \mathcal{L}_{\mathrm{an}}^F$.

Proof of Key Lemma 6.2. Let a regular triple (μ, U, X) be given. We proceed by induction on $\dim X$, where the one-dimensional case is taken care of by

Lemma 6.4. By the definition of $\mathscr{C}_{\mathcal{L}}^{\exp}(U)$, μ is a finite sum of terms of the form

$$x \in U \mapsto c\psi(f_1(x))|f_2(x)| \prod_{i=3}^{k} \operatorname{ord}(f_i(x)),$$

where the f_i are $\mathcal{L}$-definable F-valued functions and where $c \in \mathbb{C}$, and where $f_i \neq 0$ for $i > 2$. By (1) of Proposition 6.3, and up to replacing U with a dense open, we may suppose that μ equals one such term. Indeed, any such term is locally constant on a dense definable open. By working piecewise on X, we may suppose for each i that either $|f_i| > 1$ or $|f_i| \leqslant 1$ holds. Apply Theorem 2.11 to the function whose ith component is f_i if $|f_i| \leqslant 1$ on X and $1/f_i$ otherwise. By (1) and (4) of Proposition 6.3, this reduces the case to U being $(\mathcal{O}_F \setminus \{0\})^m$, $X = \mathcal{O}_F^m$, and each f_i (or $1/f_i$) being of the form (2.5.1). We may thus suppose that μ is of the form

$$\mu(x) = c\psi\big(u(x)M(x)^{\eta_1}\big) \prod_{i=1}^{m} |x_i|^{\eta_2 s_i} \operatorname{ord}(x_i)^{t_i},$$

where $c \in \mathbb{C}$, s and t lie in $\mathbb{N}^m$, η_1 and η_2 lie in $\{1, -1\}$, u is an analytic unit on $\mathcal{O}_F^m$, and M is a monomial on $\mathcal{O}_F^m$. By (6) of Theorem 2.11, we may furthermore suppose that either $M(x)$ is constant or $u(x) = 1$ for all x in $\mathcal{O}_F^m$.

In the first case that $M(x)$ is constant on $\mathcal{O}_F^m$, we are done by (2) and (3) of Proposition 6.3. Indeed, $\psi(u(x))$ is smooth and nonvanishing on $\mathcal{O}_F^m$, and $\mu(x)/\psi(u(x))$ is a Cartesian product situation with one-dimensional Cartesian factors, each of which falls under Lemma 6.4. Similarly, one treats the case that M is nonconstant and $\eta_1 = 1$. Let us now treat the final case that $u(x) = 1$ and $M(x)$ is nonconstant on $\mathcal{O}_F^m$, and $\eta_1 = -1$. We first regularize μ to a $\mathscr{C}_{\mathcal{L}}^{\exp}$-class distribution ξ on $\mathcal{O}_F^m$. We will then show, by working locally, that ξ is $\mathcal{L}$-WF-holonomic. Recall that U is $(\mathcal{O}_F \setminus \{0\})^m$ and that X is $\mathcal{O}_F^m$.

Let p be the function $\mathcal{O}_F^m \to \mathcal{O}_F^m$ that sends x to $(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m)$, where i is the minimal number in $\{1, \ldots, m\}$ such that $|x_i|$ is minimal among the $|x_j|$ for j in $\{1, \ldots, m\}$. Let φ_0 be a locally constant function on $U^c := \mathcal{O}_F^m \setminus U$ (with the subset topology) and with compact support. Associate to φ_0 a function $L(\varphi_0)$ on $\mathcal{O}_F^m$ by sending $x \in \mathcal{O}_F^m$ to $\varphi_0(p(x))$. Clearly $L(\varphi_0)$ is a Schwartz–Bruhat function on $\mathcal{O}_F^m$.

For φ, a Schwartz–Bruhat function on $\mathcal{O}_F^m$, let φ^c be the restriction of φ to U^c, and consider the lift $L(\varphi^c)$. Let $\widetilde{\varphi}$ be the restriction to U of $\varphi - L(\varphi^c)$. Then by construction, $\widetilde{\varphi}$ is a Schwartz–Bruhat function on U. Moreover, the function sending φ to $\widetilde{\varphi}$ is a $\mathbb{C}$-linear map from $\mathcal{S}(\mathcal{O}_F^m)$ to $\mathcal{S}(U)$, which is the identity on $\mathcal{S}(U)$. Now let ξ be the distribution on $\mathcal{O}_F^m$ sending a Schwartz–Bruhat function φ

on $\mathcal{O}_F^m$ to the evaluation of the distribution μ at $\widetilde{\varphi}$. Clearly, ξ is of $\mathscr{C}_{\mathcal{L}}^{\exp}$-class and extends μ (use Proposition 4.2). It is only left to show that ξ is $\mathcal{L}$-WF-holonomic, which is a local property.

Fix a point a in $\mathcal{O}_F^m \setminus U$. It is enough to show that the restriction of ξ to a small neighborhood of a is $\mathcal{L}$-WF-holonomic. By Lemma 6.4, it is sufficient to treat the case that $a \neq 0$. Up to reordering the variables, we may suppose that the first coordinate a_1 of a is nonzero. If $M(x)$ does not depend on x_1, then we are done by (2) of Proposition 6.3 and induction on the dimension of X and Lemma 6.5. If $M(x)$ depends nontrivially on x_1 but on no other variable, we can finish similarly. Now suppose that $M(x)$ depends nontrivially on x_1 and, say, also nontrivially on x_2. Write $M(x) = x_1^{k_1} \cdots x_m^{k_m}$. Consider the map $\phi : \mathcal{O}_F^m \to \mathcal{O}_F^m$

$$(x_1, \ldots, x_n) \mapsto (a_1 x_1^{k_2}, x_2 x_1^{-k_1}, x_3 \ldots, x_m).$$

There exists a small ball B around $(1, a_2, \ldots, a_m)$ such that $\phi|_B$ is a proper analytic isomorphism onto an open neighborhood of a. So, it is enough to prove that $(\phi^*(\xi))|_B$ is $\mathcal{L}$-WF-holonomic.

Note that $(B, \phi^{-1}(U) \cap B, \phi^*(\mu))$ is a good triple by (2) of Proposition 6.3 and induction on the dimension of X (indeed, the pullback of M along ϕ does not involve x_1 anymore and hence one can apply (2) of 6.3). By Lemma 6.5, we are done for $(\phi^*(\xi))|_B$. □

6.2. Holonomicity: the algebraic case.
In this section, we prove Theorem 2.7 for $\mathcal{L} = \mathcal{L}^F$. We will first treat the case that the Zariski closure of X in $\mathbb{A}_F^n$ is smooth using Theorem 2.7 with $\mathcal{L} = \mathcal{L}_{an}^F$ as proved in Section 6 and by [11, Proposition 4.3.1]. The general case will follow from this smooth case by our partition of unity result and by reducing to graphs.

The following remark amends [11, Proposition 4.3.1] by making explicit the smoothness condition.

REMARK 6.6. In [11], the notion of algebraic WF-holonomicity is only defined for distributions on analytic submanifolds $X \subset F^n$ such that the Zariski closure of X in $\mathbb{A}_F^n$ is smooth. Therefore, the condition that the Zariski closure of $W_{F,y}$ in $\mathbb{A}_F^n$ is smooth should be added as an extra assumption at the start of the 'Moreover' statement of Proposition 4.3.1, for each y and F.

PROPOSITION 6.7. *Let $X \subset F^n$ be an $\mathcal{L}^F$-manifold such that the Zariski closure of X in $\mathbb{A}_F^n$ is smooth. Let ξ be a distribution on X of $\mathscr{C}_{\mathcal{L}^F}^{\exp}$-class. Suppose that ξ is strict C^1 WF-holonomic. Then ξ is $\mathcal{L}^F$-WF-holonomic and thus algebraically WF-holonomic (see Definitions 2.2 and 2.6).*

Proof. The result is a special form of [**11**, Proposition 4.3.1] and Remark 6.6. □

Proof of Theorem 2.7 for $\mathcal{L} = \mathcal{L}^F$. We may suppose that $X \subset F^n$. If the Zariski closure of X in $\mathbb{A}^n_F$ is smooth, then we are done by Proposition 6.7 and by the above proved case of Theorem 2.7 for $\mathcal{L}^F_{\mathrm{an}}$ (which contains $\mathcal{L}^F$). Indeed, $\mathcal{L}^F_{\mathrm{an}}$-WF-holonomicity implies strict C^1 WF-holonomicity. Now let $X \subset F^n$ be a general $\mathcal{L}^F$-manifold. By Lemma 4.5 and Proposition 4.4, we may suppose that X is of dimension n. Indeed, the pieces given by Lemma 4.5 can be taken clopen and disjoint by Proposition 4.4. Hence, we are done by the previous case since the Zariski closure of X now equals $\mathbb{A}^n_F$, which is smooth. This finishes the proof of Theorem 2.7. □

7. Smooth loci and zero loci

A zero locus of a $\mathscr{C}^{\exp}_{\mathcal{L}}$-class function equals the zero locus of a bounded function of the same class, as follows.

PROPOSITION 7.1. *Let g be in $\mathscr{C}^{\exp}_{\mathcal{L}}(X)$ for some $\mathcal{L}$-definable set X. Then there is a function h in $\mathscr{C}^{\exp}_{\mathcal{L}}(X)$ such that $|h(x)|_{\mathbb{C}}$ is bounded on X and such that the zero locus of h equals the zero locus of g.*

Proof. We will construct h by multiplying g with a function f in $\mathscr{C}^{\exp}_{\mathcal{L}}(X)$ such that f takes positive real values. Such a product clearly preserves the zero locus. Write g as a finite sum of products of generators T_i of the forms (1), (2), and (3) of Definition 2.4. In each generator T_i, there occurs an F-valued definable function, say, t_i. For each $x \in X$, let $\alpha(x)$ be the maximum of 0 and the sum over i of the values $\mathrm{ord}\, t_i(x)$, where i is such that $0 < |t_i(x)| \leqslant 1$. Now let $f(x)$ be $q_F^{-\alpha(x)}$ for $x \in X$. Then clearly, $h = f(x)g(x)$ is as required. □

The following result about local constancy is more simple than Theorem 6.1.

PROPOSITION 7.2 [**10**]. *Let X be an $\mathcal{L}$-definable set and let g be in $\mathscr{C}^{\exp}_{\mathcal{L}}(X)$. Then there is a finite partition of X into $\mathcal{L}$-manifolds D_i such that the restriction of g to D_i is locally constant for each i.*

Proof. Up to partitioning X into finitely many $\mathcal{L}$-manifolds by [**6**, Proposition 1.5.3] and restricting g to the pieces, we may suppose that X is an $\mathcal{L}$-manifold. By Theorem 6.1, it follows that g is locally constant on the complement of an $\mathcal{L}$-definable set $D \subset X$ of dimension less than $\dim X$. (Alternatively, this follows by the more basic [**10**, Theorem 4.4.3] and its proof.) Up to making D larger if necessary, we may suppose that $X \smallsetminus D$ is an $\mathcal{L}$-manifold,

by the stratification result [**6**, Proposition 1.5.3]. Now we can finish by induction on $\dim X$ by working with the restriction of g to D. ☐

Propositions 7.1 and 7.2 allow us to prove our result on loci (Theorem 2.9).

Proof of Theorem 2.9. Let X and g satisfy the assumptions of the theorem, with $X \subset F^n$. By Proposition 7.1, and up to replacing g without changing its zero locus, we may suppose that the complex norm $|g(x)|_{\mathbb{C}}$ is bounded on X. Apply Proposition 7.2 to g to find $\mathcal{L}$-manifolds D_i. For each i, let μ_{D_i} be the canonical measure on D_i coming from the submanifold structure $D_i \subset F^n$; see [**11**, Section 2.3]. Note that μ_{D_i} gives a distribution on X. Now $\xi := \sum_i g\mu_{D_i}$ is as desired. ☐

8. Uniformity in the local field and in definable families

In this section, F is no longer fixed and is no longer assumed to be of characteristic zero. On the contrary, we focus on uniformity over all local fields with only the restriction that if F has positive characteristic, then F is assumed to have characteristic at least M for some M, which may become bigger when needed. Until the end of the paper, we use terminology and notation from [**11**, Section 3.1], without recalling that section in full. In particular, this fixes uniform notions of

- functions of $\mathscr{C}^{\exp}$-class;

- definable sets; and

- definable functions;

where uniformity is in all local fields F (with structure from the generalized Denef–Pas language) of characteristic zero and of positive characteristic at least M for some M, denoted together by Loc_M. Furthermore, Loc'_M denotes the collection of pairs (F, ψ) of F in Loc_M and ψ an additive character on F, which is trivial on $\mathcal{M}_F$ and nontrivial on $\mathcal{O}_F$. A definable set X is now a collection $(X_F)_{F \in \mathrm{Loc}_M}$ for some M, and VF stands for the definable set $(F)_{F \in \mathrm{Loc}_M}$.

We use the following notion of $\mathscr{C}^{\exp}$-families of distributions.

DEFINITION 8.1 ($\mathscr{C}^{\exp}$-families of distributions). Let Y and $X \subset Y \times \mathrm{VF}^n$ be definable sets. Suppose that for each $F \in \mathrm{Loc}'_M$ for some $M > 0$ and each $y \in Y_F$, the set $X_{F,y} \subset F^n$ is an analytic manifold, where $X_{F,y} = \{x \in F^n \mid (y, x) \in X\}$. Let, for each $F \in \mathrm{Loc}'_M$ and each $y \in Y_F$, a distribution $\xi_{F,y}$ be given on $X_{F,y}$.

Then we call the collection of distributions $\xi_{F,y}$ for $F \in \mathrm{Loc}'_M$ and $y \in Y_F$ a $\mathscr{C}^{\exp}$-family of distributions on X over Y if the collection of B-functions $D_{\xi_{F,y}}$ is of $\mathscr{C}^{\exp}$-class.

Denote by $\mathcal{S}^*_{\mathscr{C}^{\exp}/Y}(X)$ the space of $\mathscr{C}^{\exp}$-families of distributions on X over Y.

Denote by $\mathcal{S}_{\mathscr{C}^{\exp}/Y}(X)$ the space of $\mathscr{C}^{\exp}$-functions φ on X such that $\varphi_F(y, \cdot) : X_{F,y} \to \mathbb{C}$ is a Schwartz–Bruhat function on $X_{F,y}$ for each $F \in \mathrm{Loc}'_M$ for some M and each $y \in Y_F$.

The following result generalizes Theorem 2.7 to uniformity in the local field F and in definable families with parameter y in a definable set Y. It gives two results: the uniform description of witnessing manifolds for the WF-holonomicity (the W_i) and algebraic holonomicity also for local fields F of positive characteristic larger than some M.

THEOREM 8.2 (Uniform holonomicity). *Let Y and $X \subset Y \times \mathrm{VF}^n$ be definable sets. Suppose that $X_{F,y}$ is an analytic manifold for each $F \in \mathrm{Loc}'_M$ and each $y \in Y_F$. Let ξ be in $\mathcal{S}^*_{\mathscr{C}^{\exp}/Y}(X)$. Then there exist M' and finitely many definable sets $W_i \subset X$, such that $W_{i,F,y}$ (when nonempty) is an analytic submanifold of $X_{F,y}$ for each $F \in \mathrm{Loc}'_{M'}$, each i, and each $y \in Y_F$, and such that the wave front set of $\xi_{F,y}$ is contained in*

$$\bigcup_i CN^{X_{F,y}}_{W_{i,F,y}}.$$

Hence, $\xi_{F,y}$ is algebraically WF-holonomic for each F in $\mathrm{Loc}'_{M'}$ (see Definition 2.2).

Proof. This follows from our holonomicity result (Theorem 2.7), from [11, Section 4], and Remark 6.6 used similarly as in the proof of Theorem 2.7 for $\mathcal{L} = \mathcal{L}^F$. □

For $W_{i,F}$ as in Theorem 8.2 but with $Y = \{0\}$, see the appendix of [9] to get the extra information that the Zariski closures of the $W_{i,F}$ in $\mathbb{A}^n_F$ are defined over a number field F_0 independently from F, in the case that the initial data are defined over F_0 as well.

The following is a family version, uniform in the local field and in definable families, of our Regularization Theorem 2.8.

THEOREM 8.3 (Uniform regularization). *Let Y and $U \subset X \subset Y \times \mathrm{VF}^n$ be definable sets. Suppose that $X_{F,y}$ is an analytic submanifold of F^n and that $U_{F,y}$ is a nonempty open of $X_{F,y}$ for each $F \in \mathrm{Loc}'_M$ for some M and each $y \in Y_F$.*

Then the restriction map

$$\mathcal{S}^*_{\mathscr{C}^{\mathrm{exp}}/Y}(X) \to \mathcal{S}^*_{\mathscr{C}^{\mathrm{exp}}/Y}(U)$$

admits a linear section (in particular, it is onto).

Let us also adapt Proposition 4.6 to the uniform setting.

PROPOSITION 8.4. *Let Y and $Z \subset X \subset Y \times \mathrm{VF}^n$ be definable sets. Suppose that $X_{F,y}$ is an analytic manifold and that $Z_{F,y}$ is a closed analytic submanifold of $X_{F,y}$ for each $F \in \mathrm{Loc}'_M$ and each $y \in Y_F$, of lower dimension than the dimension of $X_{F,y}$. Then the restriction map*

$$\mathcal{S}_{\mathscr{C}^{\mathrm{exp}}/Y}(X) \to \mathcal{S}_{\mathscr{C}^{\mathrm{exp}}/Y}(Z)$$

admits a linear section.

We also give the following uniform (partial) converse to [**11**, Theorem 3.4.1].

THEOREM 8.5 (Uniform correspondence of loci). *Let Y and $X \subset Y \times \mathrm{VF}^n$ be definable sets. Suppose that $X_{F,y}$ is an analytic manifold for each $F \in \mathrm{Loc}'_M$ and each $y \in Y_F$. Let g be in $\mathscr{C}^{\mathrm{exp}}(X)$. Suppose that for each $F \in \mathrm{Loc}'_M$ and each $y \in Y_F$, the zero locus of $g_{F,y}$ is dense open in $X_{F,y}$. Then there exist M' and ξ in $\mathcal{S}^*_{\mathscr{C}^{\mathrm{exp}}/Y}(X)$ such that for each $F \in \mathrm{Loc}'_{M'}$ and each $y \in Y_F$, the zero locus of $g_{F,y}$ equals the smooth locus of $\xi_{F,y}$.*

Most proofs above directly apply to the uniform setting. Let us show how to adapt the statement of Lemma 4.5. It is important that the occurring sets in the covers form moreover a definable family, in order to generalize the proof techniques for fixed F above to our uniform setting. Recall from [**11**, Section 3.1] that $\mathrm{RF}_{N,F}$ stands for the finite ring $\mathcal{O}_F/N\mathcal{M}_F$ for F a local field.

LEMMA 8.6. *Let Y and $X \subset Y \times \mathrm{VF}^n$ be definable sets. Suppose that $X_{F,y}$ is an analytic submanifold of F^n of dimension m for each $F \in \mathrm{Loc}'_M$ and each $y \in Y_F$. Then there are N and a definable bijection $\sigma : X \to \sigma(X) \subset X \times \mathrm{RF}_N^k$ for some k, which makes a commutative diagram with the projection $p : \sigma(X) \to X$ and such that each nonempty fiber of the projection $\sigma(X) \to Y \times \mathrm{RF}_N^k$ is an open of X, which equals the graph of a definable analytic function from an open subset $V_i \subset L_i$ of a linear subspace of VF^n of dimension m to a linear complement of this subspace. (Here, analytic means for each F in $\mathrm{Loc}_{M'}$ for some M' and each y.)*

Proof. This follows by a reasoning as for Lemma 4.5, where one uses [**13**, Section 6.3] instead of [**23**] and [**18**] for the properties about analyticity. $\square$

Proofs of Theorem 8.3 and Proposition 8.4. Clearly, the proofs of Theorems 2.8 and 4.6 work uniformly in F and in $y \in Y_F$, using Lemma 8.6 instead of Lemma 4.5. □

Proof of Theorem 8.5. The proofs of Theorem 2.9 and (the statement and the proof of) Proposition 7.1 adapt naturally to the uniform case. □

REMARK 8.7. We leave it to the reader to implement the uniform results also in the analytic framework, using [11, Remark 4.3.3] and our results above for $\mathcal{L}_{\text{an}}^F$ for fixed F. Similarly, we let the (definable) strict C^1 analogues to the reader. Let us also note that within the axiomatic framework of Hensel minimality from [12] for languages on local fields, one can investigate a generalization of the results of this paper to that framework, with definable strict C^1 manifolds and functions instead of definable analytic ones.

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