Proceedings of the Royal Society of Edinburgh, **150**, 2849–2870, *2020*
DOI:10.1017/prm.2019.49

Uniqueness and time oscillating behaviour of finite points blow-up solutions of the fast diffusion equation

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(MS Received 11 September 2018; accepted 18 July 2019)

Let $n \geqslant 3$ and $0 < m < (n-2)/n$. We extend the results of Vazquez and Winkler (2011, *J. Evol. Equ.* 11, no. 3, 725–742) and prove the uniqueness of finite points blow-up solutions of the fast diffusion equation $u_t = \Delta u^m$ in both bounded domains and $\mathbb{R}^n \times (0,\infty)$. We also construct initial data such that the corresponding solution of the fast diffusion equation in bounded domain oscillates between infinity and some positive constant as $t \to \infty$.

Keywords: uniqueness; fast diffusion equation; time oscillating behaviour

2010 *Mathematics subject classification:* Primary: 35K65
Secondary: 35B51; 35B40

1. Introduction

The equation

$$u_t = \Delta u^m \tag{1.1}$$

arises in many physical and geometrical models [**1**, **4**, **19**, **20**]. When $m > 1$, (1.1) is called the porous medium equation which appears in the modelling of the flow of gases through porous media and oil passing through sand etc. Equation (1.1) also arises as the large time asymptotic limit in the study of the large time behaviour of the solution of the compressible Euler equation with damping [**12**, **18**]. When $m = 1$, (1.1) is the heat equation. When $0 < m < 1$, (1.1) is called the fast diffusion equation. When $m = (n-2)/(n+2)$ and $n \geqslant 3$, (1.1) arises in the study of Yamabe flow on $\mathbb{R}^n$ [**5**, **6**, **8**]. Note that the metric $g_{ij} = u^{4/(n+2)}\,dx^2$, $u > 0$, $n \geqslant 3$, is a solution of the Yamabe flow [**6**, **8**],

$$\frac{\partial g_{ij}}{\partial t} = -Rg_{ij} \quad \text{in } \mathbb{R}^n \times (0,T)$$

if and only if u is a solution of

$$u_t = \frac{n-1}{m}\Delta u^m$$

in $\mathbb{R}^n \times (0,T)$ with $m=(n-2)/(n+2)$ where $R(\cdot,t)$ is the scalar curvature of the metric $g_{ij}(\cdot,t)$. Recently, Golse and Salvarani [9] and Choi and Lee [2] have shown that (1.1) also appears as the nonlinear diffusion limit for the generalized Carleman models.

Although there is a lot of study on (1.1) for the case $m>(n-2)_+/n$, there are not many results of (1.1) for the case $0<m<(n-2)/n$, $n\geqslant 3$. When $0<m\leqslant (n-2)/n$ and $n\geqslant 3$, existence of positive smooth solutions of

$$\begin{cases} u_t = \Delta u^m, u\geqslant 0, & \text{in } \mathbb{R}^n\times(0,T)\\ u(x,0)=u_0 & \text{in } \mathbb{R}^n\end{cases}$$

for any $0\leqslant u_0\in L^p_{\text{loc}}(\mathbb{R}^n)$, $p>(1-m)n/2$, satisfying the condition,

$$\liminf_{R\to\infty}\frac{1}{R^{n-(2/(1-m))}}\int_{|x|\leqslant R}u_0\,\mathrm{d}x\geqslant C_1T^{1/(1-m)}$$

for some constant $C_1>0$ was proved by Hsu in [11].

Let $\Omega\subset\mathbb{R}^n$ be a smooth bounded domain. When $0<m\leqslant(n-2)/n$, $n\geqslant 3$ and $0\in\Omega$, existence of singular solutions and asymptotic large time behaviour of (1.1) in $(\Omega\backslash\{0\})\times(0,\infty)$ which blow up at $\{0\}\times(0,\infty)$ when the initial value u_0 satisfies

$$c_1|x|^{-\gamma_1}\leqslant u_0(x)\leqslant c_2|x|^{-\gamma_2}\quad\forall x\in\Omega\backslash\{0\}$$

for some constants $c_1>0$, $c_2>0$ and $\gamma_2\geqslant\gamma_1>2/(1-m)$ were proved by Vazquez and Winkler in [21]. Uniqueness of singular solutions of (1.1) in $(\Omega\backslash\{0\})\times(0,\infty)$ that blow up at $\{0\}\times(0,\infty)$ and existence of singular initial data such that the corresponding singular solution of (1.1) in $(\Omega\backslash\{0\})\times(0,\infty)$ oscillates between infinity and some positive constant as $t\to\infty$ were proved by Vazquez and Winkler in [22].

When $0<m\leqslant(n-2)/n$ and $n\geqslant 3$, existence of singular solutions of (1.1) in $(\mathbb{R}^n\backslash\{0\})\times(0,\infty)$ which blow up at $\{0\}\times(0,\infty)$ when the initial value u_0 satisfies

$$c_1|x|^{-\gamma}\leqslant u_0(x)\leqslant c_2|x|^{-\gamma}\quad\forall x\in\mathbb{R}^n\backslash\{0\}$$

for some constants $c_1>0$, $c_2>0$ and $2/(1-m)<\gamma<(n-2)/m$ was proved by Hui and Kim in [14]. Asymptotic large time behaviour of such solution was also proved by Hui and Kim in [14] when $2/(1-m)<\gamma<n$.

Let $a_1,a_2,\dots,a_{i_0}\in\Omega$, $\widehat{\Omega}=\Omega\backslash\{a_1,a_2,\dots,a_{i_0}\}$ and $\widehat{\mathbb{R}^n}=\mathbb{R}^n\backslash\{a_1,a_2,\dots,a_{i_0}\}$. For any $\delta>0$, let $\Omega_\delta=\Omega\backslash(\cup_{i=1}^{i_0}B_\delta(a_i))$ and $\mathbb{R}^n_\delta=\mathbb{R}^n\backslash(\cup_{i=1}^{i_0}B_\delta(a_i))$ where $B_R(x_0)=\{x\in\mathbb{R}^n:|x-x_0|<R\}$, $B_R=B_R(0)$, $\widehat{B}_R(x_0)=B_R(x_0)\backslash\{x_0\}$ and $\widehat{B}_R=\widehat{B}_R(0)$ for any $x_0\in\mathbb{R}^n$ and $R>0$. Let $\delta_0(\Omega)=1/3\min_{1\leqslant i,j\leqslant i_0}(\mathbf{dist}(a_i,\Omega),|a_i-a_j|)$ and $\delta_0(\mathbb{R}^n)=1/3\min_{1\leqslant i,j\leqslant i_0}|a_i-a_j|$. For any $0<\delta\leqslant\delta_0(\Omega)$, let $D_\delta=\{x\in\Omega:\ \text{dist}\,(x,\partial\Omega)<\delta\}$. Let $R_0>0$ be such that $a_1,\dots,a_{i_0}\in B_{R_0}$. For any $R>R_0$ and $0<\delta\leqslant\delta_0(\mathbb{R}^n)$, let $\Omega_{\delta,R}=B_R\backslash\cup_{i=1}^{i_0}B_\delta(a_i)$. When there is no ambiguity we will drop the parameters Ω, $\mathbb{R}^n$, and write δ_0 instead of $\delta_0(\Omega)$ or $\delta_0(\mathbb{R}^n)$. Unless stated otherwise we will assume that $0<m<(n-2)/n$ and $n\geqslant 3$ for the rest of this paper.

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Existence of singular solutions of (1.1) in $\widehat{\Omega}\times(0,T)$ which blow up at $\{a_1,a_2,\dots,a_{i_0}\}\times(0,T)$ was proved by Hui and Kim in [13] when the initial value u_0 satisfies

$$u_0(x)\approx|x-a_i|^{-\gamma_i}\quad\text{for } x\approx a_i\quad\forall i=1,2,\dots,i_0$$

for some constants $\gamma_i>\max(n/2m,(n-2)/m)$ for any $i=1,2,\dots,i_0$. When $0\leqslant f\in L^\infty(\partial\Omega\times[0,\infty))$ and the initial value $0\leqslant u_0\in L^p_{\text{loc}}(\overline{\Omega}\backslash\{a_1,\dots,a_{i_0}\})$ ($L^p_{\text{loc}}(\widehat{\mathbb{R}^n})$ respectively) for some constant $p>n(1-m)/2$ satisfies

$$u_0(x)\geqslant\frac{\lambda_i}{|x-a_i|^{\gamma_i}}\quad\forall 0<|x-a_i|<\delta_1,\quad i=1,\dots,i_0 \tag{1.2}$$

for some constants $0<\delta_1<\min(1,\delta_0),\lambda_1,\dots,\lambda_{i_0}\in\mathbb{R}^+$ and $\gamma_1,\dots,\gamma_{i_0}\in(2/(1-m),\infty)$, existence of singular solutions of

$$\begin{cases} u_t=\Delta u^m & \text{in } \widehat{\Omega}\times(0,\infty)\\ u=f & \text{on } \partial\Omega\times(0,\infty)\\ u(a_i,t)=\infty & \forall t>0,\ \ i=1,\dots,i_0\\ u(x,0)=u_0(x) & \text{in } \widehat{\Omega}\end{cases} \tag{1.3}$$

and

$$\begin{cases} u_t=\Delta u^m & \text{in } \widehat{\mathbb{R}^n}\times(0,\infty)\\ u(a_i,t)=\infty & \forall i=1,\dots,i_0,\ \ t>0\\ u(x,0)=u_0(x) & \text{in } \widehat{\mathbb{R}^n}\end{cases} \tag{1.4}$$

respectively was proved by Hui and Kim in [15]. It was proved in [15] that the singular solutions of (1.3) and (1.4) constructed in [15] have the property that for any $T>0$ and $\delta_2\in(0,\delta_1)$ there exists a constant $C_1>0$ such that

$$u(x,t)\geqslant\frac{C_1}{|x-a_i|^{\gamma_i}}\quad\forall 0<|x-a_i|<\delta_2, 0<t<T, i=1,2,\dots,i_0. \tag{1.5}$$

Moreover [15] if the initial value u_0 also satisfies

$$u_0(x)\leqslant\frac{\lambda_i'}{|x-a_i|^{\gamma_i'}}\quad\forall 0<|x-a_i|<\delta_1, i=1,\dots,i_0, \tag{1.6}$$

for some constants $\lambda_1',\dots,\lambda_{i_0}'\in\mathbb{R}^+$, and $\gamma_i'\geqslant\gamma_i$ for all $i=1,\dots,i_0$, then for any $T>0$ and $\delta_2\in(0,\delta_1)$ there exists a constant $C_2>0$ such that the singular solutions

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of (1.3) and (1.4) constructed in [**15**] satisfy

$$u(x,t) \leqslant \frac{C_2}{|x-a_i|^{\gamma_i'}} \quad \forall 0<|x-a_i|<\delta_2, 0<t<T, i=1,2,\dots,i_0. \tag{1.7}$$

When $f \geqslant \mu_0$ and $u_0 \geqslant \mu_0$ for some constant $\mu_0 > 0$, the singular solutions of (1.3) and (1.4) constructed in [**15**] also satisfy

$$u(x,t) \geqslant \mu_0 \quad \forall x \in \widehat{\Omega} \quad (\widehat{\mathbb{R}^n} \text{ respectively}), t>0. \tag{1.8}$$

Asymptotic large time behaviour of such singular solutions was also proved by Hui and Kim in [**15**].

In this paper, we extend the results of Vazquez and Winkler [**22**] and prove the uniqueness of singular solutions of (1.3) and (1.4). We also construct initial data u_0 such that the corresponding singular solution of (1.3) with $f = \mu_0 > 0$ oscillates between infinity and some positive constant as $t \to \infty$. More precisely we will prove the following results.

Theorem 1.1. *Let $n \geqslant 3$, $0<m<(n-2)/n$, $0<\delta_1<\min(1,\delta_0)$, $\mu_0>0$, $f_1, f_2 \in C^3(\partial\Omega \times (0,\infty)) \cap L^\infty(\partial\Omega \times (0,\infty))$ be such that $f_2 \geqslant f_1 \geqslant \mu_0$ on $\partial\Omega \times (0,\infty)$ and*

$$\mu_0 \leqslant u_{0,1} \leqslant u_{0,2} \in L^p_{\text{loc}}(\overline{\Omega}\backslash\{a_1,\dots,a_{i_0}\}) \quad \text{for some constant } p > \frac{n(1-m)}{2} \tag{1.9}$$

be such that

$$\frac{\lambda_i}{|x-a_i|^{\gamma_i}} \leqslant u_{0,1}(x) \leqslant u_{0,2} \leqslant \frac{\lambda_i'}{|x-a_i|^{\gamma_i'}} \quad \forall 0<|x-a_i|<\delta_1, i=1,\dots,i_0 \tag{1.10}$$

holds for some constants $\lambda_1, \dots, \lambda_{i_0}, \lambda_1', \dots, \lambda_{i_0}' \in \mathbb{R}^+$ and

$$\gamma_i' \geqslant \gamma_i > \frac{2}{1-m} \quad \forall i=1,2,\dots,i_0. \tag{1.11}$$

Suppose u_1, u_2 are the solutions of (1.3) with $u_0 = u_{0,1}, u_{0,2}$, $f = f_1, f_2$, respectively which satisfy

$$u_j(x,t) \geqslant \mu_0 \quad \forall x \in \widehat{\Omega}, t>0, j=1,2 \tag{1.12}$$

such that for any constants $T>0$ and $\delta_2 \in (0,\delta_1)$ there exist constants $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, such that

$$\frac{C_1}{|x-a_i|^{\gamma_i}} \leqslant u_j(x,t) \leqslant \frac{C_2}{|x-a_i|^{\gamma_i'}}$$
$$\forall 0<|x-a_i|<\delta_2, 0<t<T, i=1,2,\dots,i_0, j=1,2. \tag{1.13}$$

Suppose u_1, u_2 also satisfy

$$\|u_i(\cdot,t) - u_{0,i}\|_{L^1(\Omega_\delta)} \to 0 \quad \text{as } t \to 0 \quad \forall 0<\delta<\delta_0, i=1,2. \tag{1.14}$$

Then

$$u_1(x,t) \leqslant u_2(x,t) \quad \forall x \in \widehat{\Omega}, t>0. \tag{1.15}$$

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THEOREM 1.2. *Let $n \geqslant 3$, $0 < m < (n-2)/n$, $0 < \delta_1 < \min(1, \delta_0)$, $\mu_0 > 0$, $\mu_0 \leqslant f_1 \leqslant f_2 \in L^\infty(\partial\Omega \times (0,\infty))$ and (1.9), (1.10), hold for some constants $\lambda_1, \ldots, \lambda_{i_0}$, $\lambda'_1, \ldots, \lambda'_{i_0} \in \mathbb{R}^+$ and*

$$\frac{2}{1-m} < \gamma_i \leqslant \gamma'_i < \frac{n-2}{m} \quad \forall i = 1, 2, \ldots, i_0. \tag{1.16}$$

Suppose u_1, u_2 are the solutions of (1.3) with $u_0 = u_{0,1}, u_{0,2}$, $f = f_1, f_2$, respectively which satisfy (1.12) and (1.14) such that for any constants $T > 0$ and $\delta_2 \in (0, \delta_1)$ there exist constants $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, such that (1.13) holds. Then (1.15) holds.

THEOREM 1.3. *Let $n \geqslant 3$, $0 < m < (n-2)/n$, $0 < \delta_1 < \min(1, \delta_0)$, $\mu_0 > 0$, $R_1 > R_0$ and $\mu_0 \leqslant u_{0,1} \leqslant u_{0,2} \in L^p_{\mathrm{loc}}(\widehat{\mathbb{R}^n})$ for some constant $p > n(1-m)/2$ such that*

$$\int_{\mathbb{R}^n \setminus B_{R_1}} |u_{0,j} - \mu_0| \, \mathrm{d}x < \infty \quad \forall j = 1, 2. \tag{1.17}$$

Let (1.10) hold for some constants $\lambda_1, \ldots, \lambda_{i_0}$, $\lambda'_1, \ldots, \lambda'_{i_0} \in \mathbb{R}^+$ and

$$\frac{2}{1-m} < \gamma_i \leqslant \gamma'_i < n \quad \forall i = 1, 2, \ldots, i_0. \tag{1.18}$$

Suppose u_1, u_2 are the solutions of (1.4) with $u_0 = u_{0,1}, u_{0,2}$ respectively which satisfy

$$u_j(x,t) \geqslant \mu_0 \quad \forall x \in \widehat{\mathbb{R}^n}, \quad t > 0, \quad j = 1, 2 \tag{1.19}$$

and

$$\int_{\widehat{\mathbb{R}^n}} |u_j(x,t) - \mu_0| \, \mathrm{d}x \leqslant \int_{\widehat{\mathbb{R}^n}} |u_{0,j} - \mu_0| \, \mathrm{d}x \quad \forall t > 0, \quad j = 1, 2 \tag{1.20}$$

such that for any constants $T > 0$ and $\delta_2 \in (0, \delta_1)$ there exist constants $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, such that (1.13) holds. Then

$$u_1(x,t) \leqslant u_2(x,t) \quad \forall x \in \widehat{\mathbb{R}^n}, \quad t > 0. \tag{1.21}$$

THEOREM 1.4. *Let $n \geqslant 3$, $0 < m < (n-2)/n$, $0 < \delta_1 < \min(1, \delta_0)$ and $\mu_0 > 0$. There exists $u_0 \in L^p_{\mathrm{loc}}(\overline{\Omega} \setminus \{a_1, \ldots, a_{i_0}\})$ for some constant $p > n(1-m)/2$, $u_0 \geqslant \mu_0$ in $\widehat{\Omega}$, such that*

$$\frac{\lambda_i}{|x - a_i|^{\gamma_i}} \leqslant u_0(x) \leqslant \frac{\lambda'_i}{|x - a_i|^{\gamma'_i}} \quad \forall 0 < |x - a_i| < \delta_1, i = 1, \ldots, i_0 \tag{1.22}$$

for some constants satisfying (1.11) and $\lambda_1, \ldots, \lambda_{i_0}$, $\lambda'_1, \ldots, \lambda'_{i_0} \in \mathbb{R}^+$ such that

$$\begin{cases} u_t = \Delta u^m & \text{in } \widehat{\Omega} \times (0,\infty) \\ u = \mu_0 & \text{on } \partial\Omega \times (0,\infty) \\ u(a_i, t) = \infty & \forall t > 0, i = 1, \ldots, i_0 \\ u(x,0) = u_0(x) & \text{in } \widehat{\Omega} \end{cases} \tag{1.23}$$

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has a unique solution u with the property that u oscillates between μ_0 and infinity as $t \to \infty$.

This paper is organized as follows. In § 2 we will prove the uniqueness of singular solutions of (1.3) and (1.4). In § 3 we will prove the existence of initial data such that the corresponding solution of (1.3) with $f = \mu_0 > 0$ oscillates between infinity and some positive constant as $t \to \infty$.

We start with some definitions. For any $0 \leqslant f \in L^\infty(\partial\Omega \times (0,\infty))$ and $0 \leqslant u_0 \in L^1_{\mathrm{loc}}(\widehat{\Omega})$, we say that u is a solution of

$$\begin{cases} u_t = \Delta u^m & \text{in } \widehat{\Omega} \times (0,\infty) \\ u = f & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \widehat{\Omega} \end{cases} \tag{1.24}$$

if $u \in L^\infty_{\mathrm{loc}}(\overline{\Omega}\backslash\{a_1,\dots,i_0\} \times (0,\infty))$ is positive in $\widehat{\Omega} \times (0,\infty)$ and satisfies (1.1) in $\widehat{\Omega} \times (0,\infty)$ in the classical sense with

$$\|u(\cdot,t) - u_0\|_{L^1(K)} \to 0 \quad \text{as } t \to 0 \tag{1.25}$$

for any compact set $K \subset \widehat{\Omega}$ and

$$\int_{t_1}^{t_2}\int_{\widehat{\Omega}} (u\eta_t + u^m\Delta\eta)\,\mathrm{d}x\mathrm{d}t = \int_{t_1}^{t_2}\int_{\partial\Omega} f^m \frac{\partial\eta}{\partial\nu}\,\mathrm{d}\sigma\mathrm{d}t + \int_{\widehat{\Omega}} u(x,t_2)\eta(x,t_2)\,\mathrm{d}x \\ - \int_{\widehat{\Omega}} u(x,t_1)\eta(x,t_1)\,\mathrm{d}x \tag{1.26}$$

for any $t_2 > t_1 > 0$ and $\eta \in C^2_c((\overline{\Omega}\backslash\{a_1,\dots,a_{i_0}\}) \times (0,\infty))$ satisfying $\eta \equiv 0$ on $\partial\Omega \times (0,T)$. We say that u is a solution of (1.3) if u is a solution of (1.24) and satisfies

$$u(x,t) \to \infty \quad \text{as } x \to a_i \quad \forall t > 0, i = 1,\dots,i_0. \tag{1.27}$$

For any $0 \leqslant u_0 \in L^1_{\mathrm{loc}}(\widehat{\mathbb{R}^n})$ we say that u is a solution of (1.4) if $u \in L^\infty_{\mathrm{loc}}(\widehat{\mathbb{R}^n} \times (0,\infty))$ is positive in $\widehat{\mathbb{R}^n} \times (0,\infty)$ and satisfies (1.1) in $\widehat{\mathbb{R}^n} \times (0,\infty)$ in the classical sense and (1.25), (1.27), hold for any compact set $K \subset \widehat{\mathbb{R}^n}$.

For any set $A \subset \mathbb{R}^n$, we let χ_A be the characteristic function of the set A. For any $a \in \mathbb{R}$, we let $a_+ = \max(0,a)$.

2. Uniqueness of solution

In this section, we will prove the uniqueness of singular solutions of (1.3) and (1.4).

LEMMA 2.1. *Let $n \geqslant 3$, $0 < m < (n-2)/n$, $0 < \delta_1 < \delta_0$, $0 \leqslant f \in L^\infty(\partial\Omega \times [0,\infty))$ and $0 \leqslant u_0 \in L^p_{\mathrm{loc}}(\overline{\Omega}\backslash\{a_1,\dots,a_{i_0}\})$ for some constant $p > n(1-m)/2$ be such that (1.2) holds for some constants $\lambda_1,\ \dots,\ \lambda_{i_0} \in \mathbb{R}^+$ and $\gamma_1,\dots,\gamma_{i_0} \in (2/(1-m),\infty)$.*

Let u be the solution of (1.3) *constructed in theorem 1.1 of [**15**]. Then there exists a constant $C>0$ such that*

$$\int_{D_\delta} u(x,t)\,\mathrm{d}x \leqslant \left\{\left(\int_{D_{2\delta}} u_0\,\mathrm{d}x\right)^{1-m} + Ct\right\}^{1/(1-m)} + |D_\delta|\|f\|_{L^\infty} \quad \forall t>0, \quad 0<\delta<\delta_0/2 \tag{2.1}$$

holds.

Proof. We will use a modification of the proof of theorem 2.2 of [**10**] to prove this lemma. For any $0<\varepsilon<1$, let $k>\|f\|_{L^\infty}+\varepsilon$. Let $0<\delta<\delta_0/2$. We choose $\phi\in C^\infty(\overline{\Omega})$ such that $0\leqslant\phi\leqslant 1$ in $\overline{\Omega}$, $\phi(x)=0$ for any $x\in\Omega\backslash D_{2\delta}$ and $\phi(x)=1$ for any $x\in\overline{D_\delta}$. Let $\alpha>2/(1-m)$ and $\eta(x)=\phi(x)^\alpha$. Then by direct computation,

$$C_\eta := \left(\int_\Omega (\eta^{-m}|\Delta\eta|)^{1/(1-m)}\,\mathrm{d}x\right)^{1-m} < \infty.$$

For any $0<\varepsilon<1$ and $M>0$, let

$$\begin{cases} u_{0,M}(x) = \min(u_0(x),M) \\ u_{0,\varepsilon,M}(x) = \min(u_0(x),M)+\varepsilon \\ f_\varepsilon(x,t) = f(x,t)+\varepsilon \quad \forall (x,t)\in\partial\Omega\times(0,\infty) \end{cases}$$

and let u_M and $u_{\varepsilon,M}$ be the solutions of

$$\begin{cases} u_t = \Delta u^m & \text{in } \Omega\times(0,\infty) \\ u(x,t) = f_\varepsilon & \text{on } \partial\Omega\times(0,\infty) \\ u(x,0) = u_{0,\varepsilon,M} & \text{in } \Omega. \end{cases}$$

and

$$\begin{cases} u_t = \Delta u^m & \text{in } \Omega\times(0,T_M) \\ u(x,t) = f & \text{on } \partial\Omega\times(0,T_M) \\ u(x,0) = u_{0,M} & \text{in } \Omega \end{cases}$$

respectively constructed in [**15**] for some maximal time $T_M>0$ of existence. By the result in [**15**] $T_M\to\infty$ as $M\to\infty$. Moreover $u_{\varepsilon,M}$ decreases and converges to u_M in $\Omega\times(0,T_M)$ uniformly in $C^{2,1}(K)$ for every compact subset K of Ω as $\varepsilon\to 0$ and u_M increases and converges to u in $\widehat{\Omega}\times(0,\infty)$ uniformly in $C^{2,1}(K)$ for every compact subset K of $\widehat{\Omega}$ as $M\to\infty$. By approximation we may assume without loss of generality that $u_{\varepsilon,M}\in C^2(\overline{\Omega}\times(0,\infty))$. Then by the Kato inequality ([**4**, **16**]),

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\int_\Omega (u_{\varepsilon,M}-k)_+\eta\,\mathrm{d}x\right) \\ &\quad\leqslant \int_\Omega (u_{\varepsilon,M}^m-k^m)_+\Delta\eta\,\mathrm{d}x \\ &\quad\leqslant C\int_\Omega (u_{\varepsilon,M}-k)_+^m|\Delta\eta|\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\leqslant C\left(\int_\Omega (u_{\varepsilon,M}-k)_+\eta\,\mathrm{d}x\right)^m\left(\int_\Omega (\eta^{-m}|\Delta\eta|)^{1/(1-m)}\,\mathrm{d}x\right)^{1-m}\\
&= C_1\left(\int_\Omega (u_{\varepsilon,M}-k)_+\eta\,\mathrm{d}x\right)^m \quad \forall t>0, 0<\varepsilon<1, M>0 \qquad (2.2)
\end{aligned}
$$

for some constants $C>0$, $C_1>0$. Integrating (2.2) over $(0,t)$ and letting $\varepsilon\to 0$, $M\to\infty$ and $k\to\|f\|_{L^\infty}$,

$$
\begin{aligned}
&\int_{D_\delta}(u(x,t)-\|f\|_{L^\infty})_+\,\mathrm{d}x\\
&\quad\leqslant\left\{\left(\int_{D_{2\delta}}(u_0-\|f\|_{L^\infty})_+\,\mathrm{d}x\right)^{1-m}+C_1(1-m)t\right\}^{1/(1-m)} \qquad \forall t>0
\end{aligned}
$$

and (2.1) follows. □

Proposition 2.2. *Let $n\geqslant 3$, $0<m<(n-2)/n$, $0<\delta_1<\delta_0$, $0\leqslant f\in L^\infty(\partial\Omega\times[0,\infty))$ and $0\leqslant u_0\in L^p_{\mathrm{loc}}(\overline{\Omega}\backslash\{a_1,\dots,a_{i_0}\})$ for some constant $p>n(1-m)/2$ be such that (1.2) holds for some constants $\lambda_1,\ \dots,\ \lambda_{i_0}\in\mathbb{R}^+$ and $\gamma_1,\dots,\gamma_{i_0}\in(2/(1-m),\infty)$. Let u be the solution of (1.3) constructed in theorem 1.1 of [**15**]. Then*

$$
\|u(\cdot,t)-u_0\|_{L^1(\Omega_\delta)}\to 0 \quad \text{as } t\to 0 \quad \forall 0<\delta<\delta_0. \qquad (2.3)
$$

Proof. Let $0<\delta<\delta_0$ and $0<\delta'<\delta_0/2$. Then by lemma 2.1,

$$
\begin{aligned}
\|u(\cdot,t)-u_0\|_{L^1(\Omega_\delta)}&\leqslant\|u(\cdot,t)-u_0\|_{L^1(\Omega_\delta\backslash D_{\delta'})}+\|u(\cdot,t)\|_{L^1(D_{\delta'})}+\|u_0\|_{L^1(D_{\delta'})}\\
&\leqslant\|u(\cdot,t)-u_0\|_{L^1(\Omega_\delta\backslash D_{\delta'})}+\left(\|u_0\|^{1-m}_{L^1(D_{2\delta'})}+Ct\right)^{1/(1-m)}\\
&\quad+|D_{\delta'}|\|f\|_{L^\infty}+\|u_0\|_{L^1(D_{\delta'})}. \qquad (2.4)
\end{aligned}
$$

Letting $t\to 0$ in (2.4),

$$
\begin{aligned}
&\limsup_{t\to 0}\|u(\cdot,t)-u_0\|_{L^1(\Omega_\delta)}\leqslant|D_{\delta'}|\|f\|_{L^\infty}+2\|u_0\|_{L^1(D_{2\delta'})} \quad \forall 0<\delta'<\delta_0/2\\
\Rightarrow\quad &\lim_{t\to 0}\|u(\cdot,t)-u_0\|_{L^1(\Omega_\delta)}=0 \quad \text{as } \delta'\to 0
\end{aligned}
$$

and (2.3) follows. □

Proof of theorem 1.1. We will use a modification of the proof of theorem 6 of [**22**] to prove the theorem. Let $0<\delta<\delta_0$ and $t_1>t_0>0$. Since $u_1,u_2\in L^\infty_{\mathrm{loc}}(\overline{\Omega}\backslash\{a_1,\dots,i_0\}\times(0,\infty))$ there exists a constant $M_1>0$ such that

$$
u_j(x,t)\leqslant M_1 \quad \forall x\in\Omega_\delta, \quad t_0\leqslant t\leqslant t_1, \quad j=1,2. \qquad (2.5)
$$

By (1.12) and (2.5), equation (1.1) for u_1 and u_2 are uniformly parabolic on every compact subset of $\overline{\Omega}\backslash\{a_1,\dots,i_0\}\times(0,\infty)$. Hence by the parabolic Schauder estimates [**17**], $u_1,u_2\in C^{2,1}(\overline{\Omega}\backslash\{a_1,\dots,i_0\}\times(0,\infty))$.

We choose a nonnegative monotone increasing function $\phi \in C^\infty(\mathbb{R})$ such that $\phi(s) = 0$ for any $s \leqslant 1/2$ and $\phi(s) = 1$ for any $s \geqslant 1$. For any $0 < \delta < \delta_0$, let $\phi_\delta(x) = \phi(|x|/\delta)$. Then $|\nabla \phi_\delta| \leqslant C/\delta$ and $|\Delta \phi_\delta| \leqslant C/\delta^2$. Let $\alpha > \max(2 + n, \gamma_1, \gamma_2, \dots, \gamma_{i_0}) - n$. We choose $0 < \psi \in C^\infty(\overline{\Omega} \backslash \{a_1, \dots, a_{i_0}\})$ such that $\psi(x) = |x - a_i|^\alpha$ for any $x \in \cup_{i=1}^{i_0} B_{\delta_0}(a_i)$. Let $\delta_2 = \delta_1/2$ and $T > 0$. Then there exists a constant $c_1 > 0$ such that

$$\psi(x) \geqslant c_1 \quad \forall x \in \overline{\Omega} \backslash \cup_{i=1}^{i_0} B_{\delta_2}(a_i). \tag{2.6}$$

By (1.13) and the choice of α, for any $i = 1, \dots, i_0$,

$$\begin{aligned} &\int_{B_{\delta_2}(a_i)} |x - a_i|^\alpha (u_1 - u_2)_+(x, t) \,\mathrm{d}x \\ &\qquad \leqslant C_T \int_0^{\delta_2} \rho^{\alpha + n - \gamma_i - 1} \,\mathrm{d}\rho = C_T' \delta_2^{\alpha + n - \gamma_i} < \infty \quad \forall 0 < t < T \end{aligned} \tag{2.7}$$

for some constants $C_T > 0$, $C_T' > 0$. Since $u_1, u_2 \in L^\infty_{\text{loc}}(\overline{\Omega} \backslash \{a_1, \dots, i_0\} \times (0, \infty))$, by (1.14) and (2.7) for any $T > 0$ there exists a constant $C_0(T) > 0$ such that

$$\int_{\widehat{\Omega}} \psi(x)(u_1 - u_2)_+(x, t) \,\mathrm{d}x \leqslant C_0(T) < \infty \quad \forall 0 < t < T. \tag{2.8}$$

Let

$$w_\delta(x) = \Pi_{i=1}^{i_0} \phi_\delta(x - a_i).$$

By the Kato inequality ([**4**, **16**]) for any $0 < \delta < \delta_1$, $t > 0$,

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\int_{\widehat{\Omega}} (u_1 - u_2)_+ \psi w_\delta \,\mathrm{d}x\right) \\ &\quad \leqslant \int_{\widehat{\Omega}} (u_1^m - u_2^m)_+ \Delta(\psi w_\delta) \,\mathrm{d}x \\ &\quad = \int_{\widehat{\Omega}} \{w_\delta \Delta\psi + 2\nabla\psi \cdot \nabla w_\delta + \psi \Delta w_\delta\} (u_1^m - u_2^m)_+ \,\mathrm{d}x \\ &\quad \leqslant \int_{\widehat{\Omega}} (u_1^m - u_2^m)_+ w_\delta \Delta\psi \,\mathrm{d}x \\ &\qquad + \frac{C}{\delta} \sum_{i=1}^{i_0} \int_{\delta/2 \leqslant |x - a_i| \leqslant \delta} |x - a_i|^{\alpha - 1} (u_1^m - u_2^m)_+(x, t) \,\mathrm{d}x \\ &\qquad + \frac{C}{\delta^2} \sum_{i=1}^{i_0} \int_{\delta/2 \leqslant |x - a_i| \leqslant \delta} |x - a_i|^{\alpha} (u_1^m - u_2^m)_+(x, t) \,\mathrm{d}x \\ &\quad \leqslant \int_{\widehat{\Omega}} (u_1^m - u_2^m)_+ w_\delta \Delta\psi \,\mathrm{d}x \\ &\qquad + C \sum_{i=1}^{i_0} \int_{\delta/2 \leqslant |x - a_i| \leqslant \delta} |x - a_i|^{\alpha - 2} (u_1^m - u_2^m)_+(x, t) \,\mathrm{d}x. \end{aligned} \tag{2.9}$$

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By direct computation,

$$\Delta\psi = \Delta|x-a_i|^\alpha = \alpha(\alpha+n-2)|x-a_i|^{\alpha-2} \quad \forall x \in \widehat{B_{\delta_0}}(a_i),\ i=1,\ldots,i_0. \tag{2.10}$$

By (1.12) and the mean value theorem,

$$(u_1^m - u_2^m)_+(x,t) \leqslant m\mu_0^{m-1}(u_1-u_2)_+(x,t) \quad \forall x \in \widehat{\Omega},\ t>0. \tag{2.11}$$

By (2.6), (2.9), (2.10) and (2.11),

$$\begin{aligned}\frac{\partial}{\partial t}\left(\int_{\widehat{\Omega}}(u_1-u_2)_+\psi w_\delta\,\mathrm{d}x\right) &\leqslant C\int_{\Omega\setminus\cup_{i=1}^{i_0}B_{\delta_2}(a_i)}(u_1^m-u_2^m)_+(x,t)\,\mathrm{d}x\\ &\quad + C\int_{\cup_{i=1}^{i_0}B_{\delta_2}(a_i)}|x-a_i|^{\alpha-2}(u_1^m-u_2^m)_+(x,t)\,\mathrm{d}x\\ &\leqslant C\int_{\widehat{\Omega}}(u_1-u_2)_+(x,t)\psi(x)\,\mathrm{d}x\\ &\quad + C\int_{\cup_{i=1}^{i_0}B_{\delta_2}(a_i)}|x-a_i|^{\alpha-2}(u_1^m-u_2^m)_+(x,t)\,\mathrm{d}x.\end{aligned} \tag{2.12}$$

By (1.13) and the mean value theorem for any $|x-a_i|\leqslant\delta_2$, $0<t<T$, $i=1,\ldots,i_0$,

$$\begin{aligned}|x-a_i|^{\alpha-2}(u_1^m-u_2^m)_+(x,t) &\leqslant m|x-a_i|^{\alpha-2}u_2(x,t)^{m-1}(u_1-u_2)_+(x,t)\\ &\leqslant mC_1(T)^{m-1}|x-a_i|^{(1-m)\gamma_i-2+\alpha}(u_1-u_2)_+(x,t)\\ &\leqslant mC_1(T)^{m-1}\delta_0^{(1-m)\gamma_i-2}|x-a_i|^\alpha(u_1-u_2)_+(x,t)\\ &\leqslant mC_1(T)^{m-1}\delta_0^{(1-m)\gamma_i-2}\psi(x)(u_1-u_2)_+(x,t).\end{aligned} \tag{2.13}$$

By (2.12) and (2.13),

$$\frac{\partial}{\partial t}\left(\int_{\widehat{\Omega}}(u_1-u_2)_+\psi w_\delta\,\mathrm{d}x\right) \leqslant C_T\int_{\widehat{\Omega}}(u_1-u_2)_+(x,t)\psi(x)\,\mathrm{d}x. \tag{2.14}$$

Integrating (2.14) over $(0,t)$, by (1.14) and (2.8),

$$\begin{aligned}&\int_{\widehat{\Omega}}(u_1-u_2)_+(x,t)\psi(x)w_\delta(x)\,\mathrm{d}x\\ &\quad\leqslant C_T\int_0^t\int_{\widehat{\Omega}}(u_1-u_2)_+(x,t)\psi(x)\,\mathrm{d}x\,\mathrm{d}t \quad \forall 0<t<T\\ \Rightarrow &\int_{\widehat{\Omega}}(u_1-u_2)_+(x,t)\psi(x)\,\mathrm{d}x\\ &\quad\leqslant C_T\int_0^t\int_{\widehat{\Omega}}(u_1-u_2)_+(x,t)\psi(x)\,\mathrm{d}x\,\mathrm{d}t \quad \forall 0<t<T \quad \text{as } \delta\to 0.\end{aligned} \tag{2.15}$$

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By (2.15),

$$u_1(x,t) \leqslant u_2(x,t) \quad \forall x \in \widehat{\Omega}, 0 < t < T. \tag{2.16}$$

Letting $T \to \infty$ in (2.16) we get (1.15) and the theorem follows. □

By theorem 1.1, lemmas 2.3 and 2.15 of [**15**] and theorem 1.1 and proposition 2.2, we have the following result.

THEOREM 2.3. *Let $n \geqslant 3$, $0 < m < (n-2)/n$, $0 < \delta_1 < \min(1, \delta_0)$, $\mu_0 > 0$, $f \in C^3(\partial\Omega \times (0,\infty)) \cap L^\infty(\partial\Omega \times (0,\infty))$ be such that $f \geqslant \mu_0$ on $\partial\Omega \times (0,\infty)$ and $\mu_0 \leqslant u_0 \in L^p_{\text{loc}}(\overline{\Omega}\backslash\{a_1,\dots,a_{i_0}\})$ for some constant $p > n(1-m)/2$ be such that (1.2) and (1.6) hold for some constants satisfying (1.11) and $\lambda_1, \dots, \lambda_{i_0}, \lambda'_1, \dots, \lambda'_{i_0} \in \mathbb{R}^+$. Then there exists a unique solution u of (1.3) which satisfies (1.8) and (2.3) such that for any constants $T > 0$ and $\delta_2 \in (0, \delta_1)$ there exist constants $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, depending only on $\lambda_1, \dots, \lambda_{i_0}, \lambda'_1, \dots, \lambda'_{i_0}, \gamma_1, \dots, \gamma_{i_0}, \gamma'_1, \dots, \gamma'_{i_0}$, such that*

$$\frac{C_1}{|x-a_i|^{\gamma_i}} \leqslant u(x,t) \leqslant \frac{C_2}{|x-a_i|^{\gamma'_i}} \quad \forall 0 < |x-a_i| < \delta_2, 0 < t < T, i = 1, 2, \dots, i_0 \tag{2.17}$$

holds.

Proof of theorem 1.2. We will use a modification of the proof of lemma 2.3 of [**3**] to prove the theorem. Let

$$A = A(x,t) = \begin{cases} \dfrac{u_1(x,t)^m - u_2(x,t)^m}{u_1(x,t) - u_2(x,t)} & \forall x \in \widehat{\Omega}, t > 0 \text{ satisfying } u_1(x,t) \neq u_2(x,t) \\ m u_1(x,t)^{m-1} & \forall x \in \widehat{\Omega}, t > 0 \text{ satisfying } u_1(x,t) = u_2(x,t) \\ 0 & \forall x = a_i,\ i = 1, \dots, i_0, t > 0. \end{cases}$$

For any $k \in \mathbb{Z}^+$, let

$$\alpha_k(x,t) = \begin{cases} \dfrac{|u_1(x,t)^m - u_2(x,t)^m|}{|u_1(x,t) - u_2(x,t)| + (1/k)} & \forall x \in \widehat{\Omega}, t > 0 \\ 0 & \forall x = a_i,\ i = 1, \dots, i_0, t > 0 \end{cases}$$

and $A_k = A_k(x,t) = \alpha_k(x,t) + k^{-1}$. We choose a nonnegative monotone increasing function $\phi \in C^\infty(\mathbb{R})$ such that $\phi(s) = 0$ for any $s \leqslant 0$ and $\phi(s) = 1$ for any $s \geqslant 1$. Let $0 < \delta_2 \leqslant \delta_1/2$. For any $\delta \in (0, \delta_2/2)$ and $j \geqslant 2/\delta_2$, let $\phi_j(x) = \phi(j(|x| - \delta))$. Let $t_1 > t_0 > 0$ and $0 \leqslant h \in C_0^\infty(\Omega_{\delta_2})$. For any $k \in \mathbb{Z}^+$ and $0 < \delta \leqslant \delta_2/2$, let $\psi_{k,\delta}$ be the solution of

$$\begin{cases} \psi_t + A_k \Delta\psi = 0 & \text{in } \Omega_\delta \times (0, t_1) \\ \psi(x,t) = 0 & \text{on } \partial\Omega_\delta \times (0, t_1) \\ \psi(x, t_0) = h(x) & \text{in } \Omega_\delta \end{cases} \tag{2.18}$$

and

$$w_j(x) = \Pi_{i=1}^{i_0} \phi_j(x - a_i).$$

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Then $|\nabla w_j| \leqslant Cj$ and $|\Delta w_j| \leqslant Cj^2$ for some constant $C > 0$. By the maximum principle, $0 \leqslant \psi_{k,\delta} \leqslant \|h\|_{L^\infty}$ in $\Omega_\delta \times (0, t_1)$. Hence $\partial\psi_{k,\delta}/\partial\nu \leqslant 0$ on $\partial\Omega \times (0, t_1)$. Then

$$\begin{aligned}
&\int_{\Omega_\delta} (u_1(x,t_1) - u_2(x,t_1))h(x)\,\mathrm{d}x \\
&= \int_{\Omega_\delta} (u_1(x,t_0) - u_2(x,t_0))\psi_{k,\delta}(x,t_0)w_j(x)\,\mathrm{d}x + \int_{t_0}^{t_1}\int_{\partial\Omega} (f_2^m - f_1^m)\frac{\partial\psi_{k,\delta}}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\quad + \int_{t_0}^{t_1}\int_{\Omega_\delta} (u_1 - u_2)\{w_j(\partial_t\psi_{k,\delta} + A\Delta\psi_{k,\delta}) \\
&\quad + A\nabla w_j \cdot \nabla\psi_{k,\delta} + A\psi_{k,\delta}\Delta w_j\}\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \|h\|_{L^\infty}\int_{\Omega_\delta} (u_1(x,t_0) - u_2(x,t_0))_+\,\mathrm{d}x + \int_{t_0}^{t_1}\int_{\Omega_\delta} |u_1 - u_2||A - A_k||\Delta\psi_{k,\delta}|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\sum_{i=1}^{i_0}\int_{t_0}^{t_1}\int_{\delta\leqslant|x-a_i|\leqslant\delta+j^{-1}} |u_1^m - u_2^m|\{j|\nabla|x-a_i|\cdot\nabla\psi_{k,\delta}| + j^2\psi_{k,\delta}\}\,\mathrm{d}x\,\mathrm{d}t \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{2.19}$$

We will now use a modification of the proof of theorem 2.1 of [7] to estimate the derivative of $\psi_{k,\delta}$ on $\cup_{i=1}^{i_0}\partial B_\delta(a_i) \times (0, t_1)$. Let

$$q_i(x) = \frac{\delta^{2-n} - |x-a_i|^{2-n}}{\delta^{2-n} - \delta_2^{2-n}} \cdot \|h\|_{L^\infty} \quad \forall i = 1, \ldots, i_0. \tag{2.20}$$

Then for any $i = 1, \ldots, i_0$, q_i satisfies

$$\begin{cases}
q_t + A_k\Delta q = 0 & \text{in } (B_{\delta_2}(a_i)\backslash\overline{B_\delta(a_i)}) \times (0,t_1) \\
q = 0 & \text{on } \partial B_\delta(a_i) \times (0,t_1) \\
q = \|h\|_{L^\infty} & \text{on } \partial B_{\delta_2}(a_i) \times (0,t_1) \\
q \geqslant 0 & \text{on } B_{\delta_2}(a_i)\backslash\overline{B_\delta(a_i)}
\end{cases} \tag{2.21}$$

Since $\psi_{k,\delta}$ is a subsolution of (2.21), by the maximum principle,

$$0 \leqslant \psi_{k,\delta}(x,t) \leqslant q_i(x) \quad \forall \delta \leqslant |x-a_i| \leqslant \delta_2, 0 < t \leqslant t_1, i = 1, \ldots, i_0 \tag{2.22}$$

$$\Rightarrow \left|\frac{\partial\psi_{k,\delta}}{\partial\nu}\right| \leqslant \left|\frac{\partial q_i}{\partial\nu}\right| = \frac{(n-2)\delta^{1-n}}{\delta^{2-n} - \delta_2^{2-n}}\|h\|_{L^\infty} \quad \text{on } \partial B_\delta(a_i) \times (0,t_1) \quad \forall i = 1, \ldots, i_0. \tag{2.23}$$

By (2.20) and the mean value theorem,

$$q_i(x) \leqslant \frac{(n-2)j^{-1}\delta^{1-n}}{\delta^{2-n} - \delta_2^{2-n}}\|h\|_{L^\infty} \quad \forall \delta \leqslant |x-a_i| \leqslant \delta + j^{-1}, i = 1, \ldots, i_0. \tag{2.24}$$

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By (1.13), (1.16), (2.19), (2.22), (2.23) and (2.24),

$$
\begin{aligned}
I_3 &\leqslant C\sum_{i=1}^{i_0} j\int_{t_0}^{t_1}\int_{\delta\leqslant|x-a_i|\leqslant\delta+j^{-1}}|u_1^m-u_2^m| \\
&\quad\times\left\{|\nabla|x-a_i|\cdot\nabla\psi_{k,\delta}|+\frac{(n-2)\delta^{1-n}}{\delta^{2-n}-\delta_2^{2-n}}\|h\|_{L^\infty}\right\}\mathrm{d}x\,\mathrm{d}t \\
&= C\sum_{i=1}^{i_0}\int_{t_0}^{t_1}\int_{\partial B_\delta(a_i)}|u_1^m-u_2^m| \\
&\quad\times\left\{\left|\frac{\partial\psi_{k,\delta}}{\partial\nu}\right|+\frac{(n-2)\delta^{1-n}}{\delta^{2-n}-\delta_2^{2-n}}\|h\|_{L^\infty}\right\}\mathrm{d}x\,\mathrm{d}t \quad \text{as } j\to\infty \\
&\leqslant 2C\frac{(n-2)\delta^{1-n}}{\delta^{2-n}-\delta_2^{2-n}}\|h\|_{L^\infty}\sum_{i=1}^{i_0}\int_{t_0}^{t_1}\int_{\partial B_\delta(a_i)}|u_1^m-u_2^m|\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\leqslant\frac{C'\|h\|_{L^\infty}t_1\delta^{1-n}}{\delta^{2-n}-\delta_2^{2-n}}\sum_{i=1}^{i_0}\delta^{n-1-m\gamma_i'}. \qquad (2.25)
\end{aligned}
$$

By the same argument as the proof of lemma 2.3 of [4],

$$
\lim_{k\to\infty} I_2 = 0. \qquad (2.26)
$$

Hence letting first $j\to\infty$ and then $k\to\infty$ in (2.19), by (2.25) and (2.26),

$$
\begin{aligned}
\int_{\Omega_\delta}(u_1(x,t_1)-u_2(x,t_1))h(x)\,\mathrm{d}x &\leqslant \|h\|_{L^\infty}\int_{\Omega_\delta}(u_1(x,t_0)-u_2(x,t_0))_+\,\mathrm{d}x \\
&\quad+\frac{C'\|h\|_{L^\infty}t_1\delta^{1-n}}{\delta^{2-n}-\delta_2^{2-n}}\sum_{i=1}^{i_0}\delta^{n-1-m\gamma_i'}. \qquad (2.27)
\end{aligned}
$$

Letting $t_0\to 0$ in (2.27), by (1.14) and (1.16),

$$
\begin{aligned}
&\int_{\Omega_\delta}(u_1(x,t_1)-u_2(x,t_1))h(x)\,\mathrm{d}x \leqslant \frac{C'\|h\|_{L^\infty}t_1\delta^{2-n}}{\delta^{2-n}-\delta_2^{2-n}}\sum_{i=1}^{i_0}\delta^{n-2-m\gamma_i'} \\
&\Rightarrow \int_{\widehat{\Omega}}(u_1(x,t_1)-u_2(x,t_1))h(x)\,\mathrm{d}x = 0 \qquad \forall t_1>0 \quad \text{as } \delta\to 0. \qquad (2.28)
\end{aligned}
$$

We now choose a sequence of smooth functions $0\leqslant h_i\in C_0^\infty(\Omega_{\delta_2})$ such that $h_i(x)\to\chi_{\{u_1>u_2\}\cap\Omega_{\delta_2}}(x)$ for any $x\in\Omega_{\delta_2}$ as $i\to\infty$. Putting $h=h_i$ in (2.28) and letting $i\to\infty$,

$$
\begin{aligned}
&\int_{\Omega_{\delta_2}}(u_1(x,t_1)-u_2(x,t_1))_+\,\mathrm{d}x = 0 \quad \forall t_1>0, \quad 0<\delta_2<\delta_1/2 \\
&\Rightarrow \int_{\widehat{\Omega}}(u_1(x,t_1)-u_2(x,t_1))_+\,\mathrm{d}x = 0 \quad \forall t_1>0 \quad \text{as } \delta_2\to 0
\end{aligned}
$$

and (1.15) follows. □

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By theorem 1.1, lemmas 2.3 and 2.15 of [**15**] and theorem 1.2 and proposition 2.2 we have the following result.

Theorem 2.4. *Let* $n \geqslant 3$, $0 < m < (n-2)/n$, $0 < \delta_1 < \min(1, \delta_0)$, $\mu_0 > 0$, $f \in L^\infty(\partial\Omega \times (0,\infty))$ *be such that* $f \geqslant \mu_0$ *on* $\partial\Omega \times (0,\infty)$ *and* $\mu_0 \leqslant u_0 \in L^p_{\mathrm{loc}}(\overline{\Omega}\backslash\{a_1, \dots, a_{i_0}\})$ *for some constant* $p > n(1-m)/2$ *be such that* (1.2) *and* (1.6) *hold for some constants satisfying* (1.16) *and* $\lambda_1, \dots, \lambda_{i_0}, \lambda'_1, \dots, \lambda'_{i_0} \in \mathbb{R}^+$. *Then there exists a unique solution* u *of* (1.3) *which satisfies* (2.3) *such that for any constants* $T > 0$ *and* $\delta_2 \in (0, \delta_1)$ *there exist constants* $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, *such that* (2.17) *holds.*

Proof of theorem 1.3. Since the proof is similar to the proof of theorem 1.2, we will only sketch the argument here. Let

$$A = A(x,t) = \begin{cases} \dfrac{u_1(x,t)^m - u_2(x,t)^m}{u_1(x,t) - u_2(x,t)} & \forall x \in \widehat{\mathbb{R}^n}, t > 0 \ \text{ satisfying } u_1(x,t) \neq u_2(x,t) \\ m u_1(x,t)^{m-1} & \forall x \in \widehat{\mathbb{R}^n}, t > 0 \text{ satisfying } u_1(x,t) = u_2(x,t) \\ 0 & \forall x = a_i, i = 1, \dots, i_0, t > 0. \end{cases} \tag{2.29}$$

For any $k \in \mathbb{Z}^+$, let

$$\alpha_k(x,t) = \begin{cases} \dfrac{|u_1(x,t)^m - u_2(x,t)^m|}{|u_1(x,t) - u_2(x,t)| + (1/k)} & \forall x \in \widehat{\mathbb{R}^n}, t > 0 \\ 0 & \forall x = a_i, \ i = 1, \dots, i_0, t > 0 \end{cases} \tag{2.30}$$

and $A_k = A_k(x,t) = \alpha_k(x,t) + k^{-1}$. Let $0 < \delta_2 \leqslant \delta_1/2$. For any $\delta \in (0, \delta_2/2)$ and $j \geqslant 2/\delta_2$, let ϕ, ϕ_j and w_j be as in the proof of theorem 1.2. Let $t_1 > t_0 > 0$, $R'_0 > R_1 + 1$, $R > 2R'_0$ and $h \in C_0^\infty(\Omega_{\delta_2, R'_0})$. For any $k \in \mathbb{Z}^+$ and $0 < \delta \leqslant \delta_2/2$, let $\psi_{k,\delta,R}$ be the solution of

$$\begin{cases} \psi_t + A_k \Delta\psi = 0 & \text{in } \Omega_{\delta,R} \times (0, t_1) \\ \psi(x,t) = 0 & \text{on } \partial\Omega_{\delta,R} \times (0, t_1) \\ \psi(x, t_0) = h(x) & \text{in } \Omega_{\delta,R} \end{cases} \tag{2.31}$$

By the maximum principle, $0 \leqslant \psi_{k,\delta,R} \leqslant \|h\|_{L^\infty}$ in $\Omega_{\delta,R} \times (0, t_1)$. Hence $\partial\psi_{k,\delta,R}/\partial\nu \leqslant 0$ on $\partial B_R \times (0, t_1)$. Then by an argument similar to the proof of theorem 1.2,

$$\begin{aligned} &\int_{\Omega_{\delta,R}} (u_1(x,t_1) - u_2(x,t_1)) h(x)\, \mathrm{d}x \\ &\quad \leqslant \|h\|_{L^\infty} \int_{\Omega_{\delta,R}} (u_1(x,t_0) - u_2(x,t_0))_+ \, \mathrm{d}x \\ &\qquad + \int_{t_0}^{t_1} \int_{\Omega_{\delta,R}} |u_1 - u_2||A - A_k||\Delta\psi_{k,\delta,R}| \, \mathrm{d}x \, \mathrm{d}t \end{aligned}$$

$$+\int_{t_0}^{t_1}\int_{\partial B_R}|u_1(x,t)^m-u_2(x,t)^m|\left|\frac{\partial\psi_{k,\delta,R}}{\partial\nu}\right|\,\mathrm{d}\sigma\,\mathrm{d}t$$
$$+\frac{C\delta^{2-n}}{\delta^{2-n}-\delta_2^{2-n}}\|h\|_{L^\infty}\sum_{i=1}^{i_0}\delta^{n-2-m\gamma_i'}. \tag{2.32}$$

Let

$$Q(x)=\frac{|x|^{2-n}-R^{2-n}}{(R/2)^{2-n}-R^{2-n}}\|h\|_{L^\infty}.$$

Then Q satisfies

$$\begin{cases} q_t+A_k\Delta q=0 & \text{in } (B_R\backslash\overline{B_{R/2}})\times(0,t_1)\\ q=0 & \text{on } \partial B_R\times(0,t_1)\\ q=\|h\|_{L^\infty} & \text{on } \partial B_{R/2}\times(0,t_1)\\ q\geqslant 0 & \text{on } B_R\backslash B_{R/2}\end{cases} \tag{2.33}$$

Since $\psi_{k,\delta,R}$ is a subsolution of (2.33), by the maximum principle,

$$0\leqslant\psi_{k,\delta,R}(x,t)\leqslant Q(x)\quad\forall R/2\leqslant|x|\leqslant R, 0<t\leqslant t_1$$
$$\Rightarrow\left|\frac{\partial\psi_{k,\delta,R}}{\partial\nu}\right|\leqslant\left|\frac{\partial Q}{\partial\nu}\right|=\frac{(n-2)R^{1-n}}{(R/2)^{2-n}-R^{2-n}}\|h\|_{L^\infty}\leqslant\frac{C}{R}\|h\|_{L^\infty}\quad\text{on } \partial B_R\times(0,t_1). \tag{2.34}$$

By (2.32) and (2.34),

$$\int_{\Omega_{\delta,R}}(u_1(x,t_1)-u_2(x,t_1))h(x)\,\mathrm{d}x$$
$$\leqslant\|h\|_{L^\infty}\int_{\Omega_{\delta,R}}(u_1(x,t_0)-u_2(x,t_0))_+\,\mathrm{d}x$$
$$+\int_{t_0}^{t_1}\int_{\Omega_{\delta,R}}|u_1-u_2||A-A_k||\Delta\psi_{k,\delta,R}|\,\mathrm{d}x\,\mathrm{d}t$$
$$+\frac{C\|h\|_{L^\infty}}{R}\int_{t_0}^{t_1}\int_{\partial B_R}|u_1(x,t)^m-u_2(x,t)^m|\,\mathrm{d}\sigma\,\mathrm{d}t$$
$$+\frac{C\delta^{2-n}}{\delta^{2-n}-\delta_2^{2-n}}\|h\|_{L^\infty}\sum_{i=1}^{i_0}\delta^{n-2-m\gamma_i'}. \tag{2.35}$$

Letting first $k\to\infty$ and then $t_0\to 0$, $\delta\to 0$ in (2.35), by the proof of lemma 2.3 of [4] and similar argument as the proof of theorem 1.2, the first term, second term and the last term on the right hand side of (2.35) vanish. This together with the

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mean value theorem and (1.19) implies that

$$\begin{aligned}
&\int_{\widehat{B}_R}(u_1(x,t_1)-u_2(x,t_1))h(x)\,\mathrm{d}x\\
&\quad\leqslant \frac{C\|h\|_{L^\infty}}{R}\int_{t_0}^{t_1}\int_{\partial B_R}|u_1(x,t)^m-u_2(x,t)^m|\,\mathrm{d}\sigma\,\mathrm{d}t\\
&\quad\leqslant \frac{mC\mu_0^{m-1}\|h\|_{L^\infty}}{R}\int_0^{t_1}\int_{\partial B_R}|u_1(x,t)-u_2(x,t)|\,\mathrm{d}\sigma\,\mathrm{d}t\\
&\quad\leqslant \frac{C'\|h\|_{L^\infty}}{R}\left\{\int_0^{t_1}\int_{\partial B_R}|u_1(x,t)-\mu_0|\,\mathrm{d}\sigma\,\mathrm{d}t+\int_0^{t_1}\int_{\partial B_R}|u_1(x,t)-\mu_0|\,\mathrm{d}\sigma\,\mathrm{d}t\right\}.
\end{aligned} \tag{2.36}$$

By (1.20) there exists a sequence $\{R_j\}_{j=2}^\infty\subset(2R_0',\infty)$, $R_j\to\infty$ as $j\to\infty$, such that

$$\int_0^{t_1}\int_{\partial B_{R_j}}(|u_1(x,t)-\mu_0|+|u_2(x,t)-\mu_0|)\,\mathrm{d}\sigma\,\mathrm{d}t\to 0\quad\text{as } j\to\infty. \tag{2.37}$$

Putting $R=R_j$ in (2.36) and letting $j\to\infty$, by (2.37),

$$\int_{\widehat{\mathbb{R}^n}}(u_1(x,t_1)-u_2(x,t_1))h(x)\,\mathrm{d}x=0\quad\forall t_1>0. \tag{2.38}$$

By (2.38) and an argument similar to the proof of theorem 1.2,

$$\int_{\widehat{\mathbb{R}^n}}(u_1(x,t_1)-u_2(x,t_1))_+\,\mathrm{d}x=0\quad\forall t_1>0$$

and (1.21) follows. □

By theorem 1.2, lemmas 2.3 and 2.15 and the proof of theorem 1.6 of [**15**] and theorem 1.3 we have the following result.

THEOREM 2.5. *Let $n\geqslant 3$, $0<m<(n-2)/n$, $0<\delta_1<\min(1,\delta_0)$, $\mu_0>0$ and $\mu_0\leqslant u_0\in L^p_{\mathrm{loc}}(\widehat{\mathbb{R}^n}\setminus\{a_1,\ldots,a_{i_0}\})$ for some constant $p>n(1-m)/2$ be such that (1.10) holds for some constants satisfying (1.18) and $\lambda_1,\ldots,\lambda_{i_0},\lambda_1',\ldots,\lambda_{i_0}'\in\mathbb{R}^+$. Suppose (1.17) also holds for some constant $R_1>R_0$. Then there exists a unique solution u of (1.4) which satisfies*

$$u(x,t)\geqslant\mu_0\quad\forall x\in\widehat{\mathbb{R}^n},t>0$$

and

$$\int_{\widehat{\mathbb{R}^n}}|u(x,t)-\mu_0|\,\mathrm{d}x\leqslant\int_{\widehat{\mathbb{R}^n}}|u_0-\mu_0|\,\mathrm{d}x\quad\forall t>0$$

such that for any constants $T>0$ and $\delta_2\in(0,\delta_1)$ there exist constants $C_1=C_1(T)>0$, $C_2=C_2(T)>0$, such that (2.17) holds.

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3. Existence of highly oscillating solution

In this section, we will prove the existence of initial data such the corresponding solution of (1.23) oscillates between infinity and some positive constant as $t \to \infty$. We start with a stability result for the solutions of (1.23).

LEMMA 3.1. *Let* $n \geqslant 3$, $0 < m < (n-2)/n$, $0 < \delta_1 < \min(1, \delta_0)$, $\mu_0 > 0$. *Let* $\{u_{0,j}\}_{j=1}^{\infty} \subset L^p_{\mathrm{loc}}(\overline{\Omega}\backslash\{a_1, \ldots, a_{i_0}\})$ *for some constant* $p > n(1-m)/2$ *be a sequence of functions satisfying*

$$u_{0,j} \geqslant \mu_0 \quad on\ \overline{\Omega}\backslash\{a_1, \ldots, a_{i_0}\} \quad \forall j \in \mathbb{Z}^+ \tag{3.1}$$

such that

$$\frac{\lambda_i}{|x - a_i|^{\gamma_i}} \leqslant u_{0,j}(x) \leqslant \frac{\lambda_i'}{|x - a_i|^{\gamma_i'}} \quad \forall 0 < |x - a_i| < \delta_1, \quad i = 1, \ldots, i_0, j \in \mathbb{Z}^+ \tag{3.2}$$

holds for some constants satisfying (1.11), $\lambda_1, \ldots, \lambda_{i_0}, \lambda_1', \ldots, \lambda_{i_0}' \in \mathbb{R}^+$. *Let* $\mu_0 \leqslant u_0 \in L^p_{\mathrm{loc}}(\overline{\Omega}\backslash\{a_1, \ldots, a_{i_0}\})$ *be such that* (1.22) *holds. Let* u, u_j, $j \in \mathbb{Z}^+$, *be the unique solutions of* (1.23) *with initial value* u_0, $u_{0,j}$ *respectively, given by theorem 2.3. Suppose*

$$u_{0,j} \to u_0 \quad in\ L^p_{\mathrm{loc}}(\overline{\Omega}\backslash\{a_1, \ldots, a_{i_0}\}) \quad as\ j \to \infty. \tag{3.3}$$

Then u_j *converges to* u *uniformly in* $C^{2,1}(\Omega_\delta \times (t_1, t_2))$ *as* $j \to \infty$ *for any* $0 < \delta < \delta_0$ *and* $t_2 > t_1 > 0$.

Proof. Let $0 < \delta' < \delta < \delta_0$ and $t_2 > t_1 > 0$. By (3.3) there exists a constant $M_1 > 0$ such that

$$\|u_{0,j}\|_{L^p(\Omega_{\delta'})} \leqslant M_1 \quad \forall j \in \mathbb{Z}^+. \tag{3.4}$$

By (3.4) and lemma 2.9 of [**15**] there exists a constant $M_2 > 0$ depending on M_1 and μ_0 such that

$$\|u_j\|_{L^\infty(\Omega_\delta \times (t_1/2, t_2))} \leqslant M_2 \quad \forall j \in \mathbb{Z}^+. \tag{3.5}$$

By theorem 2.3,

$$u_j \geqslant \mu_0 \quad \text{in } \overline{\Omega}\backslash\{a_1, \ldots, a_{i_0}\} \times (0, \infty) \quad \forall j \in \mathbb{Z}^+. \tag{3.6}$$

By (3.5) and (3.6) equation (1.1) for u_j are uniformly parabolic on every compact subset of $\overline{\Omega}\backslash\{a_1, \ldots, a_{i_0}\} \times (0, \infty)$. Hence by the Ascoli theorem, diagonalization argument, and an argument similar to the proof of lemma 2.11 of [**15**] and theorem 1.1 of [**13**] the sequence $\{u_j\}_{j=1}^{\infty}$ has a subsequence $\{u_{j_k}\}_{k=1}^{\infty}$ that converges uniformly in $C^{2,1}(\Omega_\delta \times (t_1, t_2))$ to a solution v of (1.23) as $k \to \infty$ for any $0 < \delta < \delta_0$ and $t_2 > t_1 > 0$ and

$$v \geqslant \mu_0 \quad \text{in } \widehat{\Omega} \times (0, \infty). \tag{3.7}$$

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Since by theorem 2.3 for any $T > 0$ there exists constants $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, such that (2.17) holds for any u_j, putting $u = u_{j_k}$ in (2.17) and letting $k \to \infty$,

$$\frac{C_1}{|x-a_i|^{\gamma_i}} \leqslant v(x,t) \leqslant \frac{C_2}{|x-a_i|^{\gamma_i'}} \quad \forall 0 < |x-a_i| < \delta_2, 0 < t < T, i = 1,2,\dots,i_0. \tag{3.8}$$

By lemma 2.1 there exists a constant $C > 0$ such that

$$\begin{aligned}\int_{D_{\delta_1}} u_j(x,t)\,dx &\leqslant \left\{\left(\int_{D_{2\delta_1}} u_{0,j}\,\mathrm{d}x\right)^{1-m} + Ct\right\}^{1/(1-m)} \\ &\quad + |D_{\delta_1}|\mu_0 \quad \forall t > 0, 0 < \delta_1 < \delta_0/2, j \in \mathbb{Z}^+ \\ &\leqslant \left\{\left(|D_{2\delta_1}|^{1-1/p}\|u_{0,j}\|_{L^p(D_{2\delta_1})}\right)^{1-m} + Ct\right\}^{1/(1-m)} \\ &\quad + |D_{\delta_1}|\mu_0 \quad \forall t > 0, 0 < \delta_1 < \delta_0/2, j \in \mathbb{Z}^+. \end{aligned} \tag{3.9}$$

Let $\varepsilon > 0$. By (3.3) there exists $j_0 \in \mathbb{Z}^+$ such that

$$\|u_{0,j}\|_{L^p(D_{\delta_1})} \leqslant \|u_0\|_{L^p(D_{\delta_1})} + \varepsilon \quad \forall 0 < \delta_1 < \delta_0, j \geqslant j_0. \tag{3.10}$$

By (3.9), (3.10) and Holder's inequality,

$$\begin{aligned}\int_{\Omega_\delta} |u_j(x,t) - u_{0,j}(x)|\,\mathrm{d}x &\leqslant \int_{\Omega_\delta \setminus D_{\delta_1}} |u_j(x,t) - u_{0,j}(x)|\,\mathrm{d}x \\ &\quad + \left\{(|D_{2\delta_1}|^{1-1/p}(\|u_0\|_{L^p(D_{2\delta_1})} + \varepsilon))^{1-m} + Ct\right\}^{1/(1-m)} \\ &\quad + |D_{\delta_1}|\mu_0 + |D_{\delta_1}|^{1-1/p}(\|u_0\|_{L^p(D_{\delta_1})} + \varepsilon) \\ &\quad \forall 0 < \delta_1 < \delta_0/2, t > 0, j \geqslant j_0. \end{aligned} \tag{3.11}$$

Letting $j = j_k \to \infty$ in (3.11),

$$\begin{aligned}\int_{\Omega_\delta} |v(x,t) - u_0(x)|\,\mathrm{d}x &\leqslant \int_{\Omega_\delta \setminus D_{\delta_1}} |v(x,t) - u_0(x)|\,\mathrm{d}x \\ &\quad + \left\{\left(|D_{2\delta_1}|^{1-1/p}(\|u_0\|_{L^p(D_{2\delta_1})} + \varepsilon)\right)^{1-m} + Ct\right\}^{1/(1-m)} \\ &\quad + |D_{\delta_1}|\mu_0 + |D_{\delta_1}|^{1-1/p}(\|u_0\|_{L^p(D_{\delta_1})} + \varepsilon) \\ &\quad \forall 0 < \delta_1 < \delta_0/2, t > 0. \end{aligned} \tag{3.12}$$

Letting first $t \to 0$ and then $\delta_1 \to 0$ in (3.12),

$$\lim_{t\to 0} \int_{\Omega_\delta} |v(x,t) - u_0(x)|\,\mathrm{d}x = 0 \quad \forall 0 < \delta < \delta_0. \tag{3.13}$$

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By (3.7), (3.8), (3.13) and theorem 2.3, $v = u$ in $(\overline{\Omega}\backslash\{a_1, \ldots, a_{i_0}) \times (0, \infty)$. Hence u_j converges to u uniformly in $C^{2,1}(\Omega_\delta \times (t_1, t_2))$ as $j \to \infty$ for any $0 < \delta < \delta_0$ and $t_2 > t_1 > 0$ and the lemma follows. □

We next recall two results from [**15**].

THEOREM 3.2 (cf. theorem 1.3 of [**15**]). *Suppose that* $n \geqslant 3,\ 0 < m < (n-2)/n$ *and* $\mu_0 > 0$. *Let* $\mu_0 \leqslant u_0 \in L^p_{\text{loc}}(\overline{\Omega}\backslash\{a_1, \ldots, a_{i_0}\})$ *for some constant* $p > n(1-m)/2$ *satisfy* (1.22) *for some constants satisfying* (1.18) *and* $\lambda_1, \ldots, \lambda_{i_0}, \lambda'_1, \ldots, \lambda'_{i_0} \in \mathbb{R}^+$. *Let* u *be the solution of* (1.23) *given by theorem 2.3. Then*

$$u(x,t) \to \mu_0 \quad \text{in } C^2(K) \quad \text{as } t \to \infty \tag{3.14}$$

for any compact subset K *of* $\overline{\Omega}\backslash\{a_1, \ldots, a_{i_0}\}$.

THEOREM 3.3. *Suppose that* $n \geqslant 3,\ 0 < m < (n-2)/n$ *and* $\mu_0 > 0$. *Let* $\mu_0 \leqslant u_0 \in L^p_{\text{loc}}(\overline{\Omega}\backslash\{a_1, \ldots, a_{i_0}\})$ *for some constant* $p > n(1-m)/2$ *satisfy* (1.2) *for some constants satisfying*

$$\gamma_1 > \frac{n-2}{m}, \quad \gamma_i > \frac{2}{1-m} \quad \forall i = 2, \ldots, i_0, \tag{3.15}$$

and $0 < \delta_1 < \delta_0$, $\lambda_1, \ldots, \lambda_{i_0} \in \mathbb{R}^+$. *Let* u *be the solution of* (1.23) *given by theorem 2.3. Then*

$$u(x,t) \to \infty \quad \text{on } K \quad \text{as } t \to \infty \tag{3.16}$$

for any compact subset K *of* $\widehat{\Omega}$.

Proof of theorem 1.4. We will use a modification of the proof of theorem 1 of [**22**] to construct the oscillating solution u of (1.23) as the limit of a sequence of solutions u_j of (1.23) with initial value $u_{0,j}$ that satisfies appropriate blow-up condition at the points $a_1, \ldots, a_{i_0}$. Let

$$\alpha_1 > \frac{n-2}{m}, \quad \alpha_2 = \frac{2/(1-m) + n}{2},$$

and let K be a compact subset of $\widehat{\Omega}$. We choose $j_1 \in \mathbb{Z}^+$ such that $j_1 > \max(\delta_0^{-1}, \mu_0^{1/\alpha_1}, \mu_0^{1/\alpha_2})$. Let

$$u_{0,1}(x) = \begin{cases} j_1^{\alpha_2} & \forall x \in \overline{\Omega}\backslash \cup_{i=1}^{i_0} B_{1/j_1}(a_i) \\ |x-a_i|^{-\alpha_2} & \forall x \in B_{1/j_1}(a_i),\ i = 1, \ldots, i_0. \end{cases}$$

Then $u_{0,1}(x) \geqslant \mu_0$ for any $x \in \overline{\Omega}\backslash\{a_1, \ldots, a_{i_0}\}$. By theorem 2.3 there exists a unique solution u_1 of (1.23) which satisfies $u_1 \geqslant \mu_0$ in $(\overline{\Omega}\backslash\{a_1, \ldots, a_{i_0}\}) \times (0, \infty)$.

By theorem 3.2,

$$u_1(x,t) \to \mu_0 \quad \text{in } C^2(K) \quad \text{as } t \to \infty.$$

Hence there exists a constant $t_1 > 1$ such that

$$\mu_0 \leqslant u_1(x,t_1) \leqslant \mu_0 + \frac{1}{2} \quad \forall x \in K. \tag{3.17}$$

For any $j \in \mathbb{Z}^+$, $j > j_1$, let

$$u_{0,1,j}(x) = \begin{cases} u_{0,1}(x) & \forall x \in \overline{\Omega} \backslash B_{1/j}(a_1) \\ |x-a_1|^{-\alpha_1} & \forall x \in B_{1/j}(a_1). \end{cases} \tag{3.18}$$

Then $u_{0,1,j}(x) \geqslant \mu_0$ for any $x \in \overline{\Omega} \backslash \{a_1, \ldots, a_{i_0}\}$ and $j > j_1$. For any $j > j_1$, let $u_{2,j}$ be the unique solution of (1.23) with $u_0 = u_{0,1,j}$ given by theorem 2.3 which satisfies $u_{2,j} \geqslant \mu_0$ in $(\overline{\Omega} \backslash \{a_1, \ldots, a_{i_0}\}) \times (0, \infty)$. Since $u_{0,1,j}$ converges to $u_{0,1}$ in $L^p_{\text{loc}}(\overline{\Omega} \backslash \{a_1, \ldots, a_{i_0}\})$ as $j \to \infty$, by lemma 3.1 $u_{2,j}(x,t_1)$ converges to $u_1(x,t_1)$ uniformly in K as $j \to \infty$. Hence there exists $j_2 \in \mathbb{Z}^+$,$j_2 > j_1$, such that

$$|u_{2,j_2}(x,t_1) - u_1(x,t_1)| \leqslant \frac{1}{4} \quad \forall x \in K. \tag{3.19}$$

Let $u_2 = u_{2,j_2}$ and $u_{0,2} = u_{0,1,j_2}$. By (3.17) and (3.19),

$$\mu_0 \leqslant u_2(x,t_1) \leqslant \mu_0 + \frac{3}{4} \quad \forall x \in K.$$

By (3.18) and theorem 3.3,

$$u_2(x,t) \to \infty \quad \text{in } C^2(K) \quad \text{as } t \to \infty.$$

Hence there exists a constant $t_2 > t_1 + 1$ satisfying

$$u_2(x,t_2) \geqslant 3 \quad \forall x \in K.$$

Repeating the above argument we get sequences $\{u_{0,k}\}_{k=1}^{\infty} \subset L^p_{\text{loc}}(\overline{\Omega} \backslash \{a_1, \ldots, a_{i_0}\})$, $\{j_k\}_{k=1}^{\infty} \subset \mathbb{Z}^+$ and $\{t_k\}_{k=1}^{\infty} \subset \mathbb{R}^+$, such that $j_{k+1} > j_k$ and $t_{k+1} > t_k + 1$ for all $k \in \mathbb{Z}^+$, which satisfy

$$\begin{aligned} &u_{0,k}(x) \geqslant \mu_0 \quad \text{in } \overline{\Omega} \backslash \{a_1, \ldots, a_{i_0}\} \quad \forall k \in \mathbb{Z}^+, \\ &u_{0,k}(x) = u_{0,k-1}(x) \quad \forall x \in \overline{\Omega} \backslash B_{1/j_k}(a_1), k \geqslant 2 \end{aligned}$$

and

$$u_{0,k}(x) = \begin{cases} |x-a_1|^{-\alpha_2} & \forall x \in B_{1/j_k}(a_1) \quad \text{if } k \geqslant 1 \text{ is odd} \\ |x-a_1|^{-\alpha_1} & \forall x \in B_{1/j_k}(a_1) \quad \text{if } k \geqslant 2 \text{ is even} \end{cases}$$

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and if u_k is the solution of (1.23) with $u_0 = u_{0,k}$ given by theorem 2.3, then u_k satisfies

$$\begin{aligned} \mu_0 \leqslant u_k(x,t_l) &\leqslant \mu_0 + \frac{1}{2^l} + \cdots + \frac{1}{2^k} \\ &\leqslant \mu_0 + \frac{1}{2^{l-1}} \quad \forall x \in K, 1 \leqslant l \leqslant k \text{ and } l \text{ is odd,} \end{aligned} \tag{3.20}$$

$$u_k(x,t_l) > l + \frac{3}{2} - \left(\frac{1}{2^l} + \cdots + \frac{1}{2^k}\right) > l \quad \forall x \in K, 1 \leqslant l \leqslant k \text{ and } l \text{ is even} \tag{3.21}$$

and

$$|u_k(x,t_l) - u_{k+1}(x,t_l)| < \frac{1}{2^k} \quad \forall x \in K, 1 \leqslant l \leqslant k, k \in \mathbb{Z}^+.$$

Let

$$u_0(x) = \begin{cases} j_1^{\alpha_2} & \forall x \in \overline{\Omega} \setminus \cup_{i=1}^{i_0} B_{1/j_1}(a_i) \\ |x-a_i|^{-\alpha_2} & \forall x \in B_{1/j_1}(a_i), i = 2, \ldots, i_0 \\ u_{0,k}(x) & \forall 1/j_{k+1} \leqslant |x-a_1| \leqslant 1/j_k, k \geqslant \mathbb{Z}^+. \end{cases}$$

Then $u_0 \geqslant \mu_0$ in $\overline{\Omega} \setminus \{a_1, \ldots, a_{i_0}\}$,

$$u_0(x) = \begin{cases} |x-a_1|^{-\alpha_2} & \forall 1/j_{k+1} \leqslant |x-a_1| \leqslant 1/j_k, k \geqslant \mathbb{Z}^+ \quad \text{and } k \text{ is odd} \\ |x-a_1|^{-\alpha_1} & \forall 1/j_{k+1} \leqslant |x-a_1| \leqslant 1/j_k, k \geqslant \mathbb{Z}^+ \quad \text{and } k \text{ is even,} \end{cases}$$

and $u_{0,k}$ converges to u_0 in $L^p_{\text{loc}}(\overline{\Omega} \setminus \{a_1, \ldots, a_{i_0}\})$ as $k \to \infty$. let u be the unique solution of (1.23) given by theorem 2.3. Then by lemma 3.1 u_k converges to u on every compact subset of $\widehat{\Omega} \times (0,\infty)$ as $k \to \infty$. letting $k \to \infty$ in (3.20) and (3.21) we have

$$\begin{aligned} \mu_0 \leqslant u(x,t_l) \leqslant \mu_0 + \frac{1}{2^{l-1}} \quad &\forall x \in K, l \in \mathbb{Z}^+ \text{ and } l \text{ is odd} \\ \Rightarrow \lim_{k\to\infty} u(x,t_{2k-1}) = \mu_0 \quad &\text{uniformly in } K \end{aligned}$$

and

$$\begin{aligned} u(x,t_l) \geqslant l \quad &\forall x \in K, l \in \mathbb{Z}^+ \text{ and } l \text{ is even} \\ \Rightarrow \lim_{k\to\infty} u(x,t_{2k}) = \infty \quad &\text{uniformly in } K \end{aligned}$$

and the theorem follows. □

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