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ASYMPTOTIC BEHAVIOUR OF THE STOCHASTIC MAKI–THOMPSON MODEL WITH A FORGETTING MECHANISM ON OPEN POPULATIONS

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Abstract

Rumours have become part of our daily lives, and their spread has a negative impact on a variety of human affairs. Therefore, how to control the spread of rumours is an important topic. In this paper, we extend the classic Maki–Thompson model from a deterministic framework to a stochastic framework with a forgetting mechanism, because real-world person-to-person communications are inevitably affected by random factors. By constructing suitable stochastic Lyapunov functions, we show that the asymptotic behaviour of the stochastic rumour model is governed by the basic reproductive number. If this number is less than one, then the solution of the stochastic rumour model oscillates around the rumour-free equilibrium under extra mild conditions, indicating the extinction of the rumour with a probability of one. Otherwise, the solution always fluctuates around the endemic equilibrium under certain parametric restrictions, implying that the rumour will continually persist. In addition, we discuss a possible intervention strategy that stops the spread of rumours by strengthening the intensity of white noise, which is very different from the deterministic rumour model without white noise. Also, numerical simulations are conducted to support our analytical results.

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1. Introduction

Rumour is defined as a type of social phenomenon in which a remark with questionable veracity is spread over a large scale within a short amount of time via different channels of communication [7]. Owing to a lack of accuracy, rumours have strong negative effects on social life, including reputation damage, economic losses, political consequences, and social panic and instabilities [23]. Compared with traditional face-to-face communications, the use of internet-based media such as Facebook,

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Twitter and Microblog enables rumours to spread more quickly and widely around the world. For example, after the nuclear leak that resulted from the 2011 Tohoku earthquake, the Chinese overwhelmingly purchased large amounts of iodised salt from supermarkets as a result of an explosively spread rumour that stated the benefits of iodised salt for guarding people against radiation exposure [35]. Subsequently, with the use of the aforementioned media, rumour spreading can incur social panic and affect the economy at different levels within only a few days. Therefore, to combat social and economic concerns, there is a more urgent need than ever to investigate how to effectively impede rumour propagation.

The investigation of rumour propagation has greatly benefited from the established research on epidemics of infectious diseases, owing to their similarities, as noted by Daley and Kendall [3]. For example, both cases involve transmission processes from individuals to subpopulations (ignorants in the case of rumours) via similar contacts, namely, infective–susceptible contacts for epidemics and spreader–ignorant contacts for rumours. There are some differences between epidemic infections and rumour spreading. Susceptibles become immune through death, isolation or recovery at a rate proportional to their number in the epidemic model, whereas the production of stiflers occurs either through the common encounters of spreaders or through the encounter of a spreader with a stifler in the rumour-mongering process [22]. However, the differences are so insignificant that Daley and Kendall commented on the relation between epidemics and rumours: "Investigation soon showed the parallel to be a misleading one, and in fact the two phenomena could hardly be more different". Therefore, epidemiological models are generally used to study the spread of rumours both deterministically and stochastically.

By modifying the Daley–Kendall model [3], Maki and Thompson [17] proposed that rumours are disseminated through direct contacts of spreaders with other individuals, and that spreader–spreader contact would convert the initiating spreader to a stifler. Belen and Pearce [2] extended the Daley–Kendall and Maki–Thompson rumour models, and studied the proportion of the population starting from a general initial condition. Belen et al. [1] further envisaged a rumour model with impulsive control. Thompson et al. [29] considered both active and passive personalities in the ignorant and spreader classes, respectively, and reported the existence of a rumour-free equilibrium. Huo and Song [11] studied the stability of the equilibria and persistence of the rumour spreading system by considering the effect of the dissemination of scientific knowledge on preventing rumour transition. Furthermore, a rumour transmission model with Holling type II functional response was proposed by Huo et al. [10] to explain the impact of scientific knowledge on rumour spreading. The phenomenon of rumour-mongering repeatedly on social media was also explored by Yao et al. [33] using a susceptible-dangerous-infective-latent-recovered (SDILR) rumour spreading model, and analyses of the local and global asymptotic stabilities were presented. Although the existing models may adequately describe the rumour spreading process, they have yet to consider the impact of uncertain environmental factors on rumour spreading. Hence, a study that incorporates random fluctuations into population dynamics models of rumours could provide new practical insights.

The rumour spreading process can be affected by many uncertainty factors (considered as noise in the rumour model) in real life. For example, both educational background and legal consciousness can affect public attitudes and responses to rumours and affect rumour spreading. A rumour spreads faster if its subject is an event that is believed to be important and attractive. Moreover, emergence of other public events related to the rumour may attract more attention and lead to faster spreading of the rumour, whereas unrelated events shift public attention and impede rumour spreading. Several studies on the effects of uncertainty factors on rumour spreading have been conducted to provide guidelines for social network analysis. Dauhoo et al. [4] studied a random effect that incubators exert on ignorants using a stochastic model, and established the uniqueness of the solution and the conditions for rumour extinction. Jia and Lv [14] formulated a stochastic rumour propagation model with Gaussian white noise and investigated the sufficient conditions related to extinction and persistence of rumour. They also explored a stochastic rumour model with Levy noise [15]. On the other hand, recent studies have revealed the impact of stochastic factors on complex social networks [13, 38]. Given the scarcity of relevant studies, research on how a rumour spreads in a noisy environment using a realistic model would both enrich the current literature and offer theoretical insight.

In this paper, we propose a stochastic rumour spreading model based on the widely used Maki–Thompson model [17] for open populations, and a forgetting mechanism that has been considered in the scientific literature is also introduced into the stochastic model to conform it to realistic situations. Compared with previous studies on the deterministic models of rumour spreading without considering white noise, this study investigates the asymptotic behaviour of the stochastic rumour propagation model, by establishing the corresponding stochastic Lyapunov functions. First, the model has a unique positive global solution for any given positive initial value, and the solution is stochastically ultimately bounded. Second, the asymptotic behaviour of the stochastic rumour propagation model can be determined based on the threshold value. If the value is less than one, the model is asymptotically stable under extra mild conditions, resulting in extinction of a rumour with a probability of one. Meanwhile, a threshold value greater than one indicates that the solution of the stochastic propagation model always fluctuates around the endemic equilibrium under complex parametric restrictions, leading to the stochastic persistence of a rumour. Finally, we present a sufficient condition for the extinction of a rumour by intensifying the perturbation of the spreading rate between ignorants and spreaders, whereas the rumour always persists without the perturbation. These results may be useful for researchers in applied sciences interested in modelling information propagation phenomena and related processes.

This paper is organised as follows. Section 2 is devoted to proving a unique positive global solution of the stochastic rumour model which has a stochastically ultimately bounded solution. The asymptotic behaviours around the rumour-free equilibrium and the rumour endemic equilibrium are shown in Sections 3 and 4, respectively. Section 5 focuses on numerical simulations to illustrate the theoretical results, and the paper concludes with a brief discussion in Section 6.

2. The stochastic rumour model

As in the Maki–Thompson model [17], which is a variant of the Daley–Kendall model, we consider a population consisting of ignorants, spreaders and stiflers, represented by X, Y and Z, respectively. The rumour spreads by "directed" contact between spreaders and the rest of the population, following the law of mass action:

- when a spreader successfully contacts an ignorant, the ignorant becomes a spreader with probability β, namely, the "spreading rate";
- when a spreader contacts another spreader or a stifler, the initiating spreader becomes a stifler with probability α, which is defined as the "stifling rate".

In addition, we reasonably assume that inflows into the ignorant class and outflows from each rumour class occur at constant rates Λ and μ, respectively. In reality, during the course of rumour propagation, a spreader may cease further transmission as a result of losing interest in the rumour or forgetting to tell others. Similar to the work by Nekovee et al. [21], we incorporate this important mechanism into the model by assuming that spreaders may spontaneously cease spreading a rumour with probability δ. In light of the rumour spreading process elaborated above, a general flow diagram of the rumour spreading model is presented in Figure 1. Accordingly, the deterministic rumour model can be described by the following ordinary differential equations:

$$\begin{cases} \dfrac{dX(t)}{dt} = \Lambda - \beta X(t)Y(t) - \mu X(t), \\ \dfrac{dY(t)}{dt} = \beta X(t)Y(t) - \alpha Y(t)(Y(t)+Z(t)) - (\delta+\mu)Y(t), \\ \dfrac{dZ(t)}{dt} = \alpha Y(t)(Y(t)+Z(t)) + \delta Y(t) - \mu Z(t). \end{cases} \tag{2.1}$$

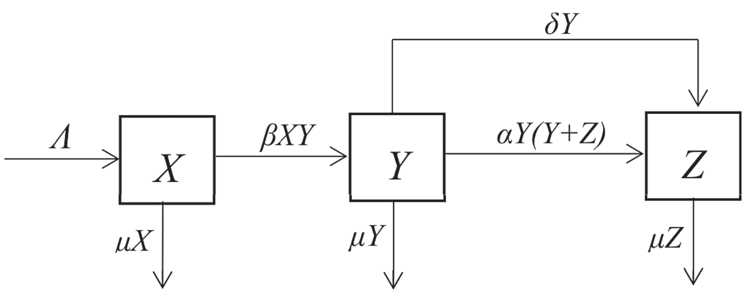

FIGURE 1. Structure of the rumour spreading process: X, Y and Z represent the number of ignorants, spreaders and stiflers, respectively.

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The state space is the first quadrant,

$$R^3_+ = \{(X, Y, Z) \in R^3 \mid X > 0, Y > 0, Z > 0\},$$

and all the parameters are positive. Applying the formula presented by Driessche and Watmough [5], we calculate the basic reproductive number

$$R_0 = \frac{\Lambda\beta}{\mu(\mu + \delta)},$$

which is the average number of secondary transmissions of a rumour, when a spreader is introduced into a population full of ignorants. The dynamics of model (2.1) are completely determined by the threshold value R_0 [9]. If $R_0 < 1$, then model (2.1) has a unique rumour-free equilibrium $E_0 = (\Lambda/\mu, 0, 0)$, which is globally asymptotically stable in the positively invariant set

$$D = \{(X, Y, Z) \in R^3_+ \mid 0 \le X + Y + Z \le \Lambda/\mu\}. \tag{2.2}$$

This implies that the rumour will become extinct, and that the entire population contains only the ignorant class. If $R_0 > 1$, E_0 is unstable, and the global asymptotical stability of the endemic equilibrium $E^* = (X^*, Y^*, Z^*)$, where

$$X^* = \frac{\alpha\Lambda + \mu(\delta + \mu)}{\mu(\beta + \alpha)}, \qquad Y^* = \frac{\mu^2(\mu + \delta)(R_0 - 1)}{\alpha\beta\Lambda + \mu\beta(\delta + \mu)},$$

$$Z^* = \frac{(\mu + \delta)(R_0 - 1)(\alpha\beta\Lambda + \delta\mu\beta - \mu^2\alpha)}{\mu(\beta + \alpha)\{\alpha\beta\Lambda + \mu\beta(\delta + \mu)\}},$$

is established, then the rumour always exists at an endemic level. From the above, we can conclude that reducing the key threshold parameter R_0 to less than one is a useful way to stop the spread of rumours.

Collective behaviour on online social networks is conducted by rational individuals, who make strategic choices influenced by the amount of information and emotion passed with the messages [27]. However, owing to the existence of uncertainty in the utility selection of the individuals as a result of their incomplete knowledge about others and the stochastic properties of individual behaviour [31], the contact behaviour between individuals also experiences some fluctuations. Similar to inclusion of the fluctuation in epidemic models, fluctuation of the spreading rate needs to be incorporated into rumour models, to enable a more realistic estimation of the rumour transmission behaviour [20]. In practice, the fluctuation of the spreading rate between ignorants and spreaders is usually estimated by an error term added to an averaged value. A commonly employed assumption is that the error term can be treated as a Gaussian white noise process [25]. From a mathematical perspective, Gaussian white noise is the formal derivative of the Wiener process (or Brownian motion). Thus, the spreading rate of the rumour is expressed as

$$\beta \to \beta + \sigma_4 \dot{B_4}(t),$$

where $B_4(t)$ is the Brownian motion and σ_4 is the intensity of the white noise that measures the amplitude of fluctuations. Therefore, the following stochastic differential equations (SDE) are obtained:

$$\begin{cases} dX(t) = \{\Lambda - \beta X(t)Y(t) - \mu X(t)\}\,dt - \sigma_4 X(t)Y(t)\,dB_4(t), \\ dY(t) = \{\beta X(t)Y(t) - \alpha Y(t)\{Y(t) + Z(t)\} - (\delta + \mu)Y(t)\}\,dt \\ \qquad\qquad + \sigma_4 X(t)Y(t)\,dB_4(t), \\ dZ(t) = \{\alpha Y(t)\{Y(t) + Z(t)\} + \delta Y(t) - \mu Z(t)\}\,dt. \end{cases} \tag{2.3}$$

In addition to parameter perturbation, another approach was proposed by Jia and Lv [14] to include stochastic perturbations in a rumour model by following similar ideas to those used in biological models, as discussed by Imhof and Walcher [12] and Jiang et al. [16]. Thus, we consider a white noise type of stochastic perturbations that are directly proportional to $X(t)$, $Y(t)$ and $Z(t)$, and that influence $\dot{X(t)}$, $\dot{Y(t)}$ and $\dot{Z(t)}$ in the rumour model (2.3). Then, the stochastic rumour model (2.3) can be written in the form:

$$\begin{cases} dX(t) = \{\Lambda - \beta X(t)Y(t) - \mu X(t)\}\,dt + \sigma_1 X(t)\,dB_1(t) - \sigma_4 X(t)Y(t)\,dB_4(t), \\ dY(t) = \{\beta X(t)Y(t) - \alpha Y(t)\{Y(t) + Z(t)\} - (\delta + \mu)Y(t)\}\,dt + \sigma_2 Y(t)\,dB_2(t) \\ \qquad\qquad + \sigma_4 X(t)Y(t)\,dB_4(t), \\ dZ(t) = \{\alpha Y(t)\{Y(t) + Z(t)\} + \delta Y(t) - \mu Z(t)\}\,dt + \sigma_3 Z(t)\,dB_3(t), \end{cases} \tag{2.4}$$

where $B_i(t)$ $(i = 1, 2, 3, 4)$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the basic conditions (that is, it is increasing and right continuous, while $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets), is a mutually independent standard Brownian motion with $B_i(0) = 0$ $(i = 1, 2, 3, 4)$ [18, page 15]. The parameter σ_i $(i = 1, 2, 3, 4)$ represents the intensity of the white noise.

For SDE with Gaussian white noise, the Itô [6] and Stratonovich integrations [26] are the two most studied interpretations. Stratonovich integration is preferable in physical kinetics [32] owing to its interpretation as a Wong–Zakai small correlation time limit of solutions of differential equations. The Itô interpretation is preferable in population biology [30], where the SDE is obtained as a continuous time limit of a discrete time problem. Furthermore, the Itô formulation has martingale close connections with diffusion processes and the advantage of preserving the property of Brownian motion; this provides many theoretical advantages, including the existence and uniqueness of solutions and stability in probability [18]. Considering that the spreading of rumours is in many ways similar to the spreading of epidemic infections [3], the Itô formula is used to investigate the spread of rumour in this work.

Here, we present several auxiliary statements, which were introduced by Mao [18, page 109]. Consider the d-dimensional SDE

$$dx(t) = f(x(t), t)\,dt + g(x(t), t)\,dB(t) \quad \text{on } t \geq t_0 \tag{2.5}$$

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with initial value $x(t_0) = x_0 \in R_+^d$. Note that $B(t)$ denotes the m-dimensional standard Brownian motion defined on the above probability space. We define the differential operator L associated with equation (2.5) for a function $V \in C^{2,1}(R_+^d \times [t_0, +\infty); R_+)$, by

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \tfrac{1}{2}\mathrm{trac}[g^T(x,t)V_{xx}(x,t)g(x,t)],$$

where $\mathrm{trac}(A)$ denotes the trace of a square matrix $A = (a_{ij})_{dxd}$, that is, $\mathrm{trace}A = \sum_{1\leqslant i\leqslant d} a_{ii}$. By the Itô formula [6],

$$dV(x,t) = LV(x(t),t)\,dt + V_x(x(t),t)g(x(t),t)\,dB(t).$$

In order to have a unique global (that is, no explosion in a finite time) solution for model (2.4) with any given initial data, the coefficients of the equation generally must satisfy the linear growth condition and local Lipschitz condition [18, page 69]. However, the coefficients of model (2.4) do not satisfy the linear growth condition, although they are locally Lipschitz continuous. Thus, the solution of model (2.4) may explode within a finite time. It is therefore useful to establish that the solution to model (2.4) not only is positive but also will not explode to infinity in any finite time [19, Theorem 2.1].

Theorem 2.1. *For any given initial conditions $(X(0), Y(0), Z(0)) \in R_+^3$, there is a unique solution $(X(t), Y(t), Z(t))$ for model (2.4) on $t \geq 0$, and the solution remains in R_+^3 with probability one, namely, $(X(t), Y(t), Z(t)) \in R_+^3$ for all $t \geq 0$ almost surely (a.s).*

Proof. Since the coefficients of model (2.4) are locally Lipschitz continuous, for given initial conditions $(X(0), Y(0), Z(0)) \in R_+^3$, there is a unique local solution $(X(t), Y(t), Z(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To justify that this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $n_0 > 0$ be sufficiently large such that $X(0)$, $Y(0)$ and $Z(0)$ lie within the interval $[1/n_0, n_0]$. For each integer $n \geq n_0$, define the stopping time

$$\tau_n = \inf\{t \in [0, \tau_e) \mid \min\{X(t), Y(t), Z(t)\} \leq 1/n \ \text{ or } \ \max\{X(t), Y(t), Z(t)\} \geq n\},$$

where, unless otherwise noted, $\inf \emptyset = \infty$ (as usual, $\emptyset$ denotes the empty set). Clearly, τ_n increases as $n \to \infty$. Set $\tau_\infty = \lim_{n\to\infty} \tau_n$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $(X(t), Y(t), Z(t)) \in R_+^3$ a.s. for all $t \geq 0$. If the statement is violated, we assume that there exists a constant $T > 0$ for any $\epsilon \in (0,1)$ such that $P\{\tau_\infty \leq T\} > \varepsilon$. Consequently, there is a positive integer $n_1 \geq n_0$ such that

$$P\{\tau_n \leq T\} \geq \epsilon \quad \text{for all } n \geq n_1. \tag{2.6}$$

We define a nonnegative function $V_1 : R_+^3 \to R_+$ as

$$V_1(X,Y,Z) = \left(X - A - A\log\frac{X}{A}\right) + \left(Y - C - C\log\frac{Y}{C}\right) + (Z - 1 - \log Z),$$

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where A and C are positive constants, which will be defined later. Note that for any fixed constant y, we have $f(x) = x - y - y\log(x/y) \geq 0$ for all $x > 0$. Hence, the nonnegativity of the function V_1 is ensured. By the Itô formula,

$$\begin{aligned} dV_1 &= \Big(1 - \frac{A}{X}\Big)dX + \frac{A}{2X^2}(dX)^2 + \Big(1 - \frac{C}{Y}\Big)dY + \frac{C}{2Y^2}(dY)^2 \\ &\quad + \Big(1 - \frac{1}{Z}\Big)dZ + \frac{1}{2Z^2}(dZ)^2 \\ &= LV_1\,dt + (X - A)(\sigma_1\,dB_1(t) - \sigma_4 Y\,dB_4(t)) \\ &\quad + (Y - C)(\sigma_2\,dB_2(t)\sigma_4 X\,dB_4(t)) + (Z - 1)\sigma_3\,dB_3(t), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} LV_1 &= \Big(1 - \frac{A}{X}\Big)(\Lambda - \beta XY - \mu X) + \Big(1 - \frac{C}{Y}\Big)(\beta XY - \alpha Y(Y + Z) - (\delta + \mu)Y) \\ &\quad + \Big(1 - \frac{1}{Z}\Big)(\alpha Y(Y + Z) + \delta Y - \mu Z) + \frac{A}{2}(\sigma_1^2 + \sigma_4^2 Y^2) + \frac{C}{2}(\sigma_2^2 + \sigma_4^2 X^2) + \frac{\sigma_3^2}{2} \\ &= \Lambda + A\mu + C(\delta + \mu) + \mu + \frac{A}{2}(\sigma_1^2 + \sigma_4^2 Y^2) + \frac{C}{2}(\sigma_2^2 + \sigma_4^2 X^2) + \frac{\sigma_3^2}{2} \\ &\quad + (-\mu - C\beta)X + (-\mu + A\beta - \alpha + C\alpha)Y + (-\mu + C\alpha)Z - \frac{\Lambda A}{X} - \frac{\alpha Y^2}{Z} - \frac{\delta Y}{Z}. \end{aligned}$$

Let $A = \alpha/\beta$ and $C = \mu/\alpha$. Direct computation yields

$$LV_1(X, Y, Z) \leq \Lambda + \frac{\alpha\mu}{\beta} + \frac{\mu(\mu + \delta)}{\alpha} + \mu + \frac{\alpha}{2\beta}(\sigma_1^2 + \sigma_4^2 Y^2) + \frac{\mu}{2\alpha}(\sigma_2^2 + \sigma_4^2 X^2) + \frac{\sigma_3^2}{2} := K.$$

Therefore, it follows from (2.7) that

$$\begin{aligned} dV_1 &\leq K\,dt + \Big(X - \frac{\alpha}{\beta}\Big)(\sigma_1\,dB_1(t) - \sigma_4 Y\,dB_4(t)) + \Big(Y - \frac{\mu}{\alpha}\Big)(\sigma_2\,dB_2(t) \\ &\quad + \sigma_4 X\,dB_4(t)) + (Z - 1)\sigma_3\,dB_3(t). \end{aligned} \tag{2.8}$$

Integrating both sides of (2.8) from 0 to $\tau_n \wedge T$ yields

$$\begin{aligned} &\int_0^{\tau_n \wedge T} dV_1(X(u), Y(u), Z(u)) \\ &\quad \leq \int_0^{\tau_n \wedge T} \Big(X(u) - \frac{\alpha}{\beta}\Big)\{\sigma_1\,dB_1(u) - \sigma_4 Y(u)\,dB_4(u)\} \\ &\qquad + \Big(Y(u) - \frac{\mu}{\alpha}\Big)\{\sigma_2\,dB_2(u) + \sigma_4 X(u)\,dB_4(u)\} \\ &\qquad + (Z(u) - 1)\sigma_3\,dB_3(u) + \int_0^{\tau_n \wedge T} K\,du, \end{aligned} \tag{2.9}$$

where $\tau_n \wedge T = \min\{\tau_n, T\}$. Taking the expectation of both sides of (2.9) yields

$$E[V_1(X(\tau_n \wedge T), Y(\tau_n \wedge T), Z(\tau_n \wedge T))] \leq V_1(X(0), Y(0), Z(0)) + KT.$$

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Let $\Omega_n = \{\tau_n \le T\}$ for all $n \ge n_1$. Then, $P(\Omega_n) \ge \epsilon$ according to (2.6). Note that there exists at least $X(\tau_n \wedge T), Y(\tau_n \wedge T)$ and $Z(\tau_n \wedge T)$, which are equal to either n or $1/n$ for every $\omega \in \Omega_n$. Consequently,

$$\begin{aligned} V_1(X(0), Y(0), Z(0)) + KT &\ge E[\chi_{\Omega_n}(\omega) V_1(X(\tau_n \wedge T), Y(\tau_n \wedge T), Z(\tau_n \wedge T))] \\ &\ge \epsilon\Big\{(n - 1 - \log n) \wedge \Big(\frac{1}{n} - 1 - \log\frac{1}{n}\Big)\Big\}, \end{aligned}$$

where χ_{Ω_n} is the indicator function of Ω_n. Letting $n \to \infty$ yields

$$\infty > V_1(X(0), Y(0), Z(0)) + KT = \infty,$$

which is a contradiction. Thus, $\tau_\infty = \infty$. This implies that $X(t)$, $Y(t)$ and $Z(t)$ will not explode in a finite time a.s., which completes the proof. □

Theorem 2.1 signifies not only that model (2.4) has a unique global solution, but also that the solution remains within R^3_+ with a probability of one, whenever it starts from there, although we still do not know whether the population will grow to infinity in the long term. The following theorem shows that this situation can be prevented. We use the definition of a stochastically ultimately bounded solution to extend our result further [19, Definition 5.1].

DEFINITION 2.2. The solution U(t)=(X(t), Y(t), Z(t)) of model (2.4) is stochastically ultimately bounded if for any $\epsilon \in (0, 1)$ there is a positive constant $\eta = \eta(\epsilon)$ such that for any initial value $U(0) \in R^3_+$, the solution $U(t)$ of model (2.4) has the property

$$\limsup_{t\to\infty} P(\|U(t)\| > \eta) \le \epsilon.$$

Now, we obtain the following theorem.

THEOREM 2.3. *For the given initial value $U(0) = (X(0), Y(0), Z(0)) \in R^3_+$, the solution to model (2.4) is stochastically ultimately bounded.*

PROOF. It follows from Theorem 2.1 that the solution of model (2.4) stays in R^3_+ with probability one for all $t \ge 0$. Similar to Theorem 2.1, the stopping time is defined as

$$\tau_n = \inf\{t \in [0, \infty) \mid \min\{X(t), Y(t), Z(t)\} \le 1/n \text{ or } \max\{X(t), Y(t), Z(t)\} \ge n\},$$

because $\tau_e = \infty$. Then, $\tau_n \to \infty$ a.s. as $n \to \infty$. Denote the total population by

$$N(t) = X(t) + Y(t) + Z(t).$$

Applying the Itô formula to $e^{\mu t}N$ yields

$$de^{\mu t}N = \Lambda e^{\mu t}\,dt + e^{\mu t}\{\sigma_1 X\,dB_1(t) + \sigma_2 Y\,dB_2(t) + \sigma_3 Z\,dB_3(t)\}. \tag{2.10}$$

Integrating both sides of (2.10) from 0 to $t \wedge \tau_n$ and then taking the expectation of both sides yields

$$E[e^{\mu(t\wedge\tau_n)}N(t \wedge \tau_n)] = N(0) + \int_0^{t\wedge\tau_n} \Lambda e^{\mu t}\,dt \le N(0) + \frac{\Lambda}{\mu}(e^{\mu t} - 1).$$

Let $n \to \infty$, which means that

$$E[N(t)] \leq N(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}).$$

Consequently,

$$\limsup_{t\to\infty} E[N(t)] \leq \frac{\Lambda}{\mu}.$$

Here, we use the fact that $\| U \| = \sqrt{X^2 + Y^2 + Z^2} \leq N$; thus,

$$\limsup_{t\to\infty} E[\| U \|] \leq \frac{\Lambda}{\mu}.$$

According to Markov's inequality, we choose $\eta = \Lambda/\mu\varepsilon$ for any given $\epsilon > 0$; then,

$$\limsup_{t\to\infty} P(\| U \| > \eta) \leq \limsup_{t\to\infty} \frac{E[\| U \|]}{\eta} \leq \frac{\Lambda}{\mu\eta} = \epsilon,$$

which is the required assertion. □

Both Theorems 2.1 and 2.3 show that model (2.4) has a positive solution that will not explode to infinity in a finite time and, in fact, will be stochastically ultimately bounded. In other words, we show that the noise will not spoil these good properties.

3. Asymptotic behaviour around the rumour-free equilibrium

As shown in model (2.1), the rumour-free equilibrium $E_0 = (\Lambda/\mu, 0, 0)$ is globally asymptotically stable when $R_0 < 1$. From the perspective of rumour spreading, a rumour will vanish over time. However, E_0 is no longer an equilibrium of the stochastic rumour model (2.4), namely, the unique solution is not ultimately convergent to E_0 owing to the existence of random effects. Therefore, it is quite interesting to consider how white noise affects the stochastic dynamic behaviour of the model (2.4); this matter is explored in this section.

THEOREM 3.1. *We assume that*

$$R_0 = \frac{\Lambda\beta}{\mu(\mu + \delta)} < 1, \quad \sigma_1^2 < \mu, \quad \sigma_2^2 < 2\mu, \quad \sigma_3^2 < 2(\mu - \mu^2).$$

Then, the solution $(X(t), Y(t), Z(t))$ *to model* (2.4) *for any given initial value* $(X(0), Y(0), Z(0)) \in R_+^3$ *has the property*

$$\limsup_{t\to\infty} \frac{1}{t} E\Big[\int_0^t \Big(X(u) - \frac{\Lambda}{\mu} \Big)^2 + Y^2(u) + Z^2(u)\, du \Big] \leq \frac{\Lambda^2(1 + \sigma_1^2)}{m_1\mu^2},$$

where $m_1 = \min\{\mu - \sigma_1^2, \mu - \sigma_2^2/2, \quad \mu - \mu^2 - \sigma_3^2/2\}$.

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PROOF. Define the C^3 function

$$V_2(X,Y,Z) = \frac{1}{2}\Big(X - \frac{\Lambda}{\mu} + Y + Z\Big)^2 + aY,$$

where a is a positive constant to be determined later. According to the Itô formula, we can compute

$$\begin{aligned} dV_2(X,Y,Z) &= \Big(X - \frac{\Lambda}{\mu} + Y + Z\Big)(dX + dY + dZ) + \frac{1}{2}(dX + dY + dZ)^2 + a\,dY \\ &= LV_2\,dt + \Big(X - \frac{\Lambda}{\mu} + Y + Z\Big)\{\sigma_1 X\,dB_1(t) + \sigma_2 Y\,dB_2(t) + \sigma_3 Z\,dB_3(t)\} \\ &\quad + a\sigma_2 Y\,dB_2(t) + a\sigma_4 XY\,dB_4(t), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} LV_2 &= -\mu\Big(X - \frac{\Lambda}{\mu}\Big)^2 - \Big(\mu - \frac{\sigma_2^2}{2}\Big)Y^2 - \Big(\mu - \frac{\sigma_3^2}{2}\Big)Z^2 - (2\mu - a\beta)\Big(X - \frac{\Lambda}{\mu}\Big)Y \\ &\quad - 2\mu XZ - 2\mu YZ + 2\Lambda Z + \frac{1}{2}\sigma_1^2 X^2 + a\Big\{\frac{\beta\Lambda}{\mu} - (\delta + \mu)\Big\}Y - a\alpha Y(Y + Z) \\ &\le -\mu\Big(X - \frac{\Lambda}{\mu}\Big)^2 - \Big(\mu - \frac{\sigma_2^2}{2}\Big)Y^2 - \Big(\mu - \frac{\sigma_3^2}{2}\Big)Z^2 - (2\mu - a\beta)\Big(X - \frac{\Lambda}{\mu}\Big)Y \\ &\quad + 2\Lambda Z + \frac{1}{2}\sigma_1^2 X^2 + a\Big\{\frac{\beta\Lambda}{\mu} - (\delta + \mu)\Big\}Y. \end{aligned}$$

Actually, we can obtain some inequalities as follows:

(1) $\frac{1}{2}\sigma_1^2 X^2 \le \sigma_1^2 (X - \Lambda/\mu)^2 + \sigma_1^2(\Lambda/\mu)^2$, since $(x + y)^2 \le 2x^2 + 2y^2$ for any $x, y \in R_+$.
(2) $(\beta\Lambda)/\mu < \delta + \mu$, under $R_0 < 1$.
(3) $2\Lambda Z \le (\Lambda/\mu)^2 + \mu^2 Z^2$.

Using inequalities (1), (2) and (3), we obtain

$$\begin{aligned} LV_2 &\le -(\mu - \sigma_1^2)\Big(X - \frac{\Lambda}{\mu}\Big)^2 - \Big(\mu - \frac{\sigma_2^2}{2}\Big)Y^2 - \Big(\mu - \mu^2 - \frac{\sigma_3^2}{2}\Big)Z^2 \\ &\quad + \Big(\frac{\Lambda}{\mu}\Big)^2(1 + \sigma_1^2) - (2\mu - a\beta)\Big(X - \frac{\Lambda}{\mu}\Big)Y. \end{aligned}$$

Taking $a = 2\mu/\beta$ leads to

$$LV_2 \le -(\mu - \sigma_1^2)\Big(X - \frac{\Lambda}{\mu}\Big)^2 - \Big(\mu - \frac{\sigma_2^2}{2}\Big)Y^2 - \Big(\mu - \mu^2 - \frac{\sigma_3^2}{2}\Big)Z^2 + \Big(\frac{\Lambda}{\mu}\Big)^2(1 + \sigma_1^2).$$

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Integrating both sides of (3.1) from 0 to t and then taking the expectation yields

$$\limsup_{t\to\infty} \frac{1}{t} E\Big[\int_0^t (\mu - \sigma_1^2)\Big(X(u) - \frac{\Lambda}{\mu}\Big)^2 + \Big(\mu - \frac{\sigma_2^2}{2}\Big) Y^2(u) + \Big(\mu - \mu^2 - \frac{\sigma_3^2}{2}\Big) Z^2(u)\, du\Big] \leq \Big(\frac{\Lambda}{\mu}\Big)^2 (1 + \sigma_1^2).$$

Then,

$$\limsup_{t\to\infty} \frac{1}{t} E\Big[\int_0^t \Big\{\Big(X(u) - \frac{\Lambda}{\mu}\Big)^2 + Y(u)^2 + Z(u)^2\Big\}\, du\Big] \leq \frac{\Lambda^2 (1 + \sigma_1^2)}{m_1 \mu^2},$$

where $m_1 = \min\{\mu - \sigma_1^2, \mu - \sigma_2^2/2, \mu - \mu^2 - \sigma_3^2/2\}$. This completes the proof. □

Theorem 3.1 reveals that the solution to model (2.4) always oscillates around the rumour-free equilibrium E_0, and that the amplitude of oscillation is directly proportional to the intensity of the white noise σ_1. That is, a smaller σ_1 will be closer to E_0 and will have a much greater positive impact on the extinction of a rumour under other parametric restrictions. On the other hand, a threshold R_0 less than one necessarily contributes to weakening the rumour propagation, similar to model (2.1). Thus, we conclude that rumour propagation should be controlled by suitable protection measures in society to reduce the spreading rate β, at which ignorant individuals interact with spreader individuals. Every individual in society needs to develop the habit of sensible discussion such that they can determine the veracity or falsity of a rumour via common sense and evidence analysis. In addition, improving rational analysis abilities is also necessary to prevent rumour mongering. Rumour spreading can also be kept under control by increasing the forgetting rate δ. We recommend using other more interesting information to distract individuals and cause them to ignore a rumour. For instance, the collision of two high-speed trains running along the Yongtaiwen railway line on a bridge in the suburbs of Wenzhou, Zhejiang Province, on July 23, 2011, distracted public attention from the Meimei Guo event, where a Chinese woman named Meimei Guo showed off her luxurious life online and announced that she was the general manager of the Chinese Red Cross Society, causing a great stir on June 20, 2011. The public soon forgot the Meimei Guo event [36].

However, Theorem 3.1 has some limitations: when $R_0 < 1$, a conclusion regarding the stochastically asymptotic stability cannot be drawn. Next, we consider the simplified case of a negligible forgetting rate δ. This assumption is relaxed via a numerical study, and we show that the numerical results are consistent with the analytical findings of the case of the negligible forgetting rate summarised in Theorem 3.3. First, we present a lemma [18, page 112].

LEMMA 3.2. *If there exists a positive-definite, radially unbounded function* $V(x, t) \in C^{2,1}(R^d \times [t_0, +\infty); R_+)$ *such that* $LV(x, t)$ *is negative definite, then the solution of model* (2.4) *is stochastically asymptotically stable.*

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THEOREM 3.3. *If* $\Lambda\beta \le \mu^2$, $\sigma_1 = 0$ *and* $\max\{\sigma_2^2, \sigma_3^2\} < 2\mu$, *then the solution* $(\Lambda/\mu, 0, 0)$ *of model* (2.4) *is stochastically asymptotically stable for any given initial value* $(X(0), Y(0), Z(0)) \in R_+^3$.

PROOF. Similar to the proof of Theorem 3.1, define a C^3 function

$$V_3(X, Y, Z) = \frac{1}{2}\Big(X - \frac{\Lambda}{\mu} + Y + Z\Big)^2 + b(Y + Z),$$

where b is a positive constant to be determined later. Then,

$$\begin{aligned} dV_3(X, Y, Z) = LV_3\,dt &+ \Big(X - \frac{\Lambda}{\mu} + Y + Z\Big)\{\sigma_1 X\,dB_1(t) + \sigma_2 Y\,dB_2(t) + \sigma_3 Z\,dB_3(t)\} \\ &+ b\{\sigma_2 Y\,dB_2(t) + \sigma_4 XY\,dB_4(t) + \sigma_3 Z\,dB_3(t)\}, \end{aligned}$$

where

$$\begin{aligned} LV_3 &= -\mu\Big(X - \frac{\Lambda}{\mu}\Big)^2 - \Big(\mu - \frac{\sigma_2^2}{2}\Big)Y^2 - \Big(\mu - \frac{\sigma_3^2}{2}\Big)Z^2 - (2\mu - b\beta)\Big(X - \frac{\Lambda}{\mu}\Big)Y \\ &\quad - (ub - 2\Lambda)Z + b\Big(\frac{\beta\Lambda}{\mu} - \mu\Big)Y - 2\mu XZ - 2\mu YZ \\ &\le -\mu\Big(X - \frac{\Lambda}{\mu}\Big)^2 - \Big(\mu - \frac{\sigma_2^2}{2}\Big)Y^2 - \Big(\mu - \frac{\sigma_3^2}{2}\Big)Z^2 - (2\mu - b\beta)\Big(X - \frac{\Lambda}{\mu}\Big)Y \\ &\quad - (ub - 2\Lambda)Z + b\Big(\frac{\beta\Lambda}{\mu} - \mu\Big)Y. \end{aligned}$$

We choose $b = 2\mu/\beta$ such that $(2\mu - b\beta)(X - \Lambda/\mu)Y = 0$. In addition, $\beta\Lambda \le \mu^2$ implies $ub - 2\Lambda = 2(\mu^2 - \Lambda\beta)/\beta > 0$. Hence, we obtain

$$LV_3 \le -\mu\Big(X - \frac{\Lambda}{\mu}\Big)^2 - \Big(\mu - \frac{\sigma_2^2}{2}\Big)Y^2 - \Big(\mu - \frac{\sigma_3^2}{2}\Big)Z^2 \le 0.$$

Then, the solution $(\Lambda/\mu, 0, 0)$ to model (2.4) is stochastically asymptotically stable, based on Lemma 3.2. □

4. Asymptotic behaviour around the rumour endemic equilibrium

In the previous section, we showed that the endemic equilibrium $E^* = (X^*, Y^*, Z^*)$ of the deterministic rumour model (2.1) is globally attractive when the basic reproductive number R_0 is greater than one, which signifies that a rumour will persist in the population. Similar to the discussion on the asymptotic behaviour of the rumour-free equilibrium, E^* is not the endemic equilibrium for the stochastic rumour model (2.4) as a result of stochastic perturbation. However, we can still estimate the average number of oscillations around E^* over time to determine whether a rumour persists in the absence of white noise σ_1, σ_2 and σ_3. Therefore, the stochastic rumour model (2.4)

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can be reduced to the form of model (2.3), that is,

$$\begin{cases} dX(t) = \{\Lambda - \beta X(t)Y(t) - \mu X(t)\}\,dt - \sigma_4 X(t)Y(t)\,dB_4(t), \\ dY(t) = \{\beta X(t)Y(t) - \alpha Y(t)\{Y(t)+Z(t)\} - (\delta+\mu)Y(t)\}\,dt \\ \qquad\qquad + \sigma_4 X(t)Y(t)\,dB_4(t), \\ dZ(t) = \{\alpha Y(t)\{Y(t)+Z(t)\} + \delta Y(t) - \mu Z(t)\}\,dt. \end{cases} \tag{4.1}$$

From Theorem 2.1, it is tempting to conclude that there is a unique global solution $(X(t), Y(t), Z(t)) \in R^3_+$ a.s. to model (4.1) on $t \geq 0$ for any given initial condition $(X(0), Y(0), Z(0)) \in D$, where

$$D = \{(X, Y, Z) \in R^3_+ \mid 0 \leq X + Y + Z \leq \Lambda/\mu\}$$

defined in (2.2) is a positive invariant set of the deterministic rumour model (2.1). Furthermore, we ascertain that the set D is an a.s. positively invariant set of the stochastic rumour model (4.1), namely, if $(X(0), Y(0), Z(0)) \in D$, then $P((X(t), Y(t), Z(t)) \in D) = 1$ for all $t \geq 0$. We use this definition to establish the following result.

THEOREM 4.1. *For any given initial value $(X(0), Y(0), Z(0)) \in D$, there is a unique global solution $(X(t), Y(t), Z(t))$ of model (4.1) on $t \geq 0$, and the solution remains in D with a probability of one.*

PROOF. Let $(X(0), Y(0), Z(0)) \in D$. Summing up the three equations in model (4.1) and denoting $N(t) = X(t) + Y(t) + Z(t)$, we have $dN(t) = (\Lambda - \mu N)\,dt$. Then, by integration,

$$N(t) = \frac{\Lambda}{\mu} + \Big(N(0) - \frac{\Lambda}{\mu}\Big)e^{-\mu t} \leq \frac{\Lambda}{\mu}.$$

Hence, when $(X(0), Y(0), Z(0)) \in D$, we obtain $0 < N(t) = X(t) + Y(t) + Z(t) \leq \Lambda/\mu$, which completes the proof. □

The next step is devoted to determining what impact the noise interference of the spreading rate has on the asymptotic behaviour of model (4.1) around the rumour endemic equilibrium $E^* = (X^*, Y^*, Z^*)$.

THEOREM 4.2. *Let $(X(t), Y(t), Z(t))$ be the solution to model (4.1) for any given initial value $(X(0), Y(0), Z(0)) \in D$. If $R_0 > 1$, $u^2 > \alpha\Lambda$ and $\sigma_4^2 < \alpha/X^*$ are satisfied, we have*

$$\limsup_{t\to\infty} \frac{1}{t} E\Big[\int_0^t \{(X(u) - X^*)^2 + (Y(u) - Y^*)^2 + (Z(u) - Z^*)^2\}\,du\Big] \leq \frac{\rho}{m_2},$$

where (X^*, Y^*, Z^*) is the unique endemic equilibrium of the deterministic model (2.1), and

$$\rho = \sigma_4^2(X^*Y^{*2} + X^{*2}Y^*) + (\alpha Z^* + \delta)\Big\{\Big(\frac{\Lambda}{\mu}\Big)^3 + Y^{*2}Z^*\Big\},$$

$$m_2 = \min\Big\{\frac{\mu}{X^*} - Y^*\sigma_4^2, \quad \alpha - X^*\sigma_4^2, \quad \frac{\alpha}{\alpha Z^* + \delta}\Big(\mu - \frac{\alpha\Lambda}{\mu}\Big)\Big\}.$$

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PROOF. When $R_0 > 1$, there exists an endemic equilibrium $E^* = (X^*, Y^*, Z^*)$ of model (2.1). Then,

$$\begin{aligned}\Lambda - \beta X^* Y^* - \mu X^* &= 0,\\ \beta X^* Y^* - \alpha Y^*(Y^* + Z^*) - (\delta + \mu) Y^* &= 0,\\ \alpha Y^*(Y^* + Z^*) + \delta Y^* - \mu Z^* &= 0.\end{aligned} \tag{4.2}$$

Consider the following C^3 function:

$$\begin{aligned}V_4(X, Y, Z) &= \Big(X - X^* - X^* \log \frac{X}{X^*}\Big) + e\Big(Y - Y^* - Y^* \log \frac{Y}{Y^*}\Big) + \frac{f}{2}(Z - Z^*)^2\\ &= V_{4a} + eV_{4b} + fV_{4c},\end{aligned}$$

where e and f are positive constants to be defined later. Applying the Itô formula,

$$\begin{aligned}dV_4(X, Y, Z) &= LV_{4a}\,dt + eLV_{4b}\,dt + fLV_{4c}\,dt - (X - X^*)\sigma_4 Y\,dB_4(t)\\ &\quad + e(Y - Y^*)\sigma_4 X\,dB_4(t),\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}LV_{4a} &= (X - X^*)\Big(\frac{\Lambda}{X} - \beta Y - \mu\Big) + \frac{X^*\sigma_4^2}{2} Y^2,\\ LV_{4b} &= (Y - Y^*)\{\beta X - \alpha(Y + Z) - (\delta + \mu)\} + \frac{Y^*\sigma_4^2}{2} X^2,\\ LV_{4c} &= (Z - Z^*)\{\alpha Y(Y + Z) + \delta Y - \mu Z\}.\end{aligned}$$

Using (4.2), Theorem 4.1 and the inequality $(x + y)^2 \le 2x^2 + 2y^2$, we obtain

$$\begin{aligned}LV_{4a} &= (X - X^*)\Big[\frac{\Lambda}{X} - \frac{\Lambda}{X^*} - \beta(Y - Y^*)\Big] + \frac{X^*\sigma_4^2}{2} Y^2,\\ &= -\frac{\Lambda}{X \cdot X^*}(X - X^*)^2 - \beta(X - X^*)(Y - Y^*) + \frac{X^*\sigma_4^2}{2}(Y - Y^* + Y^*)^2,\\ &\le -\frac{\mu}{X^*}(X - X^*)^2 - \beta(X - X^*)(Y - Y^*) + X^*\sigma_4^2(Y - Y^*)^2 + X^*\sigma_4^2 Y^{*2},\\ LV_{4b} &= (Y - Y^*)[\beta(X - X^*) - \alpha(Y - Y^*) - \alpha(Z - Z^*)] + \frac{Y^*\sigma_4^2}{2}(X - X^* + X^*)^2\\ &\le \beta(X - X^*)(Y - Y^*) - \alpha(Y - Y^*)^2 - \alpha(Y - Y^*)(Z - Z^*)\\ &\quad + Y^*\sigma_4^2(X - X^*)^2 + Y^*\sigma_4^2 X^{*2},\\ LV_{4c} &= (Z - Z^*)[\alpha(Y^2 - Y^{*2}) + \alpha(YZ - Y^*Z^*) + \delta(Y - Y^*) - \mu(Z - Z^*)]\\ &= \alpha(Y^2 - Y^{*2})(Z - Z^*) + \alpha(Z - Z^*)(YZ - YZ^* + YZ^* - Y^*Z^*)\\ &\quad + \delta(Y - Y^*)(Z - Z^*) - \mu(Z - Z^*)^2\end{aligned}$$

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$$
\begin{aligned}
&\leq \alpha(Y^2Z + Y^{*2}Z^*) + \alpha Y(Z - Z^*)^2 + \alpha Z^*(Y - Y^*)(Z - Z^*) \\
&\quad + \delta(Y - Y^*)(Z - Z^*) - \mu(Z - Z^*)^2 \\
&\leq \alpha\Big[\Big(\frac{\Lambda}{\mu}\Big)^3 + Y^{*2}Z^*\Big] + \Big(\frac{\alpha\Lambda}{\mu} - \mu\Big)(Z - Z^*)^2 + (\alpha Z^* + \delta)(Y - Y^*)(Z - Z^*).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
LV_4 &= LV_{4a} + eLV_{4b} + fLV_{4c} \\
&\leq -\Big(\frac{\mu}{X^*} - eY^*\sigma_4^2\Big)(X - X^*)^2 - (e\alpha - X^*\sigma_4^2)(Y - Y^*)^2 - f\Big(\mu - \frac{\alpha\Lambda}{\mu}\Big)(Z - Z^*)^2 \\
&\quad + (-\beta + e\beta)(X - X^*)(Y - Y^*) + \{-e\alpha + f(\alpha Z^* + \delta)\}(Y - Y^*)(Z - Z^*) \\
&\quad + \sigma_4^2(X^*Y^{*2} + eX^{*2}Y^*) + f\alpha\Big[\Big(\frac{\Lambda}{\mu}\Big)^3 + Y^{*2}Z^*\Big].
\end{aligned}
$$

Taking $e = 1$ and $f = \alpha/(\alpha Z^* + \delta)$ yields

$$
\begin{aligned}
LV_4 &\leq -\Big(\frac{\mu}{X^*} - Y^*\sigma_4^2\Big)(X - X^*)^2 - (\alpha - X^*\sigma_4^2)(Y - Y^*)^2 \\
&\quad - \frac{\alpha}{\alpha Z^* + \delta}\Big(\mu - \frac{\alpha\Lambda}{\mu}\Big)(Z - Z^*)^2 + \sigma_4^2(X^*Y^{*2} + X^{*2}Y^*) \\
&\quad + \frac{\alpha^2}{\alpha Z^* + \delta}\Big[\Big(\frac{\Lambda}{\mu}\Big)^3 + Y^{*2}Z^*\Big] \\
&= -k_1(X - X^*)^2 - k_2(Y - Y^*)^2 - k_3(Z - Z^*)^2 + \rho,
\end{aligned}
$$

where

$$
\begin{aligned}
&k_1 = \frac{\mu}{X^*} - Y^*\sigma_4^2, \quad k_2 = \alpha - X^*\sigma_4^2, \quad k_3 = \frac{\alpha}{\alpha Z^* + \delta}\Big(\mu - \frac{\mu}{\alpha\Lambda}\Big), \\
&\rho = \sigma_4^2(X^*Y^{*2} + X^{*2}Y^*) + (\alpha Z^* + \delta)\Big\{\Big(\frac{\Lambda}{\mu}\Big)^3 + Y^{*2}Z^*\Big\}.
\end{aligned}
$$

To satisfy $k_1, k_2, k_3 > 0$, we must also assume that

$$
\mu^2 > \alpha\Lambda, \quad \text{and} \quad \sigma_4^2 < \frac{1}{X^*}\min\Big\{\frac{\mu}{Y^*}, \alpha\Big\} = \frac{\alpha}{X^*} \quad \text{owing to } R_0 > 1.
$$

Hence, from (4.3), we obtain

$$
\begin{aligned}
dV_4 &= LV_4\, dt - (X - X^*)\sigma_4 Y\, dB_4(t) + (Y - Y^*)\sigma_4 X\, dB_4(t) \\
&\leq [-k_1(X - X^*)^2 - k_2(Y - Y^*)^2 - k_3(Z - Z^*)^2 + \rho]\, dt \\
&\quad - (X - X^*)\sigma_4 Y\, dB_4(t) + (Y - Y^*)\sigma_4 X\, dB_4(t).
\end{aligned} \tag{4.4}
$$

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Integrating both sides of (4.4) from 0 to t and taking the expectation yields

$$E\Big[\int_0^t k_1(X(u)-X^*)^2 + k_2(Y(u)-Y^*)^2 + k_3(Z(u)-Z^*)^2\,du\Big] \le E[V(X(0),Y(0),Z(0))] + \rho t.$$

Therefore,

$$\limsup_{t\to\infty}\frac{1}{t}E\Big[\int_0^t k_1(X(u)-X^*)^2 + k_2(Y(u)-Y^*)^2 + k_3(Z(u)-Z^*)^2\,du\Big] \le \rho.$$

When $m_2 = \min\{k_1, k_2, k_3\}$, we can obtain

$$\limsup_{t\to\infty}\frac{1}{t}E\Big[\int_0^t (X(u)-X^*)^2 + (Y(u)-Y^*)^2 + (Z(u)-Z^*)^2\,du\Big] \le \frac{\rho}{m_2}.$$

This completes the proof of the theorem. □

Theorem 4.2 indicates that the unique stochastic solution to model (4.1) fluctuates around $E^* = (X^*, Y^*, Z^*)$ for a long time under some extra parametric conditions, implying that the rumour always persists and becomes endemic to a certain extent, that is, the stochastic perturbation of the spreading rate strongly drives rumour propagation. In addition, the following numerical experiments relax the assumption, namely, σ_1, σ_2 and σ_3 are nonnegative, to illustrate that the rumour still prevails in the population.

The above discussion shows that the basic reproductive number R_0 has a vital role in determining the extinction or persistence of a rumour regardless of whether white noise is introduced. Specifically, when $R_0 > 1$, a rumour does not vanish from the population, and there is a negative influence on attempts to stop the propagation of rumours. Therefore, the following theorem shows that if the noise is sufficiently large, the solution of the associated model (4.1) will be extinct with a probability of one, even though the solution of the original model (2.1) may be persistent. Before proving this theorem, we first present a known lemma [18, page 12].

LEMMA 4.3 (Strong law of large numbers). *Let $M = M_{t\ge 0}$ be a real-valued continuous local martingale vanishing at t=0. Then,*

$$\lim_{t\to\infty}\langle M, M\rangle_t = \infty \ a.s. \ \Rightarrow \ \lim_{t\to\infty}\frac{M_t}{\langle M, M\rangle_t} = 0 \ a.s.$$

and

$$\limsup_{t\to\infty}\frac{\langle M, M\rangle_t}{t} < \infty \, a.s. \ \Rightarrow \ \lim_{t\to\infty}\frac{M_t}{t} = 0 \ a.s.$$

THEOREM 4.4. *For any given initial value $(X(0), Y(0), Z(0)) \in R^3_+$, we can obtain:*

(1) $\limsup_{t\to\infty}(lnY(t)/t) < \beta^2/2\sigma_4^2 - (\delta+\mu)$ *a.s.*
(2) *if $\beta^2/2\sigma_4^2 < \delta+\mu$, then $Y(t)$ approaches zero exponentially with probability one.*

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PROOF. Define a function $V_6(Y(t)) = \ln Y(t)$. By the Itô formula, we have

$$dV_6 = \Big[\beta X - \alpha(Y+Z) - (\delta+\mu) - \frac{\sigma_4^2 X^2}{2}\Big]dt + \sigma_4 X\, dB_4(t).$$

Integrating both sides from 0 to t yields

$$\begin{aligned}
\ln Y(t) &= \ln Y(0) + \int_0^t \Big[\beta X(u) - \alpha\{Y(u)+Z(u)\} - (\delta+\mu) - \frac{\sigma_4^2 X^2(u)}{2}\Big]du \\
&\quad + \int_0^t \sigma_4 X(u)\, dB_4(u) \\
&\le \ln Y(0) + \int_0^t \Big(\beta X(u) - \frac{\sigma_4^2 X^2(u)}{2}\Big) - (\delta+\mu)\, du + \int_0^t \sigma_4 X(u)\, dB_4(u) \\
&\le \ln Y(0) + \Big[\frac{\beta^2}{2\sigma_4^2} - (\delta+\mu)\Big]t + \int_0^t \sigma_4 X(u)\, dB_4(u). \qquad (4.5)
\end{aligned}$$

Setting $M(t) = \int_0^t \sigma_4 X(u)\, dB_4(u)$ yields

$$\frac{\langle M, M\rangle_t}{t} = \frac{1}{t}\int_0^t \sigma_4^2 X^2(u)\, du \le \sigma_4^2\Big(\frac{\Lambda}{\mu}\Big)^2 < \infty\, a.s.$$

From Lemma 4.3, we obtain $\limsup_{t\to\infty} M_t/t = 0$ *a.s.* Dividing both sides of (4.5) by t and letting $t \to \infty$,

$$\limsup_{t\to\infty} \frac{\ln Y(t)}{t} \le \frac{\beta^2}{2\sigma_4^2} - (\delta+\mu)\, a.s.,$$

which completes the proof. □

Theorem 4.4 shows that the rumour will vanish regardless of the size of the basic reproductive number R_0, given that σ_4 is sufficiently large such that $\beta^2/2\sigma_4^2 < (\delta+\mu)$. We can outline strategies to strengthen random fluctuations between the ignorant class and spreader class. For instance, the event in which individuals suddenly went out to purchase salt, mentioned in the previous section, was subdued after authorities confirmed the scientific knowledge, stating that salt is almost useless in countering radiation and that eating too much salt is harmful to one's health. Meanwhile, technicians can eliminate the source of rumours on the web to prevent further rumour transmission.

5. Numerical simulation

To understand the above analytical results, we present some numerical simulations to illustrate the different dynamic consequences of both the deterministic rumour model (2.1) and the stochastic rumour model (2.4) under the same set of parameters. For model (2.4), the numerical simulations are obtained by following Milstein's higher-order method [8]; hence, the discretisation equations can be rewritten as:

$$\begin{cases} X_{k+1} = X_k + \Delta t(\Lambda - \beta X_k Y_k - \mu X_k) + X_k(\sigma_1 \xi_{1k} \sqrt{\Delta t} + \frac{1}{2}\sigma_1^2(\xi_{1k}^2 - 1)\Delta t) \\ \qquad - X_k Y_k(\sigma_4 \xi_{4k} \sqrt{\Delta t} + \frac{1}{2}\sigma_4^2(\xi_{4k}^2 - 1)\Delta t), \\ Y_{k+1} = Y_k + \Delta t(\beta X_k Y_k - \alpha Y_k(Y_k + Z_k) - (\delta + \mu)Y(t)) + Y_k(\sigma_2 \xi_{2k} \sqrt{\Delta t} \\ \qquad + \frac{1}{2}\sigma_2^2(\xi_{2k}^2 - 1)\Delta t) + X_k Y_k(\sigma_4 \xi_{4k} \sqrt{\Delta t} + \frac{1}{2}\sigma_4^2(\xi_{4k}^2 - 1)\Delta t), \\ Z_{k+1} = Z_k + \Delta t(\alpha Y_k(Y_k + Z_k) + \delta Y_k - \mu Z_k) + Z_k(\sigma_3 \xi_{3k} \sqrt{\Delta t} + \frac{1}{2}\sigma_3^2(\xi_{3k}^2 - 1)\Delta t), \end{cases}$$

where Δt is the time increment and $\xi_{1k}, \xi_{2k}, \xi_{3k}$ and ξ_{4k} $(k = 1, 2, \ldots, n)$ are independent Gaussian random variables, with $N(0, 1)$ [24]. The initial value and the parameters in this section are as follows:

$$X(0) = 0.6, \quad Y(0) = 0.1, \quad Z(0) = 0.4, \quad \Lambda = 0.9, \quad \mu = 0.7, \quad \delta = 0.4, \quad \alpha = 0.5.$$

Using MATLAB R2014a, the main goal was to show, via some numerical examples, how the intensities of the white noise influence rumour transmission, that is, the asymptotic behaviour of the stochastic solution, which has been analysed in previous theoretical works. Moreover, the size of the basic reproductive number R_0 can be controlled by adjusting the size of the spreading rate β.

First, Theorem 3.1 is simulated by setting the spreading rate $\beta = 0.4$ such that $R_0 < 1$ and by choosing the white noise $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.2$, as shown in Figure 2. These values of the parameters satisfy all conditions of Theorem 3.1. We see that the solution to the stochastic rumour model (2.4) oscillates around the rumour-free equilibrium of the deterministic rumour model (2.1) as t goes to infinity. In addition, the parameters chosen in Figure 3 are the same as those in Figure 2, except for $\sigma_1 = 0$. We can conclude from Figures 2 and 3 that decreasing the intensities of the white noise weakens the fluctuation around the rumour-free equilibrium. From the rumour spreading point of view, this implies that a rumour will vanish when the noise interference impact and the spreading rate are simultaneously reduced, which is a useful strategy for controlling the spread of rumours.

Second, we choose $\beta = 0.9$ and $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.1$, which satisfy the complex conditions of Theorem 4.2. Figure 4 reveals that the numbers of ignorants, spreaders and stiflers are always distributed around the endemic equilibrium of model (2.1), which is in agreement with the result of Theorem 4.2. This result means that under the conditions of Theorem 4.2, a rumour always prevails in the population, which is detrimental to stopping rumour transmission. Hence, we must find ways to mitigate this situation.

Finally, $\beta = 0.9$, $\sigma_1 = 0.1, \sigma_2 = 0.2, \sigma_3 = 0.1$ and $\sigma_4 = 0.6$ are chosen in Figure 5 to validate Theorem 4.4. From Figure 5, we see that the population of spreaders will become extinct despite a threshold value $R_0 > 1$, in contrast to the corresponding model (2.1) without noise. As a result, when the spreading rate becomes large, we can still fully utilise external interference to prevent the spread of a rumour. These effective methods are widely used by us.

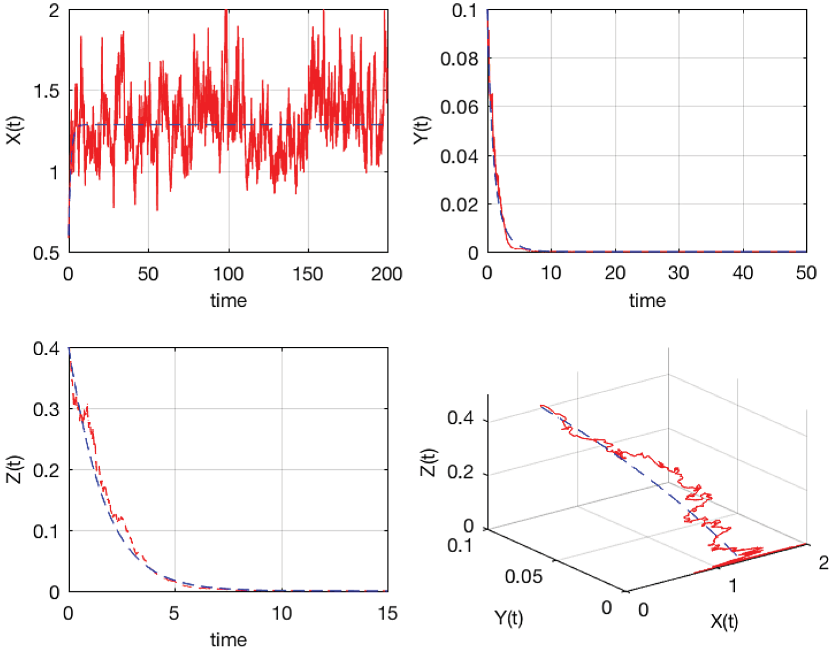

FIGURE 2. The solutions of model (2.1) without noise and model (2.4) with noise are represented by blue dashed lines and red solid lines, respectively. The initial condition and parameter values are $X(0) = 0.6$, $Y(0) = 0.1$, $Z(0) = 0.4$, $\Lambda = 0.9$, $\mu = 0.7$, $\delta = 0.4$, $\alpha = 0.5$, $\beta = 0.4$ and $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.2$. (Colour is available online).

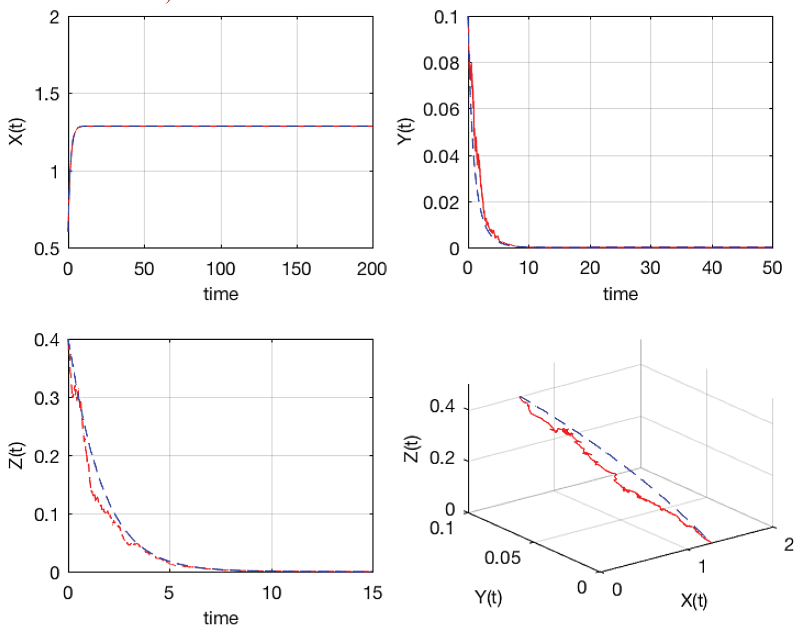

FIGURE 3. The solutions of model (2.1) without noise and model (2.4) with noise are represented by blue dashed lines and red solid lines, respectively. The initial condition and parameter values are $X(0) = 0.6$, $Y(0) = 0.1$, $Z(0) = 0.4$, $\Lambda = 0.9$, $\mu = 0.7$, $\delta = 0.4$, $\alpha = 0.5$, $\beta = 0.4$, $\sigma_1 = 0$ and $\sigma_2 = \sigma_3 = \sigma_4 = 0.2$. (Colour is available online).

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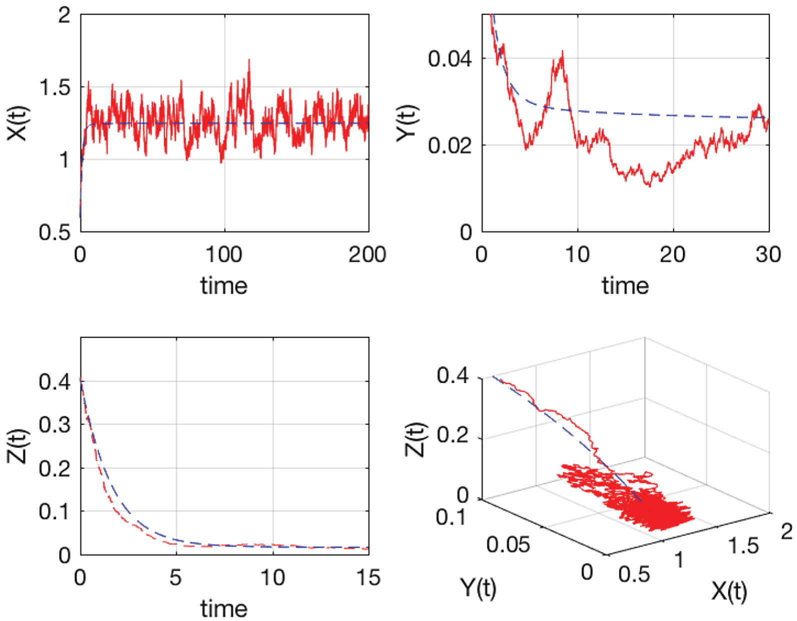

FIGURE 4. The solutions of model (2.1) without noise and model (2.4) with noise are represented by blue dashed lines and red solid lines, respectively. The initial condition and parameter values are $X(0) = 0.6$, $Y(0) = 0.1, Z(0) = 0.4, \Lambda = 0.9, \mu = 0.7, \delta = 0.4, \alpha = 0.5, \beta = 0.9$ and $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.1$. (Colour is available online).

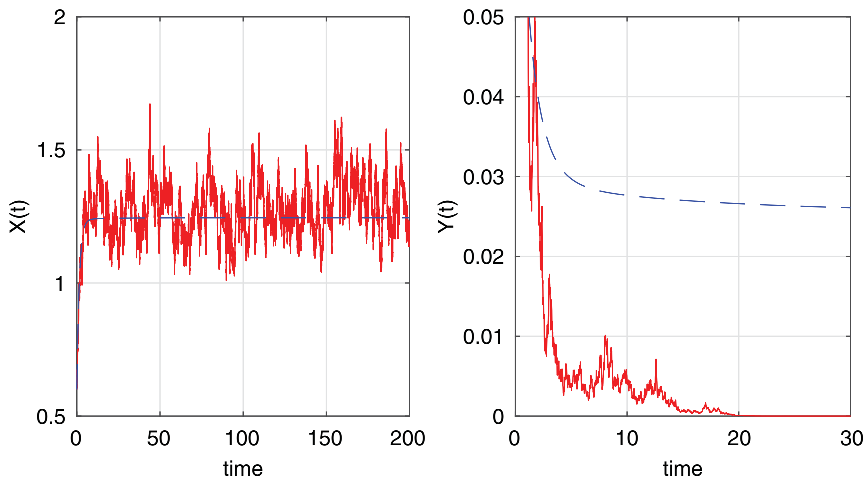

FIGURE 5. The solutions of model (2.1) without noise and model (2.4) with noise are represented by blue dashed lines and red solid lines, respectively. The initial condition and parameter values are $X(0) = 0.6, Y(0) = 0.1, Z(0) = 0.4, \Lambda = 0.9, \mu = 0.7, \delta = 0.4, \alpha = 0.5, \beta = 0.9, \sigma_1 = 0.1, \sigma_2 = 0.2, \sigma_3 = 0.1$ and $\sigma_4 = 0.6$. (Colour is available online).

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6. Conclusions

In this paper, by incorporating random effects, we extend the classic Maki–Thompson model to a stochastic rumour model with a forgetting mechanism. The random effects are divided into two types: the case in which the spreading rate between ignorants and spreaders is subject to white noise; and the case in which the stochastic perturbations are a type of white noise that is directly proportional to the numbers of ignorants, spreaders and stiflers. This paper focuses on the asymptotic behaviour of a stochastic rumour model by formulating the corresponding stochastic Lyapunov functions. First, a basic property is established, that is, a unique solution which is stochastically ultimately bounded exists for any given positive initial value. Then, when $R_0 < 1$ and the additional conditions regarding the intensity of the white noise are satisfied, the solution oscillates around the rumour-free equilibrium of the deterministic rumour model without noise. In addition, white noise with a low intensity results in smaller fluctuations of the solution. From the perspective of rumour spreading, this implies that a rumour will go extinct, similar to the case of the deterministic rumour model. Furthermore, when the intensity of the white noise is sufficiently small, the solution of the stochastic rumour model will approach the solution of the deterministic one. Once again, the asymptotic behaviour around the endemic equilibrium is ascertained under $R_0 > 1$ and some complex conditions, which suggests that spreaders cannot be completely eliminated from the population, that is, the rumour always persists. Finally, although the rumour will prevail in the deterministic situation without noise when $R_0 > 1$, we can still cause the rumour to go extinct by strengthening the white noise. Thus, this work contributes to the analysis of rumour propagation models and to measures for controlling rumour dissemination. In practice, we can take actions to intensify environmental interference factors such as government propaganda, official confirmation, and arousing mass consciousness to change individuals' behaviours and to reduce the effective interactions between ignorants and spreaders.

The proposed model provides a framework for further research. Here, we consider a simplified stochastic rumour model based on the Maki–Thompson model. As a result, there are a number of possible extensions to this work. The rumour dynamics can be subject to random environmental perturbations from other related processes such as the inflow rate, outflow rate, forgetting rate and stifling rate. In future, we will introduce these relative processes into the stochastic rumour model, and consider the incubator class that can determine the validity of a rumour [4]. In the current work, the asymptotic behaviour around the endemic equilibrium is not a general result because of the absence of white noise σ_1, σ_2 and σ_3. Therefore, we may further demonstrate a more general practice when σ_1, σ_2 and σ_3 are not equal to zero, and study whether the stationary distribution of model (2.4) can be obtained by the method of Zhou et al. [37]. Our results are illustrated through some numerical simulations. As a result, we can investigate the link between our findings and empirical data from several cases, which will be beneficial for the government to mitigate rumour propagation. In addition, the methodologies proposed by Yuan and Ao [34] and Tang et al. [28] can be applied to systems with arbitrary noise intensities through A-type stochastic integration. It could be worthwhile to further investigate the stochastic rumour spreading model using these methodologies in comparison with the current investigation in future research. We leave these investigations for our future work.

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