Proceedings of the Edinburgh Mathematical Society (2020) **63**, 807–829
doi:10.1017/S0013091520000176

ON THE BIERI–NEUMANN–STREBEL–RENZ INVARIANTS OF RESIDUALLY FREE GROUPS

DESSISLAVA H. KOCHLOUKOVA[1] AND FRANCISMAR FERREIRA LIMA[2]

[1] *State University of Campinas, São Paulo, Brazil* (desi@ime.unicamp.br)
[2] *Federal University of Technology, Paraná, Brazil* (francismarf@utfpr.edu.br)

(Received 7 May 2019; first published online 21 July 2020)

Abstract We calculate the Bieri–Neumann–Strebel–Renz invariant $\Sigma^1(G)$ for finitely presented residually free groups G and show that its complement in the character sphere $S(G)$ is a finite union of finite intersections of closed sub-spheres in $S(G)$. Furthermore, we find some restrictions on the higher-dimensional homological invariants $\Sigma^n(G,\mathbb{Z})$ and show for the discrete points $\Sigma^2(G)_{\text{dis}}$, $\Sigma^2(G,\mathbb{Z})_{\text{dis}}$ and $\Sigma^2(G,\mathbb{Q})_{\text{dis}}$ in $\Sigma^2(G)$, $\Sigma^2(G,\mathbb{Z})$ and $\Sigma^2(G,\mathbb{Q})$ that we have the equality $\Sigma^2(G)_{\text{dis}} = \Sigma^2(G,\mathbb{Z})_{\text{dis}} = \Sigma^2(G,\mathbb{Q})_{\text{dis}}$.

Keywords: homological finiteness properties of groups; residually free groups

2010 *Mathematics subject classifications:* Primary 20J05

1. Introduction

Let D be an integral domain. Bieri [9] defined a group G to be of (homological) type $FP_n(D)$ if there is a projective resolution

$$\mathcal{P} : P_n \to \cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to D \to 0$$

of the trivial DG-module D such that each projective DG-module P_i is finitely generated for $0 \leq i \leq n$. If $D = \mathbb{Z}$, G is said to be of type FP_n and, in this case, G is of type $FP_n(R)$ for any integral domain R. A homotopical version F_n of this property was earlier defined by Wall [41]. A group G is of homotopical type F_n if there is a classifying space Y, i.e. a CW-complex $Y = K(G,1)$, with finite-dimensional n-skeleton. Obviously every group of type F_n is of type FP_n but, as shown by Bestvina and Brady [8], the converse does not hold for $n \geq 2$.

In this paper we are interested in subdirect products of groups and their Σ-invariants. Finiteness conditions of fibre products and subdirect products of groups were studied by Baumslag et al. [7], Baumslag and Roseblade [5], Bridson et al. [17, 18] and Kuckuck [30]. The first example of a group that is finitely presented but not of type FP_3 was discovered by Stallings [40] and is a subdirect product of three copies of the free group of rank 2.

If H is a subgroup of a direct product $G_1 \times \cdots \times G_m$, we say that H is virtually surjective on n-tuples if for every $1 \leq j_1 < \cdots < j_n \leq m$ the canonical projection $p_{j_1,\ldots,j_n} : H \to G_{j_1} \times \cdots \times G_{j_n}$ has the property that $p_{j_1,\ldots,j_n}(H)$ has finite index in $G_{j_1} \times \cdots \times G_{j_n}$. If $n = 2$, we say that H is virtually surjective on pairs (VSP). Bridson et al. [18] proved the following result.

VSP criterion. *Let H be a subgroup of a direct product $G_1 \times \cdots \times G_m$ of finitely presented groups $G_1, \ldots, G_m$. If H is VSP then H is finitely presented.*

Kuckuck [30] suggested the following generalization of the VSP criterion.

Virtual surjection conjecture. *Let $n \leq m$ be positive integers and let H be a subgroup of a direct product $G_1 \times \cdots \times G_m$, where G_i is of type F_n for $1 \leq i \leq m$. If H is virtually surjective on n-tuples then H is of type F_n.*

The virtual surjection conjecture is still an open problem, but some cases of the conjecture and its homological version were proved by Kochloukova and Lima [28] and Kuckuck [30]. The homological version of the virtual surjection conjecture is obtained from the original virtual surjection conjecture, substituting everywhere homotopical type F_n with homological type FP_n. Kochloukova and Lima [28] proved using spectral sequence techniques that the homological version of the virtual surjection conjecture holds for $n = 2$.

In 1980, Bieri and Strebel classified all finitely presented metabelian groups G in terms of their one-dimensional Σ-invariant [13]. Later on, this invariant was generalized by Bieri, Neumann, Strebel and Renz to higher-dimensional homological and homotopical invariants [12, 14]. We will discuss the precise definition of the Σ-invariants in § 3. In all cases these invariants are open subsets of the character sphere $S(G)$, where by a character of G we mean a non-zero homomorphism $G \to \mathbb{R}$ and $S(G)$ is the set of equivalence classes of characters of G with two characters falling in the same equivalence class precisely when one is a positive real multiple of the other. The importance of the Σ-invariants lies in the fact that they control the finiteness properties F_n and FP_n of the subgroups of G above the commutator subgroup [12, 35].

In general it is not easy to calculate the Σ-invariants, but in some cases their description or some information about them is known: these cases include right-angled Artin groups (RAAGs) [31], some Artin groups that are not RAAGs [2, 3], the Thompson group F [15], generalized Thompson groups $F_{n,\infty}$ [27, 43], free-by-cyclic groups [20], Poincare duality groups of dimension 3 [25] and limit groups [26].

Bieri and Renz [12] defined higher-dimensional invariants $\Sigma^n(G, M)$ for $\mathbb{Z}(G)$-modules M. Again, $\Sigma^n(G, M)$ is an open subset of the character sphere $S(G)$. The homotopical invariant $\Sigma^n(G)$ is defined for groups G of homotopical type F_n. Note that $\Sigma^n(G, \mathbb{Z})$ is defined for any group G, but if $\Sigma^n(G, \mathbb{Z}) \neq \emptyset$ then G is of homological type FP_n. In general $\Sigma^1(G) = \Sigma^1(G, \mathbb{Z})$ and $\Sigma^n(G) = \Sigma^2(G) \cap \Sigma^n(G, \mathbb{Z})$ for $n \geq 2$; the latter is a monoidal version of the fact that a group is of type F_n if and only if it is finitely presented (i.e. of type F_2) and of type FP_n. The description of the Σ-invariants of RAAGs by Meier, Meinert and Van Wyk implies that the inclusion $\Sigma^n(G) \subseteq \Sigma^n(G, \mathbb{Z})$ can be strict for $n \geq 2$ [31].

The main results in this paper are on the Σ-invariants of finitely presented residually free groups. This requires the study of subdirect products of non-abelian limit groups.

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

In general a subgroup $H \le G_1 \times \cdots \times G_m$ is called a *subdirect product* if each projection $H \to G_i$ is surjective. If, furthermore, each $H \cap G_i \ne 1$, H is called a *full subdirect product*.

The limit groups were defined by Sela and were studied independently by Kharlampovich and Myasnikov, who considered them as finitely generated fully residually free groups. Sela defined a special class of limit groups, the ω-residually free towers, which are built inductively from free groups, free abelian groups and surface groups of Euler characteristics less than -1, and defined a limit group as a finitely generated subgroup of a ω-residually free tower. Limit groups were used in the solution of the Tarski problem on the elementary theory of non-abelian free groups of finite rank by Kharlampovich and Myasnikov [**24**] and Sela [**39**]. The importance of the subdirect products of limit groups can be seen in the result of Baumslag *et al.* in [**6**] that every finitely generated residually free group is a subdirect product of finitely many limit groups.

Limit groups are FP_∞, finitely presented and of finite cohomological dimension. Wilton [**42**] proved that every finitely generated subgroup of a limit group is a virtual retract. In the case of free groups, this is a well-known result of Hall [**23**]. Limit groups are commutative transitive as this holds for fully residually free groups [**4**]. Bridson and Howie [**16**] proved that a subgroup G of a limit group is itself a limit group if and only if $H_1(G, \mathbb{Q})$ is finite dimensional. Alibegovic [**1**] and Dahmani [**19**] independently proved that limit groups are relatively hyperbolic with respect to their maximal abelian subgroups.

Our first result is a necessary condition for the elements of $\Sigma^n(H, \mathbb{Q})$, where H is a full subdirect product of limit groups. For a character $\chi : H \to \mathbb{R}$ we set $H_\chi = \{h \in H | \chi(h) \ge 0\}$. $[\chi]$ is the equivalence class of χ, i.e. $[\chi]$ is the set of characters $\mathbb{R}_{>0}\chi$ and $S(H) = \{[\chi] | \chi \text{ is a character of } H\}$. By definition,

$$\Sigma^n(H, \mathbb{Q}) = \{[\chi] \in S(H) | \mathbb{Q} \text{ is of homological type } FP_n \text{ over } \mathbb{Q}H_\chi\},$$

where H_χ acts trivially on $\mathbb{Q}$.

Theorem A. *Let $H \le L_1 \times \cdots \times L_m$ be a finitely presented full subdirect product of non-abelian limit groups $L_1, \ldots, L_m$ with $m \ge 1$. Let n be a fixed integer such that $1 \le n \le m$. If $[\chi] \in \Sigma^n(H, \mathbb{Q})$, then*

$$p_{j_1,\ldots,j_n}(H_\chi) = p_{j_1,\ldots,j_n}(H) \quad \text{for all } 1 \le j_1 < \cdots < \cdots < j_n \le m,$$

where $p_{j_1,\ldots,j_n} : H \to L_{j_1} \times \cdots \times L_{j_n}$ is the canonical projection.

Note that whenever $\Sigma^n(H, \mathbb{Q}) \ne \emptyset$ the group H is of homological type $FP_n(\mathbb{Q})$, and by [**26**] this implies that $[L_{j_1} \times \cdots \times L_{j_n} : p_{j_1,\ldots,j_n}(H)] < \infty$. We will show in Lemma 5.10 that $p_{i_1,\ldots,i_n}(H) = p_{i_1,\ldots,i_n}(H_\chi)$ if and only if $\chi(\cap_{1\le j\le n} Ker(p_{i_j})) \ne 0$.

Theorem B. *Let $H \le L_1 \times \cdots \times L_m$ be a finitely presented full subdirect product of non-abelian limit groups $L_1, \ldots, L_m$ with $m \ge 1$. Then,*

$$\Sigma^1(H) = \Sigma^1(H, \mathbb{Q}) = \{[\chi] \in S(H) \mid p_i(H_\chi) = L_i \text{ for every } 1 \le i \le m\}.$$

Theorem B shows that the converse of Theorem A holds for $n = 1$. It is an interesting question whether the converse of Theorem A holds for $n \ge 2$, but this is still an open

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

problem. As we have explained before, the finite presentability of subdirect products of limit groups is completely understood [17, 18]; however, the higher homological and homotopical properties in dimensions at least 3 are not well understood except in particular cases (see [28] and [30]). The proof of the VSP criterion depends on the 1-2-3 theorem of fibre products [17]. Unfortunately, we do not see a natural monoidal version of the 1-2-3 theorem for fibre products of groups when one of the groups is a non-abelian limit group L (i.e. a version that calculates the homological invariants of the fibre product in dimensions 1 and 2), as in this case $\Sigma^1(L,\mathbb{Q}) = \Sigma^1(L,\mathbb{Z}) = \emptyset$. Still, we believe that the converse of Theorem A holds and suggest the following conjecture.

Monoidal virtual surjection conjecture. *Let n and m be positive integers such that $1 \leq n \leq m$. Let $H \leq L_1 \times \cdots \times L_m$ be a full subdirect product of non-abelian limit groups $L_1, \ldots, L_m$ such that H is of type FP_n and finitely presented. Then,*

$$[\chi] \in \Sigma^n(H,\mathbb{Q}) = \Sigma^n(H,\mathbb{Z}) = \Sigma^n(H)$$

if and only if

$$p_{j_1,\ldots,j_n}(H_\chi) = p_{j_1,\ldots,j_n}(H) \quad \text{for all } 1 \leq j_1 < \cdots < j_n \leq m, \tag{1.1}$$

where $p_{j_1,\ldots,j_n} : H \to L_{j_1} \times \cdots \times L_{j_n}$ is the canonical projection.

Remark. By [18, Thm. 5.1], if H is FP_2, then H is finitely presented. By Theorem 2.3, since H is FP_n we have that $p_{j_1,\ldots,j_n}(H)$ has finite index in $L_{j_1} \times \cdots \times L_{j_n}$ for all $1 \leq j_1 < \cdots < j_n \leq m$.

The name of the above conjecture is derived from the fact that we view it as a monoidal version of the virtual surjection conjecture applied to subdirect products of limit groups. It is not clear what a more general version for subdirect products of groups of homotopical type F_n should be. In our consideration of the special case of subdirect products of limit groups, crucial parts are played by the structure theory of finitely presented subdirect products of limit groups developed in [17] and the fact that for every non-abelian limit group L we have that $\Sigma^1(L,\mathbb{Q}) = \emptyset$ [26]. It is plausible that for more general subdirect products, where $L_1, \ldots, L_m$ are not supposed to be non-abelian limit groups but just groups of homotopical type F_n (respectively, homological type FP_n), condition (1.1) would imply that $[\chi] \in \Sigma^n(H)$ (respectively, $[\chi] \in \Sigma^n(H,\mathbb{Z})$), but the converse is unlikely to hold.

The case $n = m$ of the monoidal virtual surjection conjecture is easy to establish; see Lemma 4.1. In addition, Theorems A and B imply that the monoidal virtual surjection conjecture holds for $n = 1$. As a corollary, we obtain a result for finitely presented residually free groups; see Corollary C. For a subset M of H we set $S(H,M) = \{[\chi] \in S(H) \mid \chi(M) = 0\}$, called the sub-sphere in $S(H)$, and upper index c denotes the complement in the character sphere $S(H)$. A character $\chi : H \to \mathbb{R}$ is discrete if $Im(\chi) \simeq \mathbb{Z}$.

Corollary C. *Let H be a finitely presented residually free group; hence we can realise H as a full subdirect product $H \leq L_1 \times \cdots \times L_m$ of limit groups $L_1, \ldots, L_m$ for some $m \geq 1$, $L_1 \simeq \cdots \simeq L_s \simeq \mathbb{Z}$, and L_j is non-abelian for $j \geq s+1$, where $Z(H) = H \cap (L_1 \times$*

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

$\cdots \times L_s) \simeq \mathbb{Z}^s$ *for some* $s \geq 0$*. Then,*

$$[\chi] \in \Sigma^1(H) = \Sigma^1(H, \mathbb{Z}) = \Sigma^1(H, \mathbb{Q})$$

if and only if

$$\text{either } \chi(Z(H)) \neq 0 \text{ or } \chi(Z(H)) = 0 \quad \text{and} \quad p_j(H_\chi) = p_j(H) = L_j \quad \text{for } s+1 \leq j \leq m,$$

i.e.

$$\Sigma^1(H, \mathbb{Q})^c = \Sigma^1(H, \mathbb{Z})^c = \Sigma^1(H)^c = \cup_{s+1 \leq j \leq m}(S(H, Z(H)) \cap S(H, Ker(p_j)))$$

is a finite union of intersections of sub-spheres in the character sphere $S(H)$*. In particular,*

$$\Sigma^1(H)^c = -\Sigma^1(H)^c$$

and $\{[\chi] \in \Sigma^1(H)^c \mid \chi \text{ is discrete}\}$ *is a dense subset of* $\Sigma^1(H)^c$*.*

Bieri *et al.* [**14**] constructed a finitely generated subgroup G of piecewise linear automorphisms of a closed interval such that $\Sigma^1(G)^c$ has precisely two non-discrete points; in this case, $\{[\chi] \in \Sigma^1(G)^c \mid \chi \text{ is discrete}\} = \emptyset$ is not dense in $\Sigma^1(G)^c$.

Note that for a finitely generated metabelian group G, Bieri and Groves [**11**] showed that $\Sigma^1(G)^c = S(G) \setminus \Sigma^1(G)$ is a finite union of finite intersections of closed, *rationally* defined semi-spheres of $S(G)$, i.e. the defining vector of every such semi-sphere is a *rational* vector. Obviously, every sub-sphere $S(G, M)$, where M is a subgroup of an arbitrary finitely generated group G, is a finite intersection of closed *rational* semi-spheres of $S(G)$. This applied to $G = H$ and $M = Z(H)$ or $M = Ker(p_j)$, together with the description of $\Sigma^1(H)^c$ given in Corollary C, implies that $\Sigma^1(H)^c$ is a finite union of finite intersections of closed, rationally defined semi-spheres of $S(G)$. The rationality implies that $\{[\chi] \in \Sigma^1(H)^c \mid \chi \text{ is discrete}\}$ is dense in $\Sigma^1(H)^c$. The antipodality condition $\Sigma^1(H)^c = -\Sigma^1(H)^c$ that appeared in Corollary C is known to hold for other classes of groups as 3-manifold groups [**14**] or Artin groups; for Artin groups it is a simple consequence of the fact that there is an automorphism whose restriction on the abelianization is the antipodal map.

In the following result we show that the virtual surjection conjecture implies the discrete case of the monoidal virtual surjection conjecture.

Theorem D. *If the virtual surjection conjecture holds in dimension* n *and* χ *is a discrete character then the monoidal virtual surjection conjecture holds for* χ*. In particular, if* $n = 2$ *and* χ *is a discrete character then the monoidal virtual surjection conjecture holds for* χ*. Thus,*

$$\Sigma^2(H)_{\text{dis}} = \Sigma^2(H, \mathbb{Z})_{\text{dis}} = \Sigma^2(H, \mathbb{Q})_{\text{dis}},$$

where for $T \in \{\Sigma^2(H, \mathbb{Z}), \Sigma^2(H, \mathbb{Q}), \Sigma^2(H)\}$ *we write* $T_{\text{dis}} = \{[\chi] \in T | \chi \text{ is discrete}\}$*.*

Corollary E. *Let* H *be a finitely presented residually free group; hence we can realise* H *as a full subdirect product* $H \leq L_1 \times \cdots \times L_m$ *of limit groups* $L_1, \ldots, L_m$ *for some*

$m \geq 1$, $L_1 \simeq \cdots \simeq L_s \simeq \mathbb{Z}$, and L_j is non-abelian for $j \geq s+1$, where $Z(H) = H \cap (L_1 \times \cdots \times L_s) \simeq \mathbb{Z}^s$ for some $s \geq 0$. Then, for a discrete character $\chi : H \to \mathbb{Z}$, we have

$$[\chi] \in \Sigma^2(H, \mathbb{Z}) = \Sigma^2(H, \mathbb{Q}) = \Sigma^2(H)$$

if and only if either $\chi(Z(H)) \neq 0$ or

$$\chi(Z(H)) = 0 \quad \text{and} \quad p_{j_1,j_2}(H_\chi) = p_{j_1,j_2}(H) \quad \text{for all } s+1 \leq j_1 < j_2 \leq m.$$

Finally, we consider the Bieri–Stallings groups $G_m \subseteq F_2^m$, where F_2 is the free group of rank 2 and $G_m = Ker(\rho)$, where $\rho : F_2^m \to \mathbb{Z}$ is the epimorphism that sends a fixed free basis $\{x_i, y_i\}$ of the ith copy of F_2 to 1. By [9], G_m is FP_{m-1} but not FP_m and is finitely presented for $m \geq 3$. The case $m = 3$ was first considered by Stallings [40]; this is the first known example of a group that is finitely presented but not of type FP_3.

Theorem F. *Let $m \geq 3$ be a natural number. The monoidal virtual surjection conjecture holds for the Bieri–Stallings group $H = G_m$ and for a character χ if $n \leq m-2$ and χ is any character, or χ is a discrete character and $n = m-1$.*

The proof of Theorem F depends on various decompositions of G_m as the HNN extension given in Lemma 7.2. In these decompositions the base subgroup is isomorphic to F_2^{m-1} and the associated subgroups are isomorphic to G_{m-1}, allowing the use of inductive argument on m. Theorem F still holds for any character χ in the case $n = m-1$; this will be resolved in [29] with homological techniques different from the ones used in this paper.

2. Preliminaries on subdirect products of limit groups

Both free and orientable surface groups are limit groups. The class of limit groups coincides with the class of finitely generated fully residually free groups, i.e. finitely generated groups G such that for each finite subset $X \subseteq G$ there is a free group F (depending on X) and a group homomorphism $\varphi_X : G \to F$ such that the restriction of φ_X to X is injective.

The starting point in the study of finitely generated residually free groups is the following result.

Theorem 2.1 (see [6, Corollary 19]). *Every finitely generated residually free group H is a full subdirect product of a finite direct product of limit groups $L_1 \times \cdots \times L_m$.*

Recall that in the above theorem 'full' means that $H \cap L_i \neq 1$ and 'subdirect product' means that $p_i(H) = L_i$ for each $1 \leq i \leq m$, where $p_i : H \to L_i$ is the canonical projection map.

An important result due to Bridson et al. [18] states that, in the case of finitely generated residually free groups, the class of groups of type $FP_2(\mathbb{Q})$ is the same as the class of finitely presented groups.

Theorem 2.2 (see [18, Theorem 5.1]). *Let H be a finitely generated residually free group. Then, the following conditions are equivalent:*

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

(i) *H is finitely presented;*

(ii) *H is of type* $FP_2(\mathbb{Q})$*;*

(iii) *H can be embedded as a VSP full subdirect product in* $L_1 \times \cdots \times L_m$*, where* $H \cap L_1$ *has finite index in* L_1*,* L_1 *is an abelian limit group and each* L_i *is a non-abelian limit group for* $i \geq 2$*;*

(iv) $\dim_{\mathbb{Q}} H_2(H_0, \mathbb{Q}) < \infty$ *for each subgroup* H_0 *of H such that* $[H : H_0] < \infty$*.*

Remark. Throughout this paper, H will denote a residually free group of type $FP_2(\mathbb{Q})$. By Theorem 2.2, H is finitely presented. Using Theorem 2.1 we will often consider H as a finitely presented full subdirect product of a finite direct product of limit groups $L_1 \times \cdots \times L_m$ for some $m \geq 1$. In the case where H has trivial centre, each L_i is non-abelian. Furthermore, if H is not a limit group itself, $m \geq 2$.

Theorem 2.3 (see [26, Theorem 9]). *Let H be a finitely generated full subdirect product of a finite direct product of non-abelian limit groups* $L_1 \times \cdots \times L_m$*. If H is of type* $FP_n(\mathbb{Q})$ *for some* $n \in \{2, \ldots, m\}$*, then*

$$[L_{j_1} \times \cdots \times L_{j_n} : p_{j_1,\ldots,j_n}(H)] < \infty \quad \text{for } 1 \leq j_1 < \cdots < j_n \leq m,$$

where $p_{j_1,\ldots,j_n} : H \to L_{j_1} \times \cdots \times L_{j_n}$ *is the canonical projection map.*

3. Preliminaries on the Σ-invariants

Let G be a finitely generated group. The *character sphere* $S(G)$ is defined as

$$S(G) = \frac{Hom(G, \mathbb{R}) \setminus \{0\}}{\sim},$$

where $\chi_1 \sim \chi_2$ if there is a positive real number r such that $\chi_1 = r\chi_2$ with $\chi_1, \chi_2 \in Hom(G, \mathbb{R}) \setminus \{0\}$, and the equivalence class of χ_1 is denoted by $[\chi_1]$. Note that $S(G)$ is the unit sphere in $\mathbb{R}^n$, where n is the torsion-free rank of the abelianization of G. A non-trivial group homomorphism $\chi \in Hom(G, \mathbb{R}) \setminus \{0\}$ is called a *character* of G and is said to be *discrete* if $Im(\chi) \simeq \mathbb{Z}$.

Let $n \geq 0$, and let R be an associative ring with identity element and M a (right) R-module. Then M is of *(homological) type* FP_n if there is a projective resolution

$$\mathcal{P} : \cdots \to P_n \to \cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to M \to 0$$

such that each projective R-module P_i is finitely generated for $0 \leq i \leq n$.

Throughout this paper, D is an integral domain (most of the time it will be $\mathbb{Z}$ or $\mathbb{Q}$) and the considered DG-modules are right DG-modules. By definition,

$$\Sigma_D^n(G, M) = \{[\chi] \in S(G) | M \text{ is of type } FP_n \text{ as } DG_\chi\text{-module}\},$$

where M is a (right) DG-module. When $M = D$ is the trivial DG-module, $\Sigma_D^n(G, M) = \Sigma_D^n(G, D)$ is denoted by $\Sigma^n(G, D)$, and in this case we say that the invariant has coefficients in D.

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

Let X be a finite generating set of G, and let Γ be the Cayley graph of G with respect to this generating set, where G acts on the right. Then the vertex set $V(\Gamma) = G$ and the edge set $E(\Gamma) = X \times G$, where the edge (x, g) has beginning g and end xg. Let $\chi : G \to \mathbb{R}$ be a character and Γ_χ the subgraph of Γ spanned by $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$. Then, by definition,

$$\Sigma^1(G) = \{[\chi] \in S(G) \mid \Gamma_\chi \text{ is connected}\}.$$

The Bieri–Neumann–Renz invariant $\Sigma^1(G)$ coincides with $\Sigma^1(G, \mathbb{Z})$. This follows from [**12**, 1.3], where it is shown that $\Sigma^1(G, \mathbb{Z})$ coincides with the invariant defined in [**14**].

Theorem 3.1 (see [31, Theorem 9.3]). *Let K be a subgroup of G, M a DG-module and $\chi : G \to \mathbb{R}$ a character of G. If $[G : K] < \infty$ then*

$$[\chi|_K] \in \Sigma_D^n(K, M) \Leftrightarrow [\chi] \in \Sigma_D^n(G, M).$$

In particular, if $n = 0$, then

M is a finitely generated DG_χ-module $\Leftrightarrow$ M is a finitely generated $DK_{\chi|_K}$-module.

Bieri and Geoghegan [**10**] established a formula for the calculation of the homological invariants Σ^n of a direct product of groups with coefficients in an arbitrary field. A similar result for the Σ^n invariants with coefficients in the trivial module $\mathbb{Z}$ is wrong in both homological and homotopical settings, provided the dimension n is sufficiently big; see [**31**] and [**38**].

Theorem 3.2 (see [10, Theorem 1.3, and Proposition 5.2]). *Let be $n \geq 0$ be an integer, let G_1, G_2 be finitely generated groups and let F be a field. Then,*

$$\Sigma^n(G_1 \times G_2, F)^c = \bigcup_{p=0}^{n} \Sigma^p(G_1, F)^c * \Sigma^{n-p}(G_2, F)^c,$$

where $*$ denotes the join of subsets of $S(G_1 \times G_2)$ and ^{c} denotes the set-theoretic complement of subsets of a suitable character sphere.

The above result translates into the following statement. If $\chi : G_1 \times G_2 \to \mathbb{R}$ is a character with $\chi_i = \chi|_{G_i}$ for $i = 1, 2$ then $[\chi] \in \Sigma^n(G_1 \times G_2, F)^c = S(G_1 \times G_2, F) \setminus \Sigma^n(G_1 \times G_2, F)$ precisely when one of the following conditions holds: both χ_1 and χ_2 are non-trivial and $[\chi_1] \in \Sigma^p(G_1, F)^c = S(G_1) \setminus \Sigma^p(G_1, F)$, $[\chi_2] \in \Sigma^{n-p}(G_2, F)^c = S(G_2) \setminus \Sigma^{n-p}(G_2, F)$ for some $0 \leq p \leq n$; or one of the characters χ_1, χ_2 is trivial and for the non-trivial one, say χ_i, we have $[\chi_i] \in \Sigma^n(G_i, F)^c = S(G_i) \setminus \Sigma^n(G_i, F)$.

The following theorem follows from Gehrke's results [**21**]. In addition, it can be deduced from the description of the Σ-invariants for RAAGs [**31**] or for direct products of virtually free groups [**32**].

Theorem 3.3 (see [21], [32] and [31]). *Let n and s be natural numbers such that $0 \leq n \leq s - 1$. If $\chi : F_2^s = F_2 \times \cdots \times F_2 \to \mathbb{R}$ is a character whose restriction on $n + 1$ copies of F_2 is non-zero then $[\chi] \in \Sigma^n(F_2^s)$. In particular, if the restrictions of χ on all s copies of F_2 are non-zero then $[\chi] \in \Sigma^{s-1}(F_2^s)$.*

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

The following result follows from [33] and [37]. A slightly more general version can be found in [31, Thm. 3.2].

Theorem 3.4 (see [33], [31] and [37]). *Suppose that G is an HNN extension with base group B, stable letter t and associated subgroups A_1 and A_2. Suppose that $\chi : G \to \mathbb{R}$ is a character such that $\mu = \chi\mid_B \neq 0$, $\nu = \chi\mid_{A_1} \neq 0$, $[\mu] \in \Sigma^s(B)$ and $[\nu] \in \Sigma^{s-1}(A_1)$. Then $[\chi] \in \Sigma^s(G)$.*

As stated before, the Σ-invariants control which subgroups above the commutator have type F_n or FP_n.

Theorem 3.5 (see [35]). *Let G be a group of type F_n and N a subgroup of G that contains the commutator subgroup G'. Then N is of type F_n if and only if*

$$S(G, N) = \{[\chi] \in S(G) \mid \chi(N) = 0\} \subseteq \Sigma^n(G).$$

A homological version of the above theorem was proved by Bieri and Renz [12]. Thus, the above homotopical version follows from the case $n = 2$ [35], since for general $n \geq 2$ we have $\Sigma^n(G) = \Sigma^n(G, \mathbb{Z}) \cap \Sigma^2(G)$.

Lemma 3.6 (see [26]). *Let G be a non-abelian limit group. Then $\Sigma^1(G, \mathbb{Q}) = \emptyset$.*

4. The case $n = m$ of the monoidal virtual surjection conjecture

Lemma 4.1. *The monoidal virtual surjection conjecture holds for $n = m$, i.e. $\Sigma^m(H, \mathbb{Q}) = \emptyset$.*

Proof. Note that $p_{1,\dots,m}$ is the identity map and $H_\chi \neq H$ for any character χ of H. Thus, for $n = m$, the conjecture predicts that $\Sigma^m(H, \mathbb{Q}) = \Sigma^m(H, \mathbb{Z}) = \Sigma^m(H) = \emptyset$.

Since H has type FP_m, by Theorem 2.3 H has finite index in $D = L_1 \times \cdots \times L_m$. By Theorem 3.2 it is possible to calculate $\Sigma^m(-, \mathbb{Q})$ for a direct product of groups; hence $\Sigma^m(D, \mathbb{Q})$ is known. In this particular case, $\Sigma^m(D, \mathbb{Q}) = \emptyset$, since for each non-abelian limit group L_i we have that $\Sigma^1(L_i, \mathbb{Q}) = \emptyset$ by Lemma 3.6. Hence $\Sigma^m(D, \mathbb{Q}) = \emptyset$ by Theorem 3.2. Note that there are subgroups $\bar{L}_i$ of finite index in L_i such that $\bar{D} = \bar{L}_1 \times \cdots \times \bar{L}_m \subseteq H$, and the same argument as above shows that $\Sigma^m(\bar{D}, \mathbb{Q}) = \emptyset$. Since $\bar{D}$ is a subgroup of finite index in H, we deduce by Theorem 3.1 that $\Sigma^m(H, \mathbb{Q}) = \emptyset$. Then the proof is completed by the inclusions $\Sigma^m(H) \subseteq \Sigma^m(H, \mathbb{Z}) \subseteq \Sigma^m(H, \mathbb{Q}) = \emptyset$. □

5. Some technical results

In this section,

$$H \subseteq L_1 \times \cdots \times L_m \quad \text{for some } m \geq 2$$

is a *finitely presented* full subdirect product (i.e. $H \cap L_i \neq 1$ and $p_i(H) = L_i$ for every $1 \leq i \leq m$ with canonical projection $p_i : H \to L_i$), where each L_i is a non-abelian limit group. We will denote

$$N_{i,j} := p_j(Ker(p_i)) \lhd L_j \quad \text{for all } i, j \in \{1, \dots, m\} \text{ with } i \neq j.$$

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

Proposition 5.1. *For each $1 \le i \le m$, there exists a free normal subgroup $F_i \lhd L_i$ such that $F_i \subseteq H'$ and L_i/F_i is a polycyclic-by-finite group. In particular,*

$$N := F_1 \times \cdots \times F_m \subseteq H' \subseteq H.$$

Proof. Note that without loss of generality we can substitute each $L_1, \ldots, L_m$ with subgroups of finite index. Let us define

$$N_j := \bigcap_{i \neq j} N_{i,j}.$$

By [**17**, Lemma 6.1, Prop. 6.4], [**16**, Thm. 3.1] and Theorem 2.2, we have

$$\gamma_{m-1}(N_j) \subseteq [N_{1,j}, \ldots, N_{j-1,j}, N_{j+1,j}, \ldots, N_{m,j}] \subseteq H \quad \text{and} \quad [L_j : N_j] < \infty,$$

where $\{\gamma_t(N_j)\}_{t \ge 1}$ is the lower central series of N_j defined by $\gamma_1(N_j) = N_j$ and $\gamma_{i+1}(N_j) = [\gamma_i(N_j), N_j]$. Hence,

$$L_j/\gamma_{m-1}(N_j) \text{ is a nilpotent-by-finite group.}$$

By [**26**, Cor. 3] there exists a free normal subgroup $\bar{F}_j \lhd L_j$ such that $L_j/\bar{F}_j$ is a (torsion-free nilpotent)-by-finite group. We define

$$\widehat{F}_j := \gamma_{m-1}(N_j) \cap \bar{F}_j \subseteq H \tag{5.1}$$

and note that $L_j/\widehat{F}_j$ is also nilpotent-by-finite. Set

$$\widehat{N} := \widehat{F}_1 \times \cdots \times \widehat{F}_m \subseteq H.$$

Finally, we define the free groups

$$F_i := \widehat{F}_i \cap (\widehat{N} \cap H') = \widehat{F}_i \cap H'.$$

Since $\widehat{F}_i$ is normal in L_i and $p_i(H) = L_i$, we deduce that F_i is normal in L_i. By definition, $\widehat{F}_i/F_i \simeq \widehat{F}_i/(\widehat{F}_i \cap H')$ and the latter is isomorphic to a subgroup of H/H'; hence $\widehat{F}_i/F_i$ is a finitely generated abelian group. This implies that L_i/F_i is polycyclic-by-finite as claimed. □

Recall that an associated ring R is left (respectively, right) Noetherian if every left (respectively, right) ideal is finitely generated.

Lemma 5.2. *Let G be a polycyclic-by-finite group and let $\chi : G \to \mathbb{R}$ be a discrete character of G. Then, $\mathbb{Z}G_\chi$ and $\mathbb{Q}G_\chi$ are left and right Noetherian rings.*

Proof. By [**34**, Thm. 2.7], $\mathbb{Z}G$ and $\mathbb{Q}G$ are left and right Noetherian rings. Furthermore, since χ is a discrete character, we have $G = Ker(\chi) \rtimes \langle x \rangle$, where $\langle x \rangle \simeq \mathbb{Z}$ and $\chi(x) > 0$. Then $G_\chi = \{g \in G | \chi(g) \ge 0\}$ is a disjoint union $\cup_{k \ge 0} Ker(\chi) x^k$; hence

$$\mathbb{Z}G_\chi = \bigoplus_{\substack{g_0 \in Ker(\chi) \\ k \ge 0}} \mathbb{Z}(g_0 x^k) = \mathbb{Z}Ker(\chi)[x],$$

where $\mathbb{Z}Ker(\chi)[x]$ is a skew polynomial ring with x acting on $\mathbb{Z}Ker(\chi)$ via its conjugation action on $Ker(\chi)$. Since $Ker(\chi)$ is a polycyclic-by-finite group, again using [**34**, Thm. 2.7],

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

$\mathbb{Z}Ker(\chi)$ is a left and a right Noetherian ring. Then, by a version of Hilbert's base theorem for skew polynomial rings (see [**22**, Theorem 1.12]), we conclude that $\mathbb{Z}Ker(\chi)[x]$ is a left and a right Noetherian ring, and thus $\mathbb{Z}G_\chi$ is also left and right Noetherian. Similarly, $\mathbb{Q}G_\chi$ is a left and a right Noetherian ring. □

From now on, we fix $F_1, \dots, F_m$ and N as the groups given by Proposition 5.1. For $[\chi] \in S(H)$, since $N \subseteq H' \subseteq Ker(\chi) \subseteq H_\chi$, we set

$$H_\chi/N := \{gN \in H/N \mid g \in H_\chi\} = (H/N)_{\chi_0},$$

where

$$\chi_0 : H/N \to \mathbb{R}$$

is the character induced by χ.

Lemma 5.3. *If $\chi : H \to \mathbb{R}$ is a discrete character of H, then $\mathbb{Z}(H_\chi/N)$ and $\mathbb{Q}(H_\chi/N)$ are left and right Noetherian rings.*

Proof. Since $L_1/F_1 \times \cdots \times L_m/F_m$ is a polycyclic-by-finite group, it follows that H/N is also polycyclic-by-finite. Thus, $\mathbb{Z}(H/N)_{\chi_0} = \mathbb{Z}(H_\chi/N)$ and $\mathbb{Q}(H/N)_{\chi_0} = \mathbb{Q}(H_\chi/N)$ are left and right Noetherian rings by Lemma 5.2. □

Proposition 5.4. *Let $\chi : H \to \mathbb{R}$ be a discrete character. If $[\chi] \in \Sigma^n(H, \mathbb{Q})$, then $H_j(N, \mathbb{Q})$ is finitely generated as a right $\mathbb{Q}(H_\chi/N)$-module for $0 \le j \le n$.*

Proof. Since $[\chi] \in \Sigma^n(H, \mathbb{Q})$, there exists a free resolution of $\mathbb{Q}H_\chi$-modules

$$\mathscr{F} : \cdots \longrightarrow P_j \xrightarrow{d_j} P_{j-1} \longrightarrow \cdots \longrightarrow \mathbb{Q} \longrightarrow 0$$

of the trivial $\mathbb{Q}H_\chi$-module $\mathbb{Q}$, where P_j is a finitely generated free $\mathbb{Q}H_\chi$-module for $0 \le j \le n$. Furthermore, we have the following $\mathbb{Q}(H_\chi/N)$-module isomorphisms:

$$P_j \otimes_{\mathbb{Q}N} \mathbb{Q} \cong \left(\bigoplus_{\alpha \in I_j} (\mathbb{Q}H_\chi)_\alpha \right) \otimes_{\mathbb{Q}N} \mathbb{Q} \cong \bigoplus_{\alpha \in I_j} ((\mathbb{Q}H_\chi)_\alpha \otimes_{\mathbb{Q}N} \mathbb{Q}) \cong \bigoplus_{\alpha \in I_j} \mathbb{Q}(H_\chi/N)_\alpha,$$

where $(\mathbb{Q}H_\chi)_\alpha := \mathbb{Q}H_\chi$ and I_j is some finite index set for $j \in \{0, \dots, n\}$.

Consider the complex

$$\mathscr{F} \otimes_{\mathbb{Q}N} \mathbb{Q} : \cdots \longrightarrow P_j \otimes_{\mathbb{Q}N} \mathbb{Q} \xrightarrow{d_j \otimes id_{\mathbb{Q}}} P_{j-1} \otimes_{\mathbb{Q}N} \mathbb{Q} \longrightarrow \cdots \longrightarrow \mathbb{Q} \longrightarrow 0\ ,$$

and note that for $j \ge 1$,

$$H_j(N, \mathbb{Q}) = Tor_j^{\mathbb{Z}N}(\mathbb{Z}, \mathbb{Q}) \cong Tor_j^{\mathbb{Q}N}(\mathbb{Q}, \mathbb{Q}) = H_j(\mathscr{F} \otimes_{\mathbb{Q}N} \mathbb{Q}) = \frac{Ker(d_j \otimes id_{\mathbb{Q}})}{Im(d_{j+1} \otimes id_{\mathbb{Q}})}.$$

Since $\mathbb{Q}(H_\chi/N)$ is a right Noetherian ring, we conclude that for $0 \le j \le n$ every $\mathbb{Q}(H_\chi/N)$-submodule of $P_j \otimes_{\mathbb{Q}N} \mathbb{Q}$ is finitely generated, thus $Ker(d_j \otimes id_{\mathbb{Q}})$ is a finitely

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

generated $\mathbb{Q}(H_\chi/N)$-submodule and, consequently, $H_j(N,\mathbb{Q})$ is a finitely generated $\mathbb{Q}(H_\chi/N)$-module. □

Set

$$q_{j_1,\ldots,j_n} : H/N \to (L_{j_1}/F_{j_1}) \times \cdots \times (L_{j_n}/F_{j_n})$$

as the canonical projection map, where $1 \le j_1 < \cdots < j_n \le m$.

Lemma 5.5. *Let $\chi : H \to \mathbb{R}$ be a discrete character. If $[\chi] \in \Sigma^n(H,\mathbb{Q})$ for some $n \in \{1,\ldots,m\}$, then*

$$W_{j_1,\ldots,j_n} := (F^{ab}_{j_1} \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} (F^{ab}_{j_2} \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} (F^{ab}_{j_n} \otimes_{\mathbb{Z}} \mathbb{Q})$$

is finitely generated as a right $\mathbb{Q}q_{j_1,\ldots,j_n}(H_\chi/N)$-module.

Proof. By the Künneth formula and the fact that F_i is a free group for $1 \le i \le m$, there is an isomorphism of $\mathbb{Q}(H_\chi/N)$-modules

$$H_n(N,\mathbb{Q}) = H_n(F_1 \times \cdots \times F_m, \mathbb{Q}) \cong \tag{5.2}$$

$$\bigoplus_{j_1+\cdots+j_m=n, 0\le j_i \le 1} H_{j_1}(F_1,\mathbb{Q}) \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} H_{j_m}(F_m,\mathbb{Q}) \cong \bigoplus_{1\le j_1<\cdots<j_n\le m} W_{j_1,\ldots,j_n}.$$

By Lemma 5.3, $\mathbb{Q}(H_\chi/N)$ is a right Noetherian ring, and by Proposition 5.4, $H_n(N,\mathbb{Q})$ is a finitely generated right $\mathbb{Q}(H_\chi/N)$-module; thus $W_{j_1,\ldots,j_n}$ is a finitely generated right $\mathbb{Q}(H_\chi/N)$-submodule of $H_n(N,\mathbb{Q})$.

Note that $Ker(q_{j_1,\ldots,j_n}) \subseteq \prod_{j\in\{1,\ldots,m\}, j\ne j_1,\ldots,j_n} L_j/F_j$ acts trivially on $W_{j_1,\ldots,j_n}$ via conjugation. We conclude that the action of H_χ/N factors through $q_{j_1,\ldots,j_n}(H_\chi/N)$. Thus $W_{j_1,\ldots,j_n}$ is a finitely generated right $\mathbb{Q}(q_{j_1,\ldots,j_n}(H_\chi/N))$-submodule of $H_n(N,\mathbb{Q})$. □

From now on, we fix $1 \le j_1 < \cdots < j_n \le m$ and set

$$\psi = q_{j_1,\ldots,j_n} : H/N \to (L_{j_1}/F_{j_1}) \times \cdots \times (L_{j_n}/F_{j_n})$$

as the canonical projection map.

By [**34**, Lemma 2.5], there exists a characteristic subgroup $\widehat{Q}_i$ of L_i/F_i such that $\widehat{Q}_i$ is a torsion-free polycyclic group of finite index in L_i/F_i for $1 \le i \le m$.

Lemma 5.6. *Let $\chi : H \to \mathbb{R}$ be a character. Then $\mathbb{Q}[\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}]$ is a right $\mathbb{Q}[\psi(H_\chi/N) \cap (\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n})]$-submodule of $W_{j_1,\ldots,j_n} = (F^{ab}_{j_1} \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} (F^{ab}_{j_n} \otimes_{\mathbb{Z}} \mathbb{Q})$.*

Proof. By the proof of [**26**, Proposition 7], $\mathbb{Q}\widehat{Q}_i$ is a right $\mathbb{Q}\widehat{Q}_i$-submodule of $F^{ab}_i \otimes_{\mathbb{Z}} \mathbb{Q}$; thus

$$\mathbb{Q}[\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}] \cong \mathbb{Q}\widehat{Q}_{j_1} \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} \mathbb{Q}\widehat{Q}_{j_n}$$

is a right $\mathbb{Q}[\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}]$-submodule of $(F^{ab}_{j_1} \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} (F^{ab}_{j_n} \otimes_{\mathbb{Z}} \mathbb{Q}) = W_{j_1,\ldots,j_n}$.

By Lemma 5.5 $W_{j_1,\ldots,j_n}$ is a right $\mathbb{Q}\psi(H_\chi/N)$-module, thus

$$W_{j_1,\ldots,j_n} \text{ is a right } \mathbb{Q}[\psi(H_\chi/N) \cap (\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n})]\text{-module.}$$ □

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

5.1. More on discrete characters

Let $\chi : H \to \mathbb{R}$ be a discrete character and let $\chi_0 : H/N \to \mathbb{R}$ be the character induced by χ. Then we have disjoint unions

$$H/N = \dot{\bigcup_{\alpha\in\mathbb{Z}}} t^\alpha Ker(\chi_0) \quad \text{and} \quad (H/N)_{\chi_0} = H_\chi/N = \dot{\bigcup_{\alpha\geq 0}} t^\alpha Ker(\chi_0).$$

Applying ψ,

$$\psi(H/N) = \bigcup_{\alpha\in\mathbb{Z}} \psi(t)^\alpha \psi(Ker(\chi_0)), \quad \psi(H_\chi/N) = \bigcup_{\alpha\geq 0} \psi(t)^\alpha \psi(Ker(\chi_0)). \tag{5.3}$$

Note that the last two unions are not necessarily disjoint.

Lemma 5.7. *Suppose that $\chi : H \to \mathbb{R}$ is a discrete character. If $[\chi] \in \Sigma^n(H,\mathbb{Q})$ and $[\psi(H/N) : \psi(Ker(\chi_0))] = \infty$, then $W_{j_1,\ldots,j_n}$ and $\mathbb{Q}[\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}]$ are both finitely generated right $\mathbb{Q}[\psi(H_\chi/N) \cap (\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n})]$-modules.*

Proof. Since $[\psi(H/N) : \psi(Ker(\chi_0))] = \infty$, the unions in (5.3) are disjoint. Hence, we can define a discrete character

$$\mu : \psi(H/N) \to \mathbb{R}$$

by

$$\mu(\psi(q)) := \chi_0(q) \quad \text{for all } q \in H/N.$$

Note that $\Sigma^n(H,\mathbb{Q}) \neq \emptyset$ implies that H is FP_n; hence, by Theorem 2.3, $\psi(H/N)$ has finite index in $(L_{j_1}/F_{j_1}) \times \cdots \times (L_{j_n}/F_{j_n})$. We apply Theorem 3.1 for $D := \mathbb{Q}$, $M := W_{j_1,\ldots,j_n}$, $\chi := \mu$, $G := \psi(H/N)$ and $K := \psi(H/N) \cap (\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n})$ and conclude that

$$\begin{aligned} &W_{j_1,\ldots,j_n} \text{ is a finitely generated right } \mathbb{Q}[\psi(H/N)]_\mu\text{-module} \\ &\Leftrightarrow \\ &W_{j_1,\ldots,j_n} \text{ is a finitely generated right } \mathbb{Q}K_{\mu|_K}\text{-module.} \end{aligned}$$

Note that $[G : K] < \infty$, since $[(L_{j_1}/F_{j_1}) \times \cdots \times (L_{j_n}/F_{j_n}) : \widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}] < \infty$. Furthermore, $\psi(H_\chi/N) = [\psi(H/N)]_\mu$ and

$$\begin{aligned} \mathbb{Q}K_{\mu|_K} &= \mathbb{Q}[\psi(H/N)]_\mu \cap \mathbb{Q}[\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}] = \mathbb{Q}[\psi(H_\chi/N)] \cap \mathbb{Q}[\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}] \\ &= \mathbb{Q}[\psi(H_\chi/N) \cap (\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n})]. \end{aligned}$$

Then

$$W_{j_1,\ldots,j_n} \text{ is a finitely generated right } \mathbb{Q}\psi(H_\chi/N)\text{-module} \tag{5.4}$$

$$\Leftrightarrow$$

$$W_{j_1,\ldots,j_n} \text{ is a finitely generated right } \mathbb{Q}[\psi(H_\chi/N) \cap \widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}]\text{-module.} \tag{5.5}$$

Note that by Lemma 5.5, (5.4) holds; hence (5.5) holds.

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

Since $\mathbb{Q}K_{\mu|_K} = \mathbb{Q}[\psi(H_\chi/N) \cap (\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n})]$, K is a polycyclic-by-finite group and $\mu|_K$ is a discrete character, we conclude by Lemma 5.2 that $\mathbb{Q}[\psi(H_\chi/N) \cap (\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n})]$ is a right Noetherian ring. Thus $W_{j_1,\ldots,j_n}$ is a right Noetherian $\mathbb{Q}[\psi(H_\chi/N) \cap (\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n})]$-module and all its $\mathbb{Q}[\psi(H_\chi/N) \cap (\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n})]$-submodules are finitely generated. In particular, by Lemma 5.6, $\mathbb{Q}[\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}]$ is finitely generated as a right $\mathbb{Q}[\psi(H_\chi/N) \cap (\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n})]$-module. □

Proposition 5.8. *Suppose that $\chi : H \to \mathbb{R}$ is a discrete character and $[\chi] \in \Sigma^n(H, \mathbb{Q})$. Then $[\psi(H/N) : \psi(Ker(\chi_0))] < \infty$, where $\chi_0 : H/N \to \mathbb{R}$ is the character induced by χ.*

Proof. Assume that $[\psi(H/N) : \psi(Ker(\chi_0))] = \infty$. By Lemma 5.7, we conclude that $\mathbb{Q}[\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}]$ is a finitely generated right $\mathbb{Q}K_{\mu|_K}$-module, where

$$K = \psi(H/N) \cap (\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n})$$

and $\mu : \psi(H/N) \to \mathbb{R}$ is induced by χ. As we have also seen in the proof of Lemma 5.7, $\mathbb{Q}K_{\mu_0}$ is a right Noetherian ring, where $\mu_0 = \mu|_K$. Then $\mathbb{Q}[\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}]$ is a right Noetherian $\mathbb{Q}K_{\mu_0}$-module and its $\mathbb{Q}K_{\mu_0}$-submodule $\mathbb{Q}K$ is finitely generated. This is impossible since $\mu_0 \neq 0$. □

Corollary 5.9. *Suppose that $\chi : H \to \mathbb{R}$ is a discrete character and $[\chi] \in \Sigma^n(H, \mathbb{Q})$. Then $\psi(H_\chi/N) = \psi(H/N)$.*

Proof. By Proposition 5.8, $[\psi(H/N) : \psi(Ker(\chi_0))] < \infty$. Then there exists a non-negative integer a such that $\psi(t)^{a+1} \in \psi(Ker(\chi_0))$. Thus, using (5.3),

$$\psi(H/N) = \bigcup_{0 \le \beta \le a} \psi(t)^\beta \psi(Ker(\chi_0)) \subseteq \bigcup_{\beta \ge 0} \psi(t)^\beta \psi(Ker(\chi_0)) = \psi(H_\chi/N). \qquad \square$$

5.2. More on general characters

Lemma 5.10. *Let $\chi : H \to \mathbb{R}$ be a character. Then $p_{i_1,\ldots,i_n}(H) \neq p_{i_1,\ldots,i_n}(H_\chi)$ if and only if $\chi(\cap_{1 \le j \le n} Ker(p_{i_j})) = 0$.*

Proof. If $\chi(\cap_{1 \le j \le n} Ker(p_{i_j})) \neq 0$, choose $g_0 \in \cap_{1 \le j \le n} Ker(p_{i_j})$ such that $\chi(g_0) > 0$. Then, for an arbitrary $g \in H$ we have that $p_{i_1,\ldots,i_n}(g) = p_{i_1,\ldots,i_n}(g g_0^k)$, and for k sufficiently big we have that $\chi(g g_0^k) = \chi(g) + k\chi(g_0) > 0$, hence $g g_0^k \in H_\chi$ and thus $p_{i_1,\ldots,i_n}(H) \subseteq p_{i_1,\ldots,i_n}(H_\chi) \subseteq p_{i_1,\ldots,i_n}(H)$.

Suppose now that $\chi(\cap_{1 \le j \le n} Ker(p_{i_j})) = 0$. Suppose that $g \in H \setminus H_\chi$ and $p_{i_1,\ldots,i_n}(g) \in p_{i_1,\ldots,i_n}(H_\chi)$. Then there exists $g_1 \in H_\chi$ such that $p_{i_1,\ldots,i_n}(g) = p_{i_1,\ldots,i_n}(g_1)$. Then $g g_1^{-1} \in Ker(p_{i_1,\ldots,i_n}) = \cap_{1 \le j \le n} Ker(p_{i_j})$ and $\chi(g g_1^{-1}) \neq 0$, a contradiction. □

6. The main results of the paper

6.1. Proof of Theorem A

The case where χ is a discrete character is considered in Corollary 5.9. Assume now that χ is a non-discrete character of H that contradicts Theorem A, i.e. $[\chi] \in \Sigma^n(H, \mathbb{Q})$,

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

and for some $1 \leq i_1 < i_2 < \cdots < i_n \leq m$ we have that $p_{i_1,\ldots,i_n}(H) \neq p_{i_1,\ldots,i_n}(H_\chi)$. Let $N_0 = \cap_{1\leq j\leq n} Ker(p_{i_j})$. By Lemma 5.10, $\chi(N_0) = 0$. Thus

$$[\chi] \in S(H, N_0) = \{[\mu] \in S(H) \mid \mu(N_0) = 0\}$$

and $S(H, N_0)$ is a closed sub-sphere of the character sphere $S(H)$. Let $\bar{N}_0$ be the image of N_0 in $B = H^{ab}/tor(H^{ab})$, where $H^{ab} = H/H'$ is the abelianization of H and $tor(H^{ab})$ is the torsion part of H^{ab}. Since $\chi \neq 0$, there is a basis $y_1, \ldots, y_d$ of B as a free abelian group such that $\bar{N}_0$ is a subgroup of finite index in the abelian group generated by $y_1, \ldots, y_a$ for some $a \geq 0$ (the case $a = 0$ corresponds to $\bar{N}_0 = 1$). Then

$$S(H, N_0) = \{[\mu] \in S(H) \mid \mu(y_1) = \cdots = \mu(y_a) = 0\} \simeq S(H/N_0),$$

and the homeomorphism $S(H, N_0) \simeq S(H/N_0)$ preserves discrete characters.

By [**12**], $\Sigma^n(H, \mathbb{Q})$ is an open subset of the character sphere $S(H)$; hence there is a positive real number ϵ such that for any $[\mu] \in S(H)$, where the angle between $[\chi]$ and $[\mu]$ is at most ϵ, we have $[\mu] \in \Sigma^n(H, \mathbb{Q})$. Note that the discrete characters are dense in $S(H/N_0)$; hence the same holds in $S(H, N_0)$. Thus, there is a discrete character $\widehat{\chi}$ of H such that $[\widehat{\chi}]$ is in $S(H, N_0)$ and the angle between $[\chi]$ and $[\widehat{\chi}]$ is smaller than ϵ. Then $[\widehat{\chi}] \in \Sigma^n(H, \mathbb{Q})$ and, since Theorem A holds for discrete characters, we deduce that $p_{i_1,\ldots,i_n}(H) = p_{i_1,\ldots,i_n}(H_{\widehat{\chi}})$; hence, by Lemma 5.10, we have that $\widehat{\chi}(N_0) \neq 0$, a contradiction.

6.2. Proof of Theorem B

Note that by Theorem A, if $[\chi] \in \Sigma^1(H, \mathbb{Q})$ then $p_i(H_\chi) = p_i(H) = L_i$. Thus

$$\Sigma^1(H, \mathbb{Q}) \subseteq \{[\chi] \in S(H) \mid p_i(H_\chi) = L_i \text{ for } 1 \leq i \leq m\}. \tag{6.1}$$

For the converse, assume that $\chi : H \to \mathbb{R}$ is a character such that

$$p_i(H_\chi) = L_i \quad \text{for every } 1 \leq i \leq m. \tag{6.2}$$

By [**14**], $[\chi] \in \Sigma^1(H) = \Sigma^1(H, \mathbb{Z})$ if there is a finitely generated submonoid T of H_χ such that the commutator subgroup H' is finitely generated as an T-group, where T acts via conjugation (on the right). By (6.2) and since each L_i is finitely generated, we deduce that there is a finitely generated submonoid M of H_χ such that $p_i(M) = L_i$ for $1 \leq i \leq m$.

Recall that

$$N = F_1 \times \cdots \times F_m \subseteq H' \subseteq H \subseteq L_1 \times \cdots \times L_m,$$

where each L_i/F_i is virtually polycyclic and F_i is free. Since L_i/F_i is finitely presented, there is a finite subset X_i of F_i such that

$$F_i = \langle X_i \rangle^{L_i} = \langle X_i \rangle^M.$$

Since H'/N is a subgroup of the polycyclic-by-finite group $\prod_{1\leq i\leq m}(L_i/F_i)$, we deduce that H'/N is polycyclic-by-finite; hence there is a finite subset $\bar{X}$ of H' such that H'/N

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

is generated by the image of X. Then

$$H' = \langle (\cup_{1\leq i\leq m} X_i) \cup X\rangle^M,$$

i.e. H' is finitely generated as an M-group, where M acts via conjugation on the right. Then $[\chi] \in \Sigma^1(H) = \Sigma^1(H,\mathbb{Z})$. Thus, we have

$$\{[\chi] \in S(H) \mid p_i(H_\chi) = L_i \text{ for } 1 \leq i \leq m\} \subseteq \Sigma^1(H) = \Sigma^1(H,\mathbb{Z}) \subseteq \Sigma^1(H,\mathbb{Q}). \qquad (6.3)$$

Then (6.1) and (6.3) complete the proof of Theorem B.

6.3. Proof of Corollary C

If $s = 0$, we can apply Theorem B. If $s \geq 1$ by Theorem 2.2, we can realise H as a finitely presented full subdirect product

$$H \subseteq L_1 \times \cdots \times L_s \times L_{s+1} \times \cdots \times L_m,$$

where $L_1 \simeq \cdots \simeq L_s \simeq \mathbb{Z}$, L_i is a non-abelian limit group for $i \geq s+1$ and $Z(H) = H \cap (L_1 \times \cdots \times L_s)$ has finite index in $L_1 \times \cdots \times L_s \simeq \mathbb{Z}^s$; hence $Z(H) \simeq \mathbb{Z}^s$. If $\chi(Z(H)) \neq 0$, by [**12**] we have that $[\chi] \in \Sigma^1(H,\mathbb{Z}) \subseteq \Sigma^1(H,\mathbb{Q})$.

Suppose now that $\chi : H \to \mathbb{R}$ is a character such that $\chi(Z(H)) = 0$. Then

$$H/Z(H) \subseteq L_{s+1} \times \cdots \times L_m \qquad (6.4)$$

is a finitely presented full subdirect product. Since $Z(H) \simeq \mathbb{Z}^s$ is a group of type FP_∞, we deduce that H_χ is of type $FP_1(D)$ if and only if $H_\chi/Z(H)$ is of type $FP_1(D)$, i.e.

$$[\chi] \in \Sigma^1(H,D) \quad \text{if and only if } [\chi_1] \in \Sigma^1(H/Z(H),D), \qquad (6.5)$$

where $\chi_1 : H/Z(H) \to \mathbb{R}$ is the character induced by χ. To complete the proof it suffices to apply (6.5) for $D = \mathbb{Z}$ and $D = \mathbb{Q}$ and Theorem B for the full subdirect product (6.4).

6.4. Proof of Theorem D

By Theorem A, Theorem 2.3 and the inclusions $\Sigma^n(H) \subseteq \Sigma^n(H,\mathbb{Z}) \subseteq \Sigma^n(H,\mathbb{Q})$, it suffices to show that for a discrete character $\chi : H \to \mathbb{Z}$, if

$$p_{j_1,\ldots,j_n}(H_\chi) = p_{j_1,\ldots,j_n}(H) \quad \text{for all } 1 \leq j_1 < \cdots < j_n \leq m, \qquad (6.6)$$

then $[\chi] \in \Sigma^n(H)$.

Note that by Lemma 5.10, (6.6) is equivalent to $\chi(Ker(p_{j_1,\ldots,j_n})) \neq 0$ for all $1 \leq j_1 < \cdots < j_n \leq m$. Thus, if (6.6) holds for χ then (6.6) holds for χ substituted by $-\chi$. Then, since χ is a discrete character, by Theorem 3.5,

$$S(H, Ker(\chi)) = \{[\chi], [-\chi]\} \subseteq \Sigma^n(H)$$

if and only if $N := Ker(\chi)$ is of homotopical type F_n. Thus, to prove the result it remains to show that N is of homotopical type F_n.

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

Observe that we already know by Theorem B that N is finitely generated. Hence $p_i(N)$ is a finitely generated normal subgroup in a non-abelian limit group L_i, and thus by [**16**] $p_i(N)$ has finite index in L_i for every $1 \leq i \leq m$, i.e.

$$N \text{ is a subdirect product of } p_1(N) \times \cdots \times p_m(N) \tag{6.7}$$

and each $p_i(N)$ is a limit group. Note that limit groups are always finitely presented and of type FP_∞.

Suppose $H = N \rtimes \langle t \rangle$, where $\chi(t) > 0$. Thus $p_{j_1,\ldots,j_n}(H) = p_{j_1,\ldots,j_n}(H_\chi) = \cup_{i \geq 0} p_{j_1,\ldots,j_n}(N) p_{j_1,\ldots,j_n}(t)^i$ is a group; hence $p_{j_1,\ldots,j_n}(N)$ has finite index in $p_{j_1,\ldots,j_n}(H)$. Since H is of type FP_n, by Theorem 2.3, $p_{j_1,\ldots,j_n}(H)$ has finite index in $L_{j_1} \times \cdots \times L_{j_n}$; hence $p_{j_1,\ldots,j_n}(N)$ has finite index in $L_{j_1} \times \cdots \times L_{j_n}$. This implies that $p_{j_1,\ldots,j_n}(N)$ has finite index in $p_{j_1}(N) \times \cdots \times p_{j_n}(N)$. Then, if the virtually surjective conjecture holds in dimension n for the subdirect product (6.7), we deduce that N is of type F_n.

Finally, by [**18**], the virtual surjection conjecture holds for $n = 2$.

6.5. Proof of Corollary E

If $s = 0$, we can apply Theorem D for $n = 2$ and χ a discrete character. If $s \geq 1$ and $\chi(Z(H)) \neq 0$, by [**12**] we have that $[\chi] \in \Sigma^2(H, \mathbb{Z}) \subseteq \Sigma^2(H, \mathbb{Q})$. The same holds in its homotopical version, i.e. $[\chi] \in \Sigma^2(H)$, and it follows directly by the Renz Σ^2-criterion in [**36**].

Suppose now that $\chi : H \to \mathbb{R}$ is a character such that $\chi(Z(H)) = 0$. Recall that $Z(H) \simeq \mathbb{Z}^s$ is a group of type FP_∞. Hence we deduce that H_χ is of type $FP_2(D)$ if and only if $H_\chi / Z(H)$ is of type $FP_2(D)$ (this is a monoidal version of the fact that for a short exact sequence of groups $N \to G \to Q$, where N is FP_∞, G is FP_n if and only if Q is FP_n [**9**]), i.e.

$$[\chi] \in \Sigma^2(H, D) \quad \text{if and only if } [\chi_1] \in \Sigma^2(H/Z(H), D), \tag{6.8}$$

where $\chi_1 : H/Z(H) \to \mathbb{R}$ is the character induced by χ. To complete the proof of the homological version of the result, note that by (6.4) $H/Z(H) \subseteq L_{s+1} \times \cdots \times L_n$ is a full subdirect product; thus it suffices to apply (6.8) for $D = \mathbb{Z}$ and $D = \mathbb{Q}$ and Theorem D for the subdirect product (6.4).

To complete the proof, we need the homotopical version of the above argument, i.e.

$$[\chi] \in \Sigma^2(H) \quad \text{if and only if } [\chi_1] \in \Sigma^2(H/Z(H)). \tag{6.9}$$

This follows directly from [**33**, Cor. 4.2] since $\mathbb{Z}(H)$ is finitely presented.

7. The Bieri–Stallings groups: proof of Theorem F

Let $m \geq 3$ be a natural number. The Bieri–Stallings groups are

$$G_m = Ker(F_2^m \to \mathbb{Z}),$$

where F_2 is the free group of rank 2. The original Stallings example is the case $m = 3$. The group homomorphism $F_2^m \to \mathbb{Z}$ sends a fixed basis of F_2 to 1. The group G_m is of

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

type F_{m-1} but not of type F_m, and $G_m \subseteq F_2^m$ is a full subdirect product that maps surjectively on pairs.

We write x_i, y_i for the basis of the ith copy of F_2 in the direct product. Then, for pairwise different $1 \le i, j, t \le m$, we have $[x_i, y_i] = [x_i y_j^{-1}, y_i x_t^{-1}] \in G'_m$; hence

$$(F'_2)^m = F'_2 \times \cdots \times F'_2 = G'_m. \tag{7.1}$$

Then

$$G_m^{ab} = G_m/G'_m \simeq \mathbb{Z}^{2m-1} \subseteq \mathbb{Z}^{2m} \simeq (F_2/F'_2) \times \cdots \times (F_2/F'_2).$$

Furthermore, moving to additive notation,

$$G_m^{ab} \simeq \left\{ \sum_{1\le i\le m} z_{1,i}x_i + \sum_{1\le i\le m} z_{2,i}y_i \mid \sum_{1\le i\le m} (z_{1,i} + z_{2,i}) = 0, z_{1,i}, z_{2,i} \in \mathbb{Z} \right\}.$$

Then, for the canonical projection $p_j : G_m \to F_2$, we have

$$\begin{aligned} &Ker(p_j)G'_m/G'_m \\ &\simeq \left\{ \sum_{1\le i\ne j\le m} z_{1,i}x_i + \sum_{1\le i\ne j\le m} z_{2,i}y_i \mid \sum_{1\le i\ne j\le m} (z_{1,i} + z_{2,i}) = 0, z_{1,i}, z_{2,i} \in \mathbb{Z} \right\}. \end{aligned}$$

Then $G_m/Ker(p_j)G'_m \simeq p_j(G_m)/F'_2 = F_2^{ab} \simeq \mathbb{Z}^2$ and, by Corollary C,

$$\Sigma^1(G_m)^c = \cup_{1\le j\le m}\Delta_j,$$

where

$$\Delta_j = S(G_m, Ker(p_j)) = \{[\chi] \in S(G_m) \mid \chi(Ker(p_j)) = 0\}.$$

Theorem 7.1. *For $H = G_m \subseteq F_2^m = F_2 \times \cdots \times F_2$, the monoidal virtual surjection conjecture holds for any discrete character χ of H and any dimension $n \le m - 1$.*

Proof. Recall that H is FP_{m-1} but not FP_m. Then, the monoidal virtual surjection conjecture for H is defined only for $n \le m-1$, and we can assume from now that $n \le m-1$. Note that since H is FP_{m-1}, by Theorem 2.3, $p_{i_1,\ldots,i_n}(H)$ has finite index in F_2^n. Also by Theorem A we know that if $[\chi] \in \Sigma^n(H,\mathbb{Q})$ then $p_{i_1,\ldots,i_n}(H) = p_{i_1,\ldots,i_n}(H_\chi)$ for any $1 \le i_1 < \cdots < i_n \le m$.

The rest of the proof follows the proof of Theorem D. Fix a discrete character $\chi : H \to \mathbb{R}$. To show that the monoidal virtual surjection conjecture holds for the character χ, it suffices to show that if $p_{i_1,\ldots,i_n}(H) = p_{i_1,\ldots,i_n}(H_\chi)$ for any $1 \le i_1 < \cdots < i_n \le m$ then $N = Ker(\chi)$ is of homotopical type F_n, hence $[\chi] \in \Sigma^n(H) \subseteq \Sigma^n(H,\mathbb{Z}) \subseteq \Sigma^n(H,\mathbb{Q})$. As in the proof of Theorem D, we consider N as a subgroup of $p_1(N) \times \cdots \times p_m(N)$. The same argument from the proof of Theorem D applies here, and we deduce that $p_i(N)$ is finitely generated; hence $p_i(N)$ has finite index in F_2 for every $1 \le i \le m$. By (7.1),

$$p_1(N)' \times \cdots \times p_m(N)' \subseteq F'_2 \times \cdots \times F'_2 = G'_m \subseteq N = Ker(\chi). \tag{7.2}$$

By [**30**], the virtual surjection conjecture holds for subdirect products $N \subseteq \Gamma_1 \times \cdots \times \Gamma_m$ that contain the commutator subgroup $\Gamma'_1 \times \cdots \times \Gamma'_m$. Applying this for $\Gamma_i = p_i(N)$, we deduce that N is of homotopical type F_n if and only if $p_{i_1,\dots,i_n}(N)$ has finite index in $\varGamma_{i_1} \times \cdots \times \varGamma_{i_n}$ for any $1 \le i_1 < \cdots < i_n \le m$. In the proof of Theorem D, it was shown that $p_{i_1,\dots,i_n}(N)$ has finite index in $p_{i_1,\dots,i_n}(H)$, and we already know that $p_{i_1,\dots,i_n}(H)$ has finite index in F_2^n and $[F_2^n : \varGamma_{i_1} \times \cdots \times \varGamma_{i_n}] < \infty$. This completes the proof. □

Note that G_m is generated by the finite set

$$\{\bar{x}_i = x_i y_1^{-1}, \bar{y}_j = y_j y_1^{-1}\}_{1\le i\le m, 2\le j\le m}.$$

Lemma 7.2. *The group $H = G_m \subseteq F_2^m = F_2 \times \cdots \times F_2$ has the following decomposition as an HNN-extension:*

$$G_m = HNN(B_m, t) = \langle B_m, t \mid A_m^t = A_m \rangle,$$

where

$$\begin{aligned} A_m &= \langle \{\bar{x}_i = x_i y_1^{-1}, \bar{y}_j = y_j y_1^{-1}\}_{1\le i\le m-1, 2\le j\le m-1} \rangle \\ &= N_0 \rtimes (\langle \bar{x}_2, \bar{y}_2\rangle \times \langle \bar{x}_3, \bar{y}_3\rangle \times \cdots \times \langle \bar{x}_{m-1}, \bar{y}_{m-1}\rangle), \end{aligned}$$

N_0 is the normal closure of $\bar{x}_1$ in G_m, $t = \bar{y}_m$ and

$$\begin{aligned} B_m = {} & \langle \bar{x}_1, \bar{x}_m\rangle \times \langle \bar{x}_2\bar{x}_m^{-1}, \bar{y}_2\bar{x}_m^{-1}\rangle \times \langle \bar{x}_3\bar{x}_m^{-1}, \bar{y}_3\bar{x}_m^{-1}\rangle \times \cdots \\ & \times \langle \bar{x}_{m-1}\bar{x}_m^{-1}, \bar{y}_{m-1}\bar{x}_m^{-1}\rangle \simeq F_2^{m-1}. \end{aligned}$$

Remark. Every automorphism of G_m applied to the above HNN decomposition gives an HNN decomposition of G_m. Applying a permutation $\sigma \in Perm(\{1, 2, \dots, m\})$ we obtain an automorphism σ of F_2^m such that $G_m^\sigma = G_m$ and σ sends x_i to $x_i^\sigma = x_{\sigma(i)}$ and y_i to $y_i^\sigma = y_{\sigma(i)}$. Thus, we have a HNN decomposition

$$G_m = HNN(B_m^\sigma, t^\sigma) = \langle B_m^\sigma, t^\sigma \mid (A_m^\sigma)^{t^\sigma} = A_m^\sigma\rangle.$$

Another HNN extension is obtained by fixing $x_1, \dots, x_{m-1}, y_1, \dots, y_{m-1}$ and swapping x_m and y_m.

Proof. Note that

$$\langle \{\bar{x}_i, \bar{y}_j\}_{2\le i\le m, 2\le j\le m}\rangle \simeq \langle \bar{x}_2, \bar{y}_2\rangle \times \langle \bar{x}_3, \bar{y}_3\rangle \times \cdots \times \langle \bar{x}_m, \bar{y}_m\rangle,$$

where each group $\langle \bar{x}_i, \bar{y}_i\rangle$ for $2 \le i \le m$ is free of rank 2 and

$$\bar{x}_1^{\bar{y}_2} = \cdots = \bar{x}_1^{\bar{y}_m} = \bar{x}_1^{\bar{x}_2} = \cdots = \bar{x}_1^{\bar{x}_m} = \bar{x}_1^{y_1^{-1}}.$$

The normal closure N_0 of $\bar{x}_1$ in G_m is a free group of infinite rank. Thus

$$G_m = N_0 \rtimes (\langle \bar{x}_2, \bar{y}_2\rangle \times \langle \bar{x}_3, \bar{y}_3\rangle \times \cdots \times \langle \bar{x}_m, \bar{y}_m\rangle).$$

Note that

$$\begin{aligned} B_m &= \langle A_m, \bar{x}_m\rangle \\ &= \langle \bar{x}_1, \bar{x}_m\rangle \times \langle \bar{x}_2\bar{x}_m^{-1}, \bar{y}_2\bar{x}_m^{-1}\rangle \times \langle \bar{x}_3\bar{x}_m^{-1}, \bar{y}_3\bar{x}_m^{-1}\rangle \times \cdots \\ &\quad \times \langle \bar{x}_{m-1}\bar{x}_m^{-1}, \bar{y}_{m-1}\bar{x}_m^{-1}\rangle \simeq F_2^{m-1}, \end{aligned} \tag{7.3}$$

https://doi.org/10.1017/S0013091520000176 Published online by Cambridge University Press

and for $t = \bar{y}_m$ we have

$$[\langle \bar{x}_2, \bar{y}_2\rangle \times \langle \bar{x}_3, \bar{y}_3\rangle \times \cdots \times \langle \bar{x}_{m-1}, \bar{y}_{m-1}\rangle, t] = 1$$

and $\bar{x}_1^t = \bar{x}_1^{\bar{y}_m} = \bar{x}_1^{\bar{x}_2} = \cdots = \bar{x}_1^{\bar{x}_m}$. □

Theorem 7.3. *For $H = G_m \subseteq F_2^m = F_2 \times \cdots \times F_2$, the monoidal surjection conjecture holds for any $n \leq m-2$.*

Proof. Let $\chi : G_m \to \mathbb{R}$ be a character and let $n \leq m-2$ be such that

$$p_{i_1,\ldots,i_n}(G_m) = p_{i_1,\ldots,i_n}((G_m)_\chi) \quad \text{for every } 1 \leq i_1 < \cdots < i_n \leq m. \tag{7.4}$$

This is equivalent to $\chi(Ker(p_{i_1,\ldots,i_n})) \neq 0$ for every $1 \leq i_1 < \cdots < i_n \leq m$. We aim to prove that $[\chi] \in \Sigma^n(G_m)$.

Let B_m be the base group from Lemma 7.2. By (7.3), $B_m \simeq F_2^{m-1}$.

Step 1. Suppose that $\chi(x_i y_i^{-1}) \neq 0$ for all $1 \leq i \leq m$. Note that for $1 \leq i \leq m-1$, the ith copy of F_2 in $B_m \simeq F_2^{m-1}$ contains $x_i y_i^{-1}$; indeed, $x_i y_i^{-1} = \bar{x}_i \bar{y}_i^{-1} = (\bar{x}_i \bar{x}_m^{-1})(\bar{y}_i \bar{x}_m^{-1})^{-1}$ for $2 \leq i \leq m-1$ and $x_1 y_1^{-1} = \bar{x}_1$. Then for every permutation $\sigma \in Perm(\{1, 2, \ldots, m\})$ the restriction of χ on $B_m^\sigma \simeq F_2^{m-1}$ is a character μ such that the restriction of μ on each copy of F_2 is non-zero. Then, by Theorem 3.3 and since $m-2 \geq n$,

$$[\mu] \in \Sigma^{m-2}(B_m^\sigma) \subseteq \Sigma^n(B_m^\sigma) \quad \text{for every permutation } \sigma.$$

Step 2. Suppose that for some i, $\chi(x_i y_i^{-1}) = 0$. Without loss of generality, $\chi(x_m y_m^{-1}) = 0$, otherwise we can apply an appropriate permutation σ. We aim to show that $[\mu] \in \Sigma^n(B_m)$ for the restriction μ of χ on $B_m \simeq F_2^{m-1}$. Note that $\mu = (\mu_1, \ldots, \mu_{m-1})$, where μ_i is the restriction of μ to the ith copy of F_2 in $F_2^{m-1} \simeq B_m$. By Theorem 3.3, $[\mu] \in \Sigma^n(B_m)$ is equivalent to at least $n+1$ of the characters $\mu_1, \ldots, \mu_{m-1}$ being non-zero. Suppose that this is not true, and at most n of the characters $\mu_1, \ldots, \mu_{m-1}$ are non-zero, i.e. at least $m-n-1$ of the characters $\mu_1, \ldots, \mu_{m-1}$ are zero, say $\mu_{j_1}, \mu_{j_2}, \ldots, \mu_{j_{m-n-1}}$ for some $1 \leq j_1 < \cdots < j_{m-n-1} \leq m-1$.

Recall that

$$B_m = \langle x_1 y_1^{-1}, x_m y_1^{-1}\rangle \times \langle x_2 x_m^{-1}, y_2 x_m^{-1}\rangle \times \langle x_3 x_m^{-1}, y_3 x_m^{-1}\rangle \times \cdots \langle x_{m-1} x_m^{-1}, y_{m-1} x_m^{-1}\rangle.$$

Step 2.1. Suppose first that $j_1 \geq 2$. Without loss of generality, we can assume that $j_1 = 2$, $j_2 = 3, \ldots, j_{m-n-1} = m-n$, otherwise we apply an appropriate permutation σ. Then

$$\Delta := \{x_2 x_m^{-1}, y_2 x_m^{-1}, x_3 x_m^{-1}, y_3 x_m^{-1}, \ldots, x_{m-n} x_m^{-1}, y_{m-n} x_m^{-1}\} \subseteq Ker(\chi).$$

Since $m-2 \geq n$, we have that $Ker(p_{1,m-n+1,\ldots,m-1}) \simeq G_{m-n}$ is generated by $\Delta \cup \{y_m x_m^{-1}\} \subseteq Ker(\chi)$, where

$$p_{1,m-n+1,\ldots,m-1} : G_m \to \langle x_1, y_1\rangle \times \langle x_{m-n+1}, y_{m-n+1}\rangle \times \cdots \times \langle x_{m-1}, y_{m-1}\rangle = F_2^n$$

is the canonical projection. Hence $Ker(p_{1,m-n+1,\ldots,m-1}) \subseteq Ker(\chi)$. Then by Lemma 5.10, $p_{1,m-n+1,\ldots,m-1}(G_m) \neq p_{1,m-n+1,\ldots,m-1}((G_m)_\chi)$, a contradiction with (7.4).

Step 2.2. Suppose that $j_1 = 1$. Without loss of generality we can assume that $j_1 = 1$, $j_2 = 2, \dots, j_{m-n-1} = m-n-1$, otherwise we apply an appropriate permutation σ. Then

$$E := \{x_1y_1^{-1}, x_my_1^{-1}, x_2x_m^{-1}, y_2x_m^{-1}, \dots, x_{m-n-1}x_m^{-1}, y_{m-n-1}x_m^{-1}\} \subseteq Ker(\chi).$$

Since $m-2 \geq n$, we have that $Ker(p_{m-n,\dots,m-1}) \simeq G_{m-n}$ is generated by $E \cup \{y_mx_m^{-1}\} \subseteq Ker(\chi)$, where

$$p_{m-n,\dots,m-1} : G_m \to \langle x_{m-n}, y_{m-n}\rangle \times \cdots \times \langle x_{m-1}, y_{m-1}\rangle = F_2^n$$

is the canonical projection. Hence we have that $Ker(p_{m-n,\dots,m-1}) \subseteq Ker(\chi)$. Then by Lemma 5.10, $p_{m-n,\dots,m-1}(G_m) \neq p_{m-n,\dots,m-1}((G_m)_\chi)$, a contradiction with (7.4).

Thus we deduce that in both cases $[\mu] \in \Sigma^n(B_m)$.

Step 3. Finally, we will show for $\nu = \chi\mid_{A_m}$ that $[\nu] \in \Sigma^{n-1}(A_m)$. Recall that

$$A_m = \langle \{\bar{x}_i = x_iy_1^{-1}, \bar{y}_j = y_jy_1^{-1}\}_{1\leq i\leq m-1, 2\leq j\leq m-1}\rangle.$$

Thus

$$A_m = (\langle x_1, y_1\rangle \times \cdots \times \langle x_{m-1}, y_{m-1}\rangle) \cap G_m \simeq G_{m-1}.$$

Let

$$q_{j_1,\dots,j_{n-1}} : A_m \to L_{j_1} \times \cdots \times L_{j_{n-1}}$$

be the canonical projection, where $1 \leq j_1 < \cdots < j_{n-1} \leq m-1$, $L_j = \langle x_j, y_j\rangle$ for $1 \leq j \leq m$. Thus, if $\nu(Ker(q_{j_1,\dots,j_{n-1}})) = 0$, since

$$Ker(q_{j_1,\dots,j_{n-1}}) = Ker(p_{j_1,\dots,j_{n-1},m}),$$

where

$$p_{j_1,\dots,j_{n-1},m} : G_m \to L_{j_1} \times \cdots \times L_{j_{n-1}} \times L_m$$

is the canonical projection, we deduce that $\chi(Ker(p_{j_1,\dots,j_{n-1},m})) = 0$. Then by Lemma 5.10, $p_{j_1,\dots,j_{n-1},m}(G_m) \neq p_{j_1,\dots,j_{n-1},m}((G_m)_\chi)$, a contradiction with (7.4). Thus we conclude that $\nu(Ker(q_{j_1,\dots,j_{n-1}})) \neq 0$ and note that $n-1 \leq (m-1)-2$. Then, by induction on m, we can assume that Theorem 7.3 holds for m substituted with $m-1$, i.e. it holds for $G_{m-1} \simeq A_m$. Then $[\nu] \in \Sigma^{n-1}(A_m)$.

Finally, Steps 1, 2 and 3 together with Theorem 3.4 applied to the HNN decomposition of G_m given by Lemma 7.2 imply $[\chi] \in \Sigma^n(G_m)$. □

Acknowledgements. We thank the anonymous referee for comments and suggestions that improved the paper. The first named author was partially supported by FAPESP grant 2016/05678-3 and CNPq grant 301779/2017-1. The discrete character of case Theorem A was included in the second named author's PhD thesis at the University of Campinas, which was financed by PhD grants from the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil and CNPq grant 140718/2016-8.

References

1. E. Alibegovic, A combination theorem for relatively hyperbolic groups, *Bull. Lond. Math. Soc.* **37**(3) (2005), 459–466.
2. K. Almeida, The BNS-invariant for Artin groups of circuit rank 2, *J. Group Theory* **21**(2) (2018), 189–228.
3. K. Almeida and D. Kochloukova, The Σ^1-invariant for Artin groups of circuit rank 1, *Forum Math.* **27**(5) (2015), 2901–2925.
4. B. Baumslag, Residually free groups, *Proc. Lond. Math. Soc.* **3–17**(3) (1967), 402–418.
5. G. Baumslag and J. Roseblade, Subgroups of direct products of free groups, *J. Lond. Math. Soc.* **30** (1984), 44–52.
6. G. Baumslag, A. Myasnikov and V. Remeslennikov, Algebraic geometry over groups I: Algebraic sets and ideal theory, *J. Algebra* **219** (1999), 16–79.
7. G. Baumslag, M. R. Bridson, C. F. Miller III and H. Short, Fibre products, non-positive curvature, and decision problems, *Comment. Math. Helv.* **75** (2000), 457–477.
8. M. Bestvina and N. Brady, Morse theory and finiteness properties of groups, *Invent. Math.* **129** (1997), 445–470.
9. R. Bieri, *Homological dimension of discrete groups*, 2nd edn, Queen Mary College Mathematics Notes (Queen Mary College, University of London, 1981).
10. R. Bieri and R. Geoghegan, Sigma invariants of direct products of groups, *Groups Geom. Dyn.* **4**(2) (2010), 251–261.
11. R. Bieri and J. Groves, The geometry of the set of characters induced by valuations, *J. Reine Angew. Math.* **347** (1984), 168–195.
12. R. Bieri and B. Renz, Valuations on free resolutions and higher geometric invariants of groups, *Comment. Math. Helv.* **3**(63) (1988), 464–497.
13. R. Bieri and R. Strebel, Valuations and finitely presented metabelian groups, *Proc. Lond. Math. Soc.* **41** (1980), 439–464.
14. R. Bieri, W. D. Neumann and R. Strebel, A geometric invariant of discrete groups, *Invent. Math.* **90** (1987), 451–477.
15. R. Bieri, R. Geoghegan and D. Kochloukova, The sigma invariants of Thompson's group F, *Groups Geom. Dyn.* **4**(2) (2010), 263–273.
16. M. R. Bridson and J. Howie, Normalisers in limit groups, *Math. Ann.* **337** (2007), 385–394.
17. M. R. Bridson, J. Howie, C. F. Miller and H. Short, Subgroups of direct products of limit groups, *Ann. of Math.* **3**(170) (2009), 1447–1467.
18. M. R. Bridson, J. Howie, C. F. Miller and H. Short, On the finite presentation of subdirect products and the nature of residually free groups, *Amer. J. Math.* **4**(135) (2013), 891–933.
19. F. Dahmani, Combination of convergence groups, *Geom. Topol.* **7** (2003), 933–963.
20. F. Funke and D. Kielak, Alexander and Thurston norms, and the Bieri-Neumann-Strebel invariants for free-by-cyclic groups, *Geom. Topol.* **22** (2018), 2647–2696.
21. R. Gehrke, The higher geometric invariants for groups with sufficient commutativity, *Comm. Algebra* **26** (1998), 1097–1115.
22. K. R. Goodearl and R. H. Warfield, *An introduction to non-commutative Noetherian rings*, London Mathematical Society Student Texts (Cambridge University Press, Cambridge, 1989).
23. M. Hall, Jr., Subgroups of finite index in free groups, *Canad. J. Math.* **1** (1949), 187–190.
24. O. Kharlampovich and A. Myasnikov, Elementary theory of free nonabelian groups, *J. Algebra* **302** (2006), 451–552.

25. D. Kielak, The Bieri–Neumann–Strebel invariants via Newton polytopes, preprint (arXiv 1802.07049).
26. D. H. Kochloukova, On subdirect products of type FP_m of limit groups, *J. Group Theory* **10** (2010), 1–19.
27. D. H. Kochloukova, On the Σ^2-invariants of the generalised R. Thompson groups of type F, *J. Algebra* **371** (2012), 430–456.
28. D. H. Kochloukova and F. F. Lima, Homological finiteness properties of fibre products, *Q. J. Math.* **69**(3) (2018), 835–854.
29. D. Kochloukova and L. Mendonça, On the Bieri–Neumann–Strebel–Renz invariants of coabelian subgroups, preprint.
30. B. Kuckuck, Subdirect products of groups and the n-$(n+1)$-$(n+2)$ conjecture, *Q. J. Math.* **65**(4) (2014), 1293–1318.
31. J. Meier, H. Meinert and L. VanWyk, Higher generation subgroup sets and the Σ-invariants of graph groups, *Comment. Math. Helvet.* **73** (1998), 22–44.
32. H. Meinert, The geometric invariants of direct products of virtually free groups, *Comment. Math. Helvet.* **69**(1) (1994), 39–48.
33. H. Meinert, Actions on 2-complexes and the homotopical invariant Σ^2 of a group, *J. Pure Appl. Algebra* **119** (1997), 297–317.
34. D. S. Passman, *The algebraic structure of group rings* (Robert E. Krieger Publishing Company, 1985).
35. B. Renz, Geometrische Invarianten und Endlichkeitseigenschaften von Gruppen, PhD thesis (Johann Wolfgang Goethe-Universität Frankfurt am Main, 1988).
36. B. Renz, Geometric invariants and HNN-extensions, in *Group theory*, pp. 465–484 (de Gruyter, Berlin, 1989).
37. S. Schmitt, Über den Zusammenhang der geometrischen Invarianten von Gruppe und Untergruppe mit Hilfe von variablen Modulkoeffzienten, Diplomarbeit (Frankfurt a.M., 1991).
38. D. Schütz, On the direct product conjecture for sigma invariants, *Bull. Lond. Math. Soc.* **40**(4) (2008), 675–684.
39. Z. Sela, Diophantine geometry over groups I. Makanin–Razborov diagrams, *Publ. Math. Inst. Hautes Études Sci.* **93** (2001), 31–105.
40. J. R. Stallings, A finitely presented group whose 3-dimensional homology group is not finitely generated, *Amer. J. Math.* **85** (1963), 541–543.
41. C. T. C. Wall, Finiteness conditions for CW-complexes, *Ann. of Math.* **81**(1) (1965), 56–69.
42. H. Wilton, Hall's theorem for limit groups, *Geom. Funct. Anal.* **18** (2008), 271–303.
43. M. C. B. Zaremsky, On the Σ-invariants of generalized Thompson groups and Houghton groups, *Int. Math. Res. Not. IMRN* **19** (2017), 5861–5896.