

WHAT IS A RULE OF INFERENCE?

NEIL TENNANT

The Ohio State University

Abstract. We explore the problems that confront any attempt to explain or explicate exactly what a primitive logical rule of inference *is*, or *consists in*. We arrive at a proposed solution that places a surprisingly heavy load on the prospect of being able to understand and deal with specifications of rules that are essentially *self-referring*. That is, any rule ρ is to be understood via a specification that involves, embedded within it, reference to rule ρ itself. Just how we arrive at this position is explained by reference to familiar rules as well as less familiar ones with unusual features. An inquiry of this kind is surprisingly absent from the foundations of inferentialism—the view that meanings of expressions (especially logical ones) are to be characterized by the rules of inference that govern them.

§1. Introduction.

1.1. Historical background: the recent confluence of philosophical and proof-theoretical investigations in logic. Influential philosophers of logic and language are once again, after a long Quinean interregnum, acknowledging the *a priori* and analyticity of deductive inference.¹ The Dummettian anti-realist's account of the epistemology of linguistic understanding had addressed matters of *justification*—of particular inference rules, and, therefore, of one's choice of a preferred system of deductive proof as 'the correct' one.² Among logicians, in a technical re-orientation hospitable to the Dummettian anti-realist's philosophy of meaning, there has also been growing interest in 'proof-theoretic semantics':³

In proof-theoretic semantics, proofs are not merely treated as syntactic objects as in Hilbert's formalist philosophy of mathematics,

Received: March 17, 2018.

2020 *Mathematics Subject Classification*: 03F03, 03B47, 03B20.

Key words and phrases: Classical Logic, Intuitionistic Logic, Minimal Logic, Core Logic, proof systems, rules of inference, self-reference, impredicative definitions, inductive definition of proof, Horn clauses, Modus Ponens, discharge of assumptions, parallelized elimination rules, classical rules of negation.

¹ See, for example, Martin-Löf [21], Boghossian [3], and Wright [50].

² See the pioneering essay of Dummett [7]; and the mature book-length treatment by Dummett [8]. A leading advocate of proof-theoretic accounts of validity of arguments is Dag Prawitz. See, for example, the classic papers by Prawitz [24 and 25]; and, for a recent re-visiting of the main ideas, Prawitz [28]. In the same pioneering company of proof-theoretically informed philosophizing about the meanings of logical operators, see Martin-Löf [22].

³ This name for the field first appeared in print in Schroeder-Heister [30]. A state-of-the-art collection is Schroeder-Heister and Piecha [33]. A useful survey of proof-theoretic semantics is Schroeder-Heister [32]. See also Schroeder-Heister [31].

but as entities in terms of which meaning and logical consequence can be explained.⁴

This account of the meanings of logical operators is premised on the belief that the proper source of those operators' meanings is the rules of inference governing them. So it is important to have some grasp of the variety of such rules, and the constraints they might have to satisfy in order to confer those meanings.

Discussion among philosophers, however, frequently assumes the very notion of a (primitive) *rule* of inference as unproblematic, or somehow primitively given or understood. The focus in philosophers' discussions has been on rules no more complicated than Modus Ponens (indeed, in some cases: *only* on Modus Ponens):

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}.$$

This, however, is to miss the opportunity to investigate various subtleties, and important features that are missing in the case of Modus Ponens, that arise only when one considers rules of inference in general.

Actually, simple though Modus Ponens seems to be, at least one leading logician thought that there is more to it than meets the eye. Scott [74] undertakes an interesting examination of how one should understand this rule.⁵ Scott writes (*loc. cit.*, p. 148)

...four versions of this well-known rule are ...

$$\begin{array}{cccc} A, A \Rightarrow B \vdash B & \frac{\vdash A \Rightarrow B}{A \vdash B} & \frac{\vdash A}{\vdash A \Rightarrow B} & \frac{\vdash A}{A \vdash B} \\ \text{(i)} & \text{(ii)} & \text{(iii)} & \text{(iv)} \end{array}$$

As Scott goes on to say (p. 149), the meaning of the single turnstile \vdash here is what he calls the 'standard' one, which, he says, is the *semantical* one. So perhaps one should re-construe the displayed forms of 'Modus Ponens' above as having the double turnstile \models in place of \vdash . Be that as it may, the *connective* rendered by the double arrow \Rightarrow is clearly the object-linguistic conditional connective (which we render as \rightarrow) to be understood as governed by these four different 'forms of' Modus Ponens. One notes with interest that the *obvious* construal of Modus Ponens on the part of those scholars whose discussion of it has been adverted to above is *not* among Scott's four forms. That obvious construal is that Modus Ponens can be applied to two premises (themselves, in general, standing as conclusions of subproofs for this application) of the respective forms $A, A \Rightarrow B$ (to use Scott's notation), so as to produce a proof of the conclusion B from the combination of the premises of those two subproofs. This construal appears not to be accommodated in Scott's list (i)–(iv).

Even with *this* concurrence (with those other scholars) on the 'missing construal' of Modus Ponens as a rule of inference, one can proceed on the current investigation into the question whether we yet fully grasp what it is for Modus Ponens to *be* a rule of inference, in that sufficiently precise sense in which we seek to capture *all* deductive rules of inference *as such*.

⁴ Kahle and Schroeder-Heister [19], at p. 503.

⁵ Thanks are owed to an anonymous referee for this reference.

1.2. The focus here will be on logical rules of inference. In this study we explore some apparently under-appreciated problems that confront any attempt to explain exactly what a primitive deductive rule of inference might, *in general, be, or consist in*. Clearly it behooves any inferentialist about meaning to get clear about the fundamental notion(s) on which an inferentialist theory would have to be based. Among these are two that deserve special mention and study.

First, there are ‘material’ rules of inference, capturing conceptual inclusions and conceptual contrarities, such as, respectively,

This is red; therefore, this is colored
—formalizable as, say,

$$\frac{Rt}{Ct}$$

and

This is red (all over); this is green (all over) ... absurd!
—formalizable as, say,

$$\frac{Rt \quad Gt}{\perp}$$

Such rules are devoted to capturing meaning-relations among descriptive terms of the language being used (in this example, obviously: English). At best they articulate constraints on, or interrelationships among, meanings of descriptive terms (‘red’, ‘green’, ‘colored’) that are *already understood*. That the language-user recognize the validity of such rules is criterial for our attribution to them of mastery of the meanings in question. *But they did not acquire their mastery of those meanings simply by subscribing to the rules in question*. Rather, they acquired such mastery through perceptual experience of objects in the ‘real world’, and by being instructed as to which descriptive terms (here, color terms) it would be appropriate to apply. Subscribing to material rules such as those given above can therefore be seen as a reflective afterthought, coming in the wake of the learner’s early perceptual experience and linguistic instruction. We mention material rules of inference only to set them aside for the purposes of this particular study.

Second, there are what we would like to call *logical* rules of inference (such as the aforementioned and hackneyed example of Modus Ponens). It is on *this* kind of rule that this study will concentrate. And here matters of mastery of meaning are entirely different. Subscribing to a particular primitive logical rule governing a logical operator is no reflective afterthought coming in the wake of some other way of acquiring an understanding of the meaning of the logical operator whose behavior is governed by that rule. Rather, *that certain primitive rules govern one’s use of the (logical) expression involved in them is constitutive* of that expression’s enjoying the meaning that it does. The instruction that learners receive will be wholly directed at *their conforming their inferential behavior to the constraints imposed by the rules*. The questions that now arise are: ‘Which rules?’, and: ‘What kind of thing *is* such a rule, in general?’. This study is confined to these questions. It serves both to point to a thicket that seems to lie beneath the casually surveying gaze; and to begin with some ground-clearing.

1.3. The catholicity of our coverage. Note that the questions just mentioned are posed without any prejudicial expectation that satisfactory answers to them would

dictate which rules are ‘the right’ rules for ‘the correct’ logic to have. We are not taking sides, in advance, with intuitionists (or constructivists) against advocates of Classical Logic, or with relevantists against both advocates of Intuitionist Logic and advocates of Classical Logic. Intuitionists and constructivists take issue with the four ‘strictly classical’ rules of negation: the Law of Excluded Middle, Dilemma, Classical Reductio, and Double Negation Elimination, along with any inferences whose proof requires appeal to any of these four rules. Relevantists take issue with particular inferences that, according to them, offend against their logical intuitions. All relevantists object to the First Lewis Paradox

$$\varphi, \neg\varphi : \psi,$$

and therefore of course to the so-called Absurdity Rule, or rule of *Ex Falso Quodlibet*:

$$\frac{\perp}{\psi}$$

that precipitates it; whereas some,⁶ but not all,⁷ relevantists object also to Disjunctive Syllogism:

$$\varphi \vee \psi, \neg\varphi : \psi$$

and even to the ‘truth-tabular’ inferences

$$\psi : \varphi \rightarrow \psi \quad \text{and} \quad \neg\varphi : \varphi \rightarrow \psi.$$

for the conditional.

In this study we are not interested in any attempt to contribute to, or resolve, the perennial dispute over whether there is just one correct logic and, if so, which logic that is. Rather, we are concerned to get clear about the main fundamental notion on which all participants in this dispute rely: that of a (primitive) ‘rule of inference’, *tout court*.

1.4. A note on notation. At this early stage in our exposition we hasten to let the reader know that the style of presentation of arguments or sequents that we have just employed will be used throughout. When we write

$$\varphi_1, \dots, \varphi_n : \psi,$$

with a colon, we are referring to the argument ‘ $\varphi_1, \dots, \varphi_n$; ergo, ψ ’; we are not making a metalinguistic statement.⁸ Such an argument may also be construed as the *sequent* whose antecedent is the set $\{\varphi_1, \dots, \varphi_n\}$ and whose succedent is the singleton $\{\psi\}$. Gentzen would have written such a sequent as

$$\varphi_1, \dots, \varphi_n \rightarrow \psi,$$

or as

$$\{\varphi_1, \dots, \varphi_n\} \rightarrow \{\psi\}.$$

⁶ See, for example, Anderson and Belnap [1].

⁷ Note that the Core logician objects neither to Disjunctive Syllogism nor to either of the truth-tabular inferences for the conditional. This is because the Core logician seeks to ensure relevance for the metalinguistic turnstile ‘ \vdash ’ of deducibility, rather than for the conditional ‘ \rightarrow ’ in the object-language. Yet, ironically, Core Logic admits of the strongest ‘variable-sharing’ result for relevance that has been established for any so-called relevant logic. See Tennant [44].

⁸ Thanks to an anonymous referee for eliciting this clarification of the colon notation.

Because Gentzen used the old-fashioned horseshoe \supset for the object-language conditional, there was no danger of misreading his single arrow between antecedent and succedent of a sequent as the object-language conditional. Nowadays, however, it is conventional to use the single arrow \rightarrow for the object-language conditional. That is why contemporary writers on sequent calculi have tended to use the double-arrow notation $\Delta \Rightarrow \Gamma$ for sequents, so as not to have the sequent-arrow confused with the object-language conditional. We, however, prefer to reserve the double-arrow \Rightarrow for the *metalinguistic* conditional. We have therefore opted for the space-saving colon as our way to indicate the break between antecedent and succedent of a sequent.

We also use Greek letters, as Gentzen and Prawitz did, as placeholders for the framing of rules of inference at a high level of generality. Lowercase φ, ψ, θ , etc. will be placeholders for sentences; uppercase Δ, Γ , etc. for (finite) sets of sentences. The absurdity symbol \perp can be used in a sequent $\Delta : \perp$ to represent what would officially be the sequent $\Delta : \emptyset$ with *empty* succedent. The symbol \perp will also be used in its own right as a punctuation marker in natural deductions. It is *not* to be construed as a propositional constant eligible for embedding as a subformula within formulae. For proofs we shall use the uppercase Greek letters Π, Σ etc. Context will always determine whether an uppercase Greek letter stands for a proof or for a set of sentences.

In providing examples of proofs or steps of inference, we shall resort to uppercase letters from the Latin alphabet, such as *A, B, C*, or *P, Q, R*, etc.

1.5. The plan of the paper. We have already motivated the focus of this study, and supplied some historical background and guidance to the relevant literature. Let us now pause to anticipate the structure of the overall pursuit of our further inquiries.

We shall begin in §2 by acknowledging the sheer variety of proof systems, with an eye to concentrating, for principled reasons to be articulated, on systems of natural deduction (for which there are corresponding sequent calculi). In these systems, rules of inference are accorded a primary role. If any answer is to be had to our main question ‘What exactly *is* a primitive rule of inference?’, it will have to be one that squarely confronts all the conceptual data afforded by careful consideration of what ‘goes on’ in those systems of natural deduction that derive from the seminal work of Gentzen [10, 11], via Prawitz [23]. Gentzen himself dealt only with the standard logical expressions $\neg, \wedge, \vee, \rightarrow, \exists, \forall$, and $=$. We shall regard as also eligible for consideration—in pursuit of certain subtleties—rules of inference that have been proposed for the modal operators \Box (‘it is necessary that ...’) and \Diamond (‘it is possible that ...’), to which Prawitz extended a treatment by means of introduction and elimination rules.

In §3 we embark on an examination that will lead to a better understanding of what, exactly, primitive rules of inference *are*. What is involved in the ‘transition’ that a rule of inference represents? How does one keep track of the assumptions on which the conclusion of a rule-application depends? What is the role of ‘assumptions for the sake of argument’? What is ‘vacuous discharge’ of assumptions, and when is it permitted? Which rules prohibit vacuous discharge of assumptions? What about rules dealing with the *empty* set of assumptions? In so-called elimination rules in natural deduction, how might their ‘major premises’ feature? Answers to all of these questions, as they emerge, help one to appreciate the rich variety of features of rules that need to be taken into consideration in order to arrive at a satisfactory answer to the driving question ‘What exactly *is* a primitive rule of inference?’.

In §4 we address its titular question: Are constraints in rule-applications partly constitutive of the rules themselves? We consider the notion of *normal form* of proofs, and examine various ‘tweaks’ that can be made to Gentzen’s original formulation of certain of his primitive logical rules, in the interests of attaining certain theoretical (metalogical) results. We discuss ‘generalized’, or ‘parallelized’ forms of elimination rules in natural deduction. We then summarize with a list of features of rules that illuminate what a rule says can be done with it *when constructing proofs*.

This allows us, in §5, to formulate a serious and precise answer to the titular question ‘What is a rule of inference, really?’. We all agree, uncontroversially, that rules of inference are applied in such a way as to help construct *proofs*. And proofs, for philosophical reasons, have to be *finite*. Not only that; they must be, in principle, *effectively checkable* as having been *correctly constructed* from the primitive means available. All this is a *sine qua non* for the *epistemic* role that proofs have to play. The main function of proofs is to ensure certain knowledge of their conclusions, if one has certain knowledge of their premises (i.e., of their undischarged assumptions). They also enable us to determine what is *guaranteed* to follow from various undischarged assumptions about whose truth one might *not* be certain, but whose truth one is simply assuming ‘for the sake of argument’.

We shall therefore conceive of primitive rules as formulated for the purpose of constructing *finite* proofs.⁹ More precisely, they will be construed as *inductive clauses* in an inductive definition of the ternary notion

$$\mathcal{P}_S(\Pi, \varphi, \Delta)$$

—‘the *finitary* tree-like construction Π is a proof, in the system S , of the conclusion φ from the set Δ of undischarged assumptions’—for which the basis clause is the obvious

$$\mathcal{P}_S(\varphi, \varphi, \{\varphi\}).^{10}$$

This, however, brings in an all-important mention of the system S for the construction of whose proofs the rule has a role to play. While mindful of the resulting system-relativity in our characterization of rules of inference, we advance toward the formulation of a ‘most general form’ of any particular rule.

This theme is sustained in §6, where we express the most general form in a way that openly confesses a certain system-relativity that one can only hope will prove to be self-referentially virtuous, not vicious.

In §7 we propose how best to ‘identify’ rules of inference. With an eye to various systems (and in keeping with our earlier avowed catholicity of interest in them) we arrive at an understanding of our inductive-definitional clauses for the proof-predicate \mathcal{P} as metalinguistic *Horn clauses*. Among these clauses will be not only the rules governing the introduction (in conclusions) or the elimination (from major premises) of dominant occurrences of logical operators, but also those ‘strictly classical’ rules, such as Classical Reductio and Dilemma, that do not fit neatly into the ‘introductory/eliminative’

⁹ Thanks to an anonymous referee for pointing out the need to explain why proofs must be finitary.

¹⁰ To this basis clause there corresponds, of course, the Rule of Reflexivity $\varphi : \varphi$ of the sequent calculus. This conception of rules of natural deduction (as inductive clauses in the inductive definition of proof) was essayed in Tennant [37], pp. 49–55.

compartments, yet are clearly taken by their users as both primitive and fundamental rules of inference.¹¹

1.6. A word about the degree of technicality involved, and about the intended audience(s) for this kind of study. It should be clear by now to the reader that this is an investigation that seeks to discuss a philosophically deep, important, and fundamental concept while also being technically precise about different rules of inference and the different systems of proof they engender. We are not in the business of furnishing any proof-theoretical results about normalization of proofs, or ordinal strengths of mathematical theories, or system-embeddings, to name just a few of the usual pursuits of technically-minded proof-theorists. Rather, we are investigating *the most basic concept* that is actually *taken for granted* by all proof-theorists pursuing such metalogical results. And it is because of the importance of that concept (and, as we find from examining it more intently, its subtle *elusiveness*) that we hope this study will also engage the interest of the philosopher of logic and language—especially at this juncture, when there is growing collective interest, as documented in §1.1, in the broad approach of *inferentialism* as holding the key to a proper understanding of the meanings of the logical operators. (The present author would urge an inferentialist approach also to logico-mathematical operators, but that is a program of research requiring a further monograph or two for its adequate development.)

With acute awareness of the range of technical familiarity (and of stomach for technicalities) across the two main constituencies to which it is hoped this study will be of interest—namely, proof-theorists, and philosophers of logic and language—the author has sought to strike a balance between the technical and the prosaic. On one hand it is hoped that technically-minded proof-theorists will be satisfied both with the degree of precision attained and with the contacts made with the technical literature, and thereby intrigued to embark on the conceptual elucidation that the work invites them to undertake. On the other hand it is hoped that philosophers of logic and language, for whom conceptual analysis or explication is their bread and butter, will be intrigued enough by the various systemic variations and features described, of which they might not have been fully aware, to question whether inferentialists do, after all, have an absolutely satisfactory grasp of the central concept on which their school of thought depends.

§2. The variety of proof systems. Mathematical logicians and proof-theorists have invented a wide array of proof systems, deploying a variety of different formats for the presentation of proofs.

¹¹ There is then the further question—here set aside, in keeping with our present catholicity of intent—whether the strictly classical rules of inference contribute some extra ‘classical’ ingredient to the determination or constitution of ‘the meaning’ of the negation operator. The present author (with *noncatholic*, because anti-realist, intent) has argued that the best way to understand the role of the strictly classical rules of negation is to regard them as governing negation when it enjoys the meaning conferred on it by just its introduction and elimination rules. The classicist’s preparedness to apply the strictly classical rules ‘for’ negation expresses or manifests a semantic realist, *metaphysical*, attitude. This is to the effect that reality is so constituted as to render determinate the truth value of the ‘litmus sentence’ involved in the application of a classical rule. See Tennant [39] for further details.

Let us at the outset name two proof-formats that ought to be avoided. First there are the original Frege–Hilbert systems, which generate conclusions from premises in highly artificial ways, employing many complex axiom-schemes and highly restricted rules of inference. They have featured prominently in the literature of metalogic, going back to Gödel’s proof of the completeness of first-order logic (see Gödel [13]). Proofs in these systems, however, simply fail to do justice to the actual patterns of (correct) deductive reasoning undertaken by expert mathematicians. This remains true even if the Frege–Hilbert formal proofs, which are finite *sequences* of formula occurrences, are re-arranged into the tree-like arrays that make vivid the *partial* ordering of rule-applications involved in them.¹² The second kind of proof-format that ought to be avoided is Jeffrey’s ‘tree method’ of Jeffrey [18]. While this is an accessible format enabling proof-search with simple deductive examples for the beginner (in *Classical Logic*), Jeffrey’s ‘tree proofs’ bear little structural affinity with the lines of reasoning typically involved in mathematical proofs. It is also impossible, with the rules for tree proofs, to distinguish important subsystems of Classical Logic, such as Intuitionistic Logic. It would take one too far afield to make this objection dispositively stick; the reader is advised simply to take this dismissal on trust. As an exercise in order to be persuaded, the reader could sympathetically inquire whether the task of proving ψ from premises $\varphi_1, \dots, \varphi_n$ always takes the form of chasing contradictions along all branches of a tree that results from unraveling the new set of assumptions $\varphi_1, \dots, \varphi_n, \neg\psi$ (the tree’s ‘trunk’), in (ultimately fruitless but thorough) search for a way of satisfying them on some branch progressing from that trunk. The constructivist would dismiss such a suggestion out of hand. The classicist will dismiss it after being discombobulated in the task of, say, formalizing as such a tree proof Cantor’s simple argument establishing that every set has strictly more subsets than members. The resulting Jeffrey tree bears no structural resemblance at all to the real underlying tree of sentence occurrences that is involved in Cantor’s deductive reasoning. The latter tree really needs to be regimented, instead, as a *Gentzen-style natural deduction* (for general discussion of which, see below). This remark applies quite generally, to all informal mathematical proofs. Cantor’s informal proof is here taken as just one tractable example by means of which one can convince oneself of the general point here being made. In so far as mathematical proofs are ‘tree-like’, they are to be regimented as Gentzenian trees, not Jeffrey trees.

¹² Jaśkowski [17] was an early critic of the Frege–Hilbert proof-format:

In 1926 Prof. J. Łukasiewicz called attention to the fact that mathematicians in their proofs do not appeal to the theses of the theory of deduction [i.e. the theorems of a Frege–Hilbert deductive system—NT], but make use of other methods of reasoning. The chief means employed in their method ist [*sic*] that of an arbitrary supposition. (p. 5)

Thanks are owed to an anonymous referee for this reference. This referee also pointed out that ‘the most used interactive theorem provers are nowadays based on natural deduction systems (e.g., Agda, Coq, Lean, Mizar, Nuprl) and not on Frege–Hilbert systems.’ The present author is in full agreement. He had himself been part of this methodological convergence on natural deduction as the most appropriate proof-format for automated proof-search—see Tennant [38]. The latter work also emphasized the benefits, for automated proof-search, of having *all* elimination rules in so-called ‘generalized’ or ‘parallelized’ form (as in Schroeder-Heister [29]), *with no nontrivial proof-work above their major premises*. Further elaboration on this feature, however, would take us too far afield in the present study.

Gentzen's work (Gentzen [10, 11]) on systems of natural deduction, and their associated sequent calculi, was the great advance in this regard. At long last expert mathematical reasoning *could* be regimented, or formalized, as natural deductions. These would merely fill in suppressed details within informal but rigorous proofs, while preserving the essential 'lines of argument' within them. Natural deductions, or proofs, are tree-like arrays of sentence occurrences, whose branchings arise from applications of primitive rules of inference. Premises of the proof occupy leaf-nodes; the conclusion is at the root.

That the contrast between Gentzen's system of natural deduction and the earlier, much more artificial Frege–Hilbert systems reflects favorably on the former appears to be tacitly conceded at p. 25 in Hilbert and Ackermann [15]—the second, 'improved' (*verbesserte*) edition of *Grundzüge der theoretischen Logik* (whose first edition had appeared in 1928, before Gentzen's breakthrough work). In their second edition Hilbert and Ackermann—the former, in context and to his great credit, magnanimously—wrote¹³

We mention finally one more system, one occupying a special position, the 'calculus of natural deduction' set up by G. Gentzen, which emerged from *the endeavor to make the formal derivation of formulae resemble more closely than it has until now the contentful procedure of proof that is customary, for example, in mathematics*. The calculus contains no logical axioms, but only rules of inference, which specify which consequences can be drawn from given assumptions, as well as rules that deliver formulae while rendering them independent of [certain] assumptions. [Emphasis added.]

Note how in the final clause of this quote Hilbert and Ackermann are commending Gentzen's regimentation of the deductive method of making assumptions 'for the sake of argument' and subsequently discharging them when applying a particular rule. Examples of such rules would be the now familiar \rightarrow -I (Constructive Reductio), \rightarrow -E (Conditional Proof), \vee -E (Proof by Cases), \exists -E (Existential Elimination), and CR (Classical Reductio). *That they involve potential discharge of assumptions* is 'built into' their very nature as rules of inference. Moreover, if one replaces the usual serial elimination rules for \wedge , \rightarrow , and \forall :

$$\frac{\varphi \wedge \psi}{\varphi} \quad \frac{\varphi \wedge \psi}{\psi} \quad \frac{\varphi \rightarrow \psi \quad \varphi}{\psi} \quad \frac{\forall x \psi}{\psi^x}$$

¹³ This is the author's translation. Another translation is provided in Hilbert and Ackermann [16] at p. 30. The original German is as follows.

Wir erwähnen endlich noch als eine Sonderstellung einnehmend den von G. Gentzen aufgestellten „Kalkül des natürlichen Schließens“[fn], der aus dem Bestreben hervorgegangen ist, das formale Ableiten von Formeln mehr als bisher dem inhaltlichen Beweisverfahren, wie es z. B. in der Mathematik üblich ist, anzugleichen. Der Kalkül enthält keine logischen Axiome, sondern nur Schlußfiguren, die angeben, welche Folgerungen aus gegebenen Annahmen gezogen werden können, sowie solche, die Formeln liefern, bei denen die Abhängigkeit von den Annahmen beseitigt ist.

with their *parallelized* forms, then one has a new deductive environment calling for the use of dischargeable assumptions:

$$\begin{array}{c}
 \frac{(i) \boxed{\varphi, \psi} (i)}{\theta} \\
 \vdots \\
 \frac{\varphi \wedge \psi}{\theta}
 \end{array}
 \quad ; \quad
 \begin{array}{c}
 \frac{\boxed{\psi} (i)}{\theta} \\
 \vdots \\
 \frac{\varphi \rightarrow \psi}{\theta}
 \end{array}
 \quad ; \quad
 \begin{array}{c}
 \frac{(i) \boxed{\psi_{I_1}^x, \dots, \psi_{I_n}^x} (i)}{\theta} \\
 \vdots \\
 \frac{\forall x \psi}{\theta}
 \end{array}$$

(See §3.5 for an explanation of our use here of the box notation \square on the inference strokes.)

The winning format for the regimentation of mathematical proofs, in our carefully considered opinion, is that of natural deduction. The reader should be cautioned, however, that proof-theorists since Gentzen have introduced variations of ‘formatting’ that can distract from a focused discussion of fundamentals.

Closely related to the system of natural deduction is Gentzen’s sequent calculus.¹⁴ In the case of the sequent calculus, some, like Gentzen himself, take *sequents* $\Delta : \Gamma$ to involve *sequences* Δ and Γ of formal sentences. Yet others treat Δ and Γ as *multisets* of formal sentences. Sequences present their member-sentences in a particular linear order, allowing repetitions. Multisets are ‘in between’ sequences and sets. Like sets, they are unordered; but, like sequences, multisets register how many occurrences each member-sentence enjoys. It was this initial reluctance to treat only of *sets* Δ and Γ of sentences that occasioned the need for the rather unusual ‘structural’ rules (in sequent calculi) of interchange and contraction. This was by way of contrast with the systems of natural deduction, all of whose rules were focused strictly on the logical operators. There are no ‘structural rules’ in systems of natural deduction. In intuitionistic (hence also classical) natural deduction, however, there is the Absurdity Rule, or *Ex Falso Quodlibet*:

$$\frac{\perp}{\varphi}$$

This rule in natural deduction has the same (disastrous, because *irrelevantizing*) effect as the sequent-calculus structural rule of Thinning on the Right, in permitting proof of the First Lewis Paradox

$$\frac{A \quad \neg A}{B}$$

Gentzen’s resort to non-singleton succedents was a device to achieve *classicism* in his sequent calculus. The sequent calculus for Intuitionistic Logic needs only single-conclusion sequents. These points are well known. There is, however, a general lack

¹⁴ The exact relationship between natural deductions and sequent proofs of the same result was complicated to begin with. It involved rather complicated transformations between the original two types of construct—natural deductions not necessarily in normal form, and sequent proofs that could contain cuts and/or thinnings. But the relationship can actually be finessed into an isomorphism, by moving to the setting of (Classical) Core Logic. The isomorphism obtains between natural deductions in normal form, whose eliminations are parallelized and have their major premises standing proud; and the corresponding sequent proofs, which are both cut-free and thinning-free. For more details, see Tennant [45], §5.4.

of appreciation of the fact that Gentzen's resort to multiple succedents Γ —in sequents of the form $\Delta : \Gamma$ for his sequent calculus for Classical Logic—is not actually required *in order to get Classical Logic*. One can provide a perfectly adequate sequent calculus for Classical Logic by, for example, employing the strictly classical rules of Classical Reductio or Dilemma in a single-conclusion sequent setting (where Γ is at most a singleton). For details, see Tennant [45]. The single-conclusion sequent setting also reveals how all proofs in Core Logic, but not the strictly classical proofs to be had in Classical Core Logic, satisfy the 'subformula property': all sentences occurring in proofs occur as subsentences of a premise or of the conclusion. In the case of strictly classical proofs, there will be *negations* that do *not* occur as subsentences of a premise or of the conclusion. These negations would be the assumptions discharged by applications of Classical Reductio, or the negative-horn assumptions that are discharged by applications of Dilemma. Their easier identification makes for a more precise determination of the degree of nonconstructivity of a proof. See especially, in this connection, Tennant [43], §9.¹⁵

In this paper we shall follow the terminological tradition begun by Gentzen, and speak of *sequents* (rather than, say, *arguments*) being established by proofs. But our sequents are not to be confused with those of Gentzen.

DEFINITION 1. *A sequent, in the sense we intend, is of the form $\Delta : \varphi$ or $\Delta : \perp$, where Δ is a finite set of sentences (the premises, or undischarged assumptions), and φ (or \perp) is a single conclusion.*

This definition accommodates the outcomes of all mathematical proofs. Such a proof establishes a mathematical theorem φ from a set Δ of mathematical axioms. Alternatively, the proof might establish the *inconsistency* of those axioms (by having \perp as its conclusion). The *raison d'être* of mathematical logic, ever since Frege, has been the absolutely rigorous regimentation of mathematical proofs. Rules, and proofs constructed using them, need only have *single conclusions*. It is important, at the very outset, to be explicit about this (well-motivated) 'limitation' of focus.¹⁶

We now shelve this discussion of sequent calculus, and re-focus our attention on systems of natural deduction. For this is the setting, or way of formatting regimented proofs, that most readers with a grounding in logic would find most familiar.

§3. A variety of considerations concerning rules of inference. In this section we inquire after every feature that might be relevant for the exact statement or formulation of a logical rule of inference.

3.1. Rules involve subproofs, not just immediate premises. Exponents of natural-deduction systems typically present their logical rules of inference in an intuitively appealing, graphic form. For example, the rule of \wedge -Introduction is usually stated as follows:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}.$$

¹⁵ Thanks to an anonymous referee for eliciting this clarification of the difference between \mathbb{C} -proofs and the strictly classical \mathbb{C}^+ -proofs in the single-conclusion sequent setting.

¹⁶ There is no intention here to detract from the interest and subtleties of multiple-conclusion logic. See the exquisite study by Shoesmith and Smiley [35].

Now the beginner might form the mistaken impression that such a rule (or, at least, the intended rule, thus presented) may be applied only at the very outset of a proof, to two of the available premises, φ and ψ , of the argument to be proved. If so, one can correct that impression by adding another visual clue:

$$\frac{\begin{array}{c} \vdots \\ \varphi \end{array} \quad \begin{array}{c} \vdots \\ \psi \end{array}}{\varphi \wedge \psi}$$

This reassures the learner that the rule can be applied not just at the outset ('to two given premises'), but also 'anywhere in the middle' of the proof; indeed, even 'right at the end'. But a better way to put the matter is to observe that rules of inference, fundamentally, serve to produce new, more complex, proofs from immediate subproofs that have already been constructed. The rule of \wedge -Introduction, despite its deceptively simple graphic statement, does exactly this.

3.2. Distinguishing the sets of assumptions of subproofs. But now another misunderstanding might arise. Must the two immediate subproofs, gestured at here by the descending dots, each use *the same set of undischarged assumptions*, or premises? In general, we want to say 'No'. To make that answer clear in advance, and thereby prevent such a misunderstanding from arising, we can embellish the graphic form of the rule further:

$$\frac{\begin{array}{c} \Delta \quad \Gamma \\ \vdots \quad \vdots \\ \varphi \quad \psi \end{array}}{\varphi \wedge \psi}$$

The different symbols Δ and Γ , standing for the respective premise *sets* of the immediate subproofs, now make it clear that those sets can be distinct. It is left to the discerning beginner now to realize that the overall set of premises on which the newly drawn conclusion $\varphi \wedge \psi$ rests is $\Delta \cup \Gamma$. Indeed, we *have* to allow in general for Δ to be not only distinct from Γ , but even, on occasion, *disjoint*. For, how else could

$$\frac{A \quad B}{A \wedge B}$$

count as a proof (which it is!) of $A \wedge B$ from the set $\{A, B\}$ of premises?

And of course the two subproofs themselves will in general be distinct. This much is already obvious from the fact that their respective conclusions, φ and ψ , can in general be distinct. So, we might as well make that consideration explicit too, by eliminating the equiform lines of dots, and supplying explicit, distinct, symbols for the immediate subproofs themselves:

$$\frac{\begin{array}{c} \Delta \quad \Gamma \\ \Pi \quad \Sigma \\ \varphi \quad \psi \end{array}}{\varphi \wedge \psi}$$

When one sets about finding a proof whose terminal step will be an application of the rule of \wedge -Introduction, one makes no 'assumptions for the sake of argument' (that is,

assumptions to be used in at least one of the *subproofs*, but to be discharged upon application of the rule). This is clear from the graphic notation. The two sets Δ and Γ sit up at the top, *with no subtractions* (i.e., with *no discharges taking place*), and one is given to understand that it will be their *union* $\Delta \cup \Gamma$ upon which the newly drawn conclusion $\varphi \wedge \psi$ finally depends.

With this more expansive mode of stating a rule such as \wedge -I, we are now incorporating explicit mention of the sets Δ and Γ of premises of the respective immediate subproofs, and making clear what the overall set of premises of the newly constructed proof will be (namely, their union $\Delta \cup \Gamma$). This contrasts with what happens in Martin-Löf's type theory, where the corresponding rule would be

$$\frac{t_1 : \varphi \quad t_2 : \psi}{\langle t_1, t_2 \rangle : \varphi \wedge \psi}.$$

Here, the t -terms to the left of the colons code proofs, which are taken to be of the conclusion-propositions to the right of the colons respectively. One would still need some way of determining (effectively) what the (three) respective premise-sets are. So, this way of framing \wedge -I in type theory is about as cryptic and not-fully-explicit as the highly condensed graphic rule

$$\frac{A \quad B}{A \wedge B}$$

that it has been our concern to make fully explicit. To the extent that a similar making-explicit can occur with Martin-Löf's rules, by interpreting them in due course as inductive clauses in an inductive definition of the notion ' t is a proof of A from ...' (with '...' eventually filled in), what emerges from this process will be roughly equivalent, or homologous, to what we shall have to say about the *ordinary* rules of natural deduction. The ordinary rules deal, conveniently, with what is *prima facie*, without adopting any high-level theoretical re-construal, an *untyped object-language*,¹⁷ and it is in this simple setting that our motivating question 'What exactly *is* a primitive rule of inference?' can both reveal its challenging complexity and be worthy of a full and clarifying investigation. So we shall opt to set Martin-Löf's type theory to one side, in order to pursue our investigations to a satisfying conclusion, while being mindful that their outcome ought to be able to furnish a methodology for answering the same question posed with specific reference to the primitive rules of Martin-Löf's type theory.¹⁸

3.3. Rules involving assumptions for the sake of argument. With another very familiar introduction rule, however, matters are different. The reader will no doubt be in mind here of the *Rule of Conditional Proof*, as it is usually known. In the more

¹⁷ I owe to an anonymous referee the observation here that

[F]rom the point of view of the proofs-as-programs correspondence (a.k.a. the Curry–Howard correspondence), a formula corresponds to a type. Thus, the standard natural deduction rules deal with a typed-language. In particular, they can be seen as rules explaining how to construct a type from other given types.

¹⁸ This, though, is to adopt the kind of 'high-level theoretical re-construal' just adverted to. The author is grateful to an anonymous referee for raising the question of the status of the rules of Martin-Löf's type theory in the context of this study.

uniform rule-terminology of the natural-deduction theorist, it is called the rule of \rightarrow -Introduction. Its usual graphic statement takes either one of the following two (equivalent) forms:

$$\frac{\begin{array}{c} \text{---}(i) \\ \varphi \\ \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi} \quad , \quad \frac{\begin{array}{c} [\varphi] \\ \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi}$$

Here, we shall prefer the first of these two forms. It offers at least two advantages. First, it makes the useful suggestion that one can and should, in the construction of a natural deduction, avail oneself of the various numerals 1, 2, 3, ... in place of i in order to show which assumption-occurrences are discharged at which steps. For, any given occurrence of an assumption ‘for the sake of argument’, when it is finally discharged, is discharged by a particular application, lower down in the proof-tree, of a rule of inference such as \rightarrow -Introduction. Second, placing a discharge stroke *over* any assumption-occurrences being discharged provides an effective visual reminder that the assumption in question no longer qualifies for inclusion among the assumptions on which the newly drawn conditional conclusion depends.

The standard understanding, to which we shall adhere, is that when any assumption such as φ for \rightarrow -Introduction above gets discharged, it is discharged *at its every eligible occurrence* as a leaf-node of the proof-tree. So the sentence φ is in effect subtracted from the set of sentences on which the subordinate conclusion ψ , according to the subordinate proof, depends. This is what discharging φ consists in.

The usual graphic form for \rightarrow -Introduction of course falls short of being fully explicit, in much the same ways as did the rule for \wedge -Introduction. Indeed, matters are now subtly more complicated by the fact that discharge of assumptions may take place. When we supply the extra symbols for maximal explicitness, we need to stress that the set Δ of *undischarged* assumptions (*after* the rule has been applied) does *not* contain the assumption φ , since it is being discharged¹⁹:

$$\frac{\frac{\Delta, \varphi}{\Pi} \text{---}(1)}{\varphi \rightarrow \psi} \text{---}(i)$$

But it is clear also that the set of undischarged assumptions of the immediate subproof Π (of the conclusion ψ) is $\Delta \cup \{\varphi\}$ (*if* indeed φ has been used as an assumption—see below).

¹⁹ Troelstra and Schwichtenberg [47], §2.1.9 at p. 43, call this the ‘complete discharge convention’, according to which one is asked to ‘discharge always all open assumptions of the same form, whenever possible’. The author is grateful to an anonymous referee for supplying this reference.

Of course, Δ itself *may* be empty. But it is important to realize that in general Δ may be *nonempty*. That is why it is advisable to make Δ explicit. For one wants no misunderstanding on this score.

3.4. The possibility of vacuous discharge of assumptions. What beginners often fail to realize for themselves is that an application of \rightarrow -Introduction may be licit even when *there is no use made of the assumption φ* . Thus, for example, the following one-step proof involves an application of \rightarrow -Introduction:

$$\frac{\psi}{\varphi \rightarrow \psi} .$$

There is no assumption of the form φ within the degenerate proof ψ (of conclusion ψ from premise set $\{\psi\}$) that serves as the immediate subproof Π for this application of \rightarrow -Introduction. The step is said to involve *vacuous discharge of assumptions*; that is to say, it does not discharge any assumptions at all.

This vacuous-discharge step can be followed by another application of \rightarrow -Introduction, this time discharging the assumption ψ (which *has* been used), in order to obtain a proof of the *logical theorem* $\psi \rightarrow (\varphi \rightarrow \psi)$. It is called a logical theorem because the proof produces it as a conclusion ‘from’ the *empty* set of assumptions (i.e., from no assumptions at all):

$$\frac{\frac{\text{---}(1)}{\psi}}{\varphi \rightarrow \psi} (1)}{\psi \rightarrow (\varphi \rightarrow \psi)} (1) .$$

So: the rule of \rightarrow -Introduction (in Gentzen’s original formulation) is to be construed as permitted in application *without* any assumption-occurrence of φ having been used (in order to derive ψ in the subordinate proof) and therefore being available for discharge. This permission (for rule-application) ought to be marked explicitly in our graphic notation for stating the rule. We shall opt for a use of the diamond symbol as follows:

$$\frac{\frac{\Delta \cdot \varphi}{\Pi} \quad \diamond \text{---}(i)}{\varphi \rightarrow \psi} (i) .$$

We choose the symbol \diamond because of its connotation, from deontic logic, of *permission*. The rule of \rightarrow -Introduction is permitted to be applied even without the antecedent φ of the conditional having been used as an assumption in the subproof Π .

3.5. On rules with nonvacuous discharge of assumptions. In deontic logic, there is another symbol, \square , which represents *obligatoriness*; and this makes it a suitable choice as a marker on the discharge stroke when it is *required* that the assumption in question should actually have been used, and be available for discharge, when the rule in question is applied. Put another way: the rule-applier is *obliged* to have made use of the (thus far, undischarged) assumption in question in the immediate subproof concerned. Thus

the standard graphic formulation of *Proof by Cases*, or \vee -Elimination:

$$\frac{\begin{array}{ccc} \text{---}(i) & \text{---}(i) & \\ \varphi & \psi & \\ \vdots & \vdots & \vdots \\ \varphi \vee \psi & \theta & \theta \end{array}}{\theta} (i)$$

could be made fully explicit, and have its case-assumptions annotated, as follows²⁰:

$$\frac{\begin{array}{ccc} \Xi & \square \text{---}(i) & \square \text{---}(i) \\ \Omega & \underbrace{\Delta, \varphi}_{\Pi} & \underbrace{\Gamma, \psi}_{\Sigma} \\ \varphi \vee \psi & \theta & \theta \end{array}}{\theta} (i)$$

The reason for imposing this requirement on the case-assumptions is that it would be pointlessly unproductive to violate it. If, for example, the subproof Π did *not* involve φ among its undischarged assumptions, then Π would establish the sequent $\Delta : \theta$ —a potentially *stronger result* than the sequent $\Xi, \Delta, \Gamma : \theta$ that would be established by applying the rule.²¹

Prawitz [23] took over from Gentzen [10, 11] the formulations of both \vee -E and \exists -E as permitting vacuous discharge of assumptions (the formulation that in the graphic notation proposed here would involve annotating the discharge strokes with \diamond). Prawitz was aware, however, that a requirement of non-vacuous discharge (which we indicate instead with \square) could be imposed on these rules without loss of completeness. At pp. 49–50 he wrote:

By a *normal* deduction, I shall ... mean a deduction that contains no maximum segment and no redundant applications of \vee E or \exists E. An application of \vee E or \exists E in a deduction is said to be redundant if it has a minor premiss at which no assumption is discharged; obviously, such applications are superfluous ... We now have:

THEOREM 1. *If $\Gamma \vdash A$ holds in the system for intuitionistic or minimal logic, then there is a normal deduction in this system of A from Γ .*

²⁰ Note that in Core Logic, major premisses for eliminations *stand proud*, with no non-trivial proof-work above them. This means that the subproof Ω in this statement of \vee -Elimination would be $\varphi \vee \psi$ itself. But here we are following the more conventional presentation of natural deduction (established by Gentzen [10, 11] and Prawitz [23]), which allows for nontrivial proofs of major premisses of eliminations. Another point of contrast is that in Core Logic *all* elimination rules are *parallelized*, whereas the Gentzen-Prawitz tradition allows for *serial* forms of the elimination rules for \wedge , \rightarrow and \forall . See Tennant [45] for more details. We return to this point in §3.7.

²¹ We say *potentially* here because it cannot be guaranteed that the premise set Δ is logically *weaker* than the premise set Ξ, Δ, Γ . For it *might* be the case that Δ logically implies every member of Ξ, Γ . In cases where this is *not* so, however, the sequent $\Delta : \theta$ would be logically *stronger* than the sequent $\Xi, \Delta, \Gamma : \theta$.

Indeed, there is no loss of completeness in the case of classical logic either, if one insists on nonvacuous discharge with \vee -E or \exists -E.

We turn finally to another kind of rule-restriction that could be imposed in the classical case without any loss of completeness. An anonymous referee raised the following interesting question. Consider this restricted form of the \vee -E rule:

$$\frac{\begin{array}{c} \Xi \\ \Omega \\ \varphi \vee \psi \end{array} \quad \frac{\begin{array}{c} \square \text{---}(i) \\ \Delta, \varphi \\ \Pi \\ \perp \end{array} \quad \frac{\begin{array}{c} \square \text{---}(i) \\ \Gamma, \psi \\ \Sigma \\ \perp \end{array}}{(i)}}{\perp}$$

In Classical Core Logic \mathbb{C}^+ one can prove Disjunctive Syllogism as follows:

$$\frac{A \vee B \quad \frac{\neg A \quad A^{(1)}}{\perp} \quad \frac{\neg B \quad B^{(2)}}{\perp}}{(CR) \frac{\perp}{B} (2)}$$

The question is: Can one make do, in \mathbb{C}^+ , with \vee -Eliminations only in this restricted form? The answer is affirmative. This is a feature of *classical* reasoning akin to being able to obtain all desired classical results using only *terminal* applications of Classical Reductio (i.e., using CR only at the *final* step of the proof)—see §5.2; or using only *atomic* applications of Dilemma (in the propositional case)—again, see §5.2.

3.6. Empty sets of assumptions. We noted above that the set Δ of assumptions for conditional proof may be empty. Now there are *some* rules where one wishes to insist, by contrast, that there be no assumptions at all, i.e. that Δ be *empty*. Perhaps the best known example of such a rule is the Rule of Necessitation in modal logic: when one has deduced φ from *no assumptions*, then one may infer $\square\varphi$. Many an author has observed that it would be very unwise to formulate this rule graphically as

$$\frac{\varphi}{\square\varphi}.$$

This is because such a cryptic formulation does not advert in any way to the requirement that φ should have been deduced from no assumptions at all. In order to remedy this, a fully explicit graphic formulation of the Rule of Necessitation would be

$$\frac{\emptyset \quad \Pi \quad \varphi}{\square\varphi}.$$

Here, \emptyset is the empty set.

3.7. Major premises of eliminations can stand proud. In connection with Proof by Cases (\vee -Elimination) we spoke of *sequents* being established by proofs. A sequent, in this sense, as explained in Definition 1, is of the form $\Delta : \varphi$ or $\Delta : \perp$, where Δ is a finite

set of sentences (the premises, or undischarged assumptions) and φ (or \perp) is a single conclusion.

The introduction and elimination rules of natural deduction have analogs in the sequent calculus. These latter rules are known as *Right logical rules* and *Left logical rules* respectively. It would be instructive at this juncture to inspect just the Left logical rule for \vee , which is what corresponds to the rule of \vee -Elimination in natural deduction:

$$\frac{\varphi, \Delta : \theta \quad \psi, \Gamma : \theta}{\varphi \vee \psi, \Delta, \Gamma : \theta}.$$

With the left logical rule for \vee stated in this form, it is to be understood that φ is not a member of Δ , and ψ is not a member of Γ . Note that the formulation of \vee -Elimination that would *really* correspond directly to this sequent rule is

$$\frac{\frac{\frac{\Delta, \varphi}{\Pi} \quad \square \text{---}(i)}{\theta} \quad \frac{\frac{\Gamma, \psi}{\Sigma} \quad \square \text{---}(i)}{\theta}}{\varphi \vee \psi, \theta} (i)$$

where the *absence* of any symbol for a proof above the major premise $\varphi \vee \psi$ is taken to indicate that *there is no proof-work at all above $\varphi \vee \psi$* . That is, the major premise $\varphi \vee \psi$ *stands proud*, as a leaf-node of the proof-tree that is created by applying the rule. To be sure, it is a very ‘low down’ leaf node, on a shortest possible twig leading up from the bottommost indicated occurrence of θ ; but it is a leaf node nonetheless.

§4. Are constraints in rule-applications partly constitutive of the rules themselves?

It is an important fact, not widely registered, that the elimination rules of a system of natural deduction can always be stated in parallelized form (see the examples on p. 28) and indeed *also* in the more austere form just mentioned, in which major premises stand proud, with no proof-work above them. This constraint ensures that all the resulting proofs will be in *normal form*.

In the three orthodox systems **M** of Minimal Logic, **I** of Intuitionistic Logic, and **C** of Classical Logic (for which, most conveniently, see Prawitz [23]), the rules \wedge -E, \rightarrow -E and \forall -E are stated in their *serial* forms; and the rules \neg -I, \rightarrow -I, and Classical Reductio (for **C**) *permit vacuous discharge*. Moreover, in these three orthodox systems, all major premises for eliminations—be they serial eliminations, or parallelized ones, such as \vee -E or \exists -E—are allowed to stand as conclusions of nontrivial proof-work.

It is nevertheless the case that every proof in these orthodox systems (in which major premises for eliminations are permitted to stand as conclusions of nontrivial proofs) can be effectively transformed into a proof in normal form. And *these normal-form proofs can be re-written in such a way that all eliminations are parallelized, with their major premises standing proud*. Thus normal proofs in this more exigent form still furnish a complete proof system.

The systems of Core Logic and Classical Core Logic differ from the three aforementioned orthodox systems in certain crucial ways. First, all eliminations are

parallelized. Second, all major premises for eliminations must stand proud. Third, there is no rule of Ex Falso Quodlibet. This is in order to ensure relevance of premises to conclusions. Fourth, certain rules are subtly but importantly tweaked.

The tweaks that the Core systems impose, of which there are three, are as follows. First, \neg -I does not allow vacuous discharge. This holds for both the Core systems. Likewise, in Classical Core Logic, Classical Reductio does not allow vacuous discharge.

Second, the rule of \rightarrow -I is liberalized, by being furnished with a *new part*:

$$\frac{\frac{\Delta, \varphi}{\Pi} \quad \square _ (i)}{\varphi \rightarrow \psi} \quad \perp (i)$$

This new part of \rightarrow -I allows one to prove, without forced recourse to EFQ, that *if a conditional's antecedent is false then the conditional is true* (which is what we are told by the third and fourth lines of the truth table for \rightarrow):

$$\neg A : A \rightarrow B.$$

Lest the reader think that this reference to truth tables renders the point being made provincial to Classical Logic, the author hastens to point out that we are using here only the 'left-to-right' readings of the lines of the truth tables, and these are intuitionistically valid.²²

Third, the rule \vee -E (Proof by Cases) is also liberalized, so as to allow the following: if one of the case-proofs has \perp as its conclusion, then one can bring down, as the main conclusion, the conclusion of the other case-proof. This liberalized form of \vee -E can be represented graphically as follows:

$$\frac{\frac{\Delta, \varphi}{\Pi} \quad \square _ (i) \quad \frac{\Gamma, \psi}{\Sigma} \quad \square _ (i)}{\varphi \vee \psi \quad \theta / \perp \quad \theta / \perp (i)} \quad \theta / \perp (i)$$

This liberalized form of \vee -E allows one to prove Disjunctive Syllogism

$$A \vee B, \neg A : B$$

without forced recourse to EFQ. Here is the proof:

$$\frac{A \vee B \quad \frac{\neg A \quad A^{(1)}}{\perp} \quad B^{(1)}}{B} \quad (1)$$

The foregoing tweaks for the rules of the core systems \mathbb{C} and \mathbb{C}^+ are made in order to ensure relevance of premises to conclusions, without sacrificing

²² Thanks are owed to an anonymous referee for raising the need for this clarification.

completeness. Importantly, the two core systems do *not* contain the rule Ex Falso Quodlibet.²³

Now for the conceptual question that cries out for an answer here: are we justified in thinking that we are ‘merely tweaking’ a pre-existing rule, which somehow *maintains its identity as the rule that it ‘is’*, rather than formulating a *completely distinct rule*, masquerading, one might say, under the former respective label (\neg -I; CR; \rightarrow -I; or \vee -E)? When are ‘slightly’ altered conditions of applicability serious enough not to count as mere strategic constraints on the application of prior, stably identifiable rules, but rather as such important constitutive ingredients in the formulation of ‘the’ rule that it has become *a different rule altogether* from its prior namesake?

This worry about rule-identifications can arise, in principle, for *any* of the variable aspects canvassed above: serial v. parallelized form of elimination; requiring major premises of eliminations to stand proud; requiring non-vacuous discharge; and allowing a more ‘liberalized’ range of application. It would seem, at first glance, as though the ‘stable rule but with tweaks’ conception can be sustained only by concentrating on one particular (but ‘double-barreled’) central aspect, as solely constitutive of the ‘identity’ of the rule in question, which is: *Does the rule concern just one occurrence of ‘its’ logical operator? And does that operator occur immediately below the inference stroke (in the conclusion)?; or immediately above it (in the major premise)?; or—so as to accommodate the classicist—only in assumptions that applications of the rule discharge?*

At this point we should address a question raised by an anonymous referee, which bears on the identification problem that we are raising here. The referee invites one to consider two different conditional operators \rightarrow and \Rightarrow . They both enjoy Modus Ponens as their elimination rule. But they differ in their introduction rules. The conditional \rightarrow is governed by the usual rule of Conditional Proof, *permitting* vacuous discharge of the antecedent as an assumption. The otherwise similar-looking introduction rule for the conditional \Rightarrow , however, *prohibits* vacuous discharge. It is an easy exercise to deduce $A \rightarrow B$ from $A \Rightarrow B$, and vice versa. Thus the two conditionals satisfy Belnap’s ‘uniqueness condition’ (Belnap [2], p. 133). The referee asks ‘Should one then conclude that \rightarrow and \Rightarrow are the same connective and that their rules are thus fundamentally the same?’. The referee answers in the negative, giving as reason that ‘Belnap’s uniqueness condition is a condition concerning the provability (or derivability) level—since it concerns the inter-derivability of $A \rightarrow B$ and $A \Rightarrow B$ —and not the proof level.’ We agree with this assessment; and would point out also that one should invoke here the criterion of synonymy (in this case, of the two conditionals under discussion) that Smiley formulated quite generally for connectives in Smiley [36]. For synonymy, the two conditionals would need to be interreplaceable, *salva veritate*, in *all* (metalinguistic) statements of deducibility. And in general, their mere interdeducibility is not a sufficient condition for this to hold. This can be seen in the present instance from the simple facts that

$$B \vdash A \rightarrow B \quad \text{but} \quad B \not\vdash A \Rightarrow B,$$

²³ More details about Core Logic \mathbb{C} and Classical Core Logic \mathbb{C}^+ are to be found in Tennant [42–45]. It is important to appreciate, however, that the constraint that major premises of eliminations should stand proud can also be imposed on natural-deduction formulations of the three orthodox systems \mathbb{M} of Minimal Logic, \mathbb{I} of Intuitionistic Logic, and \mathbb{C} of Classical Logic, with all elimination rules parallelized. See Tennant [45] for those details.

which together show that by Smiley’s criterion the two conditionals are *not* synonymous.

4.1. Rules can have parts. In every logical system dealing with disjunction, the rule of \vee -Introduction is stated in two parts:

$$\frac{\varphi}{\varphi \vee \psi} ; \quad \frac{\psi}{\varphi \vee \psi} .$$

It is difficult to say exactly what the criteria are for identifying the parts of a rule. Would one have succeeded, for example, in stating the rule of \vee -Introduction in ‘just one part’ if one were to write

$$\frac{\theta}{\varphi \vee \psi} \text{ where } \theta = \varphi \text{ or } \theta = \psi \quad \dots?$$

In the other direction—of fission of a single whole into parts, rather than fusion of parts into a single whole—one might suggest that the earlier rule of \rightarrow -Introduction:

$$\frac{\begin{array}{c} \diamond \text{---}(i) \\ \Delta, \varphi \\ \hline \Pi \\ \hline \psi \text{---}(i) \\ \hline \varphi \rightarrow \psi \end{array}}{\varphi \rightarrow \psi} .$$

could (and should) be re-written in two parts, thus:

$$(\rightarrow -I)_1 \frac{\begin{array}{c} \text{---}(i) \\ \Delta, \varphi \\ \hline \Pi \\ \hline \psi \text{---}(i) \\ \hline \varphi \rightarrow \psi \end{array}}{\varphi \rightarrow \psi} ; \quad \text{and} \quad (\rightarrow -I)_2 \frac{\begin{array}{c} \Delta \\ \Pi \\ \hline \psi \\ \hline \varphi \rightarrow \psi \end{array}}{\varphi \rightarrow \psi} , \text{ where } \varphi \notin \Delta .$$

Our earlier two-step proof could have each of its steps annotated as follows, in order to make clear which *part* of the rule is being applied:

$$\frac{\begin{array}{c} \text{---}(1) \\ (\rightarrow -I)_2 \frac{\psi}{\varphi \rightarrow \psi} \\ \hline (\rightarrow -I)_1 \frac{\varphi \rightarrow \psi}{\psi \rightarrow (\varphi \rightarrow \psi)} \end{array}}{\psi \rightarrow (\varphi \rightarrow \psi)} .$$

4.2. Generalized or parallelized forms of the elimination rules. The apartness of the two usual parts of \vee -Introduction is much more entrenched than the apparent apartness of the two usual parts of the *serial* form of the rule of \wedge -Elimination, which is usually the favored form:

$$\frac{\varphi \wedge \psi}{\varphi} ; \quad \frac{\varphi \wedge \psi}{\psi} .$$

But, as we pointed out earlier, the rule of \wedge -Elimination can also be stated in the single *generalized* or *parallelized* form

$$\frac{\begin{array}{c} (i) \text{---} \boxed{\text{---}} (i) \\ \underbrace{\varphi, \psi} \\ \vdots \\ \varphi \wedge \psi \quad \theta (i) \end{array}}{\theta}$$

where the box indicates that at least one of φ, ψ must have been used as an assumption and thereby be available for discharge. Of course, the rule applies when both φ and ψ have been used. In such a circumstance, both of them are available for discharge, and get discharged.

The rule of \vee -Elimination (Proof by Cases), as we have seen, is already in generalized or parallelized form, as is the rule of \neg -Elimination:

$$\frac{\begin{array}{c} \vdots \\ \neg\varphi \quad \varphi \end{array}}{\perp}$$

The remaining standard connective to discuss in this regard is \rightarrow . As already mentioned, the time-honored rule of Modus Ponens is the usual (but serial) form of \rightarrow -Elimination (which we now state with its major premise leftmost, for uniformity):

$$\frac{\varphi \rightarrow \psi \quad \varphi}{\psi}$$

And as we saw on p. 10, this rule too can be parallelized; but in this case, unlike the case involving \wedge , the re-formulation does not involve fusion of erstwhile separate parts of the rule:

$$\frac{\begin{array}{c} \varphi \rightarrow \psi \quad \varphi \quad \theta (i) \\ \vdots \quad \vdots \\ \psi \quad \theta (i) \end{array}}{\theta}$$

The reader can check that in natural deduction it is the parallelized forms rather than the serial forms of elimination rules that make for homology with the corresponding ‘Left logical rules’ in the sequent calculus. Consider, for example, the fully explicit form of parallelized \rightarrow -Elimination:

$$\frac{\begin{array}{c} \Delta \quad \Gamma, \psi \\ \Pi \quad \Sigma \\ \varphi \rightarrow \psi \quad \varphi \quad \theta (i) \end{array}}{\theta}$$

The Left logical rule of the sequent calculus that corresponds to this is

$$\frac{\Delta : \varphi \quad \Gamma, \psi : \theta}{\varphi \rightarrow \psi, \Delta, \Gamma : \theta}, \text{ where } \psi \notin \Gamma.$$

(One can even go so far as to insist, in both the natural-deduction rule and the corresponding sequent-rule, that the conditional $\varphi \rightarrow \psi$ also not be a member of Γ . One cannot, however, exclude $\varphi \rightarrow \psi$ from Δ , on pain of incompleteness for intuitionistic consequence.)

Only one natural-deduction rule (for a connective) has thus far escaped mention, and that is the rule of constructive *Reductio ad Absurdum*, or \neg -Introduction. Its usual graphic statement is as follows:

$$\frac{\begin{array}{c} \text{---}(i) \\ \varphi \\ \vdots \\ \perp \end{array}}{\neg\varphi}(i)$$

The reader will by now be aware of the potential shortcomings in such a cryptic (i.e., graphically under-specific) statement of the rule. The most important question to be settled is this: *Does it permit vacuous discharge of assumptions?* If the answer is affirmative—that is to say, if the graphic statement of the rule, when made fully explicit, is

$$\frac{\frac{\Delta, \varphi}{\Pi} \quad \diamond\text{---}(i)}{\neg\varphi}(i)$$

—then any proof system that contains this rule will allow one to prove the negative form of Lewis’s First Paradox:

$$\frac{\neg\psi \quad \psi}{\neg\varphi}.$$

The proof is this simple:

$$\frac{\neg\psi \quad \psi}{\perp} \quad \neg\varphi.$$

An application of \neg -Elimination is followed here by an application of \neg -Introduction with *vacuous discharge*.

This provides the relevantist—who seeks to avoid both the positive and the negative forms of the first Lewis Paradox—strong motivation to make the rule of \neg -Introduction

a ‘no vacuous discharge’ rule:

$$\frac{\frac{\Delta, \varphi}{\Pi} \quad \square\text{—}(i)}{\perp\text{—}(i)} \cdot \neg\varphi$$

4.3. Summary of features of rules that need to be specified. Our discussion thus far has revealed the following features of rules of inference that need to be specified explicitly in order to identify exactly what the rule *is*—or, to put it another way, in order to identify exactly what the rule *says can be done when constructing proofs*.

1. We have to specify the immediate subproofs involved in any application of the rule.
2. We have to specify those subproofs’ sets of undischarged assumptions.
3. We have to specify the status of assumptions for the sake of argument, which are to be discharged when the rule is applied.
4. We have to specify whether major premises for eliminations have to stand proud, or can instead stand as conclusions of nontrivial proof-work.
5. We have to choose between serial and parallelized forms of certain elimination rules.

§5. What is a rule of inference, really? Our concern here is not so much to argue for the choice of any particular logical system (i.e., any particular collection of primitive rules of inference), as it is to get clear about *what kind of thing*, exactly, a primitive rule of inference *is*, from a metalogical or proof-theoretic point of view. The problematic here presents itself quite generally, to anyone thinking about rules of inference. But it is especially pressing for *inferentialists*, since they are the theorists of meaning who are taking the notion of a rule of inference as central and basic for their theorizing.²⁴

Tennant [37] characterized primitive rules of inference in natural deduction as inductive clauses in an inductive definition of the notion of proof-in-a-system- \mathcal{S} . These inductive clauses collectively define the ternary notion²⁵

$$\mathcal{P}_{\mathcal{S}}(\Pi, \varphi, \Delta)$$

—‘the finitary tree-like construction Π is a proof, in the system \mathcal{S} , of the conclusion φ from the set Δ of undischarged assumptions’—for which the basis clause is the obvious

$$\mathcal{P}_{\mathcal{S}}(\varphi, \varphi, \{\varphi\}).$$

²⁴ Thanks are owed to an anonymous referee for pointing out the need to emphasize this point.

²⁵ The proof-theoretic reader will note here a similarity to Kreisel’s inductively defined binary notion $\Pi(c, \ulcorner A \urcorner)$ of [20], for ‘ c is a proof of the formula A ’. Kreisel’s notion is binary and not ternary because it makes no mention of any possible premise-set Δ . He also suppresses mention of any specific system \mathcal{S} . Whether Kreisel’s treatment of proofs as untyped objects leads to a satisfactory theory avoiding any impredicativity, circularity, or inconsistency, is not at all clear. In this regard, see Weinstein [49] and Dean and Kurokawa [6]. Thanks are owed to an anonymous referee for these observations.

In Gentzen [10, 11], the definition of a ‘Herleitung’ (in accordance with his stated rules) is not an explicitly inductive one, although it would invite being recast as an inductive definition were it to be regimented more rigorously. Such regimentation was undertaken by Prawitz [23]. He contented himself, however, with stating only what we are here calling the *graphic* forms of rules. He did not make the explicit point that these graphic forms are really only convenient ways of picturing to oneself the metalinguistic effect of the respective inductive-clause forms of the rules in question. This understanding of proof-constructional matters is arrived at (*op. cit.*, p. 24) rather circuitously in Prawitz’s presentation, by way of a distinction (pp. 22–23) between what he calls ‘proper’ inference rules (ones not involving any potential discharge of assumptions) and ‘deduction’ rules (ones that do involve potential discharge of assumptions). This distinction, though, works against seeing immediately that they are *all* inference rules, *tout court*; and that foundational questions about the nature of inference rules would affect them all equally.

Here we re-visit the more unifying explanatory conception of rules of inference as inductive clauses in a metalinguistic definition of what counts as a proof, in a system, of a conclusion from a set of undischarged assumptions. We do so in order to explain the conception thoroughly, and in order to explore further subtleties of which the inferentialist must nevertheless be mindful. Even if our implied conclusions end up being pessimistic about the prospects for complete clarification of the notion of a logical rule of inference, we nevertheless remain convinced that it is the *inductive clause* conception that holds any prospect of serving the inferentialist’s conceptual and foundational needs when it comes to the problem of characterizing exactly what kind of thing a primitive rule of inference *is*.

The sought clarity about primitive rules will be attained by exploring the implications of our having described any rule ρ , in general, as *saying what can be done when constructing proofs* in a given system (containing ρ).

This pushes our considerations in a possibly holist direction. *Proofs in which particular system?*, one might ask. Consider the straightforward answer ‘Whichever system S one happens to be using, of which the rule ρ in question is a rule’.

5.1. Rules of inference as inductive-definitional clauses in an inductive definition of proof. Suppose the system one happens to be using is intuitionistic logic, for a language whose only logical operators are \neg and \wedge . The proof system $\mathbf{I}_{\{\neg, \wedge\}}$ of intuitionistic logic for these two connectives consists of just those proofs Π generated by the following inductive definition of the ternary relation $\mathcal{P}_{\mathbf{I}_{\{\neg, \wedge\}}}(\Pi, \varphi, \Delta)$ —‘ Π is a proof of φ , in the system $\mathbf{I}_{\{\neg, \wedge\}}$, from the set Δ of undischarged assumptions’. We shall use here the *parallelized* form of \wedge -E, not the serial form. We shall also suppress any overly complex system subscript when it can be understood from the context.

DEFINITION 2 (of $\mathcal{P}(\Pi, \varphi, \Delta)$ for $\mathbf{I}_{\{\neg, \wedge\}}$).

Basis clause: $\mathcal{P}(\varphi, \varphi, \{\varphi\})$.

Inductive clauses (in the form of metalinguistic inference rules, and with convenient labels):

$$(\neg\text{-I}) \quad \frac{\mathcal{P}(\Pi, \perp, \Delta)}{\mathcal{P}\left(\frac{\Pi}{\neg\varphi}, \neg\varphi, \Delta \setminus \{\varphi\}\right)}$$

$$\begin{aligned}
 (\neg\text{-}E) \quad & \frac{\mathcal{P}(\Pi, \varphi, \Delta)}{\mathcal{P}\left(\frac{\neg\varphi}{\perp}, \Pi, \Delta \cup \{\neg\varphi\}\right)} \\
 (\wedge\text{-}I) \quad & \frac{\mathcal{P}(\Pi_1, \varphi_1, \Delta_1) \quad \mathcal{P}(\Pi_2, \varphi_2, \Delta_2)}{\mathcal{P}\left(\frac{\Pi_1 \quad \Pi_2}{\varphi_1 \wedge \varphi_2}, \varphi_1 \wedge \varphi_2, \Delta_1 \cup \Delta_2\right)} \\
 (\wedge\text{-}E) \quad & \frac{\mathcal{P}(\Pi, \theta, \Delta) \quad \Delta \cap \{\varphi_1, \varphi_2\} \neq \emptyset}{\mathcal{P}\left(\frac{\varphi_1 \wedge \varphi_2 \quad \Pi}{\theta}, \theta, \Delta \setminus \{\varphi_1, \varphi_2\}\right)} \\
 (EFQ) \quad & \frac{\mathcal{P}(\Pi, \perp, \Delta)}{\mathcal{P}\left(\frac{\Pi}{\varphi}, \varphi, \Delta\right)}
 \end{aligned}$$

Closure clause: If $\mathcal{P}(\Pi, \varphi, \Delta)$, then this can be shown from the preceding clauses.

The attentive reader will have noticed that we are doing something new here, by bringing into the conclusions of these metalinguistic rules a way of referring to newly constructed proofs—such as

$$\frac{\Pi}{\neg\varphi}; \quad \frac{\neg\varphi \quad \Pi}{\perp}; \quad \frac{\Pi_1 \quad \Pi_2}{\varphi_1 \wedge \varphi_2}; \quad \frac{\varphi_1 \wedge \varphi_2 \quad \Pi}{\theta}; \quad \text{and} \quad \frac{\Pi}{\varphi}$$

—that calls for an explanation of the *proof-term notation* being employed. This is very simple, and is given by the following inductive definition.

DEFINITION 3. *Basis clause:* Any sentence is a proof-term.

Inductive clause: If Π_1, \dots, Π_n are proof-terms, then so too is $\frac{\Pi_1, \dots, \Pi_n}{\theta}$, where θ is either a sentence or the absurdity symbol \perp .

Closure clause: Any proof-term can be shown to be such by means of the Basis and Inductive clauses.

It is obvious that fully expanded proof-terms are tree-like arrays of (i) occurrences of sentences or of \perp , and (ii) horizontal inference-strokes. Which of these proof-terms Π succeed in denoting *proofs* will be determined, for any proof-system \mathcal{S} , by the definition of the ternary notion $\mathcal{P}(\Pi, \varphi, \Delta)$.²⁶

Now the thing about Definition 2 is that it defines the notion of proof in a *very particular system*. We have called that system $\mathbf{I}_{\{\neg, \wedge\}}$.²⁷ The ternary relation $\mathcal{P}(\Pi, \varphi, \Delta)$

²⁶ There are other proof-term notations in the literature. See, for example, the somewhat more complicated notation of Troelstra and Schwichtenberg [47], §2.2 at pp. 45–48. We opt for the slightly simpler notation given by Definition 3 because it is wholly adequate for our purposes, while affording very succinct statements of rules of inference in their eventual, fully explicit forms.

²⁷ An anonymous referee raised the question whether there is any particular reason for choosing the intuitionistic rather than the classical system here, given that the two systems have the

that Definition 2 defines is not one that captures the notion of a *proof* Π in general—that is, in any system whatsoever; rather, we should regard it as capturing only the provincial relation $\mathcal{P}_{\mathbf{I}_{\{\neg, \wedge\}}}(\Pi, \varphi, \Delta)$. Alternatively, the ‘system-specificity subscript’ could be elevated so as to occupy an explicit, fourth argument place in the inductively defined proof-relation:

$$\mathcal{P}(\Pi, \varphi, \Delta, \mathbf{I}_{\{\neg, \wedge\}}).$$

This now puts any particular clause in Definition 2, construed as a rule of inference, in an interesting new light. Let us focus on $(\wedge\text{-I})$. In the context of this particular proof-system, it needs to be understood as the newly illuminated inductive-definitional clause

$$(\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge\}}} \frac{\mathcal{P}(\Pi_1, \varphi_1, \Delta_1, \mathbf{I}_{\{\neg, \wedge\}}) \quad \mathcal{P}(\Pi_2, \varphi_2, \Delta_2, \mathbf{I}_{\{\neg, \wedge\}})}{\mathcal{P}\left(\frac{\Pi_1 \quad \Pi_2}{\varphi_1 \wedge \varphi_2}, \varphi_1 \wedge \varphi_2, \Delta_1 \cup \Delta_2, \mathbf{I}_{\{\neg, \wedge\}}\right)}.$$

Yet we *want* to be able to say that a rule such as \wedge -Introduction is somehow *the very same rule* within any proof-system in which it is included, regardless of what other rules (for other logical operators) might be included with it. Surely, the intuition goes, we are dealing with the very same rule of \wedge -Introduction, whichever system of proof is having its proofs defined. Otherwise we shall be beset by an inferentialist Babel of captious characterizations of the rule in question as, respectively,

$$(\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge\}}}, (\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \vee\}}}, (\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \rightarrow\}}}, (\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \vee, \rightarrow\}}},$$

$$(\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \exists\}}}, (\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \vee, \exists\}}}, (\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \rightarrow, \exists\}}}, (\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \vee, \rightarrow, \exists\}}},$$

$$(\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \forall\}}}, (\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \vee, \forall\}}}, (\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \rightarrow, \forall\}}}, (\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \vee, \rightarrow, \forall\}}},$$

$$(\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \exists, \forall\}}}, (\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \vee, \exists, \forall\}}}, (\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \rightarrow, \exists, \forall\}}}, (\wedge\text{-I})_{\mathbf{I}_{\{\neg, \wedge, \vee, \rightarrow, \exists, \forall\}}}, \dots$$

Enough of this unruly menagerie! one wishes to say. *Shouldn't we be permitted to simply construe \wedge -Introduction as*

$$(\wedge\text{-I})_{\mathcal{S}}, \text{ for any proof-system } \mathcal{S} \text{ that ...?}$$

For any proof-system \mathcal{S} that *what*, exactly? That contains *it* (‘*the*’ rule)? But then what exactly is the ‘*it*’ in question here? What exactly is *the* rule? Are we dealing here with a surprising but essential case of *impredicativity* in our frustrated attempt to say, clearly, what a rule of inference *is*?

same theorems:

$$\vdash_{\mathbf{I}_{\{\neg, \wedge\}}} \varphi \Leftrightarrow \vdash_{\mathbf{C}_{\{\neg, \wedge\}}} \varphi.$$

The answer is affirmative, because systems can be distinguished from one another not just by their theorems, but, more generally, by their deducibilities (involving nonempty sets of premises). Note that $\psi \wedge \neg\neg\varphi \vdash_{\mathbf{C}_{\{\neg, \wedge\}}} \varphi$; whereas $\psi \wedge \neg\neg\varphi \not\vdash_{\mathbf{I}_{\{\neg, \wedge\}}} \varphi$.

It would appear that it is impossible to identify a rule such as \wedge -Introduction except as an inductive-definitional clause of the form

$$(\wedge\text{-I})_{\mathcal{S}} \quad \frac{\mathcal{P}(\Pi_1, \varphi_1, \Delta_1, \mathcal{S}) \quad \mathcal{P}(\Pi_2, \varphi_2, \Delta_2, \mathcal{S})}{\mathcal{P}\left(\frac{\Pi_1 \quad \Pi_2}{\varphi_1 \wedge \varphi_2}, \varphi_1 \wedge \varphi_2, \Delta_1 \cup \Delta_2, \mathcal{S}\right)}$$

that is eligible for inclusion in the definition of any proof-system \mathcal{S} .

This last suggestion, however, won't quite do. The rule displays a degree of uniformity in its formulation that is not generally attainable. Note how the system \mathcal{S} is mentioned (at the fourth argument place) *both* in the premises, *and* in the conclusion, of this definitional rule. It is the mention in the conclusion that determines what proof-system it is that contains the proofs that result from applying the rule. But the mention in the premise(s) might well be of some proper *subsystem* of \mathcal{S} , rather than \mathcal{S} itself. Examples to be offered in §5.3 and §5.4 will make this point clear. But first, in §5.2, we shall emphasize how rules are to be identified by how they contribute to the variety of proofs, rather than by the field of the deducibility relation that those proofs generate.

5.2. Extension of Intuitionistic Logic by atomic applications of Dilemma. At the propositional level, if one adds to Intuitionistic Logic **I** the *strictly classical* rule of Dilemma:

$$(Dil) \quad \frac{\begin{array}{c} \text{---}(i) \\ \varphi \\ \vdots \\ \psi \end{array} \quad \begin{array}{c} \text{---}(i) \\ \neg\varphi \\ \vdots \\ \psi \end{array}}{\psi \text{---}(i)}$$

one obtains full Classical Logic **C**. Here, φ may be any sentence. So let us call this 'version' of the rule *General Dilemma*.

One also obtains **C** even if one *restricts* φ (the 'positive-horn' assumption) to be *atomic*. Let us call this 'version' of the rule *Atomic Dilemma*. In the following statement of this rule, A is taken to be an atom:²⁸

$$(Atomic\ Dil) \quad \frac{\begin{array}{c} \text{---}(i) \\ A \\ \vdots \\ \psi \end{array} \quad \begin{array}{c} \text{---}(i) \\ \neg A \\ \vdots \\ \psi \end{array}}{\psi \text{---}(i)}$$

If rules are to be identified only by their net effects on the deducibility relation, then General Dilemma and Atomic Dilemma would have to count as the same rule. Each of them extends Intuitionistic Logic **I** to Classical Logic **C**. But intuitively they are rather different rules. The former is applicable in situations in which the latter is not. The conclusion that forces itself upon one is that rules are to be identified *at*

²⁸ The use of A for atom is so alliteratively compelling that we are inclined to make this sole exception to using lowercase Greek letters as placeholders for sentences in statements of rules of inference.

least by considering the *variety of proofs* that they enable us to construct, and not just by the grosser measure of the *variety of arguments* $\Delta : \varphi$ for which they enable us to supply some proof or other. Otherwise, not only would General Dilemma and Atomic Dilemma count as one and the same rule; so too, either form of Dilemma would be identifiable with the Law of Excluded Middle:

$$(LEM) \quad \frac{}{\varphi \vee \neg\varphi} ;$$

with Classical Reductio:

$$(CR) \quad \frac{\overbrace{\Delta, \neg\varphi}^{\diamond \text{---}(i)} \quad \perp(i)}{\varphi} ;$$

and with Double Negation Elimination:

$$(DNE) \quad \frac{\neg\neg\varphi}{\varphi} .$$

This is because any one of the latter ‘classical negation’ rules, appended to **I**, yields **C**.²⁹ Yet these classical negation rules are all very distinct from one another, given their individually idiosyncratic ‘local conditions’ for application. That they respectively produce the same overall systemic effect appears to be neither here nor there, as far as the problem of *rule-identification* is concerned.

5.3. Extension of Intuitionistic Logic by terminal applications of Classical Reductio.

We turn now to the first of our examples promised above, of how the systems mentioned in the premises of an inductive clause in the definition of proof can be (proper) *subsystems* of the system of proof in question.

Consider the proof-system for classical propositional logic $C_{\{\neg, \wedge, \vee, \rightarrow\}}$ that has, as its only classical negation rule, the following:

$$(CR)_{C_{\{\neg, \wedge, \vee, \rightarrow\}}} \frac{\mathcal{P}(\Pi, \perp, \Delta, I_{\{\neg, \wedge, \vee, \rightarrow\}})}{\mathcal{P}(\frac{\Pi}{\varphi}, \varphi, \Delta \setminus \{\neg\varphi\}, C_{\{\neg, \wedge, \vee, \rightarrow\}})} .$$

This is the rule of *Classical Reductio ad Absurdum*, confined to subordinate *reductio* proofs of $\neg\varphi$ that are *intuitionistic*:

$$\left. \begin{array}{c} \overbrace{\Delta', \neg\varphi}^{\diamond \text{---}(i)} \\ \Pi \\ \perp(i) \\ \varphi \end{array} \right\} \text{intuitionistic.}$$

²⁹ Note that (LEM) is a zero-premise rule.

It is well known that every *classically* inconsistent set of sentences in a first-order language based on $\{\neg, \wedge, \vee, \rightarrow, \exists\}$ is *intuitionistically* inconsistent.³⁰ It follows that, despite its restriction to intuitionistic subproofs, the foregoing rule of *Classical Reductio* yields the full classical system (of propositional logic, and indeed of first-order logic with \exists as its only quantifier) as an extension of the intuitionistic one. The formulation of this rule (and of the intuitionistic I- and E-rules) ensures that in any strictly classical proof, *Classical Reductio* need only be applied *once*, and—if all other rules require *their* subordinate proofs to be intuitionistic—only as its *terminal* step.

5.4. The Rule of Necessitation in S0.5. A second example of the same kind of trouble will be useful here. In modal (propositional) logic the Rule of Necessitation mentioned in §3.6 is standardly thought of as applying to subproofs in the system to produce new proofs in the very same system. Graphically:

$$\left. \begin{array}{l} \emptyset \\ \Pi \\ \frac{\varphi}{\Box\varphi} \end{array} \right\} \text{in system } \mathcal{S} \left. \vphantom{\begin{array}{l} \emptyset \\ \Pi \\ \frac{\varphi}{\Box\varphi} \end{array}} \right\} \text{in system } \mathcal{S}.$$

The Rule of Necessitation in this form gives one a so-called *normal* modal logic. (This modal-logical notion of normality, by the way, has nothing at all to do with the proof-theorist’s notion of a proof being in normal form.)

Now, the Lemmon system S0.5 is not a normal modal logic. But it does have a Rule of Necessitation of sorts. The rule in question says that any classical propositional *tautology* (or theorem) may be necessitated. Graphically:

$$\left. \begin{array}{l} \emptyset \\ \Pi \\ \frac{\varphi}{\Box\varphi} \end{array} \right\} \text{in nonmodal propositional logic} \left. \vphantom{\begin{array}{l} \emptyset \\ \Pi \\ \frac{\varphi}{\Box\varphi} \end{array}} \right\} \text{in modal system S0.5 .}$$

Formulated alternatively as an inductive-definitional rule, the Rule of Necessitation in S0.5 would be

$$(N)_{S0.5\{\neg, \wedge, \vee, \rightarrow, \Box\}} \frac{\mathcal{P}(\Pi, \varphi, \emptyset, C_{\{\neg, \wedge, \vee, \rightarrow\}})}{\mathcal{P}\left(\frac{\Pi}{\Box\varphi}, \Box\varphi, \emptyset, S0.5_{\{\neg, \wedge, \vee, \rightarrow, \Box\}}\right)}.$$

Once again we have a rule in a system that is restricted in its applications to subproofs from a properly contained subsystem.

5.5. Towards a satisfactory ‘most general’ form of any particular rule. In general, we can expect there to be rules ρ whose formulation as metalinguistic inductive-definitional clauses will be of the following overall form, for various subsystems S_1, \dots, S_n of \mathcal{S} :

$$(\rho)_S \frac{\mathcal{P}(\Pi_1, \varphi_1, \Delta_1, S_1) \quad \dots \quad \mathcal{P}(\Pi_n, \varphi_n, \Delta_n, S_n) \quad C(\Delta_1, \varphi_1, \dots, \Delta_n, \varphi_n)}{\mathcal{P}\left(\frac{\Pi_1, \dots, \Pi_n}{\varphi}, \varphi, \Delta, S\right)}$$

³⁰ This is an easy corollary of the Glivenko–Gödel–Gentzen Theorem. The original sources are Glivenko [12], Gödel [14], and Gentzen [10]. See also Tennant [37], pp. 129–130, for the relevant Double Negation theorems, proved via an adaptation of Henkin’s method.

where $\mathcal{C}(\Delta_1, \varphi_1, \dots, \Delta_n, \varphi_n)$ is an effectively decidable relational property on the indicated arguments; and where the overall premise set Δ will be of some effectively computable form

$$\delta(\Delta_1, \varphi_1, \dots, \Delta_n, \varphi_n).$$

Presumably one would also be allowed to ‘minimize’ \mathcal{S} by setting

$$\mathcal{S} = \bigcup_{i=1}^n \mathcal{S}_i \cup \{(\rho)\}.$$

We are avoiding here the yet further complexities that would be involved in indicating how the set Δ of undischarged assumptions in the newly formed proof would indeed be effectively determinable from the various Δ_i ; and how the overall conclusion φ relates both to the various φ_i and to the various assumptions among the Δ_i that might be discharged. But, as we have indicated, there might also be an effectively decidable condition \mathcal{C} on the premises and conclusions of the subordinate proofs, which needs to be stated separately, and for whose presence one would have to check before applying the rule ρ to construct the more complex proof—we can call it $\rho(\Pi_1, \dots, \Pi_n) \mid \varphi$ —of the conclusion φ by applying the rule ρ to the immediate subproofs Π_1, \dots, Π_n .

§6. A tentative proposal about how to identify rules of inference. Our discussion thus far has sought to shake the confidence that the reader might have had at the outset in our ability to say precisely what a logical rule of inference *is*. We have encountered at least the following aspects of the identification-problem.

First, there appears to be an ineliminable *system-relativity*, hence a kind of ‘holism’, involved in characterizing any rule of inference as a means of constructing new proofs from given proofs. The ‘input’ proofs can be from various systems; and the ‘output’ proof might be in yet another system. And even if the systems in question are ‘all the same’—so that the same system-parameter \mathcal{S} occurs throughout the inductive clause for proof-formation in question—this immediately makes one worry about the possibility of some kind of *impredicativity* in our conception of such a rule. The rule is supposed to belong *to* the system \mathcal{S} , indeed is crucially constitutive *of* the system \mathcal{S} ; yet one seems to require reference *to* the system \mathcal{S} in order to so much as give expression to the rule (as in the minimization suggestion towards the end of §5.5).

It would be instructive to compare the situation here concerning proof systems and their rules of inference with a familiar and more standard example of impredicativity in mathematics.³¹ One says that a real number y is an *upper bound* to the set $\{x \mid \Phi(x)\}$ of real numbers just in case

$$\forall z(\Phi(z) \rightarrow z \leq y).$$

The *least* upper bound $\lambda x \Phi(x)$ of this set is that upper bound whose every lesser is no such upper bound. That is:

$$\lambda x \Phi(x) =_{df} \iota w (\forall z(\Phi(z) \rightarrow z \leq w) \wedge \forall v(v < w \rightarrow \neg \forall z(\Phi(z) \rightarrow z \leq v))).$$

³¹ We offer the following brief discussion of impredicativity in response to an anonymous referee’s helpful inquiry.

An unproblematic instance is where for $\Phi(x)$ we take $x < 0$. For then we have

$$\lambda x x < 0 =_{df} \text{iv}(\forall z(z < 0 \rightarrow z \leq w) \wedge \forall v(v < w \rightarrow \neg \forall z(z < 0 \rightarrow z \leq v))).$$

Note that $\lambda x(x < 0) = 0$; whence $\lambda x(x < 0) \not\leq 0$.

The problem of impredicativity arises, however, when one considers the possibility that $\Phi(\lambda x \Phi(x))$. For then the definition involves quantifying over numbers among which is the very number that is to be denoted by the defined expression.

This possibility is realized if for $\Phi(x)$ we take $x \leq 0$. For then we have $\lambda x(x \leq 0) = 0$; whence $\lambda x(x \leq 0) \leq 0$:

$$\lambda x x \leq 0 =_{df} \text{iv}(\forall z(z \leq 0 \rightarrow z \leq w) \wedge \forall v(v < w \rightarrow \neg \forall z(z \leq 0 \rightarrow z \leq v))).$$

Platonists about numbers are unruffled by such impredicativity. The least upper bound is ‘out there’, and just happens to be picked out by an expression involving quantification over a range that contains it. But what, in the context of our discussion of systems of proof and rules of inference, would be the reassuring analogue of Platonism about numbers, which could ensure that the rule-theorist could be similarly unruffled?³²

Returning to the way a rule ρ and a system \mathcal{S} invoke each other: one really has to ask oneself whether one has here a vicious or a virtuous circle of conceptual constitution. We confine ourselves to simply raising this question here. At present, we see no clear answer as to whether whatever circle is involved is vicious or virtuous. All we wish to bring to the reader’s awareness is that *there is such a circle*; and it has not, to the best of the present author’s knowledge, been brought clearly into focus before.³³ Inferentialists about meaning, especially, will now owe their audience an argument for the conclusion that the circle is virtuous, after all; whereas their opponents might see here an opportunity to discredit inferentialism by arguing, to the contrary, that the circle is somehow vicious.

Secondly, it would appear that ‘one and the same rule’ can be framed in different ‘versions’, to the same overall systemic effect (as far as deducibility is concerned), but with different effects at the level of proof-structures. If we take the latter differences

³² As an anonymous referee has pointed out, an apparently Platonistic position concerning proofs expressed by Prawitz [27] at p. 287, and commented on by Dummett [9] at p. 12, cannot easily be reconciled with the more cautious ontological position on proofs-as-objects that Prawitz expressed earlier in Prawitz [26] at p. 154. These, however, are subtleties of ontological debate that we do not have space to consider here.

³³ An anonymous referee has raised the further question whether this circle is at all analogous to the problem that intuitionists face with the BHK interpretation of the introduction rules for negation and implication. That interpretation is intended as a form of justification for the rule itself. Take, for example, the rule of \rightarrow -Introduction. According to BHK, a proof of the conditional $\varphi \rightarrow \psi$ consists in—or at least must supply—a function that maps any proof of φ to a proof of ψ . The danger of a ‘circle’ accordingly arises if among the possible proofs of φ are ones that involve appeal to proofs of $\varphi \rightarrow \psi$ itself. (See, for example, van Atten [48], at p. 3.)

The present author agrees that there is some sort of echo here of the circle that he is concerned about, involving a rule ρ and a system \mathcal{S} . But he sees that latter circle as somehow more fundamental. It arises *in advance* of the problem of potential impredicativity in the intuitionist’s interpretation, or justification, of the rule of \rightarrow -Introduction. For the present author, the problem is that of identifying what the very rule *is*; and *that* problem seems to call for prior solution before one can (as the BHK theorist does) seek to offer any kind of *justification* for ‘the rule’, from any particular methodological or philosophical standpoint.

seriously, we should be inclined to regard these different versions of (putatively) the ‘same’ rule as nothing of the sort; rather, they are simply *distinct rules*.

Thirdly, thinking of a rule of inference as licensing a certain kind of truth-preserving passage from *premises* (immediately above the inference stroke) to a *conclusion* (immediately below the inference stroke) is much too simplistic. Even if we confine our attention here to the introduction and elimination rules for the standard logical operators, the over-simplification involved in the ‘ $P_1, \dots, P_n; \text{so } Q$ ’ conception is readily apparent. Among those rules, the only ones *that* simple are \neg -E, \wedge -I, \wedge -E (serial form only), \vee -I, \rightarrow -E (serial form only, i.e., Modus Ponens), \exists -I and \forall -E. The other rules for these operators are more complicated. Some involve conditions on expressions *other than* their immediate premises and their conclusions. Such is the case with *discharge of assumptions*—as with \neg -I, \wedge -E (parallelized form), \vee -E (i.e., Proof by Cases), \rightarrow -I (i.e., Conditional Proof), \exists -E and \forall -I. And some involve conditions on *parameters*—as with \exists -E and \forall -I.

The appropriate *general* conception of what an application of any rule of inference vouchsafes is not truth-preservation from the immediate premises involved to the conclusion involved; *rather*, what is vouchsafed is *validity-preservation* from the immediate *subproofs* involved in any application of the rule, to the *overall proof* that results from that application!³⁴ This is particularly evident when we consider the Right and Left rules of the sequent calculus. These can be understood as reprising the effect of the introduction and elimination rules.

Moving on to Intuitionistic Logic, even the rule EFQ:

$$\frac{\perp}{\varphi}$$

is hard to reckon to the ‘ $P_1, \dots, P_n; \text{so } Q$ ’ mold. To be sure, we can take n to be 1; but can we *really* take \perp as a premise P ? This author thinks of \perp only as a punctuation marker in proofs, not eligible to be regarded as a sentence in its own right, and *a fortiori* not able to feature as a *subsentence* of any sentence.³⁵ (Hence it is ill-advised to define $\neg\varphi$ as $\varphi \rightarrow \perp$.)

Finally, moving on to Classical Logic: *LEM* has to be construed as a zero-premise rule; and *DNE* is a genuine single-premise rule. But both *Dil* and *CR* involve discharge of assumptions, and therefore they too fail to conform to the ‘ $P_1, \dots, P_n; \text{so } Q$ ’ mold.

As we survey these problems, a certain line of resolution has considerable appeal. As our discussion of the third problem reveals, the ‘ $P_1, \dots, P_n; \text{so } Q$ ’ conception is way too impoverished. Far better is the conception, already advanced above, of rules as *particular means of building up new proofs from given proofs*; that is, the conception of rules as given by, or corresponding to, *inductive clauses in an inductive definition* of the ternary notion ‘ Π is a proof of the conclusion φ from the set Δ of premises (i.e., undischarged assumptions)’. The latter notion smoothly subsumes the likes of \wedge -I with those of \exists -E (to choose but two rules with striking enough points of contrast to make the matter clear). Moreover, the *semantic* property to be preserved is not truth-of-sentences, but rather the Prawitzian one of *validity-of-argument*. (When a proof is of

³⁴ The stress on sequent- or argument-validity as the property that is preserved by correct inferences is due to Prawitz [24]. See especially Section A, ‘Validity of derivations’. Prawitz’s notion of validity is analyzed in Schroeder-Heister [31].

³⁵ See, in this connection, Tennant [40]. See also Cook and Cogburn [5].

the conclusion Q from (exactly) the premises P_1, \dots, P_n , then the argument it proves is the *sequent* $P_1, \dots, P_n : Q$.)

Note that one can endorse this Prawitzian conception of preservation-of-argument-validity with the catholicity of intent already remarked on in §1.3—that is, without advancing any particular conception of validity that might lead to a preference for one logical system over another (such as for Intuitionistic Logic over Classical Logic). Now of course Prawitz’s own recursive definition of validity of argument, given the conceptual materials that he brought to bear, *did* lead to a preference for Intuitionistic Logic over Classical Logic.³⁶ The professed ‘catholicity of intent’, however, can be exhibited by offering an *alternative* conception of validity of argument, formed and applied in a manner analogous to that of Prawitz, but employing, rather than the epistemic notion of a *canonical warrant for assertion*, the notion of a *model-relative truthmaker*; and proceeding, thus equipped, to a demonstration that one’s rules of inference preserve *argument validity*, rather than truth-of-sentences. A first foray in this direction can be found in Chapter 9 of Tennant [45] (with an important correction in Brauer and Tennant [4]). For the logical theory of truthmakers and falsitymakers there deployed, see Tennant [41 and 46].

One can also ‘uniformize’ in search of a solution to the holism or impredicativity problem—indeed, perhaps even turn it to one’s advantage. Let us set aside those challenging pathologies of ‘rules’ we pointed out above, such as terminal *CR*, Atomic Dilemma, and Necessitation. Let us assume instead that the system-parameter S will be held constant throughout. When characterizing a rule ρ the only assumption one needs to make is that ρ will belong to S , whatever that system S might turn out to be. The rule ρ must be formulated, or identified, in such a way as to carry within it its own seeds of *extendability to ever-enlarged systems S to which it might subsequently be taken to belong*. In this way we come to see rules of inference as *open-textured*. Whether one confines one’s attention to the fragment of intuitionistic propositional logic based on \neg and \wedge (say), or the full system of classical first-order logic with all the usual operators primitive, we can focus on the rule of \wedge -I (just to pick an easy illustration of the main idea) and say that it would be *essentially the same rule* in each of those two systems, because it is to be understood as the following *virtuously self-referential* claim:

(\wedge -I) For any system S containing (\wedge -I) :

$$\frac{\mathcal{P}(\Pi_1, \varphi_1, \Delta_1, S) \quad \mathcal{P}(\Pi_2, \varphi_2, \Delta_2, S)}{\mathcal{P}\left(\frac{\Pi_1 \quad \Pi_2}{\varphi_1 \wedge \varphi_2}, \varphi_1 \wedge \varphi_2, \Delta_1 \cup \Delta_2, S\right)} .$$

Note how—rather ironically—we have ascended to the metalevel (in formulating our definition of the ternary relation \mathcal{P}) and have used there the *two-premise, atomic inference rule*

$$\frac{\mathcal{P}(\Pi_1, \varphi_1, \Delta_1, S) \quad \mathcal{P}(\Pi_2, \varphi_2, \Delta_2, S)}{\mathcal{P}\left(\frac{\Pi_1 \quad \Pi_2}{\varphi_1 \wedge \varphi_2}, \varphi_1 \wedge \varphi_2, \Delta_1 \cup \Delta_2, S\right)}$$

³⁶ Thanks to an anonymous referee for pointing out the need here for further qualification.

And we can make do with this kind of format even when the rule in question is *not*, like \wedge -I, a rule that invites simplistic representation as a two-premise rule not involving discharge of assumptions, and not involving any more ‘global’ pre-conditions on applicability. Consider, for example, the following re-formulation in the new format of the inductive clause for \exists -E, where the premises of the metalinguistic inference are now arranged vertically, in order to avoid sideways spread:

$$\begin{array}{l}
 (\exists\text{-E}) \text{ For any system } \mathcal{S} \text{ containing } (\exists\text{-E}): \\
 \mathcal{P}(\Pi, \exists x\varphi, \Delta, \mathcal{S}) \\
 \mathcal{P}(\Sigma, \psi, \Gamma, \mathcal{S}) \\
 \varphi_a^x \in \Gamma \text{ and } a \text{ does not occur in } \exists x\varphi, \psi, \text{ or } \Gamma \setminus \{\varphi_a^x\}. \\
 \hline
 \mathcal{P}\left(\frac{\Pi_1 \quad \Pi_2}{\psi}, \psi, \Delta \cup (\Gamma \setminus \{\varphi_a^x\}), \mathcal{S}\right)
 \end{array}$$

Note that the third premise for the metalinguistic inference rule is of the form $\mathcal{C}(\Delta, \exists x\varphi, \Gamma, \psi)$, and is effectively decidable, since all the sets involved are finite.

It is clear from these two examples (namely, the rules \wedge -I and \exists -E) that the suggested general form of a rule in the new format is the following:

(ρ) For any system \mathcal{S} containing (ρ) [one may reason thus about proofs]:

$$\frac{\mathcal{P}(\Pi_1, \varphi_1, \Delta_1, \mathcal{S}) \quad \dots \quad \mathcal{P}(\Pi_n, \varphi_n, \Delta_n, \mathcal{S}) \quad \mathcal{C}(\Delta_1, \varphi_1, \dots, \Delta_n, \varphi_n)}{\mathcal{P}\left(\frac{\Pi_1, \dots, \Pi_n}{\varphi}, \varphi, \delta(\Delta_1, \varphi_1, \dots, \Delta_n, \varphi_n), \mathcal{S}\right)} .$$

The ascent to the metalevel here appears to be essential for our purposes. A rational agent fully apprised of *what rule is being applied* in any given inferential step would need to have this kind of *metalinguistic awareness* of what is guiding the inferential transition. We attain the (not over-)simplifying case of transitions of the form ‘ P_1, \dots, P_n ; so, Q ’ only by this metalinguistic ascent. This is what allows us to take in our stride all the complications of discharge of assumptions, checking for correct parametric behavior with quantifier moves, etc. But the price is that *the rule becomes self-referring* (as indicated by the two occurrences of ‘ ρ ’), in order for its statement to carry within it the seeds of its own future applicability within whatever further extension of the proof system it continues to play ‘its’ part. That is what we meant by rules being ‘open-textured’.

To repeat: our framing of the general form (ρ) here would have to be *virtuously self-referential* in order to serve our purposes. And the big question that then arises is simply: *Is it?*

§7. Identifying rules via metalinguistic inductive-definitional clauses using only literals. Recall our earlier suggested general form of an inference rule in any system \mathcal{S} containing it:

$$\frac{\mathcal{P}(\Pi_1, \varphi_1, \Delta_1, \mathcal{S}) \quad \dots \quad \mathcal{P}(\Pi_n, \varphi_n, \Delta_n, \mathcal{S}) \quad \mathcal{C}(\Delta_1, \varphi_1, \dots, \Delta_n, \varphi_n)}{\mathcal{P}\left(\frac{\Pi_1, \dots, \Pi_n}{\varphi}, \varphi, \delta(\Delta_1, \varphi_1, \dots, \Delta_n, \varphi_n), \mathcal{S}\right)} .$$

It now needs to be stressed not only that the \mathcal{P} -predications are *atomic* (in the metalanguage); but also that the condition(s) \mathcal{C} need to be spelled out by means of

(metalinguistic) *literals*—that is, atomic statements or negations thereof. In particular, one wishes to *avoid using explicitly disjunctive conditions* at all costs. The maxim guiding rule-formulations is to keep matters as simple as possible; and, in order to do so, we must state these rules in what turn out to be, *inferentially*, the most *basic* way possible.

Let us illustrate by considering the rule \vee -I. As already pointed out in §4.1, it is always stated in two parts, which we shall render here in our usual graphic way with all the earlier suppressed details filled in:

$$\frac{\Delta}{\Pi} \quad ; \quad \frac{\Delta}{\Pi} \\ \frac{\varphi}{\varphi \vee \psi} \quad ; \quad \frac{\psi}{\varphi \vee \psi}$$

The corresponding metalinguistic inductive-definitional clauses (for the notion \mathcal{P} of proof) are the following. Note how they instantiate the general form just reiterated. The condition \mathcal{C} in each part is the *null condition*, by virtue of the separation of this rule into its two parts:

$$\frac{\mathcal{P}(\Pi, \varphi, \Delta, \mathcal{S})}{\mathcal{P}\left(\frac{\Pi}{\varphi \vee \psi}, \varphi \vee \psi, \Delta, \mathcal{S}\right)} \quad ; \quad \frac{\mathcal{P}(\Pi, \psi, \Delta, \mathcal{S})}{\mathcal{P}\left(\frac{\Pi}{\varphi \vee \psi}, \varphi \vee \psi, \Delta, \mathcal{S}\right)}$$

If we had attempted to frame this rule by means of only one definitional clause, then we would have been forced to use a *disjunctive* condition \mathcal{C} , as follows:

$$\frac{\mathcal{P}(\Pi, \chi, \Delta, \mathcal{S}) \quad \chi = \varphi \text{ or } \chi = \psi}{\mathcal{P}\left(\frac{\Pi}{\varphi \vee \psi}, \varphi \vee \psi, \Delta, \mathcal{S}\right)}$$

This disjunctive condition is obviously *not* a literal; so the preceding null-condition clauses are to be preferred. Basically, what we are saying, methodologically, is this: our inductive-definitional clauses for the proof-predicate \mathcal{P} are to be metalinguistic *Horn clauses*. An inference of the form

$$\frac{P_1, \dots, P_n, Q \vee R}{S}$$

is *not* in Horn form; whereas each of the inferences

$$\frac{P_1, \dots, P_n, Q}{S} \quad \frac{P_1, \dots, P_n, R}{S}$$

is in Horn form. And the latter two inferences are jointly equivalent to the former one.

So much for \vee -I. Let us turn now to \vee -E—in particular, the liberalized form of \vee -E, mentioned above, that is to be found in Core Logic (and its classicized extension). There are three distinct parts to this liberalized rule. They differ from one another solely in whether their subordinate proofs have the same conclusion, or whether one of them (but not the other) has \perp as its conclusion.

$$\frac{\mathcal{P}(\Pi, \theta, \Delta, \mathcal{S}) \quad \mathcal{P}(\Sigma, \theta, \Gamma, \mathcal{S}) \quad \varphi \in \Delta \quad \psi \in \Gamma}{\mathcal{P}\left(\frac{\varphi \vee \psi}{\theta} \frac{\Pi \Sigma}{\theta}, \theta, \{\varphi \vee \psi\} \cup (\Delta \setminus \{\varphi\}) \cup (\Gamma \setminus \{\psi\}), \mathcal{S}\right)} ;$$

$$\frac{\mathcal{P}(\Pi, \theta, \Delta, \mathcal{S}) \quad \mathcal{P}(\Sigma, \perp, \Gamma, \mathcal{S}) \quad \varphi \in \Delta \quad \psi \in \Gamma}{\mathcal{P}\left(\frac{\varphi \vee \psi \quad \Pi \quad \Sigma}{\theta}, \theta, \{\varphi \vee \psi\} \cup (\Delta \setminus \{\varphi\}) \cup (\Gamma \setminus \{\psi\}), \mathcal{S}\right)} ;$$

$$\frac{\mathcal{P}(\Pi, \perp, \Delta, \mathcal{S}) \quad \mathcal{P}(\Sigma, \theta, \Gamma, \mathcal{S}) \quad \varphi \in \Delta \quad \psi \in \Gamma}{\mathcal{P}\left(\frac{\varphi \vee \psi \quad \Pi \quad \Sigma}{\theta}, \theta, \{\varphi \vee \psi\} \cup (\Delta \setminus \{\varphi\}) \cup (\Gamma \setminus \{\psi\}), \mathcal{S}\right)} .$$

Separating out these three distinct parts of the liberalized rule of \vee -E enables us to keep them in Horn form. This would *not* be the case if we were to attempt to combine them into ‘just one’ inductive-definitional clause, even though the following ‘single’ clause is inferentially equivalent to the three foregoing clauses taken together:

$$\frac{\mathcal{P}(\Pi, \chi, \Delta, \mathcal{S}) \quad \mathcal{P}(\Sigma, \eta, \Gamma, \mathcal{S}) \quad (\chi = \theta \wedge \eta = \theta) \text{ OR } (\chi = \theta \wedge \eta = \perp) \text{ OR } (\chi = \perp \wedge \eta = \theta)}{\mathcal{P}\left(\frac{\varphi \vee \psi \quad \Pi \quad \Sigma}{\theta}, \theta, \{\varphi \vee \psi\} \cup (\Delta \setminus \{\varphi\}) \cup (\Gamma \setminus \{\psi\}), \mathcal{S}\right)} .$$

We note another feature of the preferred ‘canonical’ form taken by inferential Horn clauses: they *minimize* the number of distinct variables involved in their formulation of the permitted inference(s). Note how the extra variables χ and η above are involved only in the *noncanonical* clauses, not in the canonical, because Horn, clauses.

We have finally arrived at a tentative answer to our original question. As far as the rule of Classical Reductio is concerned, it is unitary—it has only one part, and it can be expressed in canonical Horn form:

$$\frac{\mathcal{P}(\Pi, \perp, \Delta, \mathcal{S}) \quad \neg\varphi \in \Delta}{\mathcal{P}\left(\frac{\Pi}{\theta}, \theta, (\Delta \setminus \{\neg\varphi\}), \mathcal{S}\right)} .$$

As far as the rule of Dilemma is concerned, its form for Classical Core Logic \mathbb{C}^+ is in two parts, each expressible in canonical Horn form:

$$\frac{\mathcal{P}(\Pi, \theta, \Delta, \mathcal{S}) \quad \mathcal{P}(\Sigma, \theta, \Gamma, \mathcal{S}) \quad \varphi \in \Delta \quad \neg\varphi \in \Gamma}{\mathcal{P}\left(\frac{\Pi \quad \Sigma}{\theta}, \theta, (\Delta \setminus \{\varphi\}) \cup (\Gamma \setminus \{\neg\varphi\}), \mathcal{S}\right)} ;$$

$$\frac{\mathcal{P}(\Pi, \theta, \Delta, \mathcal{S}) \quad \mathcal{P}(\Sigma, \perp, \Gamma, \mathcal{S}) \quad \varphi \in \Delta \quad \neg\varphi \in \Gamma}{\mathcal{P}\left(\frac{\Pi \quad \Sigma}{\theta}, \theta, (\Delta \setminus \{\varphi\}) \cup (\Gamma \setminus \{\neg\varphi\}), \mathcal{S}\right)} .$$

As far as the introduction and elimination rules are concerned, we take each logical operator in turn, and gather together the ‘rule-parts’ of each kind. These rule-parts must be in canonical Horn form, when formulated as metalinguistic inductive-definitional clauses in the definition of the ternary proof-predicate \mathcal{P} . The various parts of a given introduction or elimination rule (ρ) can then be gathered together according to our earlier rubric for a self-referential rule, but making clear that it can consist in k distinct parts:

(ρ) For any system \mathcal{S} containing (ρ) [one may reason about proofs in any of the following k ways]:

$$\frac{\mathcal{P}(\Pi_1, \varphi_1, \Delta_1, \mathcal{S}) \dots \mathcal{P}(\Pi_n, \varphi_n, \Delta_n, \mathcal{S}) \quad \mathcal{C}_1(\Delta_1, \varphi_1, \dots, \Delta_n, \varphi_n)}{\mathcal{P}\left(\frac{\Pi_1, \dots, \Pi_n}{\varphi}, \varphi, \delta_1(\Delta_1, \varphi_1, \dots, \Delta_n, \varphi_n), \mathcal{S}\right)} ;$$

$$\vdots$$

$$\frac{\mathcal{P}(\Pi_1, \varphi_1, \Delta_1, \mathcal{S}) \dots \mathcal{P}(\Pi_n, \varphi_n, \Delta_n, \mathcal{S}) \quad \mathcal{C}_k(\Delta_1, \varphi_1, \dots, \Delta_n, \varphi_n)}{\mathcal{P}\left(\frac{\Pi_1, \dots, \Pi_n}{\varphi}, \varphi, \delta_k(\Delta_1, \varphi_1, \dots, \Delta_n, \varphi_n), \mathcal{S}\right)} .$$

This explication of rule-identity seems to cohere well with pre-theoretic intuitions, even after informing oneself of all the potentially complicating, often unfamiliar, considerations that have been broached in our investigations. We are now in a position to appreciate how and why two proponents of different logical systems can engage in a productive dispute about the right form that a *particular rule* should take. It should *not* be the case that two disputants advocating different forms for a rule (for, say, \vee -E) should be regarded as ‘talking past one another’ by allegedly talking about *altogether different rules*.

Acknowledgments. The author is grateful to two anonymous referees for their very helpful comments.

BIBLIOGRAPHY

- [1] Anderson, A. R. & Belnap Jr., N. D. (1975). *Entailment: The Logic of Relevance and Necessity*. Vol. 1. Princeton and London: Princeton University Press.
- [2] Belnap, N. (1962). Tonk, Plonk and Plink. *Analysis*, **22**, 130–134.
- [3] Boghossian, P. (2014). What is inference? *Philosophical Studies*, **1690**(1), 1–18.
- [4] Brauer, E. & Tennant, N. (2020). Transmission of verification. *The Review of Symbolic Logic*, 1–23. <http://dx.doi.org/10.1017/S1755020320000234>.
- [5] Cook, R. T. & Cogburn, J. (2000). What negation is not: Intuitionism and ‘0 = 1’. *Analysis*, **600**(1), 5–12.
- [6] Dean, J. & Kurokawa, H. (2016). Kreisel’s theory of constructions, the Kreisel-Goodman paradox, and the second clause. In Piecha, T. and Schroeder-Heister, P., editors. *Advances in Proof-Theoretic Semantics*. Berlin: Springer, pp. 27–63.
- [7] Dummett, M. (1973). The justification of deduction. *Proceedings of the British Academy*, **LIX**, 201–232.
- [8] ———. (1991). *The Logical Basis of Metaphysics*. Cambridge, MA: Harvard University Press.
- [9] ———. (2000). *Elements of Intuitionism* (second edition). Oxford, UK: Clarendon Press.
- [10] Gentzen, G. (1936). Die Widerspruchsfreiheit der reinen Zahlentheorie. *Mathematische Annalen*, **112**(1), 493–565.

- [11] ———. (1934, 1935). Untersuchungen über das logische Schliessen I. *Math. Zeitschrift*, **39**(176–210), 405–431. Translated as ‘Investigations into logical deduction’. In Szabo, M. E., editor. *The Collected Papers of Gerhard Gentzen*. North-Holland, Amsterdam, pp. 68–131.
- [12] Glivenko, V. (1929). Sur quelques points de la logique de M. Brouwer. *Bull. Soc. Math. Belg.*, **15**(5), 183–188.
- [13] Gödel, K. (1930). Die Vollständigkeit der Axiome des logischen Funktionenkalküls. *Monatshefte für Mathematik und Physik*, **37**(1), 349–360.
- [14] ———. (1933). Zur intuitionistischen Arithmetik und Zahlentheorie. *Ergebnisse eines mathematischen Kolloquiums*, **4**(1933), 34–38.
- [15] Hilbert, D. & Ackermann, W. (1938). *Grundzüge der Theoretischen Logik*, 2. Auflage. Berlin: Springer.
- [16] ———. (1950). *Principles of Mathematical Logic*. New York, NY: Chelsea Publishing Co.
- [17] Jaśkowski, S. (1934). On the rules of suppositions in formal logic. *Studia Logica*, **10**(1), 5–32.
- [18] Jeffrey, R. C. (1967). *Formal Logic: Its Scope and Limits*. New York, NY: McGraw-Hill, Inc.
- [19] Kahle, R. & Schroeder-Heister, P. (2006). Introduction: Proof-theoretic semantics. *Synthese* (Special issue: Kahle R., and Schroeder-Heister, P., editors. Proof-Theoretic Semantics), **148**(3), 503–506.
- [20] Kreisel, G. (1962). Foundations of intuitionistic logic. In Nagel, E., Suppes, P., and Tarski, A., editors. *Logic, Methodology and Philosophy of Science*. Stanford: Stanford University Press, pp. 198–210.
- [21] Martin-Löf, P. (1994). Analytic and synthetic judgments in type theory. In Parrini, P., editor. *Kant and Contemporary Epistemology*. Dordrecht: Kluwer Academic Publishers, pp. 87–99.
- [22] ———. (1996). On the meanings of the logical constants and the justifications of the logical laws. *Nordic Journal of Philosophical Logic*, **10**(1), 11–60.
- [23] Prawitz, D. (1965). *Natural Deduction: A Proof-Theoretical Study*. Stockholm: Almqvist & Wiksell.
- [24] ———. (1971). Ideas and results in proof theory. In Fenstad, J. E., editor. *Proceedings of the Second Scandinavian Logic Symposium*. Amsterdam: North-Holland, pp. 235–307.
- [25] ———. (1974). On the idea of a general proof theory. *Synthese*, **27**(0) 63–77.
- [26] ———. (1987). Dummett on a theory of meaning and its impact on logic. In Taylor, B., editor. *Michael Dummett: Contributions to Philosophy*. Nijhoff International Philosophy Series, Vol. 25. Dordrecht: Martinus Nijhoff Publishers, pp. 117–165.
- [27] ———. (1998). Comments on the papers. *Theoria*, **640**(2–3), 283–337.
- [28] ———. (2019). The seeming interdependence between the concepts of valid inference and proof. *Topoi*, **38**(3), 493–503.
- [29] Schroeder-Heister, P. (1984). A natural extension of natural deduction. *Journal of Symbolic Logic*, **49**(4), 1284–1300.
- [30] ———. (1991). Uniform proof-theoretic semantics for logical constants. *Journal of Symbolic Logic*, **56**(0), 1142.

- [31] ———. (2006). Validity concepts in proof-theoretic semantics. *Synthese* (Special issue: Kahle R., and Schroeder-Heister, P., editors. Proof-Theoretic Semantics), **148**(3), 525–571.
- [32] ———. (2018). Proof-theoretic semantics. In Zalta, E. N., editor. *The Stanford Encyclopedia of Philosophy* (spring 2018 edition). Metaphysics Research Lab, Stanford University.
- [33] Schroeder-Heister, P. & Piecha, T., editors. (2016) *Advances in Proof-Theoretic Semantics*. Berlin: Springer.
- [34] Scott, D. (1974). Rules and derived rules. In Stenlund, S., editor. *Logical Theory and Semantic Analysis*. Dordrecht: Reidel, pp. 147–161.
- [35] Shoesmith, D. J. & Smiley, T. J. (1978). *Multiple-Conclusion Logic*. Cambridge and New York: Cambridge University Press.
- [36] Smiley, T. (1962). The independence of connectives. *The Journal of Symbolic Logic*, **27**(4), 426–436.
- [37] Tennant, N. (1978). *Natural Logic*. Edinburgh: Edinburgh University Press.
- [38] ———. (1992). *Autologic*. Edinburgh: Edinburgh University Press.
- [39] ———. (1996). The law of excluded middle is synthetic a priori, if valid. *Philosophical Topics*, **24**(1), 205–229.
- [40] ———. (1999). Negation, absurdity and contrariety. In Gabbay, D. and Wansing, H., editors. *What is Negation?* Dordrecht: Kluwer, pp. 199–222.
- [41] ———. (2010). Inferential semantics. In Lear, J. and Oliver, A., editors, *The Force of Argument: Essays in Honor of Timothy Smiley*. London: Routledge, pp. 223–257.
- [42] ———. (2012). Cut for core logic. *Review of Symbolic Logic*, **50**(3), 450–479.
- [43] ———. (2015a). Cut for classical core logic. *Review of Symbolic Logic*, **80**(2), 236–256.
- [44] ———. (2015b). The relevance of premises to conclusions of core proofs. *Review of Symbolic Logic*, **80**(4), 743–784.
- [45] ———. (2017). *Core Logic*. Oxford: Oxford University Press.
- [46] ———. (2018). Logical theory of truthmakers and falsitymakers. In Glanzberg, M., editor. *Handbook on Truth*. Oxford, UK: Oxford University Press, pp. 355–393.
- [47] Troelstra, A. S. and Schwichtenberg, H. (2000). *Basic Proof Theory* (second edition). Cambridge, UK: Cambridge University Press.
- [48] van Atten, M. (2018). Predicativity and parametric polymorphism of Brouwerian implication. Available from <https://arxiv.org/pdf/1710.07704.pdf>, p. 41.
- [49] Weinstein, S. (1993). The intended interpretation of intuitionistic logic. *Journal of Philosophical Logic*, **120**(2), 261–270.
- [50] Wright, C. (2014). Comment on Paul Boghossian, “What is inference”. *Philosophical Studies*, **169**(1), 27–37.

DEPARTMENT OF PHILOSOPHY
 THE OHIO STATE UNIVERSITY
 COLUMBUS, OH 43210, USA
 E-mail: tennant9@osu.edu