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A characterization of a map whose inverse limit is an arc

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Abstract. For a continuous function $f : [0, 1] \to [0, 1]$ we define a splitting sequence admitted by f and show that the inverse limit of f is an arc if and only if f does not admit a splitting sequence.

Key words: arc, inverse limit, single bonding map
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1. *Introduction*

In this paper we solve a problem that has been open for more than 50 years about a characterization of a single bonding map on an interval whose inverse limit is an arc. Although at first glance the problem seems purely topological, it is also important in dynamical systems since, by Barge and Martin [**BM1**], every inverse limit space of an interval map can be realized as a global attractor for a homeomorphism of the plane. Therefore, our result sheds light on homeomorphisms of the plane whose attractors are arcs. In addition, on our way to proving the main result, we give dynamical properties, interesting in their own right, of a map on an interval whose inverse limit is an arc.

In 1968, Rogers [**R**] considered the class of single bonding maps on [0, 1] that are nowhere strictly monotone and showed that the inverse limit of such a function can be an arc. In the same paper Rogers asked a very natural question: *what kind of maps will yield an arc, or more specifically, what kind of single bonding map will yield an arc?*

The question turned out to be very hard and has been studied by a number of authors. In 1995, Block and Schumann [**BS**] characterized a unimodal map whose inverse limit is an arc. They showed that if f is a unimodal map then its inverse limit is an arc if and only if either f has more than one fixed point and no points of other periods, or f has a single fixed point, a point of period 2, and no points of other periods. They also gave an example which shows that their characterization for the unimodal maps cannot be extended to piecewise monotone maps. In addition, they proved that if the inverse limit of a continuous map f on the interval is an arc, then all periodic points of f either are fixed points or have period 2.

In 2004, Mo *et al* [**MSZM**] considered functions of type N on [0, 1] and gave a characterization of a single type N bonding map whose inverse limit is an arc. A continuous function $f : [0, 1] \to [0, 1]$ is called of type N if there are $a, b \in (0, 1)$, $a < b$, such that f is strictly increasing on $[0, a]$ and on $[b, 1]$, and strictly decreasing on $[a, b]$. The necessary and sufficient conditions are given in terms of mutual positions of points a, b, $f(a)$ and $f(b)$. Existence of periodic points under given conditions is also discussed.

Very recently (2020), Anušić and Činč [**AC**] obtained a characterization of a continuous surjective piecewise monotone function $f : [0, 1] \to [0, 1]$ (with finitely many monotone pieces) whose inverse limit is an arc. They consider (connected) components of the unit interval without any fixed points of f^2. The necessary and sufficient condition is that for every such component C there exists a fixed point y of f^2 such that for every $x \in C$ the sequence $\langle f^{2n}(x) : n \in \mathbb{N}\rangle$ converges to y.

We introduce the very simple notion of a tight sequence (Definition 3.1) and study a subclass of tight sequences that we call splitting sequences (Definition 3.4). We prove that the inverse limit of a continuous surjective function f on an interval is an arc if and only if f does not admit a splitting sequence (Theorem 4.7). We also prove that f admits a splitting sequence if there are two disjoint intervals whose images coincide and one of them, A, has a subinterval $D \subset A$ such that $f^k(D) = A$ for some positive integer k (Lemma 3.9). This criterion is easy to check for a large class of continuous functions (especially if k is small). Additionally, we show that if f has a periodic point of period greater than 2, then f has a splitting sequence (Lemma 3.8). This, together with our main theorem, implies the above-mentioned result from [**BS**] about a continuous map whose inverse limit is an arc (that all of its periodic points are either fixed points or have period 2).

As shown in [**BS**], an inverse limit may not be an arc even if its periodic points have period no greater than 2. There are maps that have only fixed points, but yield complex inverse limit spaces. As we show in this paper, the reason is a splitting sequence. In the Block–Schumann example a splitting sequence is easily recognized using the criterion from Lemma 3.9, as we show in Example 3.10.

The other very interesting example is the Henderson map [**H**]. It has only two fixed points and no points of other periods, but its inverse limit space is the pseudo-arc. The Henderson map is not piecewise monotone, so the criterion from [**AC**] does not work for it. But the existence of a splitting sequence for the Henderson map is not hard to prove, as we show in Example 3.7.

On our way towards the main result we also prove that a continuous function f which has at least two different periodic orbits of period 2, and has an arc as its inverse limit, also has the following very interesting property. If $\{s, t\}$ and $\{u, v\}$ are two 2-cycles with $s < t$ and $u < v$, then $s < u$ implies $v < t$ (Lemma 3.13). Moreover, f has exactly one fixed point (Lemma 3.18).

The paper is organized as follows. In §2 we give definitions and define notation required in the sequel. In §3 we define tight sequences, introduce splitting sequences and discuss properties of functions on an interval that do not admit a splitting sequence, and which are the basis for the proof of our main theorem. In §4 we prove our main theorem.

2. *Preliminaries*

A *continuum* is a non-empty compact connected metric space. Let X be a continuum and $p \in X$ a point. Then p is a *separating point* if $X \setminus \{p\}$ is disconnected. A continuum X is an *arc* if X has exactly two non-separating points called *endpoints*.

For each $n \in \mathbb{N}$, let X_n be a closed interval and $f_{n+1} : X_{n+1} \to X_n$ a continuous function. The *inverse limit* of $(f_n)_{n\in\mathbb{N}}$ is the space

$$\varprojlim(X_n, f_n) = \left\{(x_0, x_1, \ldots) \in \prod_{n\in\mathbb{N}} X_n : \text{for all } n \in \mathbb{N}, x_n = f(x_{n+1})\right\}$$

with the topology inherited from the product space $\prod_{n\in\mathbb{N}} X_n$. The functions f_n are called *bonding functions*. An inverse limit of continua is a continuum [**N**]. We are concerned with inverse limits of functions $f : [0, 1] \to [0, 1]$. Denote the inverse limit of a single bonding function f by $\varprojlim \boldsymbol{f}$. Bold symbols represent members of $[0, 1]^{\mathbb{N}}$, for example $\boldsymbol{x} = (x_0, x_1, \ldots)$. Denote the graph of a function f by $\Gamma(f)$.

Barge and Martin give the following characterization of an endpoint of an inverse limit $\varprojlim \boldsymbol{f}$ for a function $f : [0, 1] \to [0, 1]$.

THEOREM 2.1. [**BM2**, Theorem 1.4] *Let $f : [0, 1] \to [0, 1]$ be a continuous function. Then $\boldsymbol{p}$ is an endpoint of $\varprojlim \boldsymbol{f}$ if and only if for each integer n, each closed interval $J_n = [a_n, b_n]$ with $p_n \in (a_n, b_n)$, and each $\epsilon > 0$, there is a positive integer k such that if $p_{n+k} \in J_{n+k}$ and $f^k(J_{n+k}) = J_n$, then p_{n+k} does not separate*

$$(f^k \restriction J_{n+k})^{-1}([a_n, a_n + \epsilon]) \quad \textit{and} \quad (f^k \restriction J_{n+k})^{-1}([b_n - \epsilon, b_n])$$

*in $[a_{n+k}, b_{n+k}]$ (f^k is ϵ-*crooked with respect to p_{n+k}).

We also require the following result by Block and Schumann.

PROPOSITION 2.2. [**BS**, Proposition 3.1] *Let $f : [0, 1] \to [0, 1]$ be a continuous function. Then $\varprojlim \boldsymbol{f}$ is a point if and only if f admits exactly one fixed point and no periodic points.*

In order to show that the Henderson map admits a splitting sequence in Example 3.7, we will require the following lemma.

LEMMA 2.3. [**H**, Lemma 1] *There is a map $f : [0, 1] \to [0, 1]$ such that if $[a, b, c, d]$ is an increasing 4-tuple of rational numbers in $(0, 1)$ (that is, $0 < a < b < c < d < 1$), then there exists an integer m such that if $n > m$ and $[u, w]$ is an interval such that $f^n([u, w]) = [a, d]$, then $f^n \restriction [u, w]$ is crooked on $[a, b, c, d]$.*

By crooked it is meant that $f^n([u, w])$ contains $[a, d]$ and there is in $[u, w]$ either an inverse of c under f^n between two inverses of b or an inverse of b under f^n between two inverses of c. The Henderson map satisfies the above lemma.

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For each $m, n \in \mathbb{N}$, $m < n$, denote the sequence of natural numbers from m to n (inclusive) by $[m, n]$, and the sequence of natural numbers greater than or equal to m by $[m, \infty)$. Let

$$G_{m,n}(f) = \{(x_m, \ldots, x_n) \in [0, 1]^{[m,n]} : \text{for all } i \in [m, n-1],\ f(x_{i+1}) = x_i\},$$

and

$$G_{m,\infty}(f) = \{(x_m, x_{m+1}, \ldots) \in [0, 1]^{[m,\infty)} : \text{for all } i \in \mathbb{N}, i \geq m,\ f(x_{i+1}) = x_i\}.$$

We define projection functions

$$\pi_n : \varprojlim \boldsymbol{f} \to [0, 1] \text{ by } \pi_n(\boldsymbol{x}) = x_n,$$

$$\pi_{n+1,n} : \varprojlim \boldsymbol{f} \to \Gamma(f) \text{ by } \pi_{n+1,n}(\boldsymbol{x}) = (x_{n+1}, x_n),$$

if $m < n$, and define

$$\pi_{[m,n]} : \varprojlim \boldsymbol{f} \to G_{m,n}(f) \text{ by } \pi_{[m,n]}(\boldsymbol{x}) = (x_j)_{j\in[m,n]}$$

and

$$\pi_{[m,\infty)} : \varprojlim \boldsymbol{f} \to G_{m,\infty}(f) \text{ by } \pi_{[m,\infty)}(\boldsymbol{x}) = (x_j)_{j\in[m,\infty)}.$$

If f is surjective, each of these projection functions is onto.

A *basic open subset of* $\varprojlim \boldsymbol{f}$ is a set of the form

$$U = \bigcap\{\pi_{n_j}^{-1}(U_j) : j \leq k\} \cap \varprojlim \boldsymbol{f},$$

where $\{k\} \cup \{n_j : j \leq k\} \subset \mathbb{N}$ and each U_j is an open subinterval of $[0, 1]$.

3. *Splitting sequences*

In this section we define tight sequences, and splitting sequences which are a subclass of tight sequences. We prove a number of lemmas that give properties of splitting sequences required to prove our main theorem.

Definition 3.1. Let $f : [0, 1] \to [0, 1]$ be a continuous surjective function and

$$\sigma = \langle T_n \subsetneq [0, 1] : n \in \mathbb{N}\rangle$$

a sequence of closed intervals. If for each $n \in \mathbb{N}$, $f(T_{n+1}) = T_n$ and there exists $m \in \mathbb{N}$ such that for each $n > m$, T_n is non-degenerate, then σ is a *tight* sequence. The subcontinuum $\varprojlim(T_n, f \restriction T_n)$ is denoted by $L(\sigma)$.

Definition 3.2. Let $f : [0, 1] \to [0, 1]$ be a surjective continuous function. Let $\boldsymbol{p} \in \varprojlim \boldsymbol{f}$, $m \in \mathbb{N}$ and $[a, b] \subset [0, 1]$ be a non-degenerate closed interval such that $p_m \in (a, b)$. Let $C \subset \varprojlim \boldsymbol{f}$ be the component of $\pi_m^{-1}([a, b])$ containing $\boldsymbol{p}$. Then $\sigma = \langle \pi_n(C) : n \in \mathbb{N}\rangle$ is a *generated sequence*, or more specifically, the *sequence generated by* $\boldsymbol{p}$*, m and* $[a, b]$.

LEMMA 3.3. *Let $f : [0, 1] \to [0, 1]$ be a continuous surjective function. If $\boldsymbol{p} \in \varprojlim \boldsymbol{f}$, $m \in \mathbb{N}$, $[a, b] \subset [0, 1]$ is non-degenerate, $p_m \in (a, b)$ and σ is the sequence generated by $\boldsymbol{p}$, m and $[a, b]$, then σ is tight.*

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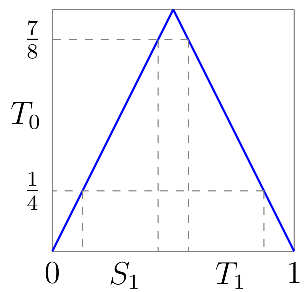

FIGURE 1. Graph of a tent map.

Proof. Let $C \subset \varprojlim \boldsymbol{f}$ be the component of $\pi_m^{-1}([a, b])$ containing $\boldsymbol{p}$. First observe that C is non-degenerate since $\boldsymbol{p} \in \mathrm{Int}_{\varprojlim \boldsymbol{f}}(\pi_m^{-1}([a, b]))$, where $\mathrm{Int}_{\varprojlim \boldsymbol{f}}(\pi_m^{-1}([a, b]))$ denotes the interior of $\pi_m^{-1}([a, b])$ in the (relative) topology of $\varprojlim \boldsymbol{f}$. As a component of $\pi_m^{-1}([a, b])$, C must also meet the boundary of $\pi_m^{-1}([a, b])$.

Since C is non-degenerate, for some $m \in \mathbb{N}$, $\pi_m(C)$ is non-degenerate. Thus if $n \geq m$ and $\pi_n(C)$ is non-degenerate, then it follows that $\pi_{n+1}(C)$ is non-degenerate since $f_{n+1}(\pi_{n+1}(C)) = \pi_n(C)$, and so by induction σ is tight. □

Definition 3.4. Let $f : [0, 1] \to [0, 1]$ be a surjective continuous function. If

$$\sigma = \langle T_n = [l_n, r_n] : n \in \mathbb{N}\rangle$$

is a tight sequence admitted by f, $N \subseteq \mathbb{N}$ an infinite set, and

$$\{S_n \subset [0, 1] : n \in N\}$$

is a collection of non-degenerate closed intervals such that for each $n \in N$, $S_n \cap T_n \subset \{l_n, r_n\}$, and $f(S_n) = f(T_n)$, then σ is a *splitting sequence admitted by* f *and witnessed by* $\{S_n : n \in N\}$.

Example 3.5. If $f : [0, 1] \to [0, 1]$ is the tent map illustrated in Figure 1, then f admits a splitting sequence. Let $T_0 = [\frac{1}{4}, \frac{7}{8}]$. If T_n has been defined let T_{n+1} be the component of $f^{-1}(T_n)$ contained in $[\frac{1}{2}, 1]$ and S_{n+1} be the component of $f^{-1}(T_n)$ contained in $[0, \frac{1}{2}]$. Then $\langle T_n : n \in \mathbb{N}\rangle$ is a splitting sequence witnessed by the sets S_n.

Example 3.6. The function $f : [0, 1] \to [0, 1]$ whose graph is shown in Figure 2 does not admit a splitting sequence. If $\boldsymbol{x} \in \varprojlim \boldsymbol{f}$ and $x_0 \neq \frac{5}{6}$, a fixed point of f, then $x_n \to 0$. Hence for any tight sequence $\langle T_n = [l_n, r_n] : n \in \mathbb{N}\rangle$ there exists $m \in \mathbb{N}$ such that $f^{-1}(r_n) < \frac{3}{4}$ for every $n > m$ and so there does not exist an interval $S_n \subset [0, 1]$ such that $|S_n \cap T_n| \leq 1$ and $f(S_n) = f(T_n)$, where $|A|$ denotes the cardinality of a set A.

Example 3.7. Let $f : [0, 1] \to [0, 1]$ be the Henderson map [**H**]. Recall that f has exactly two fixed points, 0 and 1, and for every $x \in (0, 1)$, $f(x) < x$. Its construction is rather complex, but may be described roughly as starting with $g(x) = x^2$ and notching its graph

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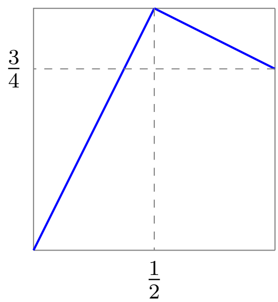

FIGURE 2. Graph of a function whose inverse limit is an arc.

with an infinite set of non-intersecting V-shaped notches which accumulate at $(1, 1)$. The map f is continuous and $\varprojlim \boldsymbol{f}$ is the pseudo-arc.

We will show that f has a splitting sequence. Let $[a_0, b_0, c_0, d_0]$ be an increasing 4-tuple of rational numbers in $(0, 1)$. Let $T_0 = [b_0, c_0]$. By Lemma 2.3, there exist increasing sequences $\langle n_k \in \mathbb{N} : k \in \mathbb{N}\rangle$, $n_k < n_{k+1}$, and $\langle [u_k, w_k] \subset (0, 1) : k \in \mathbb{N}\rangle$, $w_k < u_{k+1}$, such that $f^{n_k}([u_k, w_k]) = [a_0, d_0]$ and is crooked on $[a_0, b_0, c_0, d_0]$. Note that $f^{n_k - n_{k-1}}([u_k, w_k]) = [u_{k-1}, w_{k-1}]$. For every $k \in \mathbb{N}$ choose closed intervals T_{n_k} and S_{n_k} in $[u_k, w_k]$ such that

$$f^{n_k - n_{k-1}}(T_{n_k}) = f^{n_k - n_{k-1}}(S_{n_k}) = T_{n_{k-1}},$$

and $|T_{n_k} \cap S_{n_k}| \le 1$, and observe that $f^{n_k}(T_{n_k}) = [b_0, c_0]$. Such a choice is possible since $f^{n_k} \restriction [u_k, w_k]$ is crooked on $[a_0, b_0, c_0, d_0]$, meaning that there is in $[u_k, w_k]$ either an inverse of c_0 under f^{n_k} between two inverses of b_0 or an inverse of b_0 under f^{n_k} between two inverses of c_0. Hence for each k, $f^{-n_k}((b_0, c_0))$ has three components, and so $f^{-(n_k - n_{k-1})}(\mathrm{Int}(T_{n_{k-1}}))$ has three components.

For each $k \in \mathbb{N}$ and j, $0 < j < n_k - n_{k-1}$, let $T_{n_k - j} = f^j(T_{n_k})$. Then $\langle T_n : n \in \mathbb{N}\rangle$ is a splitting sequence witnessed by $\langle S_{n_k} : k \in \mathbb{N}\rangle$.

LEMMA 3.8. *Let $f : [0, 1] \to [0, 1]$ be a surjective continuous function. If f admits a periodic point with period m for any $m > 2$ then f admits a splitting sequence.*

Proof. In this proof we may write a closed interval $[a, b]$ if we do not know whether $a < b$ or $b < a$ and it is assumed to be the appropriate non-empty closed interval.

Suppose x_0 is a periodic point with period $m > 2$ and for each $i < m$, $f^i(x_0) = x_i$. Without loss of generality suppose that $x_0 = \min\{x_n : n < m\}$. Then $x_1 = f(x_0) > x_0$ and $f(x_{m-1}) = x_0 < x_1$.

Suppose $f(x_1) > x_1$. If $x_0 < x_{m-1} < x_1$, since $f(x_{m-1}) < f(x_0) < f(x_1)$ there are closed intervals $A \subseteq [x_0, x_{m-1}]$ and $B \subset [x_{m-1}, x_1]$ such that $f(A) = f(B) = [f(x_{m-1}), f(x_0)]$; see Figure 3. Moreover, for each $i < m$ there is a closed subinterval A_i of $[x_i, x_{i-1}]$ such that $f(A_i) = [f(x_i), f(x_{i-1})]$.

Let $T_0 = [f(x_{m-1}), f(x_0)]$ and $T_1 = A$. If $n \ge 1$ and T_n has been defined such that for some $i < m$, $T_n \subseteq [x_i, x_{i+1}]$, let T_{n+1} be a subinterval of $[x_{i-1}, x_i]$ such that

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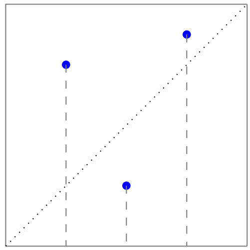

FIGURE 3. Graph showing $f(x_0)$, $f(x_1)$ and $f(x_{m-1})$ for the first case of Lemma 3.8.

$f(T_{n+1}) = T_n$. Then $\sigma = \langle T_n : n \in \mathbb{N}\rangle$ is tight. For each $n \in \mathbb{N}$ there is a set $S_{mn+1} \subset B$ such that $f(S_{mn+1}) = f(T_{mn+1})$ and $S_{mn+1} \cap T_{mn+1} \subseteq \{x_{m-1}\}$. Hence τ is a splitting sequence.

The proof in all cases is analogous. We need only show that in each case there are three points x_i, x_j, x_k in the cycle such that $x_i < x_j < x_k$ and $f(x_j)$ is either greater than or less than both $f(x_i)$ and $f(x_k)$. If $[f(x_i), f(x_j)] \subset [f(x_j), f(x_k)]$ then take the sets A and B, used to define the sets T_n and S_n, to be subintervals of $[x_i, x_j]$ and $[x_j, x_k]$ respectively, such that $f(A) = f(B) = [f(x_i), f(x_j)]$, and vice versa.

We show that we can always find three points x_i, x_j, x_k as required. If $f(x_1) > x_1$ and $x_1 < x_{m-1}$ then we can take $x_i = x_0$, $x_j = x_1$ and $x_k = x_{m-1}$. If $x_1 > f(x_1)$ and $x_{m-1} < x_1$ then we can choose $x_i = x_0$, $x_j = x_{m-1}$ and $x_k = x_1$.

Suppose $x_1 < x_{m-1}$. Then $f(x_{m-2}) = x_{m-1} > x_1$, so if $x_0 < x_{m-2} < x_1$, let $x_i = x_0$, $x_j = x_{m-2}$ and $x_k = x_1$. If $x_0 < x_1 < x_{m-1} < x_{m-2}$, let $x_i = x_0$, $x_j = x_{m-1}$ and $x_k = x_{m-2}$. Finally, if $x_0 < x_1 < x_{m-2} < x_{m-1}$, let $x_i = x_0$, $x_j = x_{m-2}$ and $x_k = x_{m-1}$. □

In the preceding proof we used a certain technique in our construction of splitting sequences. As we will frequently require it, the technique is captured in the following lemma.

LEMMA 3.9. *Let $f : [0, 1] \to [0, 1]$ be a surjective continuous function. If there exist $k >$ 0, closed subintervals A and B of* $[0, 1]$*, such that $f(A) = f(B)$, $|A \cap B| \leq 1$, and there is a non-degenerate component of $f^{-k}(A)$ in A, then f admits a splitting sequence.*

Proof. Let $T_1 = A$ (and $T_0 = f(A)$). Since there is a non-degenerate component of $f^{-k}(A)$ in A, we can choose T_{k+1} to be a subinterval of A such that $f^k(T_{k+1}) = T_1$. For $k \geq i \geq 2$ let $T_i = f(T_{i+1})$. Obviously $T_1 = f(T_2) = f^k(T_{k+1})$. Analogously, if $n > k$, $n = 0 \bmod k$ and T_{n-k+1} has been defined, let T_{n+1} be a subinterval of A such that $f^k(T_{n+1}) = T_{n-k+1}$. For $n \geq i \geq n-k+1$ let $T_i = f(T_{i+1})$. Then $\sigma = \langle T_i : i \in \mathbb{N}\rangle$ is a tight sequence. Since, for every $n > 0$, $T_{nk+1} \subseteq A$ and $f(A) = f(B) = T_0$, for every $n > 0$ we can choose an interval $S_{nk+1} \subseteq B$ such that $f(S_{nk+1}) = f(T_{nk+1})$. Thus σ is a splitting sequence. □

If A and B are intervals and $k \in \mathbb{N}$ as in Lemma 3.9, we say that *the pair* (A, B) *generates a splitting sequence of order* k.

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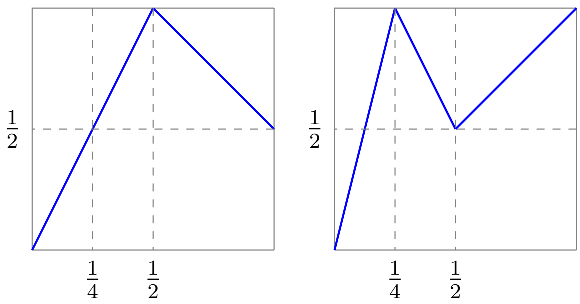

FIGURE 4. Graphs of functions f (left) and $g = f^2$ (right) whose inverse limits are the $\sin(1/x)$-continuum.

Example 3.10. We give an example from [**BS**] which shows that there exists a piecewise monotone map which has more than one fixed point and no points of other periods, but its inverse limit is not an arc.

Let $f, g : [0, 1] \to [0, 1]$ be maps whose graphs are shown in Figure 4. Obviously, the map $g = f^2$ has more than one fixed point and no points of other periods. Also, it is well known that $\varprojlim \boldsymbol{f} = \varprojlim \boldsymbol{g}$ and is homeomorphic to a $\sin(1/x)$-continuum [**N**].

By using the above criterion it is easy to determine that both maps admit a splitting sequence as follows. Let $A = [\frac{1}{2}, 1]$ and $B = [\frac{1}{4}, \frac{1}{2}]$. Then $A \cap B = \{\frac{1}{2}\}$, $f(A) = g(A) = A$ and $f(B) = g(B) = A$. Therefore, (A, B) generates a splitting sequence of order 1.

LEMMA 3.11. *Let $f : [0, 1] \to [0, 1]$ be a continuous function such that f does not admit a splitting sequence. If $0 \le d < e \le 1$ and either d and e are fixed points or $\{d, e\}$ is a 2-cycle, then there is exactly one component C of $f^{-1}((d, e))$ such that $f(C) = (d, e)$.*

Proof. Since either d and e are fixed points or $\{d, e\}$ is a 2-cycle, there exists a component $C \subseteq [d, e]$ of $f^{-1}((d, e))$ such that $f(C) = (d, e)$. Suppose that (for either case), $f^{-1}((d, e))$ has a second component D such that $f(D) = (d, e)$. Then $f(\overline{C}) = f(\overline{D}) = [d, e]$, $|\overline{C} \cap \overline{D}| \le 1$ and there is a non-degenerate component of $f^{-1}(C)$ in C. Thus the pair $(\overline{C}, \overline{D})$ generates a splitting sequence of order 1. □

COROLLARY 3.12. *Let $f : [0, 1] \to [0, 1]$ be a surjective continuous function such that f does not admit a splitting sequence. If F is the set of fixed points admitted by f and d is an accumulation point of F, then*

$$\varprojlim \boldsymbol{f} = \varprojlim([0, d], f \restriction [0, d]) \cup \varprojlim([d, 1], f \restriction [d, 1])$$

and

$$\varprojlim([0, d], f \restriction [0, d]) \cap \varprojlim([d, 1], f \restriction [d, 1]) = \{(d, d, \ldots)\}.$$

Proof. Let $\langle d_n \in [0, 1] : n \in \mathbb{N} \rangle$ be a sequence of fixed points that limits to d. We can assume that either $d_n < d$ for each n, or $d_n > d$ for each n. Assume the former (the proof is symmetrical if the latter holds), and assume that the sequence is strictly increasing.

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We first show that $f([0, d]) = [0, d]$ and $f([d, 1]) = [d, 1])$, and hence the first statement holds. Observe that, by the preceding lemma, for each $n \in \mathbb{N}$, $f([d_n, d_{n+1}]) \subset (d_{n-1}, d_{n+2})$, and $f([0, d_0]) \subset [0, d_1)$. Thus $f([0, d]) = [0, d]$.

Suppose that $x > d$ and $f(x) < d$. Then for some $m \in \mathbb{N}$, $[d_m, d_{m+1}] \subset f([d, x])$, which gives a contradiction with Lemma 3.11. Thus $f([d, 1]) = [d, 1]$.

To complete the proof observe that $f([0, d)) = [0, d)$, and hence the second statement holds. □

LEMMA 3.13. *Let $f : [0, 1] \to [0, 1]$ be a surjective continuous function such that f does not admit a splitting sequence. If f admits two 2-cycles $\{s, t\}$ and $\{u, v\}$ with $s < t$ and $u < v$, then either $s < u < v < t$ or $u < s < t < v$.*

Proof. Suppose $s < u < t < v$. Then there are closed intervals $A \subseteq [s, u]$ and $B \subset [u, t]$ such that $f(A) = f(B) = [t, v]$. Also, there is an interval $A' \subset [t, v]$ such that $f(A') = [s, u]$. Thus (A, B) generates a splitting sequence of order 2. Similarly if $s < t < u < v$, $u < s < v < t$, or $u < v < s < t$. □

LEMMA 3.14. *Let $f : [0, 1] \to [0, 1]$ be a surjective continuous function such that f does not admit a splitting sequence. If f admits two 2-cycles $\{s, t\}$ and $\{u, v\}$ with $s < u$, then there is exactly one component C of $f^{-1}([s, u])$ such that $f(C) = [s, u]$, and there is exactly one component C' of $f^{-1}([v, t])$ such that $f(C') = [v, t]$.*

Proof. Let $C \subseteq [v, t]$ be a component of $f^{-1}([s, u])$ such that $f(C) = [s, u]$. Since $f([s, u]) \supseteq [v, t]$, we can choose a non-degenerate component of $f^{-2}(C)$ in C. If $f^{-1}([s, u])$ has a second component D such that $f(D) = [s, u]$ and $|C \cap D| \leq 1$, the pair (C, D) generates a splitting sequence of order 2. The proof of the second statement is analogous. □

LEMMA 3.15. *Let $f : [0, 1] \to [0, 1]$ be a surjective continuous function. Then f admits a splitting sequence if and only if f^2 admits a splitting sequence.*

Proof. Let $N \subset \mathbb{N}$ be an infinite set, $\langle T_n : n \in \mathbb{N}\rangle$ a splitting sequence admitted by f and witnessed by $\{S_n : n \in N\}$. Let $\sigma = \langle T_{2n} : n \in \mathbb{N}\rangle$ and let $\tau = \langle T_{2n+1} : n \in \mathbb{N}\rangle$. Observe that either the set of even values in N is infinite, or the set of odd values is. If the even values are infinite then σ is a splitting sequence admitted by f^2 and witnessed by $\{S_n : n \in N,\ n \text{ is even}\}$. If the set of odd values of N is infinite then τ is a splitting sequence admitted by f^2 and witnessed by $\{S_n : n \in N,\ n \text{ is odd}\}$.

Suppose $\langle R_n : n \in \mathbb{N}\rangle$ is a splitting sequence admitted by f^2 and witnessed by $\{S_n : n \in N\}$ for some infinite set N. For each n let $T_{2n} = R_n$ and $T_{2n+1} = f(R_{n+1})$. For each $n \in N$ let $S'_{2n} = S_n$. Then $\langle T_n : n \in \mathbb{N}\rangle$ is a splitting sequence admitted by f and witnessed by $\{S'_{2n} : n \in N\}$. □

For the remainder of this paper, given a function $f : [0, 1] \to [0, 1]$, let $a = \max(f^{-1}(0))$ and $b = \min(f^{-1}(1))$.

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LEMMA 3.16. *Let* $f : [0, 1] \to [0, 1]$ *be a surjective continuous function that does not admit a splitting sequence. Let* d *be the maximum fixed point of* f. *Suppose* $a < b$. *Then the following assertions hold:*

(i) d *is the only fixed point in* $[b, 1]$*;*
(ii) $f([b, 1]) \subset (b, 1]$*;*
(iii) $f \restriction [b, 1]$ *does not admit a* 2*-cycle; and*
(iv) $\varprojlim([b, 1], f \restriction [b, 1]) = \{(d, d, \ldots)\}$.

Proof. Observe that if $f(1) = 1$ then $d = 1$.

(i) Suppose $d' \in [b, 1]$, d' is a fixed point and $d' < d$. Then there are intervals $A \subseteq [d', d]$ and $B \subset [0, b]$ such that $f(A) = f(B) = [d', d]$, contradicting Lemma 3.11.
(ii) If $b \in f([b, 1])$ then there exist $A \subseteq [b, 1]$ and $B \subseteq [a, b]$ such that $f(A) = f(B) = [b, 1]$. Since $f^{-1}([b, 1]) \supseteq [b, 1]$, by Lemma 3.9 f admits a splitting sequence, a contradiction.
(iii) The statement follows from Lemma 3.11 since if $\{p, q\}$ is a 2-cycle admitted by $f \restriction [b, 1]$, $p < q$, then there are intervals $A \subseteq [p, q]$ and $B \subset [0, b]$ such that $f(A) = f(B) = [p, q]$.
(iv) By (i), (iii), Proposition 2.2 and Lemma 3.8, $\varprojlim([b, 1], f \restriction [b, 1])$ is a singleton, and as d is a fixed point, $\varprojlim([b, 1], f \restriction [b, 1]) = \{(d, d, \ldots)\}$. □

Analogously to Lemma 3.16 we can show the next lemma.

LEMMA 3.17. *Let* $f : [0, 1] \to [0, 1]$ *be a surjective continuous function that does not admit a splitting sequence. Let* e *be the minimum fixed point of* f. *Suppose* $a < b$. *Then the following assertions hold:*

(i) e *is the only fixed point in* $[0, a]$*;*
(ii) $f([0, a]) \subset [0, a)$*;*
(iii) $f \restriction [0, a]$ *does not admit a* 2*-cycle; and*
(iv) $\varprojlim([0, a], f \restriction [0, a]) = \{(e, e, \ldots)\}$.

LEMMA 3.18. *Let* $f : [0, 1] \to [0, 1]$ *be a surjective continuous function that does not admit a splitting sequence. Suppose* $b < a$. *Let* $a' = \min(f^{-1}(0))$, $b' = \max(f^{-1}(1))$, *and let*

$$r = \max\{x \in (a', b') : f(x) \in (a', b') \text{ and } x \text{ is periodic with } \operatorname{Per}(x) \le 2\}.$$

Then the following assertions hold:

(i) *for every* $x \in [0, b']$, $f(x) > r$, *and for every* $x \in [a', 1]$, $f(x) < f(r)$*;*
(ii) *the function* f *admits exactly one fixed point; and*
(iii) f *admits a unique* 2*-cycle* $\{s, t\}$ *such that* $s < t$, *and either* $0 \le s < b'$ *or* $a' < t \le 1$.

Proof. (i) If there exists $x \in [0, b']$ such that $f(x) \le r$, then there are closed intervals $A \subseteq [b', f(r)]$ and $B \subset [x, b']$ such that $f(A) = f(B) = [r, 1]$. There is an interval $A' \subset [r, 1]$ such that $f(A') = [b', f(r)]$. Thus (A, B) generates a splitting sequence of order 2.

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Analogously, if there exists $x \in [a', 1]$ such that $f(x) \geq f(r)$, then we can obtain a splitting sequence of order 2.

(ii) Let d be a fixed point between b and a (by the definition of a and b, there exists such a fixed point). Suppose f admits a second fixed point e and $d < e$. Then by (i), $b' < d < e < a'$, and $f^{-1}([d, e])$ has a component $C \subset [b', d]$ and a component $D \subseteq [d, e]$ such that $f(C) = f(D) = [d, e]$, contradicting Lemma 3.11.

(iii) Since $f([b', a']) = [0, 1]$, the claim follows from Lemmas 3.11, 3.13 and 3.14. □

PROPOSITION 3.19. *If* $f : [0, 1] \to [0, 1]$ *is a surjective continuous function that does not admit a splitting sequence, then either*

(a) f *admits at least two fixed points and if* d *is the maximum and* e *the minimum fixed point, then* $(d, d, d, \ldots)$ *and* $(e, e, e, \ldots)$ *are endpoints of* $\varprojlim \boldsymbol{f}$*; or*

(b) f *admits a* 2*-cycle and if* $\{s, t\}$ *is a* 2*-cycle such that for any other* 2*-cycle* $\{u, v\}$*,* $s < u$*, then* $(s, t, s, t, \ldots)$ *and* $(t, s, t, s, \ldots)$ *are endpoints of* $\varprojlim \boldsymbol{f}$.

Proof. We consider two cases:

(1) $a < b$;

(2) $b < a$.

Case 1. We first show that $(d, d, \ldots)$ is an endpoint of $\varprojlim \boldsymbol{f}$, which we do by applying Lemma 3.16 and Theorem 2.1.

Let $\epsilon > 0$ and let $J_0 = [\alpha_0, \beta_0]$ be an interval such that $d \in (\alpha_0, \beta_0)$. By Lemma 3.16 (iv), for every $\boldsymbol{x} \in \varprojlim \boldsymbol{f}$ with $x_0 \in [b, 1] \setminus \{d\}$, there exists j such that $x_j < b$, and so by Lemma 3.16 (ii), $x_n < b$ for every $n > j$. Hence there exists k such that $f^{-k}(\alpha_0) \subset [0, b)$ and $f^{-k}(\beta_0) \subset [0, b)$, and therefore $f^{-k}([\alpha_0, \alpha_0 + \epsilon]) \cap [0, b] \neq \emptyset$ and $f^{-k}([\beta_0 - \epsilon, \beta_0]) \cap [0, b] \neq \emptyset$. Thus, as $[0, b] \subset [0, d)$, f^k is ϵ-crooked with respect to $(d, d, \ldots)$.

By applying Lemma 3.17 and Theorem 2.1 we can analogously establish that $(e, e, \ldots)$ is an endpoint of $\varprojlim \boldsymbol{f}$.

Case 2. Let d be a fixed point between b and a. We now show that $(s, t, s, t, \ldots)$ and $(t, s, t, s, \ldots)$ are endpoints.

Let $g = f^2$. By Lemma 3.15, g does not admit a splitting sequence. Since f admits a 2-cycle, g admits at least two fixed points, and hence by Lemma 3.18 (ii), g must satisfy the condition of case (1). Thus g admits at least three fixed points d, s' and t', such that d is the fixed point guaranteed by Lemma 3.18 (ii), s' is the minimum and t' the maximum fixed point admitted by g. Hence $s' < d < t'$. It follows from Lemma 3.18 (ii) and (iii) and Lemma 3.13 that 2-cycles $\{s', t'\}$ and $\{s, t\}$ coincide, $\{s', t'\} = \{s, t\}$.

Now the function $h : \varprojlim \boldsymbol{f} \to \varprojlim \boldsymbol{g}$ defined by

$$h((x_0, x_1, x_3, \ldots)) = (x_0, x_2, x_4, \ldots)$$

is a homeomorphism, so

$$(s, t, s, t, \ldots) = h^{-1}((s, s, \ldots)) \quad \text{and} \quad (t, s, t, s, \ldots) = h^{-1}((t, t, \ldots))$$

are endpoints of $\varprojlim \boldsymbol{f}$.

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Thus, if case (1) holds we have two fixed points that determine two endpoints of $\varprojlim \boldsymbol{f}$, and if case (2) holds we have a 2-cycle that determines two endpoints as required. □

LEMMA 3.20. *Let $f:[0,1]\to[0,1]$ be a continuous surjective function that does not admit a splitting sequence. If f admits a 2-cycle $\{s,t\}$ such that $s\in\{0,1\}$, then $(s,t,s,t,\ldots)$ and $(t,s,t,s,\ldots)$ are endpoints of $\varprojlim \boldsymbol{f}$.*

Proof. Suppose $s=1$; the proof is similar if $s=0$. Observe that $t\geq b=\min(f^{-1}(1))$, and hence f does not satisfy Lemma 3.16 (iii) which states that the function $f\restriction[b,1]$ does not admit a 2-cycle. Hence f satisfies the condition $b<a$ (case (2) in the proof of Proposition 3.19). Since $s=1$, by Lemma 3.13, $\{s,t\}$ is the 2-cycle determining the two endpoints of Proposition 3.19 (b). □

LEMMA 3.21. *Let $f:[0,1]\to[0,1]$ be a continuous surjective function such that f does not admit a splitting sequence, and f admits at least two fixed points. If $\sigma=\langle T_n=[l_n,r_n]:n\in\mathbb{N}\rangle$ is the sequence generated by a point $\boldsymbol{p}$, $m\in\mathbb{N}$ and interval $[c,d]$, and $[c,d]$ does not contain the maximum or minimum fixed point, then there exists $k\in\mathbb{N}$ such that for every $n\geq k$, $p_n\notin\{l_n,r_n\}$.*

Proof. By Lemma 3.18 (ii), f satisfies the requirement of case (1) in the proof of Proposition 3.19. Thus $a=\max f^{-1}(0)<b=\min(f^{-1}(1))$ and, by Lemma 3.16 (iv) and Lemma 3.17 (iv), there exists $j\in\mathbb{N}$ such that for each $n>j$, $T_n\subset[a,b]$.

By Lemma 3.3, σ is tight so there exists $r>j$ such that for every $n>r$, T_n is non-degenerate. Let

$$N=\{n>r:p_n\in\{l_n,r_n\}\},$$

and suppose that N is infinite. For every $n\in\mathbb{N}$, let $[l'_{n+1},r'_{n+1}]$ be the component of $f^{-1}(T_n)$ containing p_{n+1}. Since $p_m\in(c,d)=\operatorname{Int}T_m$, for every $n>m$ we have that $p_n\notin\{l'_n,r'_n\}$. If $n\in N$ then either

$$f([l'_{n+1},r'_{n+1}])=[l'_n,p_n]=T_n$$

or

$$f([l'_{n+1},r'_{n+1}])=[p_n,r'_n]=T_n.$$

Then for $n\in N$ we have that $T_n\subset[a,b]$ and $p_n\in\{l_n,r_n\}$, and hence we can choose two sets $A_{n+1}\subset[l'_{n+1},p_{n+1}]$ and $B_{n+1}\subset[p_{n+1},r'_{n+1}]$ such that $f(A_{n+1})=f(B_{n+1})=T_n$ and $A_{n+1}\cap B_{n+1}\subseteq\{p_{n+1}\}$.

Let $R_0=T_0$. If $n\geq 0$ and R_n has been defined, let R_{n+1} be a subinterval of either $T_{n+1}\cap A_{n+1}$ or $T_{n+1}\cap B_{n+1}$ if $n\in N$, otherwise let R_{n+1} be any subinterval of T_{n+1}, and in each case such that $f(R_{n+1})=R_n$. For each $n\in N$, if $T_{n+1}\subset A_{n+1}$ let S_{n+1} be a subinterval of B_{n+1}, and if $T_{n+1}\subset B_{n+1}$ let S_{n+1} be a subinterval of A_{n+1}, such that $f(S_{n+1})=R_n$. Thus $\langle R_n:n\in\mathbb{N}\rangle$ is a splitting sequence, a contradiction. □

The benefits of the lemmas proved so far will be reaped in the section that follows.

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4. *Arcs*

LEMMA 4.1. *If $f : [0, 1] \to [0, 1]$ is a continuous surjective function with exactly two fixed points, and f does not admit a splitting sequence, then $\varprojlim \boldsymbol{f}$ is an arc.*

Proof. Suppose d and e are the only fixed points, $e < d$. Since f is surjective and does not admit a splitting sequence, we have $e = 0$ or $d = 1$. Suppose that $e = 0$ and $d \neq 1$ (the proof is analogous if $e \neq 0$ and $d = 1$, or if $e = 0$ and $d = 1$). Since f is surjective, for every $x \in (0, d)$, $f(x) > x$. The conditions of case (1) in the proof of Proposition 3.19 are satisfied, so $(0, 0, \ldots)$ and $(d, d, \ldots)$ are endpoints.

Let

$$\boldsymbol{p} \in \varprojlim \boldsymbol{f} \setminus \{(0, 0, \ldots), (d, d, \ldots)\}.$$

We show that $\boldsymbol{p}$ is a separating point. Recall that $b = \min\{x \in [0, 1] : f(x) = 1\}$. By Lemma 3.16 (iv) it follows that for some $m \in \mathbb{N}$, $p_n < b$ for every $n > m$. Let

$$N = \{n > 0 : |f^{-1}(p_{n-1})| > 1\}.$$

(a) Suppose N is finite. Choose some $m \geq \max(N)$ such that $p_m < b$. Then for every $n > m$, $p_n < b$ and $f^{-1}(p_n) = \{p_{n+1}\}$, so

$$f^{-1}([0, p_n]) = [0, p_{n+1}] \quad \text{and} \quad f^{-1}([p_n, 1]) = [p_{n+1}, 1].$$

For each $n \in \mathbb{N}$ let $X_n = [0, p_{m+n}]$, $Y_n = [p_{m+n}, 1]$, $g_n = f \restriction X_n$ and $h_n = f \restriction Y_n$, and let $X = \varprojlim(X_n, g_n)$ and $Y = \varprojlim(Y_n, h_n)$. Then clearly

$$\pi_{[m,\infty)}(\varprojlim \boldsymbol{f}) = X \cup Y \quad \text{and} \quad X \cap Y = \{\boldsymbol{p}\}.$$

Let $X' = \pi^{-1}_{[m,\infty)}(X)$ and $Y' = \pi^{-1}_{[m,\infty)}(Y)$. Since $\pi^{-1}_{[m,\infty)}$ is the bijection defined by

$$(x_m, x_{m+1}, \ldots) \mapsto (f^m(x_m), \ldots, f(x_m), x_m, x_{m+1}, \ldots),$$

it follows that $\varprojlim \boldsymbol{f} = X' \cup Y'$ and $X' \cap Y' = \{\boldsymbol{p}\}$. Thus $\boldsymbol{p}$ is a separating point of $\varprojlim \boldsymbol{f}$.

(b) Suppose N is infinite. For all $\epsilon > 0$ and $i \in \mathbb{N}$ let

$$\sigma_{\epsilon,i} = \langle T_n^{\epsilon,i} = [a_n^{\epsilon,i}, b_n^{\epsilon,i}] : n \in \mathbb{N}\rangle$$

be the tight sequence generated by $\boldsymbol{p}$, i and $[p_i - \epsilon, p_i + \epsilon]$. Suppose that for some i and ϵ, there is an infinite set $M \subseteq N$ such that for every $n \in M$, $f^{-1}(T_n^{\epsilon,i}) \setminus \text{Int}\,(T_{n+1}^{\epsilon,i})$ has a component C_{n+1} with $p_n \in f(C_{n+1})$. By Lemma 3.21 we can assume that for each $n \in M$, $p_n \notin \{a_n^{\epsilon,i}, b_n^{\epsilon,i}\}$. Let $k \in M$ and let $L_k = [a_k^{\epsilon,i}, p_k]$ and $R_k = [p_k, b_k^{\epsilon,i}]$. If $j \geq k$ and L_j, R_j have been defined, let L_{j+1} and R_{j+1} be components of $T_{j+1}^{\epsilon,i}$ such that $f(L_{j+1}) = L_j$ and $f(R_{j+1}) = R_j$. Clearly each of the sets L_{k+1} and R_{k+1} contains a different endpoint of $T_{k+1}^{\epsilon,i}$. If $j \leq k$ and L_j, R_j have been defined, let $L_{j-1} = f(L_j)$ and $R_{j-1} = f(R_j)$.

Then $\tau_1 = \langle L_n : n \in \mathbb{N}\rangle$ and $\tau_2 = \langle R_n : n \in \mathbb{N}\rangle$ are tight sequences. Observe that for each $n \in M$ there is a subinterval D_{n+1} of C_{n+1} such that either $f(D_{n+1}) = L_n$ or $f(D_{n+1}) = R_n$. Then one of the sequences τ_1 or τ_2 is a splitting sequence.

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Thus we have that for every ϵ and i, $f^{-1}(p_n) \subset T^{\epsilon,i}_{n+1}$ for all but finitely many $n \in N$. For every $\epsilon > 0$ and $i \in \mathbb{N}$ such that

$$0, d, 1 \notin [p_i - \epsilon, p_i + \epsilon],$$

choose $m_{\epsilon,i}$ such that $f^{-1}(p_n) \subset T^{\epsilon,i}_{n+1}$ for all $n \geq m_{\epsilon,i}$. Thus $[0, 1] \setminus T^{\epsilon,i}_{m_{\epsilon,i}}$ has two components, $A'_{\epsilon,i}$ and $B'_{\epsilon,i}$. Let

$$A_{\epsilon,i} = \pi^{-1}_{m_{\epsilon,i}}(A'_{\epsilon,i}), \quad B_{\epsilon,i} = \pi^{-1}_{m_{\epsilon,i}}(B'_{\epsilon,i}),$$

$$A = \bigcup\{A_{\epsilon,i} : \epsilon > 0, i \in \mathbb{N} \text{ and } 0, d, 1 \notin [p_i - \epsilon, p_i + \epsilon]\},$$

and

$$B = \bigcup\{B_{\epsilon,i} : \epsilon > 0, i \in \mathbb{N} \text{ and } 0, d, 1 \notin [p_i - \epsilon, p_i + \epsilon]\}.$$

Then $\boldsymbol{p} \notin A \cup B$, $A \cap B = \emptyset$ and, since $\bigcap\{L(\sigma_{\epsilon,i}) : \epsilon > 0, i \in \mathbb{N}\} = \{\boldsymbol{p}\}$, $A \cup B \cup \{\boldsymbol{p}\} = \varprojlim \boldsymbol{f}$. Thus $\boldsymbol{p}$ is a separating point and so $\varprojlim \boldsymbol{f}$ is an arc. □

The next three lemmas reference the behaviour of a function on either side of a fixed point. We define four types of fixed point in the following definition in order to simplify the discussions.

Definition 4.2. Suppose that $f : [0, 1] \to [0, 1]$ is a continuous surjective function and $c, d, e \in [0, 1]$, $c < d < e$. If d is a fixed point of f, d is the only fixed point in the interval (c, e), either $c = 0$ or c is a fixed point, and either $e = 1$ or e is a fixed point, then d is:

- an *S-type* fixed point if for each $x \in [c, d]$, $f(x) \leq x$, and for each $x \in [d, e]$, $f(x) \geq x$;
- an *N-type* fixed point if for each $x \in [c, d]$, $f(x) \geq x$, and for each $x \in [d, e]$, $f(x) \leq x$;
- an *M-type* fixed point if for each $x \in [c, e]$, $f(x) \geq x$;
- a *W-type* fixed point if for each $x \in [c, e]$, $f(x) \leq x$.

In each case the type is *witnessed by* (c, e).

LEMMA 4.3. *Suppose that* $f : [0, 1] \to [0, 1]$ *is a continuous surjective function that does not admit a splitting sequence. If* f *admits a fixed point* d *that is S-type, M-type or W-type, then*

$$\varprojlim \boldsymbol{f} = \varprojlim([0, d], f \restriction [0, d]) \cup \varprojlim([d, 1], f \restriction [d, 1])$$

and

$$\varprojlim([0, d], f \restriction [0, d]) \cap \varprojlim([d, 1], f \restriction [d, 1]) = \{(d, d, \ldots)\}.$$

Proof. Suppose d is an S-type fixed point witnessed by (c, e). Then by the definition of S-type, c and e are fixed points. By Lemma 3.11, $f^{-1}(d) = \{d\}$ and hence the result follows.

Suppose that d is an M-type fixed point witnessed by (c, e); the proof for a W-type fixed point is analogous. Observe that, by the surjectivity of f and Lemma 3.11, c and e are fixed points.

Let $p' = \max(f([0, d])$ and let $p = \max\{x \in [0, d] : f(x) = p'\}$. By Lemma 3.11, $p' < e$. Let $q = \min\{x \in [d, 1] : f(x) = p'\}$. If $p' = d$ then the result follows from Lemma 3.11. Suppose that $p' > d$. Let $A \subseteq [p, d]$ be an interval such that $f(A) = [d, p']$. Then $([d, q], A)$ generates a splitting sequence of order 1, and hence $p' = d$. The result follows. □

LEMMA 4.4. *Suppose that $f : [0, 1] \to [0, 1]$ is a continuous surjective function that does not admit a splitting sequence. If f admits an N-type fixed point d witnessed by (c, e), then $\varprojlim([c, e], f \restriction [c, e])$ is an arc, and if c and e are fixed points, then $(d, d, \ldots)$ is a separating point of $\varprojlim \boldsymbol{f}$.*

Proof. Suppose c and e are fixed points. Let $p = \max(f([c, d])$ and $q = \min(f([d, e])$. By Lemma 3.11, $c < q$ and $p < e$. Then the functions $f \restriction [c, p]$ and $f \restriction [q, e]$ satisfy the conditions of Proposition 3.19, case (1). Each function has exactly two fixed points and so by Lemma 4.1, each of the sets $A_1 := \varprojlim([c, d], f \restriction [c, d])$ and $A_2 := \varprojlim([d, e], f \restriction [d, e])$ is an arc, and by Lemma 3.16 (iv), $A_1 \cap A_2 = \{(d, d, \ldots)\}$.

Suppose $\boldsymbol{x} \in \varprojlim([c, e], f \restriction [c, e]) \setminus \{(d, d, \ldots)\}$. If $x_0 \in [c, q)$, then for each $n \in \mathbb{N}$, $x_n \in [c, q)$. Hence $\boldsymbol{x} \in A_1$, and similarly if $x_0 \in (p, e]$ then $\boldsymbol{x} \in A_2$. Suppose $x_0 \in [q, p]$. Since $\boldsymbol{x} \neq (d, d, \ldots)$ there exists $n \in \mathbb{N}$ such that $x_n \notin [q, p]$. Let $m = \min\{n \in \mathbb{N} : x_n \notin [q, p]\}$. If $x_m \in [c, q)$, then $x_n \in [c, q)$ for each $n > m$, and hence $\boldsymbol{x} \in A_1$. Otherwise $\boldsymbol{x} \in A_2$.

Thus $\varprojlim([c, e], f \restriction [c, e]) = A_1 \cup A_2$ and $(d, d, \ldots)$ is a separating point of $\varprojlim([c, e], f \restriction [c, e])$ and hence of $\varprojlim \boldsymbol{f}$.

If e is not a fixed point, then $e = 1$, and by the surjectivity of f and Lemma 3.11, $f \restriction [c, 1]$ satisfies the condition of Proposition 3.19, case (1). Since d is an N-type fixed point, if $c \neq 0$, c is either an S-type or an M-type fixed point, or an accumulation point of the set of fixed points. Hence by Corollary 3.12 and Lemma 4.3,

$$\varprojlim \boldsymbol{f} = \varprojlim([0, c], f \restriction [0, c]) \cup \varprojlim([c, 1], f \restriction [c, 1]),$$

$$\varprojlim([0, c], f \restriction [0, c]) \cap \varprojlim([c, 1], f \restriction [c, 1]) = \{(c, c, \ldots)\},$$

and $\varprojlim([c, 1], f \restriction [c, 1])$ is an arc since $f \restriction [c, 1]$ admits exactly two fixed points.

Similarly if c is not a fixed point. □

LEMMA 4.5. *If $f : [0, 1] \to [0, 1]$ is a continuous surjective function that does not admit a splitting sequence, then $\varprojlim \boldsymbol{f}$ is an arc.*

Proof. Since $\varprojlim \boldsymbol{f}$ is an arc if and only if $\varprojlim \boldsymbol{f}^2$ is an arc, and if f satisfies the condition of case (2) of the proof of Proposition 3.19, then f^2 satisfies the condition of case (1), and we can assume that f satisfies the condition of case (1) and hence admits more than one fixed point.

Let E be the set containing the two endpoints admitted by f as in Proposition 3.19, and let d and e, $d < e$, be the two fixed points that determine the members of E. It remains to show that if $\boldsymbol{x} \in \varprojlim \boldsymbol{f} \setminus E$, then $\boldsymbol{x}$ is a separating point. So let $\boldsymbol{x} \in \varprojlim \boldsymbol{f} \setminus E$.

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By Lemma 4.1 we can assume that f admits more than two fixed points. Let F be the set of fixed points admitted by f and let

$$\boldsymbol{F} = \{(p, p, \ldots) : p \in F\}.$$

If $\boldsymbol{x} \in \boldsymbol{F}$, then by Lemma 3.11, Corollary 3.12, Proposition 3.19, and Lemmas 4.3 and 4.4, $\boldsymbol{x}$ is a separating point of $\varprojlim \boldsymbol{f}$.

Suppose $\boldsymbol{x} \notin \boldsymbol{F}$. By Lemma 3.16 (iv) and Lemma 3.17 (iv), there exists $n \in \mathbb{N}$ such that $\min(F) < x_n < \max(F)$. Then there are fixed points c, c' such that $c < x_n < c'$ and $(c, c') \cap F = \emptyset$. We consider three cases.

(a) $c = 0$ or $c' = 1$. Suppose $c' = 1$. If c is an S-type, M-type or W-type fixed point, or an accumulation point of F, then

$$\varprojlim \boldsymbol{f} = \varprojlim([0, c], f \restriction [0, c]) \cup \varprojlim([c, 1], f \restriction [c, 1])$$

and $\boldsymbol{x} \in \varprojlim([c, 1], f \restriction [c, 1])$. Since $\varprojlim([c, 1], f \restriction [c, 1])$ has exactly two fixed points and $\boldsymbol{x}$ is not one of them, $\varprojlim([c, 1], f \restriction [c, 1])$ is an arc and $\boldsymbol{x}$ is a separating point of $\varprojlim([c, 1], f \restriction [c, 1])$ and hence of $\varprojlim \boldsymbol{f}$.

If c is an N-type fixed point, witnessed by (e, c'), then e is a fixed point and e is not an N-type fixed point, so

$$\varprojlim \boldsymbol{f} = \varprojlim([0, e], f \restriction [0, e]) \cup \varprojlim([e, 1], f \restriction [e, 1]),$$

$\boldsymbol{x} \in \varprojlim([e, 1], f \restriction [e, 1])$, $\varprojlim([e, 1], f \restriction [e, 1])$ is an arc and $\boldsymbol{x}$ is not an endpoint of $\varprojlim([e, 1], f \restriction [e, 1])$. So again, $\boldsymbol{x}$ is a separating point of $\varprojlim \boldsymbol{f}$.

Similarly if $c = 0$.

(b) $c = \min(F) \neq 0$ or $c' = \max(F) \neq 1$. Suppose $c' = \max(F) \neq 1$. Then c' is an N-type fixed point and c is either an S-type or an M-type fixed point, or an accumulation point of F. In any case

$$\varprojlim \boldsymbol{f} = \varprojlim([0, c], f \restriction [0, c]) \cup \varprojlim([c, 1], f \restriction [c, 1]),$$

$$\varprojlim([0, c], f \restriction [0, c]) \cap \varprojlim([c, 1], f \restriction [c, 1]) = \{(c, c, \ldots)\},$$

$\boldsymbol{x} \in \varprojlim([c, 1], f \restriction [c, 1])$, and by Lemma 4.1, $\varprojlim([c, 1], f \restriction [c, 1])$ is an arc. Thus $\boldsymbol{x}$ is a separating point of $\varprojlim \boldsymbol{f}$.

Similarly if $c = \min(F) \neq 0$.

(c) $c \neq 0, c' \neq 1, c \neq \min(F), c' \neq \max(F)$. Either c or c' is not an N-type fixed point. Suppose c' is not. Then

$$\varprojlim \boldsymbol{f} = \varprojlim([0, c'], f \restriction [0, c']) \cup \varprojlim([c', 1], f \restriction [c', 1]),$$

$\boldsymbol{x} \in \varprojlim([0, c'], f \restriction [0, c'])$, c' is the maximum fixed point admitted by $f \restriction [0, c']$, and so the result follows as in case (b) above.

Similarly if c is not an N-type fixed point □

We now show that if f does admit a splitting sequence then $\varprojlim \boldsymbol{f}$ is not an arc.

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LEMMA 4.6. *Let $f : [0, 1] \to [0, 1]$ be a surjective continuous function. If f admits a splitting sequence then there is a non-degenerate continuum $C \subset \varprojlim \boldsymbol{f}$ and a sequence of non-degenerate continua*

$$\langle C_n \subset \varprojlim \boldsymbol{f} : n \in \mathbb{N}\rangle,$$

$C_n \neq C$, such that $C_n \to C$ in the Hausdorff metric.

Proof. Suppose $\sigma = \langle T_n = [l_n, r_n] : n \in \mathbb{N}\rangle$ is a splitting sequence and let N be an infinite subset of $\mathbb{N}$ such that for each $n \in N$ there is a non-degenerate interval $S_n \subset [0, 1]$, $S_n \cap T_n \subset \{l_n, r_n\}$ and $f(S_n) = f(T_n)$.

For each $n \in N$ let $S_n^n = T_n$. If $j \geq n$ and $S_n^j \subset [0, 1]$ has been defined, choose an interval $S_n^{j+1} \subset [0, 1]$ such that $f(S_n^{j+1}) = S_n^j$. Since f is surjective, S_n^{j+1} exists as $\Gamma(f) \cap ([0, 1] \times S_n^j)$ must have a component C such that $\pi_j(C) = S_n^j$, and so we can let $S_n^{j+1} = \pi_{j+1}(C)$.

For each $j < n$ let $S_n^j = T_j$. It follows that S_n^j is non-degenerate for each $n \in N$, $j \leq n$.

For each $n \in N$ let $S^n = \varprojlim(S_n^m, f \restriction S_n^m)$. Then $\{L(\sigma)\} \cup \{S^n : n \in \mathbb{N}\}$ is a collection of non-degenerate continua in $\varprojlim \boldsymbol{f}$. If $\boldsymbol{t} \in L(\sigma)$ then for each $n \in \mathbb{N}$ there is a point $\boldsymbol{s}^n \in S^n$ such that $s_j^n = t_j$ for every $j \leq n$ and hence any neighbourhood of $\boldsymbol{t}$ meets infinitely many sets S^n. Furthermore, any sequence $\{\boldsymbol{s}^n \in S^n : n \in \mathbb{N}\}$ has a limit point in $L(\sigma)$. It follows that $S^n \to L(\sigma)$ in the Hausdorff metric. □

THEOREM 4.7. *Suppose $f : [0, 1] \to [0, 1]$ is a continuous surjective function. Then $\varprojlim \boldsymbol{f}$ is an arc if and only if $\varprojlim \boldsymbol{f}$ does not admit a splitting sequence.*

Proof. By Lemmas 4.5 and 4.6. □

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