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NONNEGATIVE SOLUTIONS FOR WEAKLY NONLINEAR ELLIPTIC EQUATIONS

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Let $x = (x_1, \ldots, x_n)$ denote a point of Euclidean n space E^n and set $D_i = \partial/\partial x_i$ for $i = 1, \ldots, n$. Let Ω denote an exterior domain in E^n with smooth boundary and consider in Ω the formal elliptic problem:

$$(1) \quad Lu = - \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + f(x, u)u = r(x),$$

$$u = \tau \text{ on } \partial\Omega.$$

We first consider the problem of finding nonnegative generalized solutions of (1) when $\tau \geqq 0$, $\tau \not\equiv 0$, and $r(x) \equiv 0$. Under more stringent conditions on the coefficients and for suitable $r(x)$, we then show the existence of a locally bounded solution. Next, we show that, under stronger assumptions, our main criterion is also necessary. The final arguments are devoted to the consideration of illustrative examples.

One of the classical procedures employed to study problem (1), and related problems, is the construction of a solution under the assumption that suitable upper and lower solutions exist. For this and related methods we refer in particular to **[4]**, **[5]**, **[6]**, **[10]**, **[23]**, **[18]**, **[19]**, **[20]**, **[24]**, **[27]**, **[8]**, **[25]**, **[21]**, **[11]**, **[26]**, **[3]** and the numerous references therein. A general discussion of the various procedures employed for problem (1), and related equations, together with numerous references (up to the middle 1970's) can be found, for example, in **[14]**.

It is the purpose of this paper to show the existence of a nonnegative solution to (1) by means of a variant of the upper and lower solution method. Our procedure will relate the question of existence to the question of the solvability of a linear problem related to (1), and hence is apparently restricted to equations with a weak nonlinearity of type $f(x, u)u$. However, some specific results (e.g. Theorem 4) are valid for more general nonlinearities if we make the assumption that upper and lower solutions exist. Indeed, the primary reason for our restriction to equations

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such as (1) is that we can then generate a positive supersolution, under suitable growth conditions, by solving a related linear problem. Furthermore, the solvability of the linear problem will also be shown to be necessary for the existence of a positive solution to (1), if we seek a solution u whose weighted L^2 norm is bounded in terms of a weighted L^2 norm of r. Finally, we deal with generalized solutions which, in particular, are of class $H^{1,2}_{\text{loc}}(\Omega)$. Hence we do not work with Schauder spaces, nor with $\mathrm{H}^{1,p}(\Omega)$ (for $p > n$), nor with iterative arguments. Such procedures, which have been extensively used in the literature (see, e.g. [**4-6**] and [**21-27**]) will be replaced in our considerations by adaptations of more general bounded domain results of Hess, [**18-20**], and by the introduction of suitable spaces and local arguments recently used by the author for linear problems, [**1**], [**2**].

For convenience, we do not distinguish in notation in the sequel between functions and associated Nemytskii operators and we explicitly state our conditions and results only for $n \geqq 3$. Constants whose precise value is irrelevant will be denoted by the same symbol. The same procedure will be applied to subsequences and subdomains. Finally, we will not distinguish in notation between functions and equivalence classes of functions. The nature of an inequality between the latter will be obvious from the context.

Our basic assumptions on $f(x, u)$, a_{ij} are as follows: There exists a smooth domain $\Omega^1 \supset \bar{\Omega}$ and sequence $\{r_m\}_{m=2}^{\infty}$, $r_m \uparrow \infty$, such that if we set

$$S_m = \{x| \quad |x| = r_m\}; \quad \Omega^1_m = \Omega^1 \cap \{x| \quad |x| < r_m\},$$
$$\Omega_m = \Omega \cap \{x| \quad |x| < r_m\}, \; S_0 \underline{\underline{\Delta}} \; \partial\Omega^1, \; S_1 \underline{\underline{\Delta}} \; \partial\Omega,$$

then:

(a) $f(x, u)$ satisfies the Caratheodory conditions in $\Omega_1 \times \mathbf{R}^+$;

(b) a_{ij}, $g(x) \leqq \inf_{t \geqq 0} f(x, t)$ are of class $L^2(\Omega^1_m)$;

(c) $a_{ij} = a_{ji}$, (a_{ij}) is locally uniformly positive definite in $\bar{\Omega}^1$;

(d) for every P, bounded subdomain of Ω^1, there is a constant $K = K(P) > 0$ such that the linear form $B(g, \cdot, \cdot, P)$ defined on $C_0^{\infty}(\Omega^1)$ by:

$$B(g, \phi, \psi, P) = \int_P \sum a_{ij} D_i\phi D_j\psi + g\phi\psi$$

satisfies on $C_0^{\infty}(P)$ the inequality:

(2) $\quad B(g, \phi, \phi, P) \geqq K\int_P \phi^2 dx;$

(e) each S_m has a neighbourhood N_m such that $a_{ij} \in L^\infty(N_m)$, $g(x) \in L^\infty(N_m)$;

(f) $g_1^- \in L^{n/2}(\Omega_m^1)$ for each m;

(g) $\Omega_m^1 = G_m \cup Z_m$ with Z_m sets of measure zero and G_m domains such that if $T \subset \bar{T} \subset G_m$ then $|a_{ij}| < M$, $g \in L^{\gamma/2}$ in T with M, γ functions of T and $\gamma > n$.

We do not require that $f(x, t) \geqq 0$ for $t \geqq 0$, nor that $tf(x, t)$ be monotone in t. We also note that the only "growth" condition (apart from the assumptions that $g(x) \leqq \inf_{t \geqq 0} (f(x, t))$ be reasonably well behaved locally) is the restriction placed on B in condition (d). All other conditions will be automatically satisfied if $f(x, t)$ and a_{ij} are locally suitably well behaved. These conditions are essentially special cases of the ones imposed in [**1**], [**2**] and consequently could be weakened somewhat at the expense of complicating the presentation. For example, $\{S_m\}_2^\infty$ need not be a sequence of spheres, etc. Finally we observe that if $f(x, t)t \geqq 0$ for $t \geqq 0$, then we may always choose $g(x) \equiv 0$.

Let $\tilde{L}_0$ denote a possible self-adjoint extension in L^2 of the formal expression:

$$L_0 u = -\sum D_i(a_{ij} D_j u) + gu,$$

defined on $C_0^\infty(\Omega)$ and let $\sigma(\tilde{L}_0)$ be the spectrum of $\tilde{L}_0$. If L is viewed as a perturbation of L_0 then clearly we may have $0 \in \sigma(\tilde{L}_0)$. Such singular problems have been investigated by many authors. We mention in particular the results in [**22**].

Let $S \subset \Omega^1$, $\xi \in L^1(S)$. We denote by $B^1(\xi, v, \psi, S)$ the linear form given by

$$B^1(\xi, v, \psi, S) = \int_S \sum a_{ij} D_i v D_j \psi + [\xi^+ + 1] v\psi,$$

with domain $B^1 = E \times E$, where

$$E = \{v | \quad v \in C^1(\bar{S}),\ B^1(\xi, v, v, S) < \infty\}.$$

Let $W(\xi, S)$ be the completion of E with respect to

$$\|v\|(W(\xi, S)) = (B^1(\xi, v, v, S))^{1/2}.$$

Analogously, we define $\mathring{W}(\xi, S)$ by completing $C_0^\infty(S)$, and we denote by $H^{1,p}(S)$, $\mathring{H}^{1,p}(S)$ the standard Sobolev spaces with norm

$$\|v\|_{1,p}^p(S) = \int_S \sum |D_i v|^p + |v|^p.$$

The L_p norm is denoted by $\| \quad \|_{0,p}(S)$ and we let $\langle\ ,\ \rangle$ represent the usual duality map. If S is obvious from the context we write $W(\xi)$ for $W(\xi, S)$, etc.

Finally, a solution of Problem (1) will mean a function v such that:

$$f(x, v) \in L^1(\Omega_m) \quad \text{and} \quad v \in W(f(x, v), \Omega_m) \quad \text{for all } m,$$

and further,

$$B(f(x, v), v, \phi, \Omega) = \langle r, \phi \rangle \quad \text{for all } \phi \subset C_0^\infty(\Omega)$$

and $v = \tau$ on $\partial\Omega$. In view of the definition of $W(\xi, \Omega_m)$, we may associate $W(f(x, v), \Omega_m)$ with a subspace of $H^{1,2}(\Omega_m)$, and it follows that solutions must belong to $H^{1,2}(\Omega_m)$ for all m.

THEOREM 1. *There exists a function $u \in W(g, S) \subset H^{1,2}(S)$ for all bounded domains $S \subset \bar{S} \subset \Omega^1$ such that: $B(g, u, \phi) = 0$ for $\phi \in C_0^\infty(\Omega^1)$, $u \geqq 0$ in Ω^1, $u \not\equiv 0$.*

Proof. Choose points $x_m \in S_m$, and constants $\epsilon_m > 0$ such that

$$\{x| \quad |x - x_m| < \epsilon_m\} \subset N_m \quad \text{for } m = 2, 3, \dots.$$

By the construction of [**1**] there is an a.e. positive function $v_m \in W(g, F_m)$ such that

$$B(g, v_m, \phi) = 0 \quad \text{for } \phi \in C_0^\infty(F_m),$$

where

$$F_m = \Omega^1_{m+1} - \{x| \quad |x - x_m| < \epsilon_m\}.$$

Let x_0 be a fixed point on S_1. Since $v_m \in C(N_1)$ for all m, see e.g. [**16**, p. 192], we can normalize $\{v_m\}$ by setting $v_m(x_0) = 1$. The local compactness argument of [**1**] then shows that if $\phi \in C_0^\infty(\Omega^1)$, supp (grad ϕ) $\subset UN_i$ and s, m are large, it follows that:

$$\|\phi(v_s - v_m)\|^2(W(g, \Omega^1)) \leqq K(\phi)\|v_s - v_m\|^2_{0,2} \text{ (supp (grad } \phi)).$$

Furthermore, by Harnack's inequality, $\{v_s\}$ will be uniformly bounded on supp (grad ϕ). Setting $v_m \equiv 0$ first, we conclude that $\{\phi v_s\}$ is bounded in $W(g, \Omega_1)$ and, without loss of generality, Cauchy in $L^2(\Omega^1)$. The estimate then implies that $\{\phi v_m\}$ is Cauchy in $W(g, \Omega^1)$. We next exhaust Ω^1 with a family of nested compact sets $\{K_\alpha\}_{\alpha=1}^\infty$, and let $\{\phi_\alpha\}$ denote C_0^∞ functions with

$$\text{supp } (\phi_\alpha) \subset K_{\alpha+1} \quad \text{and} \quad \phi_\alpha \equiv 1 \quad \text{in } K_\alpha.$$

The above arguments show that for any $\alpha \geqq 1$ there is a subsequence $\{v_m^\alpha\}_m$ of $\{v_m^{\alpha-1}\}_m$ such that $\{\phi_\alpha v_m^\alpha\}_m$ converges in $W(g, \Omega^1)$, where we define $v_m^0 = v_m$. The usual diagonalization procedure then shows that the sequence $\{\phi_m v_m^m\}$ converges to a limit function u with the desired properties.

As an immediate consequence it follows that there is a function $u \in W_{\text{loc}}(g, \Omega^1)$ such that $u \geqq 0$ in Ω, $u \geqq M$ on $\partial\Omega$ and $B(g, u, \phi) = 0$ for $\phi \in C_0^\infty(\Omega)$. It is possible to construct examples where the above conditions are satisfied, $0 \in \sigma_p(\widetilde{L}_0)$, ($\sigma_p$ = point spectrum), and all positive functions u such that $B(g, u, \phi) = 0$ in Ω^1, $u \in W_{\text{loc}}(g, \Omega^1)$ must actually be linear multiples of the lowest eigenvector of $\widetilde{L}_0$. Since in these cases $u = 0$ on $\partial\Omega_1$, then problem 1 (with $r \equiv 0$, $\tau \not\equiv 0$) would have no positive solution. The formulation of our conditions on the larger domain Ω^1 is designed to avoid such behaviour.

COROLLARY 2. *If $Z_m = \phi$ for all m, then $u \in L^\infty(\Omega_m)$.*

Indeed, condition (g) suffices for this, see e.g. [**17**, p. 192 and p. 197].

The function u plays the role of an uppersolution in the sequel. It would be possible to start our considerations at this point if we began with the assumption of the existence of an uppersolution u with the desired properties.

COROLLARY 3. *Assume $Z_m = \phi$ for all m. Let $\tau \geqq 0$ be the trace of a nonnegative function in $\mathring{H}^{1,2}(\Omega^1) \cap L^\infty(\Omega^1)$ (also denoted by τ), with $u \geqq \tau$ on $\partial\Omega$. If for each constant k,*

$$H(k, m) = \sup_{0 \leqq t \leqq k} |f(x, t)| \in L^1(\Omega_m),$$

then there exists a function $v_m \in H^{1,2}(\Omega_m) \cap L^\infty(\Omega_m)$ such that $0 \leqq v_m \leqq u$; $v_m = \tau$ on $\partial\Omega$, $v_m = 0$ on S_m;

$$v_m \in W(f(x, v_m), \Omega_m) \quad \text{and}$$

$$B(f(x, v_m), v_m, \phi) = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega_m).$$

Proof. Observe that u is of class $L^\infty(\Omega_m)$ as a consequence of Corollary 2. Since zero is a lowersolution, a result of Hess, [**20**], shows the existence of a function $v_m \in H^{1,2}(\Omega_m) \cap L^\infty(\Omega_m)$ such that

$$f(x, v_m)v_m \in L^1(\Omega_m) \quad \text{and}$$

$$B(f(x, v_m), v_m, \phi) = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega_m).$$

To conclude, we observe the embedding:

$$H^{1,2}(\Omega_m) \cap L^\infty(\Omega_m) \subset W(\Omega_m).$$

Indeed, let $\omega \in H^{1,2}(\Omega_m) \cap L^\infty(\Omega_m)$. Extending ω to a larger domain and using mollifiers, shows the existence of a sequence $\{u_\alpha\}$ such that $u_\alpha \in C^1(\bar{\Omega}_m)$, $u_\alpha \to \omega$ in $H^{1,2}(\Omega_m)$ and, for all α, $|u_\alpha| < K$. We observe that

$$\|u_\alpha\|^2(W(\Omega_m)) = \int_{\Omega_m} \sum a_{ij} D_i u_\alpha D_j u_\alpha + [f^+(x, v_m) + 1] u_\alpha^2 \leqq K$$

where we used the fact that $a_{ij} \in L^\infty(\Omega_m)$, $f(x, v_m) \in L^1(\Omega_m)$. We may thus assume that u_α converges weakly in W, and consequently that $\{1/\beta \sum_{\alpha=1}^{\beta} u_\alpha\}$ converges strongly in W, necessarily to ω.

THEOREM 4. *Let the conditions of Corollary* 3 *hold. Then the problem* (1), *with* $r \equiv 0$, *has a nonnegative solution* v *in* Ω, $0 \leqq v \leqq u$.

Proof. Without loss of generality, assume that $\tau \in \mathring{W}(\Omega_2^1)$ with supp $\tau \subset N_1$ and set $v_m = \tau$ in $\sim\Omega$, $v_m = 0$ in $\Omega - \Omega_m$. Let $\epsilon > 0$, m_0 be given. Choose $\phi \in C_0^\infty(E^n)$ such that

$$\Omega_{m_0} \cap \{x| \operatorname{dist}(x, S_{m_0}) > \epsilon\} \subset \{x| \phi(x) \equiv 1\},$$

while

$$(\sim\Omega_1) \cup \{x| \quad |x| > r_{m_0}\} \subset \{x| \quad \phi(x) \equiv 0\}.$$

For m large, we then have

$$\phi w_m \in \mathring{W}(f(x, v_m), \Omega_{m_0}), \quad \text{where } w_m = v_m - \tau.$$

A direct calculation shows:

$$\begin{aligned}(3) \quad B(f(x, v_m), \phi w_m, \phi w_m) &\leqq k(\phi)\|w_m\|_{0,2}^2 (\text{supp grad } \phi) \\ &+ \Big| \sum_{i,j} \langle a_{ij} D_i w_m, D_j \tau \rangle (\text{supp } \phi) \\ &+ \langle f(x, v_m) w_m, \tau \rangle (\text{supp } \phi) \Big|,\end{aligned}$$

where we have used the condition: supp $\tau \subset \{x| \quad \phi(x) \equiv 1\}$. Since w_m, $\tau \in \mathring{W}(\Omega^1)$ inequalities (2) and (3) lead to

$$\begin{aligned}(4) \quad B(f(x, v_m), \phi w_m, \phi w_m) &\leqq k(\phi)\,\|w_m\|_{0,2}^2 (\text{supp grad } \phi) \\ &+ [B(f(x, v_m), \phi w_m, \phi w_m)]^{1/2} [B(f(x, v_m), \tau, \tau)]^{1/2}\end{aligned}$$

where $f(x, v_m)$ is assumed extended to Ω^1 as a function which exceeds g. We conclude from assumption (f) and inequality (4) that for some constants K, C we have

$$\|\phi w_m\|_{1,2}^2 \leqq KB(g, \phi w_m, \phi w_m) \leqq KB(f(x, v_m), \phi w_m, \phi w_m) < C.$$

We may thus conclude that $\{w_m\}$ converges in L^q (supp ϕ) for $q < 2n/(n-2)$. Essentially repeating the calculations of (3) with $w_m - w_r$ in place of w_m gives

$$\begin{aligned} &B(g, \phi(w_m - w_r), \phi(w_m - w_r)) \\ &\qquad\qquad \leqq K(\phi)\, \|w_m - w_r\|_{0,2}^2 \text{ (supp grad } \phi) \\ &+ \int_\Omega \phi^2 |w_m - w_r|\, |(f(x, v_m) - g)w_m - (f(x, v_r) - g)w_r| \\ &+ \int_\Omega |(f(x, v_m) - f(x, v_r)\,|\tau|\phi|^2 |w_m - w_r| \\ &\leqq C\{\, \|w_m - w_r\|_{0,2}^2 \text{ (supp } \phi) + \int_\Omega \phi^2 |w_m - w_r| H\}. \end{aligned}$$

Since $-\tau \leqq w_m, w_r \leqq u < k$ on supp ϕ, and $H \in L^1(\Omega_m)$, we conclude from standard measure theory arguments that $\{\phi w_m\}$ is Cauchy in $W(g, \Omega)$. Repeating the diagonalization procedure of Theorem 1 shows the existence of a function w in $W(g, \Omega_s)$ for all s such that $w_m \to w$ in $W(g, \Omega_s)$. We may assume that $w_m \to w$ pointwise a.e. Ω and if we set $v = w + \tau$ then $0 \leqq v \leqq u$. By the Caratheodory conditions,

$$f(x, v_m(x))v_m(x) \to f(x, v(x))v(x) \quad \text{pointwise,}$$

while $f(x, v_m)v_m$ and $f(x, v)v$ are majorized locally by $kH(k, s)$. It follows that:

$$\int_\Omega \phi f(x, v_m)v_m \to \int_\Omega \phi f(x, v)v, \quad \text{for any } \phi \in C_0^\infty(\Omega),$$

and, as a consequence, that $B(f(x, v), v, \phi) = 0$, $v = \tau$ on $\partial\Omega$. Since

$$v \in H^{1,2}(\Omega_m) \cap L^\infty(\Omega_m),$$

the procedure of Corollary 3 shows that $v \in W(f(x, v), \Omega_m)$ for all m and the result follows.

Added in Revision. Since the completion of the original manuscript, a paper of Cac, [**9**], has appeared which essentially contains Theorem 4. He considered a more general nonlinearity and boundary condition under the a priori assumption that upper and lower solutions exist. Though the basic methods used in [**9**] and here are similar, we left a proof of Theorem 4. This was done because our calculations (which are also applicable to the more general nonlinearity considered in [**9**]) are somewhat different from those of [**9**], and furthermore, indicate clearly the bootstrap nature of the argument. Observe that the global uniform ellipticity, postulated in [**9**], is not needed in the arguments used.

Note that since u is determined by Ω, (a_{ij}) and g, no conclusion can be drawn concerning the global integrability of H. Observe that we also allow $H \in L^1$. Consequently, our results do not appear to be contained in the unbounded domain criterion mentioned by Hess at the end of [**20**]. Finally, it would be interesting to construct v iteratively. It is not clear how this could be done without the imposition of further restrictions on the coefficients.

If we assume somewhat more about f, for example that $f:L^\infty(S) \to L^\gamma(S)$ for $S \subset \bar{S} \subset G_m$ with $\gamma = \gamma(S) > n/2$, then it is a consequence of the above arguments and of results in [**17**, p. 189] that the solution v is actually positive a.e. Ω or identically zero.

It is not difficult to construct examples where the above assumptions are satisfied, $r(x) > 0$, but there is no positive solution. To deal with some of these cases we modify our assumptions as follows:

Assume now that the form B of assumption (d) satisfies instead of (2) the stronger assumption:

(5) $B(g, \phi, \phi) \geqq \langle p\phi, p\phi \rangle$,

where $\phi \in C_0^\infty(\Omega)$ and p is a fixed $C^\infty(\bar{\Omega})$ positive function in $\bar{\Omega}_m$ for all m, and for simplicity, that $\tau \equiv 0$.

It is well known that there are examples of problems where (2) is satisfied in $C_0^\infty(\Omega)$ but for which the only nonnegative p for which (5) holds in $C_0^\infty(\Omega)$ is $p \equiv 0$. However, in many cases the known conditions for (2) actually also suffice for the stronger condition (5). The presence of (5) also implies that conditions (a)-(g) need only be postulated for Ω, and mention of Ω^1 can be removed. Let $0 \leqq \delta < 1$ and let $r/p \in L^2(\Omega)$.

Inequality (5) and the previous considerations now imply that the linear problem: $L_0 u - \delta p^2 u = r \geqq 0$, $u = 0$ on $\partial\Omega$ has a nonnegative generalized solution u in $H^{1,2}(\Omega_m)$ such that $pu \in L^2(\Omega)$. Further, from the spectral theorem it follows that

$$\|pu\|_{0,2}(\Omega) \leqq K[\|r/p\|_{0,2}(\Omega)] \quad \text{with } 0 < K = K(\delta),$$

while if $r \in L^{\gamma/2}(\Omega_m)$ with $\gamma = \gamma(m) > n$ then again the results of [**17**, p. 184] imply that u must be locally bounded. To summarize we state:

COROLLARY 5. *Let conditions* (a)-(g) *and inequality* (5) *hold. Assume* $Z_m = \phi$ *for all m. If*:

(i) $H(k, m) \in L^1(\Omega_m)$;

(ii) $0 \leqq r \in L^{\gamma/2}(\Omega_m)$ *with* $\gamma = \gamma(m) > n$;

(iii) $r/p \in L^2(\Omega)$;

then the problem: $Lv - \delta p^2 v = r$, $v = 0$ *on* $\partial\Omega$ *has a locally bounded nonnegative solution* v *such that*

$$\|pv\|_{0,2}(\Omega) \leqq K\|r/p\|_{0,2}(\Omega).$$

Observe that Corollary 5 gives conditions for the existence of a nonnegative supersolution for the somewhat more general case: $h(x, u) \geqq g(x)u - r$.

We now show that for a class of problems the growth condition (5) is also necessary for the existence of solutions with the above properties.

THEOREM 6. *Let the conditions of Corollary* 5 *hold except for inequality* (5). *Assume further that* $g(x) = f(x, 0)$ *and that*:

$$f: L^\infty(S) \to L^\gamma(S) \quad \textit{for } S \subset \bar{S} \subset G_m \textit{ with } \gamma = \gamma(S) > n/2.$$

Then the problem: $Lv - \delta p^2 v = r$, $v = 0$ *on* $\partial\Omega$, *has locally bounded nonnegative solutions which satisfy the* $L^2(\Omega)$ *estimate stated in Corollary* 5 *if and only if*

$$B(g, \phi, \phi) \geqq \langle p\phi, p\phi\rangle \textit{ for all } \phi \in C_0^\infty(\Omega).$$

Proof. It suffices to show that $B(g, \phi, \phi) \geqq \langle p\phi, p\phi \rangle$. Let $0 \leqq \delta < 1$ and choose $r \geqq 0$ ($\not\equiv 0$) of class $C_0^\infty(\Omega)$. By assumption there exist locally bounded nonnegative solutions $\{w_m\}$ of:

$$L(w_m) - \delta p^2 w_m = r/m,$$

$$w_m = 0 \text{ on } \partial\Omega.$$

Let $\phi \in C_0^\infty(\Omega)$ be given. Choose $R > 0$ such that if $x \in \operatorname{supp} \phi$ then $B_{2R}(x) \subset \subset \Omega$, where $B_{2R}(x)$ denotes the ball with center x and radius $2R$. Since $f(x, w_m) \geqq g(x)$ and $w_m \geqq 0$, it follows that $\{w_m\}$ are subsolutions of the linear problems

$$L_0(w_m) - \delta p^2 w_m \leqq r/m.$$

It follows (see e.g. [**17**, p. 184]), that if $x \in \operatorname{supp} \phi$ then:

$$\sup_{B_R(x)} w_m \leqq C[\,\|w_m\|_{0,2}(B_{2R}(x)) + \frac{\|r\|_{0,\gamma}}{m}(B_{2R}(x))\,],$$

where C is a constant which depends on R, the coefficients, and supp ϕ but not on w_m or r. By assumption, we have

$$\|pw_m\|_{0,2}(\Omega) \leqq K\|\frac{r}{pm}\|_{0,2}(\Omega).$$

Substituting into the above inequality yields:

$$\sup_{B_R(x)} w_m \leqq \frac{C}{m}[\,\|r\|_{0,2}(\Omega) + \|r\|_{0,\gamma}(\Omega)\,].$$

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We conclude that $w_m \to 0$ uniformly on supp ϕ, and since $|f(x, w_m)| \leqq H(\tau, k)$ on supp ϕ for some constants τ, k, it follows that

$$\langle f(x, w_m)\phi, \phi\rangle \to \langle f(x, 0)\phi, \phi\rangle = \langle g\phi, \phi\rangle.$$

Since f: $L^\infty(S) \to L^\gamma(S)$, we observe that each w_m is actually positive in supp ϕ, again by [**17**, p. 189]. An application of a generalized Picone's identity (see e.g. [**1**]) shows that

$$(7) \quad B(f(x, w_m), \phi, \phi) = \int_\Omega w_m^2 \sum_{i,j} a_{ij} D_i\left(\frac{\phi}{w_m}\right) D_j\left(\frac{\phi}{w_m}\right)$$

$$+ B\left(f(x, w_m) - \delta p^2, \frac{\phi^2}{w_m}, w_m\right) + \delta\langle p\phi, p\phi\rangle.$$

Since w_m must be continuous, we conclude that $\phi^2/w_m \in \overset{\circ}{H}_{1,2}$ (supp ϕ) and it follows that

$$B\left(f(x, w_m) - \delta p^2, \frac{\phi^2}{w_m}, w_m\right) \geqq 0.$$

From equation (7) we have:

$$B(f(x, w_m), \phi, \phi) \geqq \delta\langle p\phi, p\phi\rangle.$$

Letting $m \to \infty$, we obtain

$$B(g, \phi, \phi) \geqq \delta \langle p\phi, p\phi\rangle \quad \text{for any } \delta < 1,$$

and therefore, $B(g, \phi, \phi) \geqq \langle p\phi, p\phi\rangle$.

COROLLARY 7. *Under the conditions of Theorem* 6, $(L + \mu)v = r \geqq 0$, $v = 0$ *on* $\partial\Omega$ *has a nonnegative solution for all* $r \in L^2(\Omega)$, $\mu > 0$, *with* $\|v\|_{0,2} \leqq K\|r\|_{0,2}$ *and* $K = K(\mu)$ *if and only if* $B(g, \phi, \phi) \geqq 0$.

Proof. $(L + \mu)v = r$ has such a solution for all $\mu > 0$, if and only if $(L + \mu)v - \delta\mu v = r$ has a solution for $0 < \delta < 1$ and $\mu > 0$. By Theorem 6 and the choice $p = \sqrt{\mu}$, we find that this occurs if and only if $B(g + \mu, \phi, \phi) \geqq \mu \langle \phi, \phi\rangle$, that is: if and only if $B(g, \phi, \phi) \geqq 0$.

We briefly and heuristically remark that more properties can easily be obtained by the imposition of further assumptions. For example assume that the solution u of the linear problem: $L_0 u - \delta p^2 u = r$, $u = 0$ on $\partial\Omega$ (or any other suitable a priori nonnegative uppersolution of problem (1)) is known to approach zero at ∞ then so will the solution of problem (1) which we constructed. Further, it we assume that the generalized maximum principle is valid (in particular this requires $g \geqq 0$) then such solutions of problem (1) must be unique, since the maximum of their difference in Ω_m will be achieved on S_m.

We consider next some illustrative examples. Suppose first that $a_{ij} = \delta_{ij}$, $\tau = 0$, $f(x, u) = (g(x) + g_1(x)u^{\alpha})$ with $\alpha \geqq 0$, $g_1(x) \geqq 0$ and $|x|r(x) \in L^2$. Apart from the local regularity conditions for the existence of positive solutions as above it is necessary and sufficient that inequality (5) hold, with $p = c|x|^{-1}$, for some $c > 0$. For this it suffices, for example, that

$$g^- \in L^{n/2}(\Omega) \quad \text{with} \quad \|g\|_{0,n/2}(\Omega) < Z$$

where

$$Z^2 = n\pi\,(n-2)\left\{\frac{\Gamma(n/2)}{\Gamma(n)}\right\}^{2/n}.$$

This is an optimum embedding constant calculated by Talenti, [**28**] and Aubin, [**7**]. Indeed, if $\phi \in C_0^{\infty}(\Omega)$ it follows from the Gagliardo-Nirenberg results, [**13**, p. 24], that for some constant μ, $0 \leqq \mu < 1$ we have:

$$\langle g\phi, \phi\rangle \geqq -\mu\langle -\Delta\phi, \phi\rangle.$$

Whence we conclude that:

$$\langle -\Delta\phi, \phi\rangle + \langle g\phi, \phi\rangle \geqq (1-\mu)\langle -\Delta\phi, \phi\rangle.$$

The well known inequality:

$$\langle -\Delta\phi, \phi\rangle \geqq \frac{(n-2)^2}{4}\left\|\frac{\phi}{|x|}\right\|_{0,2}^2$$

then leads to the estimate:

$$\langle -\Delta\phi, \phi\rangle + \langle g\phi, \phi\rangle \geqq \frac{(1-\mu)(n-2)^2}{4}\left\|\frac{\phi}{|x|}\right\|_{0,2}^2.$$

Therefore inequality (5) holds with

$$p = \frac{\sqrt{1-\mu}(n-2)}{2|x|}.$$

For another example, set $a_{ij} = \delta_{ij}$, $f(x, u) = [g_1(x) + g_2(x)e^u]$, r as before. This example is related to an equation considered in [**12**] (where the functions were $g_1 \equiv 0$, $g_2 \geqq 0$). If we also assume $g_2 \geqq 0$ but allow g_1 to be negative then a key growth condition again becomes: $\|g_1^-\|_{0,n/2}(\Omega) < Z$.

We note as indicated above that the conditions on $g(x)$ could be replaced by, or combined with, conditions from ordinary differential equations. For example, more general conditions than the well known Kneser condition for (2), namely $g(x) > -(n-2)^2/4|x|^2$, could combine with $\|g^-\|_{n/2} < Z$ and would lead to analogous results.

Finally, we compare the above results with some which were previously known. As mentioned in the introduction, many of these dealt only with cases of regular coefficients, C^2 (or at least $C^{1+\alpha}$) solutions, and the a priori existence of upper and lower solutions. A quite general setting was considered in [**12**], but here coerciveness was required. This was realized by the assumption of uniform ellipticity and restrictions on the sign of the nonlinearity. Analogously, in the unbounded domain case considered [**8**] the linearity was either independent of x or, at most, depended on $|x|$ and and u. Furthermore, little consideration appears to have previously been given to the necessity of the assumptions that were made. A notable exception to this rule was the paper by Swanson, [**27**], where necessary and sufficient conditions are stated for the existence of a positive solution u with $|x|^{n-2}u(x)$ bounded, but only for equations which allow one dimensional argument. It appears, therefore, that our results are not covered by any of the above.

In conclusion, we note that if $f(x, t)$ is not somewhat restricted from below then $Lu = 0$ may have no positive solutions at all. We refer to [**16**] where, in particular, conditions are given so that there are no positive solutions in E^n. Other nonexistence results can be obtained by reversing the arguments we employed above (basically now assuming that $f(x, t) \leqq g(x)$ for $t \geqq 0$) and using linear theory. A comparison with ordinary differential equations may also be used to advantage for this and related questions (see e.g. [**27**], and the references therein).

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