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# Asymptotic stability of rarefaction wave for the compressible Navier-Stokes-Korteweg equations in the half space

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In this study, we are concerned with the asymptotic stability towards a rarefaction wave of the solution to an outflow problem for the Navier-Stokes Korteweg equations of a compressible fluid in the half space. We assume that the space-asymptotic states and the boundary data satisfy some conditions so that the time-asymptotic state of this solution is a rarefaction wave. Then we show that the rarefaction wave is non-linearly stable, as time goes to infinity, provided that the strength of the wave is weak and the initial perturbation is small. The proof is mainly based on $L^2$-energy method and some time-decay estimates in $L^p$-norm for the smoothed rarefaction wave.

*Keywords:* Compressible Navier-Stokes-Korteweg equation; Rarefaction wave; Asymptotic stability; Energy method

2020 *Mathematics subject classification:* 76W05; 35B40

## 1. Introduction

In this article, we are concerned with the models of compressible fluids endowed with internal capillarity, which are supposed to govern the motion of compressible fluids such as liquid vapour mixtures. The model (called as the compressible Navier-Stokes-Korteweg equations) originates from the work of Van de Waals [46] and Korteweg [29] more than one century ago, and was actually derived in its modern form in the 1980s using the second gradient theory, see for instance [11]. The one-dimensional isentropic compressible Navier-Stokes-Korteweg equation can be described by the following system in the Eulerian coordinate

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_x + (\rho u^2 + p(\rho))_x = \mu u_{xx} + k\rho\rho_{xxx}. \end{cases} \tag{1.1}$$

Here, $\rho, u$ are unknown functions in $t$ and $x$, which stand for the density and the velocity, respectively. The time and space variables are $t, x \in \mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$. The function $p(\rho)$ is the pressure defined by $p(\rho) = k\rho^\gamma$, where $k > 0$ and $\gamma \geqslant 1$ are the gas constants. The positive constants $\mu, \kappa$ denote, respectively, the viscosity and the capillary coefficient, and $\kappa$ is also called Weber number.

One can see easily that when $\kappa = 0$, system (1.1) is reduced to the classical Navier-Stokes equations for compressible fluids.

Recently, the compressible Navier-Stokes-Korteweg equation has attracted a lot of attention of physicists and mathematicians because of its physical importance, complexity, rich phenomena and mathematical challenges. There are many studies on the global existence and uniqueness of solutions to the isentropic compressible Navier-Stokes-Korteweg equations, and we can refer to [**2**–**4**, **6**, **10**, **13**–**18**, **21**, **30**] and some references therein. In what follows, let us focus on the large-time behaviour of solutions to the isentropic compressible Navier-Stokes-Korteweg equations, which is related to our interest. When the initial data are small perturbation near the non-vacuum constant states, Wang and Tan [**47**], Tan *et al.* [**43**], and Tan and Wang [**42**] established the optimal decay rates of the global classical solutions and the global strong solutions for the isentropic compressible Navier-Stokes-Korteweg equations, respectively. Tan and Zhang [**44**] further obtained the decay rates of more derivatives of solutions when the initial perturbation also is in the $H^{-s}(\mathbb{R}^3)$ (negative Sobolev norms) with $0 \leqslant s < 3/2$. Moreover, for the initial value problem of the isentropic compressible Navier-Stokes-Korteweg equations, the large-time behaviour around nonlinear wave patterns such as the stationary wave, discontinuous wave and the rarefaction wave has been studied. More precisely, the stability of stationary states of the multi-dimensional isentropic compressible Navier-Stokes-Korteweg equations was studied by Li [**32**], and Wang and Wang [**48**] in the case with an external force, respectively, under the assumption that the states at far fields $\pm\infty$ are equal. Later, Chen [**5**] and Li and Luo [**33**] discussed asymptotic stability of the rarefaction waves for the one-dimensional compressible fluid models of Korteweg type with different gas states at far fields, respectively. Chen *et al.* [**6**] also showed asymptotic stability of the rarefaction waves for the one-dimensional compressible Naviver-Stokes-Korteweg equation with large initial data. Li and Zhu [**34**] further showed asymptotic stability of the rarefaction wave with vacuum for the one-dimensional compressible Navier-Stokes-Korteweg equations. Chen, He and Zhao [**7**] studied nonlinear stability of travelling wave solutions for the one-dimensional compressible Navier-Stokes-Korteweg equations with different gas states at far fields.

For the initial-boundary value problem, Tsyganov [**45**] discussed the global existence and time-asymptotic behaviour of weak solutions for an isothermal model with the viscosity coefficient $\mu(\rho) \equiv 1$, the capillarity coefficient $\kappa(\rho) = \rho^{-5}$ and large initial data on the interval $[0, 1]$. The global existence and exponential decay of strong solutions with small initial data to the Korteweg system in a bounded domain of $\mathbb{R}^n$ $(n \geqslant 1)$ were also obtained by Kotschote in [**31**]. Another interesting and challenging problem is to study the stability of the compressible Navier-Stokes-Korteweg equation in the half space with different gas states at boundary and far field. Recently, Chen, Li and Sheng [**9**] proved the nonlinear stability of viscous shock wave for an impermeable wall problem of the one-dimensional compressible Navier-Stokes-Korteweg equation with constant viscosity and capillarity coefficients and small initial data. Chen and Li [**8**] discussed the time-asymptotic behaviour of strong solutions to the initial-boundary value problem of the one-dimensional compressible Navier-Stokes-Korteweg equation with density-dependent viscosity and capillarity on the half-line $\mathbb{R}^+$, and showed the strong solution converges to the rarefaction wave as $t \to \infty$ for the impermeable wall problem under large initial perturbation. Hong [**19**] and Li and Zhu [**35**] showed the existence and stability of stationary solution to an outflow problem of the one-dimensional compressible Navier-Stokes-Korteweg equation with constant viscosity and capillarity coefficients, respectively.

In this article, we shall investigate large-time behaviour of the solution to an initial boundary value problem for the one-dimensional Navier-Stokes-Korteweg equations (1.1) on the half space $\mathbb{R}^+$, thus we add the following initial data

$$(\rho, u)(0, x) = (\rho_0, u_0)(x) \text{ for } x > 0, \text{ and } \inf_{x\in\mathbb{R}^+} \rho_0(x) > 0, \tag{1.2}$$

far-field states at the infinity $x = +\infty$

$$\lim_{x\to+\infty} (\rho, u)(t, x) = (\rho_+, u_+), \text{ for any } t \geqslant 0, \tag{1.3}$$

and also the boundary condition at $x = 0$

$$u(t, 0) = u_b, \ \rho_x(t, 0) = 0, \text{ for any } t \geqslant 0. \tag{1.4}$$

Here $\rho_+$, $u_+$ and $u_b$ are constants satisfying $\rho_+ > 0$. And $\rho_0(x), u_0(x)$ are given functions.

We are interested in the so-called outflow problem. For this case the boundary data of $u$ is taken as negative value, i.e.,

$$u_b < 0.$$

This means physically that the outflow exits constantly through the wall. Moreover, we also need $\rho_x(t, 0) = 0$ for the third-order capillary term in (1.1). We note that for the case that $u_b > 0$, the situation is different and the corresponding problem is called an inflow problem. In that case, for the well-posedness, one must impose one more boundary condition at $x = 0$, namely we must consider a set of boundary conditions of the form

$$\rho(t, 0) = \rho_b, \ u(t, 0) = u_b, \ \rho_x(t, 0) = 0, \quad t \geqslant 0,$$

with $\rho_b > 0$ and $u_b > 0$.

**Related literature.** There has been a huge number of papers in the literature on the large-time behaviour of the solutions for the initial-boundary value problem to the compressible Navier-Stokes equations. In this type of problems, the influence of viscosity is expected to emerge not only in the smoothing effect on discontinuous shock wave but also in the forming of a boundary layer. More precisely, Matsumura and Mei [**37**] considered the stability of viscous shock wave to the one-dimensional Navier-Stokes equation with a Dirichlet boundary condition. Matsumura and Nishihara [**38**] showed global asymptotics towards rarefaction waves for the solution of the viscous $p$-system with boundary effect. Matsumura [**36**] gave, in 2001, a classification of the large-time behaviour of the solutions in terms of the far-field state and boundary data. Kawashima, Nishibata and Zhu [**26**] investigated the asymptotic stability of the stationary solution to an outflow problem of the compressible

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Navier-Stokes equations in the half space. Matsumura and Nishihara [**39**] studied nonlinear stability of the rarefaction wave and stationary solution to an inflow problem in the half space for the isentropic compressible Navier-Stokes equations. Huang, Matsumura and Shi [**24**] obtained the nonlinear stability of viscous shock wave and boundary layer solution for an inflow problem of the isentropic compressible Navier-Stokes equations. Recently, there are lots of references about the topic for the isentropic and full Navier-Stokes equations, the interested readers are referred to, e.g., [**12**, **20**, **22**, **23**, **25**, **27**, **28**, **40**, **41**] etc.

We now turn back to the outflow problem. The purpose of this paper is to investigate the large-time behaviour of the solution to the outflow problem (1.1)–(1.4). Motivated by [**1**, **4**] and [**28**, **36**], we believe that as $t \to \infty$, the solution $(\rho, u)$ to the above problem (1.1)–(1.4) is asymptotically described by one of the following waves, such as a viscous shock wave, a stationary wave, a rarefaction wave or the superposition of a stationary wave and a rarefaction wave, which can be determined by the space-asymptotic conditions (1.3) and the boundary data $u_b$. The stability of a stationary wave has been investigated in [**19**, **35**], respectively. In this paper, we are interested particularly in the case that the corresponding time-asymptotic state is rarefaction wave. For this, we first introduce the corresponding compressible equation without viscosity and capillarity

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0. \end{cases} \tag{1.5}$$

It has two eigen-values:

$$\lambda_1(\rho, u) = u - C(\rho),\ \ \lambda_2(\rho, u) = u + C(\rho),$$

with $C(\rho) = \sqrt{K\gamma\rho^{\gamma-1}}$. Further, let us introduce $(\rho_*, u_*)$ by

$$u_* = -C(\rho_*),\ \ u_+ - u_* = \int_{\rho_*}^{\rho_+} C(s)s^{-1}\mathrm{d}s.$$

Then from the complete classification of the asymptotic states of the outflow problem to the compressible Navier-Stokes equation in [**27**, **28**, **36**], we know that when either $-C(\rho_+) < u_+ < 0$ and $u_* \leqslant u_b < u_+$, or $u_+ > 0$ and $u_* \leqslant u_b < 0$, we can choose some $\rho_- > 0$ such that $(v_-, u_b) \in R_2$ ($R_2$ is the 2-rarefaction curve, defined by $R_2 : u - u_b = -\int_{v_-}^{v} \sqrt{K\gamma}y^{-(\gamma-1/2)}\mathrm{d}y$ for $v_- > v$), here $v_- = 1/\rho_-$ and $v = 1/\rho$. That is, there exists a 2-rarefaction wave $(\rho^R, u^R)(x/t)$ with $(\lambda_2(\rho, u) \geqslant 0)$, which connects $(\rho_-, u_b)$ and $(\rho_+, u_+)$, i.e., $(\rho^R, u^R)(x/t)$ satisfies the corresponding Riemann problem:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \\ (\rho, u)(t = 0, x) = \begin{cases} (\rho_-, u_b), & x < 0, \\ (\rho_+, u_+), & x > 0. \end{cases} \end{cases} \tag{1.6}$$

Before stating our results, let us first give some notations. Throughout this paper, $C$ denotes a universal positive constant which is independent of time $t$ and may

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vary from line to line. $L^p(\mathbb{R}^+)(1 \leqslant p < \infty)$ are the spaces of measurable functions whose $p$-powers are integrable on $\mathbb{R}^+$, with the norm $\|\cdot\|_{L^p} = (\int_{\mathbb{R}} |\cdot|^p \mathrm{d}x)^{1/p}$. For the case that $p = 2$, we simply denote $\|\cdot\|_{L^2}$ by $\|\cdot\|$. And $L^\infty(\mathbb{R}^+)$ is the space of bounded measurable functions on $\mathbb{R}^+$, with the norm $\|\cdot\|_{L^\infty} = \text{ess sup}_{x\in\mathbb{R}^+}|\cdot|$. For a nonnegative integer $k$, $H^k = H^k(\mathbb{R}^+)$ denotes the usual $L^2$-type Sobolev space of order $k$. We write $\|\cdot\|_k$ for the standard norm of $H^k(\mathbb{R}^+)$. In addition, we denote by $C([0,T]; H^k(\mathbb{R}^+))$ (resp. $L^2(0,T; H^k(\mathbb{R}^+))$) the space of continuous (resp. square integrable) functions on $[0,T]$ with values taken in a Banach space $H^k(\mathbb{R}^+)$.

The main purpose of this article is to investigate the time-asymptotic stability of the rarefaction wave $(\rho^R, u^R)(x/t)$, and the main results are stated as follows.

THEOREM 1.1. *Assume that $u_b$, $u_*$ and the infinite states satisfy that $u_b < 0$, and that either (i) $-C(\rho_+) < u_+ < 0$ and $u_* \leqslant u_b < u_+$, or (ii) $u_+ > 0$ and $u_* \leqslant u_b < 0$. Suppose furthermore that $(\rho_0 - \rho_+, u_0 - u_+) \in H^2(\mathbb{R}^+) \times H^1(\mathbb{R}^+)$ such that $\varepsilon$ (is given by in (2.3)) and $\|\rho_0 - \rho_+\|_2 + \|u_0 - u_+\|_1$ are suitably small. And the compatibility conditions $u_0(0) = u_b$ and $\rho_{0x}(0) = 0$ are satisfied. Then there exists a unique global strong solution $(\rho, u)(t, x)$ to the problem (1.1)–(1.4) such that*

$$\rho - \rho^R, u - u^R \in C([0,\infty); L^2(\mathbb{R}^+)), \tag{1.7}$$

$$\rho_x, \rho_{xx}, u_x \in C([0,\infty); L^2(\mathbb{R}^+)) \cap L^2([0,\infty); L^2(\mathbb{R}^+)), \tag{1.8}$$

$$\rho_{xxx}, u_{xx} \in L^2([0,\infty); L^2(\mathbb{R}^+)). \tag{1.9}$$

*Moreover, we assert that as $t \to \infty$, the solution $(\rho, u)(t, x)$ converges to the rarefaction wave $(\rho^R, u^R)(x/t)$, that is*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^+} \left|(\rho, u)(t, x) - (\rho^R, u^R)\left(\frac{x}{t}\right)\right| = 0. \tag{1.10}$$

REMARK 1.2. In the present article we consider only that the time-asymptotic state of the out-flow problem to one-dimensional compressible Navier-Stokes-Korteweg equations is rarefaction wave. The study of the stability of other wave pattern such as a viscous shock wave or the superposition of a rarefaction wave and a stationary wave will be carried out in other papers by the authors. Further, we try to give the complete classification of the asymptotic states of the outflow problem to the compressible Navier-Stokes-Korteweg equations as [**27**, **28**, **36**] for the compressible Navier-Stokes equation. Moreover, we should mention that the corresponding inflow problem is surely more difficult, thus more interesting. Finally, we also mention that here we only focus on small perturbation of the initial data, in fact, it is interesting and plausible that we can consider the corresponding results for large perturbation. These are expected to be done in the forthcoming papers.

This article is follow-up study of [**8**, **9**, **35**]. Now we give main ideas and arguments of the proof for theorem 1.1. Applying $L^2$-energy method and some time-decay estimates in $L^p$-norm for the smoothed rarefaction wave as in [**28**], we prove the asymptotic stability of the rarefaction wave in the case that the initial data are a small perturbation of the rarefaction wave. The key ingredient in the proof of theorem 1.1 is to deduce the *a priori* estimates. The main difficulties are as follows.

The first one is the occurrence of the third order dispersion term. The second is that it is not easy to control the boundary terms $\varphi_{xx}(t,0)$, $\varphi_{xxx}(t,0)$ and $\psi_{xx}(t,0)$. To overcome the first difficulty, we need more regularities for the density and smooth rarefaction wave. We also note that the basic energy is obtained with the help of higher order estimates. For the second difficulty, we can introduce $\varphi_{xx}(t,0)^2$ by the second equation of (3.1) and integration by parts. Moreover, we can control $(\kappa\varphi_{xxx}(t,0)+\mu\psi_{xx}(t,0)/\rho(t,0))^2$ by $C\|\psi_x(t)\|_1^2$, which is derived by $(3.1)_2$ and lemma 2.2. These are the main novelty of the present paper.

The rest of the article is organized as follows. In § 2, we first review a smooth approximate rarefaction wave which tends to the rarefaction wave fan uniformly as the time $t$ tends to infinity. Then we reformulate the original problem in terms of the perturbation variables in § 3. § 4 is the key part of this article, in which we will establish the *a priori* estimates by the elaborate energy estimates. Finally, we complete the proof of theorem 1.1 in § 5.

## 2. Smooth rarefaction wave

Since the rarefaction wave $(\rho^R,u^R)(x/t)$ is not smooth, we need to construct a smooth approximation of the rarefaction wave $(\rho^r,u^r)(t,x)$. As [**38**], we start with the Riemann problem on $\mathbb{R}=(-\infty,+\infty)$ for the typical Burgers equation:

$$w_t + ww_x = 0, \tag{2.1}$$

with initial data

$$w(0,x) = w_0^R(x) = \begin{cases} w_-, & x<0 \\ w_+, & x>0, \end{cases} \tag{2.2}$$

where $w_\pm$ are given by $w_- = u_b + C(\rho_-) > 0$ and $w_+ = u_+ + C(\rho_+) > 0$, satisfying $w_- < w_+$. It is well known that the Riemann problem (2.1)–(2.2) has a unique rarefaction wave solution:

$$w^R\left(\frac{x}{t}\right) = \begin{cases} w_-, & x < w_-t, \\ \dfrac{x}{t}, & w_-t \leqslant x \leqslant w_+t, \\ w_+, & x > w_+t. \end{cases}$$

Then we can define the functions $\rho^R(t,x)$ and $u^R(t,x)$ by

$$\begin{aligned} &\lambda_2(\rho^R,u^R) = u^R + C(\rho^R) = w^R(1+t,x), \\ &\frac{du^R}{d\rho^R} = \frac{C(\rho^R)}{\rho^R}. \end{aligned}$$

It is easy to check that $\rho^R(t,x)$ and $u^R(t,x)$ satisfy

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0 \end{cases}$$

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with

$$(\rho,u)(0,x)=\begin{cases}(\rho_-,u_b), & x<0,\\ (\rho_+,u_+), & x>0.\end{cases}$$

Now we approximate the rarefaction wave $w^R(x/t)$ by the solution of the following Cauchy problem:

$$\begin{cases} w_t+ww_x=0,\\ w(0,x)=w_0^r(x)=\begin{cases} w_-, & x<0,\\ w_-+C_q\tilde{w}\int_0^{\varepsilon x}y^qe^y\mathrm{d}y, & x\geqslant 0,\end{cases}\end{cases} \tag{2.3}$$

where $\tilde{w}=w_+-w_-$, $C_q>0$ is a constant satisfying: $C_q\int_0^{+\infty}z^qe^{-z}\mathrm{d}z=1$ with $q\geqslant 10$ being a positive constant, and $\varepsilon\leqslant 1$ is a positive constant to be determined later. Then the properties of $w(t,x)$ can be summarised in the following lemma.

Lemma 2.1. *(See [8, 24]) Let* $0<w_-<w_+$*, then the Cauchy problem* (2.3) *admits a unique global smooth solution* $w(t,x)$ *satisfying:*

(*i*) $w_-<w(t,x)<w_+,\ w_x>0,\quad x\geqslant 0, t\geqslant 0.$

(*ii*) *For any* $p(1\leqslant p\leqslant +\infty)$*, there exists a constant* $C_{p,q}>0$ *such that for* $t\geqslant 0$,

$$\begin{aligned}\|w_x(t)\|_{L^p}&\leqslant C_{p,q}\min\left\{\tilde{w}\varepsilon^{1-\frac{1}{p}},\ \tilde{w}^{\frac{1}{p}}t^{-1+\frac{1}{p}}\right\},\\ \|w_{xx}(t)\|_{L^p}&\leqslant C_{p,q}\min\left\{\tilde{w}\varepsilon^{2-\frac{1}{p}},\ \tilde{w}^{\frac{1}{q}}\varepsilon^{1-\frac{1}{p}+\frac{1}{q}}t^{-1+\frac{1}{q}}\right\},\\ \|w_{xxx}(t)\|_{L^p}&\leqslant C_{p,q}\min\left\{\tilde{w}\varepsilon^{3-\frac{1}{p}},\ \tilde{w}^{\frac{2}{q}}\varepsilon^{2-\frac{1}{p}+\frac{2}{q}}t^{-1+\frac{2}{q}}\right\},\\ \|w_{xxxx}(t)\|_{L^p}&\leqslant C_{p,q}\min\left\{\tilde{w}\varepsilon^{4-\frac{1}{p}},\ \tilde{w}^{\frac{3}{q}}\varepsilon^{3-\frac{1}{p}+\frac{3}{q}}t^{-1+\frac{3}{q}}\right\}.\end{aligned}$$

(*iii*) *When* $x\leqslant w_-t$*, it holds that*

$$w(t,x)-w_-=w_x(t,x)=w_{xx}(t,x)=w_{xxx}(t,x)=0.$$

(*iv*) $\lim\limits_{t\to+\infty}\sup\limits_{x\in\mathbb{R}}\left|w(t,x)-w^R(t,x)\right|=0.$

Now, we define the smooth approximate rarefaction wave $(\rho^r,u^r)(t,x)$ of $(\rho^R,u^R)(x/t)$ as follows:

$$\begin{aligned}&\lambda_2(\rho^r,u^r)=u^R+C(\rho^r)=w(1+t,x),\\ &\frac{du^r}{d\rho^r}=\frac{C(\rho^r)}{\rho^r}.\end{aligned}$$

Therefore, from lemma 2.1, we know that $(\rho^r,u^r)(t,x)$ has the following properties:

Lemma 2.2. *Let* $\delta=|\rho_+-\rho_-|+|u_+-u_b|$*, the smooth approximation* $(\rho^r,u^r)(t,x)$ *of* $(\rho^R,u^R)$ *has the following properties:*

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(*i*) $u_x^r \geqslant 0, \quad |u_x^r| \leqslant C\varepsilon, \quad \forall\, t \geqslant 0,\, x \geqslant 0.$

(*ii*) *For any* $p$ *with* $1 \leqslant p \leqslant +\infty$, *there exists a constant* $C_{p,q} > 0$ *such that*

$$\|(\rho_x^r, u_x^r)\,(t)\|_{L^p} \leqslant C_{p,q} \min\left\{\delta\varepsilon^{1-\frac{1}{p}},\ \delta^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}\right\},$$

$$\|(\rho_{xx}^r, u_{xx}^r)\,(t)\|_{L^p} \leqslant C_{p,q} \min\left\{\delta\varepsilon^{2-\frac{1}{p}},\ \delta^{\frac{1}{q}}\varepsilon^{1-\frac{1}{p}+\frac{1}{q}}(1+t)^{-1+\frac{1}{q}}\right\},$$

$$\|(\rho_{xxx}^r, u_{xxx}^r)\,(t)\|_{L^p} \leqslant C_{p,q} \min\left\{\delta\varepsilon^{3-\frac{1}{p}},\ \delta^{\frac{2}{q}}\varepsilon^{2-\frac{1}{p}+\frac{2}{q}}(1+t)^{-1+\frac{2}{q}}\right\},$$

$$\|(\rho_{xxxx}^r, u_{xxxx}^r)\,(t)\|_{L^p} \leqslant C_{p,q} \min\left\{\delta\varepsilon^{4-\frac{1}{p}},\ \delta^{\frac{3}{q}}\varepsilon^{3-\frac{1}{p}+\frac{3}{q}}(1+t)^{-1+\frac{3}{q}}\right\}.$$

(*iii*) $(\rho^r, u^r)(t,x)\Big|_{x\leqslant\lambda_2(\rho_-,u_b)t} = (v_-, u_-),\ \frac{\partial^j}{\partial x^j}(\rho^r, u^r)(t,x)\Big|_{x\leqslant\lambda_2(\rho_-,u_b)t} = 0,$ $j = 1, 2, 3.$

(*iv*) $\lim\limits_{t\to+\infty} \sup\limits_{x\in\mathbb{R}^+} \left|(\rho^r, u^r)(t,x) - (\rho^R, u^R)\left(\frac{x}{t}\right)\right| = 0.$

## 3. Reformulation of the problem

Since it is convenient to regard the solution $(\rho, u)$ as the perturbation of $(\rho^r, u^r)$, we are going to reformulate the original problem in terms of the perturbation variables in this section. First, we define

$$\varphi(t,x) = \rho(t,x) - \rho^r(t,x),\ \ \psi(t,x) = u(t,x) - u^r(t,x).$$

Then, the original problem (1.1)–(1.4) can be rewritten as

$$\begin{cases} \varphi_t + \rho\psi_x + u\varphi_x = f, \\ \rho(\psi_t + u\psi_x) + p'(\rho)\varphi_x = \mu\psi_{xx} + \kappa\rho\varphi_{xxx} + g \end{cases} \tag{3.1}$$

with the initial boundary conditions:

$$\begin{cases} (\varphi,\psi)(0,x) = (\rho_0(x) - \rho^r(0,x), u_0(x) - u^r(0,x), \\ \psi(t,0) = 0, \\ \varphi_x(t,0) = \rho_x(t,0) - \rho_x^r(t,0) = 0, \end{cases} \tag{3.2}$$

where

$$f = -u_x^r\varphi - \rho_x^r\psi, \tag{3.3}$$

and

$$g = \mu u_{xx}^r + \kappa\rho\rho_{xxx}^r + \frac{p'(\rho^r)}{p^r}\rho_x^r\varphi - [p'(\rho) - p'(\rho^r)]\rho_x^r - \rho\psi u_x^r. \tag{3.4}$$

Therefore, we are now in a position to restate our main results in terms of the perturbed variable $(\varphi, \psi)(t,x)$ as follows.

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Theorem 3.1. *Suppose that all the assumptions of theorem* 1.1 *are met. Then there exists a unique global solution* $(\varphi, \psi)(t, x)$ *to problem* (3.1)–(3.2)*, satisfying*

$$\begin{aligned} &\varphi, \psi \in C([0, \infty); L^2(\mathbb{R}^+)), \\ &\varphi_x, \varphi_{xx}, \psi_x \in C([0, \infty); L^2(\mathbb{R}^+)) \cap L^2([0, \infty); L^2(\mathbb{R}^+)), \\ &\varphi_{xxx}, \psi_{xx} \in L^2([0, \infty); L^2(\mathbb{R}^+)), \end{aligned}$$

*and*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^+} |(\varphi, \psi)(t, x)| = 0.$$

To prove this theorem, we shall employ the standard continuation argument based on a local existence theorem in the following lemma and on *a priori* estimates stated in the following proposition. First, the local existence of the solution $(\varphi, \psi)$ to the initial-boundary value problem (3.1)–(3.2) is proved by the standard method, for example, the dual argument and iteration technique. For details, we refer [**17**, **18**, **31**, **45**].

Lemma 3.2 Local existence. *Assume that the conditions in theorem* 1.1 *hold. Then there exists a positive constant* $T_0$ *such that the initial-boundary value problem* (3.1)–(3.2) *has a unique strong solution* $(\varphi, \psi)(t, x)$ *that has the following properties:*

$$\begin{aligned} &\varphi(t, x) \in C([0, T_0]; H^2(\mathbb{R}^+)), \psi(t, x) \in C([0, T_0]; H^1(\mathbb{R}^+)), \\ &\varphi_x(t, x) \in L^2([0, T_0]; H^2(\mathbb{R}^+)),\ \ \psi_x(t, x) \in L^2([0, T_0]; H^1(\mathbb{R}^+)), \\ &\inf_{t\in[0,T_0], x\in\mathbb{R}^+} \rho(t, x) > 0. \end{aligned}$$

Next, we prove the following *a priori* estimates in Sobolev spaces, which are stated in proposition 3.3.

Proposition 3.3. *Let* $(\varphi, \psi)$ *be a solution to the initial-boundary value problem* (3.1)–(3.2) *in a time interval* $[0, T]$*, which has same regularities as in lemma* 3.2*. Then there exist constants* $\varepsilon_1 > 0$ *and* $C > 0$ *such that if*

$$N(T) := \sup_{t\in[0,T]} [\|\varphi(t)\|_2 + \|\psi(t)\|_1] \leqslant \varepsilon_1, \tag{3.5}$$

*then the following estimate holds for any* $t \in [0, T]$

$$\begin{aligned} &\|\varphi(t)\|_2^2 + \|\psi(t)\|_1^2 + \int_0^t \left(\|\varphi_x(\tau)\|_2^2 + \|\psi_x(\tau)\|_1^2 + |(\varphi, \varphi_{xx})(\tau, 0)|^2\right) d\tau \\ &\quad \leqslant C(\|\varphi_0\|_2^2 + \|\psi_0\|_1^2 + \varepsilon^{\frac{1}{8}}). \end{aligned} \tag{3.6}$$

## 4. *A priori* estimate

This section is devoted to the derivation of *a priori* estimates for the unknown function $(\varphi, \psi)(t, x)$ and their derivatives, we then show that proposition 3.3 is

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valid. To derive these *a priori* estimates, we assume that there exists a strong solution $(\varphi,\psi)(t,x)$ to problem (3.1)–(3.2), such that

$$\begin{aligned}
&\varphi(t,x)\in C([0,T];H^2(\mathbb{R}^+)),\ \ \psi(t,x)\in C([0,T];H^1(\mathbb{R}^+)),\\
&\varphi_x(t,x)\in L^2([0,T];H^2(\mathbb{R}^+)),\psi_x(t,x)\in L^2([0,T];H^1(\mathbb{R}^+)),\\
&\inf_{(t,x)\in[0,T]\times\mathbb{R}^+}(\varphi+\rho^r)(t,x)>0
\end{aligned}$$

for any $T>0$. Indeed, we may assume that $(\varphi,\psi)(t,x)$ is a classical solution from the standard mollifier arguments. From (3.5), one can see easily that there exist two positive constants $c$ and $C$ such that

$$0<c\leqslant\rho\leqslant C,\ \ |u|\leqslant C\ \ \text{for } t\in[0,T], \tag{4.1}$$

since $\rho^r\geqslant c>0$ for a positive constant $c$. To this end, we introduce

$$\Phi(\rho,\rho^r)=\int_{\rho^r}^{\rho}\frac{p(\eta)-p(\rho^r)}{\eta^2}d\eta,$$

combining this with (4.1) yields

$$c\varphi^2\leqslant\Phi(\rho,\rho^r)\leqslant C\varphi^2. \tag{4.2}$$

Next, from (3.1), the straightforward but tedious computations give

$$\begin{aligned}
&\Big[\rho\Big(\frac{1}{2}\psi^2+\Phi(\rho,\rho^r)\Big)\Big]_t+\Big[\rho u\Big(\frac{1}{2}\psi^2+\Phi(\rho,\rho^r))+(p(\rho)-p(\rho^r)\Big)\psi-\mu\psi\psi_x\Big]_x\\
&=\ -\mu\psi_x^2-[\rho\psi^2+p(\rho)-p(\rho^r)-p'(\rho)\varphi]u_x^r+\kappa\rho\varphi_{xxx}\psi+\mu u_{xx}^r\psi+\kappa\rho\rho_{xxx}^r\psi.
\end{aligned} \tag{4.3}$$

Moreover from $(3.1)_1$, we also have

$$\begin{aligned}
\kappa\rho\varphi_{xxx}\psi&=\kappa(\rho\varphi_{xx}\psi)_x-\kappa(\rho\psi)_x\varphi_{xx}\\
&=\kappa(\rho\varphi_{xx}\psi)_x+\kappa\varphi_{xx}(\varphi_t+u^r\varphi_x+u_x^r\varphi)\\
&=\kappa(\rho\varphi_{xx}\psi)_x+\kappa(\varphi_x\varphi_t)_x-\Big(\frac{\kappa}{2}\varphi_x^2\Big)_t+\frac{\kappa}{2}(\varphi_x^2u^r)_x-\frac{\kappa}{2}u_x^r\varphi_x^2+\kappa u_x^r\varphi\varphi_{xx}\\
&=\Big(\kappa\rho\varphi_{xx}\psi+\kappa\varphi_x\varphi_t+\frac{\kappa}{2}u^r\varphi_x^2+\kappa u_x^r\varphi\varphi_x\Big)_x\\
&\quad-\frac{\kappa}{2}(\varphi_x^2)_t-\frac{3\kappa}{2}u_x^r\varphi_x^2-\kappa u_{xx}^r\varphi\varphi_x,
\end{aligned}$$

which together with (4.3) implies

$$\begin{aligned}
&\Big[\rho\Big(\frac{1}{2}\psi^2+\Phi(\rho,\rho^r)\Big)+\frac{\kappa}{2}\varphi_x^2\Big]_t+R_{1x}+R_2\\
&\quad=\mu u_{xx}^r\psi+\kappa\rho\rho_{xxx}^r\psi-\frac{3\kappa}{2}u_x^r\varphi_x^2-\kappa u_{xx}^r\varphi\varphi_x,
\end{aligned} \tag{4.4}$$

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here

$$R_1 = \rho u\left(\frac{1}{2}\psi^2 + \Phi(\rho,\rho^r)\right) + (p(\rho) - p(\rho^r))\psi - \mu\psi\psi_x$$
$$- \kappa\rho\varphi_{xx}\psi - \kappa\varphi_x\varphi_t - \frac{\kappa}{2}u^r\varphi_x^2 - \kappa u_x^r\varphi\varphi_x,$$

and

$$R_2 = [\rho\psi^2 + p(\rho) + p(\rho^r) - p'(\rho)\varphi]u_x^r + \mu\psi_x^2.$$

Then we arrive at

LEMMA 4.1. *Assume that* $(\varphi,\psi)(t,x)$ *is a solution to* (3.1)–(3.2)*, satisfying the conditions in proposition* 3.3*, then the following estimate holds*

$$\|\varphi(t)\|^2 + \|\psi(t)\|^2 + \|\varphi_x(t)\|^2 + \int_0^t (\|\psi_x(\tau)\|^2 + \varphi(\tau,0)^2)d\tau$$
$$\leqslant C(\|\varphi_0\|_1^2 + \|\psi_0\|^2 + C\varepsilon^{\frac{1}{8}}) + C(\varepsilon^{\frac{1}{3}} + \varepsilon)\int_0^t \|\varphi_x(\tau)\|^2 d\tau \tag{4.5}$$

*for all* $t \in [0,T]$.

**Proof.** Integrating (4.4) with respect to $x$ over $(0,\infty)$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \left(\frac{1}{2}\rho\psi^2 + \rho\Phi\right)\mathrm{d}x + R_1\Big|_{x=0} + \int_0^\infty R_2\mathrm{d}x$$
$$= \int_0^\infty (\mu u_{xx}^r\psi + \kappa\rho\rho_{xxx}^r\psi - \frac{3\kappa}{2}u_x^r\varphi_x^2 - \kappa u_{xx}^r\varphi\varphi_x)\mathrm{d}x. \tag{4.6}$$

First, noting (4.1) and using (4.2), we easily obtain

$$\int_0^\infty \left(\frac{1}{2}\rho\psi^2 + \rho\Phi\right)\mathrm{d}x \geqslant c(\|\varphi\|^2 + \|\psi\|^2), \tag{4.7}$$

and

$$R_1|_{x=0} = -\rho u\Phi(\rho,\rho^r)|_{x=0} \geqslant c\varphi(t,0)^2 \tag{4.8}$$

with the help of $\psi(t,0) = 0 = \varphi_x(t,0)$ and $u_b < 0$. Similarly, we have

$$R_2 \leqslant C(\|\sqrt{u_x^r}\varphi\|^2 + \|\sqrt{u_x^r}\psi\|^2 + \|\sqrt{u_x^r}\varphi_x\|^2 + \|\psi_x\|^2). \tag{4.9}$$

Further, combining (4.6)–(4.9) and using (4.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty (\varphi^2 + \psi^2 + \varphi_x^2)\mathrm{d}x + \|\sqrt{u_x^r}\varphi\|^2 + \|\sqrt{u_x^r}\psi\|^2$$
$$+ \|\psi_x\|^2 + \|\sqrt{u_x^r}\varphi_x\|^2 + \varphi(t,0)^2$$
$$\leqslant C\Big|\int_0^\infty u_{xx}^r\psi\mathrm{d}x\Big| + C\Big|\int_0^\infty \rho_{xxx}^r\psi\mathrm{d}x\Big| + C\Big|\int_0^\infty u_x^r\varphi_x^2\mathrm{d}x\Big| + C\Big|\int_0^\infty u_{xx}^r\varphi\varphi_x\mathrm{d}x\Big|. \tag{4.10}$$

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Now let us estimate the terms on the right-hand side of (4.10). First, we employ Hölder inequality, the Sobolev inequality

$$\|f\|_{L^\infty} \leqslant \sqrt{2}\|f\|^{\frac{1}{2}}\|f_x\|^{\frac{1}{2}} \tag{4.11}$$

for any $f \in H^1(\mathbb{R}^+)$, lemma 2.2 and Young inequality to obtain

$$\begin{aligned}
\Big|\int_0^\infty u^r_{xx}\psi \mathrm{d}x\Big| &\leqslant \|\psi\|_{L^\infty}\|u^r_{xx}\|_{L^1} \leqslant C\|\psi\|^{\frac{1}{2}}\|\psi_x\|^{\frac{1}{2}}\|u^r_{xx}\|_{L^1} \\
&\leqslant \frac{1}{4}\|\psi_x\|^2 + C\|\psi\|^{\frac{2}{3}}\|u^r_{xx}\|^{\frac{4}{3}}_{L^1} \\
&\leqslant \frac{1}{4}\|\psi_x\|^2 + C\|\psi\|^{\frac{2}{3}}\|u^r_{xx}\|^{\frac{1}{6}}_{L^1}\|u^r_{xx}\|^{\frac{7}{6}}_{L^1} \\
&\leqslant \frac{1}{4}\|\psi_x\|^2 + C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{21}{20}}\|\psi\|^{\frac{2}{3}} \\
&\leqslant \frac{1}{4}\|\psi_x\|^2 + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{21}{20}}\|\psi\|^2 + C\varepsilon^{\frac{1}{8}}(1+t)^{-\frac{21}{20}}.
\end{aligned} \tag{4.12}$$

Similarly, we have

$$\begin{aligned}
\Big|\int_0^\infty \rho^r_{xxx}\psi \mathrm{d}x\Big| &\leqslant C\|\psi\|^{\frac{1}{2}}\|\psi_x\|^{\frac{1}{2}}\|\rho^r_{xxx}\|_{L^1} \\
&\leqslant \frac{1}{4}\|\psi_x\|^2 + C\|\psi\|^{\frac{2}{3}}\|\rho^r_{xxx}\|^{\frac{4}{3}}_{L^1} \\
&\leqslant \frac{1}{4}\|\psi_x\|^2 + C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{16}{15}}\|\psi\|^{\frac{2}{3}} \\
&\leqslant \frac{1}{4}\|\psi_x\|^2 + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{11}{10}}\|\psi\|^2 + C\varepsilon^{\frac{1}{8}}(1+t)^{-\frac{21}{20}}.
\end{aligned} \tag{4.13}$$

Next, from lemma 2.2, it is easy to obtain

$$\Big|\int_0^\infty u^r_x\varphi_x^2 \mathrm{d}x\Big| \leqslant C\varepsilon\|\varphi_x\|^2. \tag{4.14}$$

Finally, using Hölder inequality, lemma 2.2 and Young inequality, we have

$$\begin{aligned}
\Big|\int_0^\infty u^r_{xx}\varphi\varphi_x \mathrm{d}x\Big| &\leqslant C\|u^r_{xx}\|^{\frac{1}{6}}_{L^\infty}\|u^r_{xx}\|^{\frac{5}{6}}_{L^\infty}\|\varphi\|\|\varphi_x\| \\
&\leqslant C\varepsilon^{\frac{1}{3}}(1+t)^{-\frac{3}{4}}\|\varphi\|\|\varphi_x\| \\
&\leqslant C\varepsilon^{\frac{1}{3}}\|\varphi_x\|^2 + C\varepsilon^{\frac{1}{3}}(1+t)^{-\frac{3}{2}}\|\varphi\|^2.
\end{aligned} \tag{4.15}$$

Therefore, combining (4.10), (4.12)–(4.14) and (4.15), and integrating the resultant inequality with respect to $t$, then implies (4.5) provided that $C\varepsilon^{\frac{1}{4}} < \frac{1}{4}$ and $C\varepsilon^{\frac{1}{3}} < \frac{1}{4}$. This completes the proof of lemma 4.1.

Next, we derive the estimate for $\varphi_x$ and $\varphi_{xx}$.

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Lemma 4.2. *Assume that* $(\varphi,\psi)(t,x)$ *is a solution to* (3.1)–(3.2)*, satisfying the conditions in proposition* 3.3*, then the following estimate holds*

$$\|\varphi_x\|_1^2 \leqslant C(\|\varphi_0\|_1^2 + \|\psi_0\|^2 + C\varepsilon^{\frac{1}{8}}) \tag{4.16}$$

*for all* $t \in [0,T]$.

**Proof.** We first differentiate formally $(3.1)_1$ in $x$ to obtain

$$\varphi_{tx} + u\varphi_{xx} + \rho\psi_{xx} = f_x - \rho_x^r\psi_x - u_x^r\varphi_x - 2\varphi_x\psi_x. \tag{4.17}$$

Then multiplying above equation by $\frac{1}{\rho^2}\varphi_x$, and integrating the resulting equality with respect to $x$ over $\mathbb{R}^+$ by parts, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \frac{\varphi_x^2}{\rho^2}\mathrm{d}x + \int_0^\infty \frac{1}{\rho}\psi_{xx}\varphi_x\mathrm{d}x = \int_0^\infty f_1\frac{\varphi_x}{\rho^2}\mathrm{d}x, \tag{4.18}$$

with the help of $\varphi_x(t,0)=0$ and $(1.1)_1$, here

$$f_1 = f_x - \rho_x^r\psi_x - \frac{1}{2}\varphi_x\psi_x + \frac{1}{2}u_x^r\varphi_x.$$

Moreover, multiplying $(3.1)_2$ by $\frac{1}{\rho}\varphi_x$, and integrating the resulting equality with respect to $x$ over $\mathbb{R}^+$ by parts, and using $\psi(t,0)=\varphi_x(t,0)=0$ and $(3.1)_1$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \varphi_x\psi\mathrm{d}x &+ \int_0^\infty \frac{p'(\rho)}{\rho}\varphi_x^2\mathrm{d}x + \kappa\int_0^\infty \varphi_{xx}^2\mathrm{d}x \\ &= \int_0^\infty \frac{\mu}{\rho}\psi_{xx}\varphi_x\mathrm{d}x + \int_0^\infty \frac{g}{\rho}\varphi_x\mathrm{d}x + \int_0^\infty \psi_x(\rho\psi_x + u\varphi_x - f)\mathrm{d}x\end{aligned}$$

which together with (4.18) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \Big(\frac{\mu}{2\rho^2}\varphi_x^2 + \psi\varphi_x\Big)\mathrm{d}x &+ \int_0^\infty \frac{p'(\rho)}{\rho}\varphi_x^2 + \kappa\int_0^\infty \varphi_{xx}^2\mathrm{d}x \\ = \int_0^\infty \frac{\mu}{\rho^2}f_1\varphi_x\mathrm{d}x &+ \int_0^\infty \frac{g}{\rho}\varphi_x\mathrm{d}x + \int_0^\infty \psi_x(\rho\psi_x + u\varphi_x - f)\mathrm{d}x.\end{aligned} \tag{4.19}$$

Further, using (4.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty (\varphi_x^2 + \psi\varphi_x)\mathrm{d}x + \|\varphi_x(t)\|^2 + \|\varphi_{xx}(t)\|^2 \leqslant C\|\psi_x(t)\|^2 + C\sum_{i=1}^5 I_i, \tag{4.20}$$

where

$$\begin{aligned}
I_1 &= \Big|\int_0^\infty \varphi_x\psi_x\mathrm{d}x\Big| + \Big|\int_0^\infty \varphi_x^2\psi_x\mathrm{d}x\Big|,\\
I_2 &= \Big|\int_0^\infty u_x^r\varphi_x^2\mathrm{d}x\Big| + \Big|\int_0^\infty \rho_x^r\varphi_x\psi_x\mathrm{d}x\Big|,\\
I_3 &= \Big|\int_0^\infty u_{xx}^r\varphi\varphi_x\mathrm{d}x\Big| + \Big|\int_0^\infty |\rho_{xx}^r\psi\varphi_x\mathrm{d}x\Big|,\\
I_4 &= \Big|\int_0^\infty u_x^r\varphi\psi_x\mathrm{d}x\Big| + \Big|\int_0^\infty \rho_x^r\psi\psi_x\mathrm{d}x\Big| + \Big|\int_0^\infty u_x^r\psi\varphi_x\mathrm{d}x\Big| + \Big|\int_0^\infty \rho_x^r\varphi\varphi_x\mathrm{d}x\Big|,
\end{aligned}$$

and

$$I_5 = \Big| \int_0^\infty u^r_{xx}\varphi_x \mathrm{d}x \Big| + \Big| \int_0^\infty \rho^r_{xxx}\varphi_x \mathrm{d}x \Big|.$$

In the following, let us estimate $I_1 - I_4$ and $I_5$. First, from Young inequality and (3.5), it is easy to obtain

$$\begin{aligned} I_1 &\leqslant C\|\psi_x\|^2 + \frac{1}{8}\|\varphi_x\|^2 + C\|\varphi_x\|_{L^\infty}(\|\varphi_x\|^2 + \|\psi_x\|^2) \\ &\leqslant C\|\psi_x(t)\|^2 + \frac{1}{8}\|\varphi_x(t)\|^2 + C\varepsilon_1\|(\varphi_x, \psi_x)(t)\|^2. \end{aligned} \tag{4.21}$$

Similar as (4.14) and (4.15), we conclude

$$I_2 \leqslant C\varepsilon(\|\varphi_x(t)\|^2 + \|\psi_x(t)\|^2), \tag{4.22}$$

and

$$I_3 \leqslant C\varepsilon^{\frac{1}{3}}\|\varphi_x\|^2 + C\varepsilon^{\frac{1}{3}}(1+t)^{-\frac{3}{2}}\|(\varphi, \psi)(t)\|^2. \tag{4.23}$$

Finally, using lemma 2.2, Hölder inequality and Young inequality, we have

$$\begin{aligned} I_4 &\leqslant C\|u^r_x\|^{\frac{1}{4}}_{L^\infty}\|u^r_x\|^{\frac{3}{4}}_{L^\infty}(\|\varphi\|\|\psi_x\| + \|\psi\|\|\varphi_x\|) \\ &\quad + C\|\rho^r_x\|^{\frac{1}{4}}_{L^\infty}\|\rho^r_x\|^{\frac{3}{4}}_{L^\infty}(\|\varphi\|\|\varphi_x\| + \|\psi\|\|\psi_x\|) \\ &\leqslant C\varepsilon^{\frac{1}{4}}(\|\varphi_x(t)\|^2 + \|\psi_x(t)\|^2) + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{3}{2}}\|(\varphi, \psi)(t)\|^2, \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} I_5 &\leqslant \frac{1}{8}\|\varphi_x\|^2 + C\|u^r_{xx}\|^2 + C\|\rho^r_{xxx}\|^2 \\ &\leqslant \frac{1}{8}\|\varphi_x(t)\|^2 + C\varepsilon^{\frac{1}{5}}(1+t)^{-\frac{9}{5}} + C\varepsilon^{\frac{2}{5}}(1+t)^{-\frac{8}{5}}. \end{aligned} \tag{4.25}$$

Therefore, insertion of (4.21)–(4.25) into (4.20), and integrating the resultant inequality with respect to $t$ and using (4.5), yields (4.16) if $C\varepsilon^{\frac{1}{4}} < \frac{1}{4}$ and $C\varepsilon^{\frac{1}{3}} < \frac{1}{4}$, and $\varepsilon_1$ is assumed sufficiently small. This completes the proof of lemma 4.2.

With lemmas 4.1 and 4.2 in hand, we can show the fundamental energy estimate.

Corollary 4.3. *Assume that $(\varphi, \psi)(t, x)$ is a solution to (3.1)–(3.2), satisfying the conditions in proposition 3.3, then it holds that*

$$\begin{aligned} &\|\varphi(t)\|_1^2 + \|\psi(t)\|^2 + \int_0^t (\|\psi_x(\tau)\|^2 + \|\varphi_x(t)\|_1^2 + \varphi(\tau, 0)^2)d\tau \\ &\quad \leqslant C(\|\varphi_0\|_1^2 + \|\psi_0\|^2 + \varepsilon^{\frac{1}{8}}) \end{aligned} \tag{4.26}$$

*for any $t \in [0, T]$.*

Next, let us derive estimates for the derivatives of unknowns, i.e., $\varphi_{xx}$ and $\psi_x$.

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Lemma 4.4. *Assume that* $(\varphi,\psi)(t,x)$ *is a solution to* (3.1)–(3.2)*, satisfying the conditions in proposition* 3.3*, then it holds*

$$\|\psi_x(t)\|^2+\|\varphi_{xx}(t)\|^2+\int_0^t(\|\psi_{xx}(\tau)\|^2+\varphi_{xx}(\tau,0)^2)d\tau \leqslant C(\|\varphi_0\|_2^2+\|\psi_0\|_1^2+\varepsilon^{\frac{1}{8}})+C(\varepsilon^{\frac{1}{3}}+\varepsilon_1)\int_0^t\|\varphi_{xxx}(\tau)\|^2d\tau \tag{4.27}$$

*for all* $t\in[0,T]$.

**Proof.** Multiplying $(3.1)_2$ by $-\psi_{xx}$ and integrating the resultant equal over $\mathbb{R}^+$ with respect to $x$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty\rho\psi_x^2\mathrm{d}x+\mu\int_0^\infty\psi_{xx}^2\mathrm{d}x &= -\kappa\int_0^\infty(\rho\varphi_{xxx}\psi_{xx}+\rho_x\psi_x\varphi_{xxx})\mathrm{d}x+\int_0^\infty\rho u\psi_x\psi_{xx}\mathrm{d}x\\ &\quad+\int_0^\infty p'(\rho)\varphi_x\psi_{xx}\mathrm{d}x-\int_0^\infty g\psi_{xx}\mathrm{d}x\\ &\quad-\int_0^\infty\rho_x\psi_x\Big[\frac{g}{\rho}+\frac{\mu}{\rho}\psi_{xx}-\frac{p'(\rho)}{\rho}\varphi_x-u\psi_x\Big]\mathrm{d}x\\ &\quad-\frac{1}{2}\int_0^\infty\psi_x^2(\rho\psi_x+u\varphi_x+\rho u_x^r+u\rho_x^r)\mathrm{d}x,\end{aligned} \tag{4.28}$$

here we have used

$$\begin{aligned}-\int_0^\infty\rho\psi_t\psi_{xx}\mathrm{d}x &= -\rho\psi_t\psi_x\mid_0^\infty+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty\rho\psi_x^2\mathrm{d}x\\ &\quad-\frac{1}{2}\int_0^\infty\rho_t\psi_x^2\mathrm{d}x+\int_0^\infty\rho_x\psi_t\psi_x\mathrm{d}x\\ &=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty\rho\psi_x^2\mathrm{d}x-\frac{1}{2}\int_0^\infty\rho_t\psi_x^2\mathrm{d}x+\int_0^\infty\rho_x\psi_t\psi_x\mathrm{d}x\end{aligned}$$

and $\psi_t(t,0)=0$, $(1.1)_1$ and $(3.1)_2$. On the other hand, note that

$$\psi_t\varphi_{xx}=(\varphi_{xx}\psi)_t-(\psi\varphi_{tx})_x+\psi_x\varphi_{tx},$$

and

$$2\int_0^\infty\varphi_{xxx}\varphi_{xx}\mathrm{d}x=-\varphi_{xx}(t,0)^2,$$

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then multiplying $(3.1)_2$ by $-(2u_b/\rho)\varphi_{xx}$, and integrating the resulting equality over $\mathbb{R}^+$ with respect to $x$, and using (4.17) and $\varphi_{tx}(t,0)=0$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty -2u_b\psi\varphi_{xx}\mathrm{d}x - \kappa u_b\varphi_{xx}(t,0)^2 \\
&= -2u_b\int_0^\infty \left[\frac{g}{\rho}+\frac{\mu}{\rho}\psi_{xx}-\frac{p'(\rho)}{\rho}\varphi_x\right]\varphi_{xx}\mathrm{d}x \\
&\quad + 2u_b\int_0^\infty \psi_x(f_x-\rho\psi_{xx}-\rho_x^r\psi_x-u_x^r\varphi_x-2\varphi_x\psi_x)\mathrm{d}x,
\end{aligned}$$

which together with (4.28) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \left(\frac{1}{2}\rho\psi_x^2-2u_b\psi\varphi_{xx}\right)\mathrm{d}x+\mu\int_0^\infty \psi_{xx}^2\mathrm{d}x-\kappa u_b\varphi_{xx}(t,0)^2 \\
&= -\kappa\int_0^\infty(\rho\varphi_{xxx}\psi_{xx}+\rho_x\psi_x\varphi_{xxx})\mathrm{d}x+\int_0^\infty[\rho u\psi_x+p'(\rho)\varphi_x-2u_b\psi_x]\psi_{xx}\mathrm{d}x \\
&\quad -2u_b\int_0^\infty\left[\frac{\mu}{\rho}\psi_{xx}-\frac{p'(\rho)}{\rho}\varphi_x\right]\varphi_{xx}\mathrm{d}x-\frac{1}{2}\int_0^\infty\psi_x^2(\rho\psi_x+u\varphi_x+\rho u^r+u\rho_x^r)\mathrm{d}x \\
&\quad -\int_0^\infty g\psi_{xx}\mathrm{d}x-2u_b\int_0^\infty\frac{g}{\rho}\varphi_{xx}\mathrm{d}x \\
&\quad -\int_0^\infty\rho_x\psi_x\left[\frac{g}{\rho}+\frac{\mu}{\rho}\psi_{xx}-\frac{p'(\rho)}{\rho}\varphi_x-u\psi_x\right]\mathrm{d}x \\
&\quad +2u_b\int_0^\infty\psi_x(f_x-\rho_x^r\psi_x-u_x^r\varphi_x-2\varphi_x\psi_x)\mathrm{d}x. \qquad (4.29)
\end{aligned}$$

First, from (4.1) and the Young inequality, one has

$$\int_0^\infty[\rho u\psi_x+p'(\rho)\varphi_x-2u_b\psi_x]\psi_{xx}\mathrm{d}x\leqslant\frac{\mu}{8}\|\psi_{xx}(t)\|^2+C\|(\varphi_x,\psi_x)(t)\|^2, \qquad (4.30)$$

and

$$-2u_b\int_0^\infty\left[\frac{\mu}{\rho}\psi_{xx}-\frac{p'(\rho)}{\rho}\varphi_x\right]\varphi_{xx}\mathrm{d}x\leqslant\frac{\mu}{8}\|\psi_{xx}(t)\|^2+C\|(\varphi_x,\varphi_{xx})(t)\|^2. \qquad (4.31)$$

Next, using (4.1), (3.5), lemma 2.2, and the Sobolev and Young inequalities, we have

$$\begin{aligned}
&-\frac{1}{2}\int_0^\infty\psi_x^2(\rho\psi_x+u\varphi_x+\rho u^r+u\rho_x^r)\mathrm{d}x \\
&\leqslant C\left|\int_0^\infty\psi_x^3\mathrm{d}x\right|+C\left|\int_0^\infty\rho_x^r\psi_x^2\mathrm{d}x\right|+C\left|\int_0^\infty u_x^r\psi_x^2\mathrm{d}x\right|+C\left|\int_0^\infty\varphi_x\psi_x^2\mathrm{d}x\right| \\
&\leqslant C\|\psi_x\|_{L^\infty}\|\psi_x\|^2+C(\|\rho_x^r\|_{L^\infty}+\|u_x^r\|_{L^\infty})\|\psi_x\|^2+C\|\varphi_x\|_{L^\infty}\|\psi_x\|^2 \\
&\leqslant C\|\psi_x(t)\|^{\frac{5}{2}}\|\psi_{xx}\|^{\frac{1}{2}}+C(\varepsilon_1+\varepsilon)\|\psi_x(t)\|^2 \\
&\leqslant\frac{\mu}{8}\|\psi_{xx}(t)\|^2+C(\varepsilon_1+\varepsilon)\|\psi_x(t)\|^2. \qquad (4.32)
\end{aligned}$$

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Similar as (4.24) and (4.25), we have

$$
\begin{aligned}
&-\int_0^\infty g\psi_{xx}\mathrm{d}x - 2u_b\int_0^\infty \frac{g}{\rho}\varphi_{xx}\mathrm{d}x \\
&\quad\leqslant C\int_0^\infty |u_x^r\psi\psi_{xx}|\mathrm{d}x + C\int_0^\infty |\rho_x^r\varphi\psi_{xx}|\mathrm{d}x + C\int_0^\infty |u_{xx}^r\psi_{xx}|\mathrm{d}x \\
&\qquad + C\int_0^\infty |\rho_{xxx}^r\psi_{xx}|\mathrm{d}x + C\int_0^\infty |u_x^r\psi\varphi_{xx}|\mathrm{d}x + C\int_0^\infty |\rho_x^r\varphi\varphi_{xx}|\mathrm{d}x \\
&\qquad + C\int_0^\infty |u_{xx}^r\varphi_{xx}|\mathrm{d}x + C\int_0^\infty |\rho_{xxx}^r\varphi_{xx}|\mathrm{d}x \\
&\quad\leqslant \frac{\mu}{8}\|\psi_{xx}(t)\|^2 + C\|\varphi_{xx}(t)\|^2 + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{3}{2}}\|(\varphi,\psi)(t)\|^2 \\
&\qquad + C\varepsilon^{\frac{1}{4}}\|(\varphi_{xx},\psi_{xx})(t)\|^2 + C\varepsilon^{\frac{1}{5}}(1+t)^{-\frac{9}{5}} + C\varepsilon^{\frac{2}{5}}(1+t)^{-\frac{8}{5}}.
\end{aligned}
\tag{4.33}
$$

In a similar way, we can obtain

$$
\begin{aligned}
&-\int_0^\infty \rho_x\psi_x\Big[\frac{g}{\rho} + \frac{\mu}{\rho}\psi_{xx} - \frac{p'(\rho)}{\rho}\varphi_x - u\psi_x\Big]\mathrm{d}x \\
&\leqslant C\int_0^\infty (|\rho_x^r u_{xx}^r\psi_x| + |\rho_x^r\rho_{xxx}^r\psi_x|)\mathrm{d}x + C\int_0^\infty (|\rho_x^r\rho_x^r\varphi\psi_x| + |\rho_x^r u_x^r\varphi\psi_x|)\mathrm{d}x \\
&\quad + C\int_0^\infty (|\rho_x^r\psi_x\psi_{xx}| + |\rho_x^r\varphi_x\psi_x| + |u_{xx}^r\varphi_x\psi_x| + |\rho_{xxx}^r\varphi_x\psi_x|)\mathrm{d}x \\
&\quad + C\int_0^\infty (|\rho_x^r\varphi\varphi_x\psi_x| + |u_x^r\psi\varphi_x\psi_x|)\mathrm{d}x \\
&\quad + C\int_0^\infty (|\varphi_x\psi_x\psi_{xx}| + |\varphi_x\psi_x^2| + |\varphi_x^2\psi_x|)\mathrm{d}x \\
&\leqslant C(\varepsilon+\varepsilon_1)\|(\varphi_x,\psi_{xx})(t)\|^2 + C\|\psi_x(t)\|^2 + C\varepsilon(1+t)^{-2}\|\varphi(t)\|^2 \\
&\quad + C\varepsilon^{\frac{1}{5}}(1+t)^{-\frac{9}{5}} + C\varepsilon^{\frac{2}{5}}(1+t)^{-\frac{8}{5}},
\end{aligned}
\tag{4.34}
$$

and

$$
\begin{aligned}
&2u_b\int_0^\infty \psi_x\Big(f_x - \rho_x^r\psi_x - u_x^r\varphi_x - 2\varphi_x\psi_x\Big)\mathrm{d}x \\
&\quad\leqslant C\int_0^\infty (|\rho_x^r\psi_x^2| + |u_x^r\varphi_x\psi_x|)\mathrm{d}x \\
&\qquad + C\int_0^\infty (|\rho_{xx}^r\psi\varphi_x| + |u_{xx}^r\varphi\psi_{xx}|)\mathrm{d}x + C\int_0^\infty |\varphi_x\psi_x^2|\mathrm{d}x \\
&\quad\leqslant \frac{\mu}{8}\|\psi_{xx}(t)\|^2 + C(\|\varphi_x(t)\|^2 + \|\psi_x(t)\|^2) + C\varepsilon^{\frac{1}{3}}\|\varphi_x(t)\|^2 + \|\psi_{xx}(t)\|^2) \\
&\qquad + C\varepsilon^{\frac{1}{3}}(1+t)^{-\frac{3}{2}}\|(\varphi,\psi)(t)\|^2.
\end{aligned}
\tag{4.35}
$$

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Finally, using (4.17), integration by parts and $\varphi_{tx}(t,0)=0$, we have

$$\begin{aligned}
&-\kappa\int_0^\infty \rho\varphi_{xxx}\psi_{xx}\mathrm{d}x-\kappa\int_0^\infty \rho_x\psi_x\varphi_{xxx}\mathrm{d}x\\
&=\kappa\int_0^\infty \varphi_{xxx}(\varphi_{tx}+u_x\varphi_x+u\varphi_{xx}-f_x)\mathrm{d}x\\
&=\kappa\varphi_{xx}\varphi_{tx}|_0^\infty-\kappa\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \varphi_{xx}^2\mathrm{d}x+\frac{k}{2}u\varphi_{xx}^2|_0^\infty-\frac{\kappa}{2}\int_0^\infty u_x\varphi_{xx}^2\mathrm{d}x\\
&\quad+\kappa\int_0^\infty \varphi_{xxx}(u_x\varphi_x-f_x)\mathrm{d}x\\
&=-\kappa\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \varphi_{xx}^2\mathrm{d}x-\frac{\kappa}{2}u_b\varphi_{xx}(t,0)^2\\
&\quad-\frac{k}{2}\int_0^\infty u_x\varphi_{xx}^2\mathrm{d}x+\kappa\int_0^\infty \varphi_{xxx}(u_x\varphi_x-f_x)\mathrm{d}x. \qquad (4.36)
\end{aligned}$$

Similar to (4.14) and (4.32), we have

$$\begin{aligned}
-\int_0^\infty u_x\varphi_{xx}^2\mathrm{d}x&=-\int_0^\infty \psi_x\varphi_{xx}^2\mathrm{d}x-\int_0^\infty u_x^r\varphi_{xx}^2\mathrm{d}x\\
&\leqslant C\|\psi_x(t)\|_{L^\infty}\|\varphi_{xx}(t)\|^2+C\varepsilon\|\varphi_{xx}(t)\|^2\\
&\leqslant C\|\psi_x(t)\|^{\frac12}\|\psi_{xx}(t)\|^{\frac12}\|\varphi_{xx}(t)\|^2+C\varepsilon\|\varphi_{xx}(t)\|^2\\
&\leqslant \frac{\mu}{8}\|\psi_{xx}(t)\|^2+C(\varepsilon+\varepsilon_1)\|\varphi_{xx}(t)\|^2.
\end{aligned}$$

Moreover, similar as (4.14), (4.15) and (4.32), we get

$$\begin{aligned}
&\kappa\int_0^\infty \varphi_{xxx}(u_x\varphi_x-f_x)\mathrm{d}x\\
&\leqslant C\int_0^\infty |\varphi_x\psi_x\varphi_{xxx}|\mathrm{d}x+C\int_0^\infty |u_x^r\varphi_x\varphi_{xxx}|\mathrm{d}x+C\int_0^\infty |\rho_x^r\psi_x\varphi_{xxx}|\mathrm{d}x\\
&\quad+C\int_0^\infty |\rho_{xx}^r\psi\varphi_{xxx}|\mathrm{d}x+C\int_0^\infty |u_{xx}^r\varphi\varphi_{xxx}|\mathrm{d}x\\
&\leqslant C(\varepsilon_1+\varepsilon)\|(\varphi_x,\psi_x,\varphi_{xxx})(t)\|^2+C\varepsilon^{\frac13}\|\varphi_{xxx}(t)\|^2+C\varepsilon^{\frac13}(1+t)^{-\frac32}\|(\varphi,\psi)(t)\|^2.
\end{aligned}$$

Putting the above two inequalities into (4.36) yields

$$\begin{aligned}
&-\kappa\int_0^\infty \rho\varphi_{xxx}\psi_{xx}\mathrm{d}x-\kappa\int_0^\infty \rho_x\psi_x\varphi_{xxx}\mathrm{d}x\\
&\leqslant -\kappa\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \varphi_{xx}^2\mathrm{d}x-\frac{\kappa u_b}{2}\varphi_{xx}(t,0)^2+\frac{\mu}{8}\|\psi_{xx}(t)\|^2\\
&\quad+C(\varepsilon+\varepsilon_1)\|(\varphi_x,\psi_x,\varphi_{xx})(t)\|^2+C(\varepsilon^{\frac13}+\varepsilon_1)\|\varphi_{xxx}(t)\|^2\\
&\quad+C\varepsilon^{\frac13}(1+t)^{-\frac32}\|(\varphi,\psi)(t)\|^2. \qquad (4.37)
\end{aligned}$$

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Further, combining (4.29), (4.30)–(4.35) and (4.37), and using (4.1), we see

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty(\psi_x^2+\varphi_{xx}^2-\psi\varphi_{xx})\mathrm{d}x+\varphi_{xx}(t,0)^2+\|\psi_{xx}(t)\|^2\\&\quad\leqslant C(\|\varphi_x\|_1^2+\|\psi_x\|^2)+C(\varepsilon^{\frac{1}{3}}+\varepsilon_1)\|\varphi_{xxx}\|^2+C\varepsilon^{\frac{1}{3}}(1+t)^{-\frac{3}{2}}\|(\varphi,\psi)(t)\|^2\\&\qquad+C\varepsilon(1+t)^{-2}\|(\varphi,\psi)(t)\|^2+C\varepsilon^{\frac{1}{5}}(1+t)^{-\frac{9}{5}}+C\varepsilon^{\frac{2}{5}}(1+t)^{-\frac{8}{5}}.\end{aligned}$$

Therefore, integrating the above inequality with respect to $t$, and using (4.26), we obtain (4.27). This completes the proof.

Finally, we are going to establish the dissipation for $\varphi_{xxx}$.

LEMMA 4.5. *Let $(\varphi,\psi)$ be a solution to the initial boundary value problem (3.1)–(3.2), satisfying the conditions in proposition* 3.3*, then it holds that*

$$\int_0^t\|\varphi_{xxx}(\tau)\|^2d\tau\leqslant C(\|\varphi_0\|_2^2+\|\psi_0\|_1^2+\varepsilon^{\frac{1}{8}})\tag{4.38}$$

*for an arbitrary $t\in[0,T]$.*

**Proof.** We first divide $(3.1)_2$ by $\rho$, then differentiate formally the resultant equality to obtain

$$\begin{aligned}&\psi_{tx}+u\psi_{xx}+u_x\psi_x+\frac{p'(\rho)}{\rho}\varphi_{xx}+\left(\frac{p'(\rho)}{\rho}\right)_x\varphi_x\\&\quad=\frac{\mu}{\rho}\psi_{xxx}+\left(\frac{\mu}{\rho}\right)_x\psi_{xx}\varphi_{xx}+\kappa\varphi_{xxxx}+\left(\frac{g}{\rho}\right)_x,\end{aligned}$$

further, multiplying the above equality by $\varphi_{xx}$, integrating with respect to $x$ over $\mathbb{R}_+$ and using (4.17) and $\varphi_{tx}(t,0)=0$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty\psi_x\varphi_{xx}\mathrm{d}x+\int_0^\infty\frac{p'(\varphi)}{\rho}\varphi_{xx}^2\mathrm{d}x-\int_0^\infty\frac{\mu}{\rho}\psi_{xxx}\varphi_{xx}\mathrm{d}x-\kappa\int_0^\infty\varphi_{xxxx}\varphi_{xx}\mathrm{d}x\\&\quad=\int_0^\infty\rho\psi_{xx}^2\mathrm{d}x+\int_0^\infty\left(\frac{\mu}{\rho}\right)_x\psi_{xx}\varphi_{xx}\mathrm{d}x-\int_0^\infty\left(\frac{p'(\rho)}{\rho}\right)_x\varphi_x\varphi_{xx}\mathrm{d}x\\&\quad-\int_0^\infty u_x\psi_x\varphi_{xx}\mathrm{d}x-\int_0^\infty\psi_{xx}(f_x-u_x\varphi_x-\rho_x\psi_x)\mathrm{d}x+\int_0^\infty\left(\frac{g}{\rho}\right)_x\varphi_{xx}\mathrm{d}x.\end{aligned}\tag{4.39}$$

First, using (4.1), (3.5), lemma 2.2 and Cauchy inequality, it is easy to obtain

$$\begin{aligned}&\int_0^\infty\left(\frac{\mu}{\rho}\right)_x\psi_{xx}\varphi_{xx}\mathrm{d}x-\int_0^\infty\left(\frac{p'(\rho)}{\rho}\right)_x\varphi_x\varphi_{xx}\mathrm{d}x\\&\quad\leqslant C\int_0^\infty|\rho_x\psi_{xx}\varphi_{xx}|\mathrm{d}x+C\int_0^\infty|\rho_x\varphi_x\varphi_{xx}|\mathrm{d}x\\&\quad\leqslant C\int_0^\infty|(\varphi_x+\rho_x^r)\psi_{xx}\varphi_{xx}|\mathrm{d}x+C\int_0^\infty|(\varphi_x+\rho_x^r)\varphi_x\varphi_{xx}|\mathrm{d}x\\&\quad\leqslant C(\varepsilon+\varepsilon_1)(\|\varphi_x\|^2+\|\varphi_{xx}\|^2+\|\psi_{xx}\|^2).\end{aligned}\tag{4.40}$$

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Next, utilizing (3.5), lemma 2.2, Sobolev inequality and Young inequality, one gets

$$\begin{aligned} -\int_0^\infty u_x\psi_x\varphi_{xx}\mathrm{d}x &\leqslant C\int_0^\infty |u_x^r\psi_x\varphi_{xx}|\mathrm{d}x + C\int_0^\infty |\psi_x^2\varphi_{xx}|\mathrm{d}x \\ &\leqslant C\|u_x^r\|_{L^\infty}\|\psi_x\|\|\varphi_{xx}\| + C\|\varphi_{xx}\|_{L^\infty}\|\psi_x\|^2 \\ &\leqslant C\varepsilon(\|\psi_x\|^2+\|\varphi_{xx}\|^2) + C\|\varphi_{xx}\|^{\frac12}\|\varphi_{xxx}\|^{\frac12}\|\psi_x\|^2 \\ &\leqslant C\varepsilon(\|\psi_x\|^2+\|\varphi_{xx}\|^2) + C\varepsilon_1(\|\varphi_{xx}(t)\|^2+\|\varphi_{xxx}(t)\|^2). \quad (4.41) \end{aligned}$$

Similar as (4.14), (4.15) and (4.21), we have

$$\begin{aligned} &-\int_0^\infty \psi_{xx}(f_x-u_x\varphi_x-\rho_x\psi_x)\mathrm{d}x \\ &\leqslant C\int_0^\infty |\varphi_x\psi_x\psi_{xx}|\mathrm{d}x + C\int_0^\infty |u_x^r\varphi_x\psi_{xx}|\mathrm{d}x + C\int_0^\infty |\rho_x^r\psi_x\psi_{xx}|\mathrm{d}x \\ &\quad + C\int_0^\infty |\rho_{xx}^r\psi\psi_{xx}|\mathrm{d}x + C\int_0^\infty |u_{xx}^r\varphi\psi_{xx}|\mathrm{d}x \\ &\leqslant C\varepsilon_1\|(\psi_x,\psi_{xx})(t)\|^2 + C\varepsilon\|(\varphi_x,\psi_x,\psi_{xx})(t)\|^2 \\ &\quad + C\varepsilon^{\frac13}\|\psi_{xx}(t)\|^2 + C\varepsilon^{\frac13}(1+t)^{-\frac32}\|(\varphi,\psi)(t)\|^2. \end{aligned} \tag{4.42}$$

Since

$$\begin{aligned} \left(\frac{g}{\rho}\right)_x \sim\ & u_{xxx}^r + \rho_{xxxx}^r + \rho_x^r u_{xx}^r + u_{xx}^r\varphi_x + \rho_{xx}^r\psi + \rho_{xx}^r\varphi + \rho_x^r\varphi_x + \rho_x^r\rho_x^r\varphi + \rho_x^r\varphi_x\varphi \\ & + u_{xx}^r\psi + u_{xx}^r\psi_x, \end{aligned}$$

similar to (4.14), (4.15), (4.21) and (4.25), we can show

$$\begin{aligned} &\int_0^\infty \left(\frac{g}{\rho}\right)_x \varphi_{xx}\mathrm{d}x \\ &\leqslant C\Big|\int_0^\infty (u_{xxx}^r + \rho_{xxxx}^r + \rho_x^r u_{xx}^r)\varphi_{xx}\mathrm{d}x\Big| \\ &\quad + C\Big|\int_0^\infty (u_{xx}^r\varphi_x + \rho_x^r\varphi_x + \rho_x^r\varphi\varphi_x + u_{xx}^r\psi_x)\varphi_{xx}\mathrm{d}x\Big| \\ &\quad + C\Big|\int_0^\infty (\rho_{xx}^r\psi + \rho_{xx}^r\varphi + \rho_x^r\rho_x^r\varphi + u_{xx}^r\psi)\varphi_{xx}\mathrm{d}x\Big| \\ &\leqslant C(\varepsilon+\varepsilon_1)\|(\varphi_x,\psi_x)(t)\|^2 + \frac18\|\varphi_{xx}(t)\|^2 + C\varepsilon^{\frac13}(1+t)^{-\frac32}\|(\varphi,\psi)\|^2 \\ &\quad + C\varepsilon^{\frac15}(1+t)^{-\frac95} + C\varepsilon^{\frac25}(1+t)^{-\frac85} + C\varepsilon^{\frac35}(1+t)^{-\frac75}. \end{aligned} \tag{4.43}$$

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Finally, using integration by parts, one gets

$$\begin{aligned}
&\kappa\int_0^\infty \varphi_{xxxx}\varphi_{xx}\mathrm{d}x+\int_0^\infty \frac{\mu}{\rho}\psi_{xxx}\varphi_{xx}\mathrm{d}x\\
&\quad=\kappa\varphi_{xxx}\varphi_{xx}|_0^\infty+\frac{\mu}{\varphi}\psi_{xx}\varphi_{xx}|_0^\infty-\kappa\int_0^\infty\varphi_{xxx}^2\mathrm{d}x-\int_0^\infty\frac{\mu}{\rho}\psi_{xx}\varphi_{xxx}\mathrm{d}x\\
&\qquad+\int_0^\infty\frac{\mu}{\rho^2}\rho_x\psi_{xx}\varphi_{xx}\mathrm{d}x\\
&\quad\leqslant C\varphi_{xx}(t,0)^2+C\Big(\kappa\varphi_{xxx}(t,0)+\frac{\mu}{\rho(t,0)}\psi_{xx}(t,0)\Big)^2-\frac{\kappa}{2}\|\varphi_{xxx}\|^2\\
&\qquad+C\|\psi_{xx}\|^2+C(\varepsilon+\varepsilon_1)\|\varphi_{xx}\|^2\\
&\quad\leqslant C\varphi_{xx}(t,0)^2-\frac{\kappa}{2}\|\varphi_{xxx}\|^2+C\|\psi_x\|_1^2+C(\varepsilon+\varepsilon_1)\|\varphi_{xx}\|^2, \qquad (4.44)
\end{aligned}$$

here we have used

$$\left(\kappa\varphi_{xxx}(t,0)+\frac{\mu}{\rho(t,0)}\psi_{xx}(t,0)\right)^2\leqslant C\psi_x(t,0)^2\leqslant C\|\psi_x\|_1^2,$$

which is derived by $(3.1)_2$, (3.2) and lemma 2.2.

Therefore, insertion of (4.40)–(4.44) into (4.39) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty\psi_x\varphi_{xx}\mathrm{d}x+\|\varphi_{xxx}\|^2+\|\varphi_{xx}\|^2\\
&\quad\leqslant C\|\psi_x\|_1^2+C\varphi_{xx}(t,0)^2+C\varepsilon^{\frac13}(1+t)^{-\frac32}(\|\psi\|^2+\|\varphi\|^2)\\
&\qquad+C\varepsilon^{\frac15}(1+t)^{-\frac95}+C\varepsilon^{\frac25}(1+t)^{-\frac85}+C\varepsilon^{\frac35}(1+t)^{-\frac75},
\end{aligned}$$

further, integrating the above inequality with respect to $t$, and using (4.26) and (4.27), we obtain (4.38). This completes the proof.

**Proof of proposition 3.3.** Summing up the estimates (4.26), (4.27) and (4.38), we immediately have (3.6).

## 5. The proof of theorem 1.1

This section is concerned with the proof of our main theorem. From theorem 3.1, we know that there exists a unique classical global solution $(\rho,u)(t,x)$ to the problem (1.1)–(1.4), satisfying (1.7)–(1.9). Therefore, to complete the proof of theorem 1.1, we need only to investigate the large-time behaviour of the solution $(\rho,u)(t,x)$ to the initial boundary value problem (1.1)–(1.4) as time tends to infinity.

**The completion of the proof of theorem 1.1.** Based upon the energy estimates derived in the previous sections, we will complete the proof of theorem 1.1. To this

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end, we first prove that

$$\sup_{x\in\mathbb{R}^+} |(\rho-\rho^r, u-u^r)(t,x)| \to 0, \tag{5.1}$$

namely,

$$\sup_{x\in\mathbb{R}^+} |(\varphi,\psi)(t,x)| \to 0, \tag{5.2}$$

as $t\to\infty$.

This is obvious suppose that we have proved the following assertion

$$\lim_{t\to+\infty} \|(\varphi_x,\psi_x)(t)\| = 0. \tag{5.3}$$

As a matter of fact, if it were true, we infer from the Sobolev inequality that

$$\|(\varphi,\psi)\|_{L^\infty} \to 0,\ \text{as } t\to+\infty. \tag{5.4}$$

Hence, it remains to show (5.3). To this end, from the relations (4.17) and (4.39), and corollary 4.3, lemmas 4.4 and 4.5, one can show that

$$\int_0^\infty \left(\|\varphi_x\|^2 + \|\psi_x\|^2\right) \mathrm{d}\tau < +\infty, \tag{5.5}$$

and that

$$\int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_x\|^2\right| \mathrm{d}\tau < +\infty, \quad \int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_x\|^2\right| \mathrm{d}\tau < +\infty. \tag{5.6}$$

Then (5.3) follows from inequalities (5.5)–(5.6). Consequently, from (5.1) and (iv) of lemma 2.2, we prove (1.10) and complete the proof of theorem 1.1.

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