EMBEDDING THEOREMS FOR GROUPS

by C. G. CHEHATA

(Received 5th December, 1962)

1. Introduction

By a *partial endomorphism* of a group G we mean a homomorphic mapping μ of a subgroup A of G onto a subgroup B of G. If μ is defined on the whole of G then it is called a total endomorphism. We call a partial endomorphism *totally extendable* (or *extendable*) if there exists a supergroup $G^* \supseteq G$ with a total endomorphism μ^* which extends μ in the sense that $g\mu^* = g\mu$, whenever the right-hand side is defined (**3**).

In a previous paper (**2**), we derived necessary and sufficient conditions for a well-ordered set of partial endomorphisms $\mu(\alpha)$ of a group G to be extendable to a set of total endomorphisms $\mu^*(\alpha)$ of a supergroup G^* such that each $\mu^*(\alpha)$ acts as an isomorphism on $G^*[\mu^*(\alpha)]^{n(\alpha)}$, where $n(\alpha)$ is a given positive integer. These conditions are in fact a generalisation of the conditions in case of a single extension (**1**).

In this work sufficient conditions are derived for the required extension, with the same condition imposed on $\mu^*(\alpha)$, to be established in case $\mu(\alpha)$ are partial endomorphisms of certain types of subgroups. In particular sufficient conditions for the extension of partial endomorphisms of E-subgroups are given; where the subgroup H of the group G is called an *E-subgroup* if every normal subgroup of H is the intersection with H of a normal subgroup of G. This is equivalent to the fact that if N is a normal subgroup of H then $N^G \cap H = N$, where N^G is the normal closure of N in G.

We conclude by deriving necessary and sufficient conditions for a well-ordered set of partial endomorphisms of G to be all extendable to one and the same total endomorphism θ^* of a supergroup G^* such that θ^* is an isomorphism on $G^*(\theta^*)^m$ for some positive integer m.

2. Extension in a special case

We shall assume that G is a given group and $\mu(\alpha)$, where α ranges over a well-ordered set Σ, is a partial endomorphism of G mapping the subgroup $A(\alpha) \subseteq G$ onto the subgroup $B(\alpha) \subseteq G$. In (**2**) it was proved that the necessary and sufficient conditions for the existence of $G^* \supseteq G$ with total endomorphisms $\mu^*(\alpha)$ which extend $\mu(\alpha)$ such that for every α, $\mu^*(\alpha)$ acts as an isomorphism on $G^*[\mu^*(\alpha)]^{n(\alpha)}$, where $n(\alpha)$ is a positive integer are that if Ω is the semigroup freely generated by the $\mu(\alpha)$, then for every $\omega \in \Omega$ there exists a normal subgroup

https://doi.org/10.1017/S0013091500014723 Published online by Cambridge University Press

$L(\omega)$ of G such that

$$L(\omega) \subseteq L(\omega\omega_1) \text{ for all } \omega, \omega_1 \in \Omega, \quad \text{......(2.1)}$$

$$L[\{\mu(\alpha)\}^{n(\alpha)}] = L[\{\mu(\alpha)\}^{n(\alpha)+i}], \quad \text{......(2.2)}$$

for any $\alpha \in \Sigma$ and any positive integer i,

$$L[\mu(\alpha)] \cap A(\alpha) \text{ is the kernel of } \mu(\alpha), \quad \text{......(2.3)}$$

$$[L\{\mu(\alpha)\omega\} \cap A(\alpha)]\mu(\alpha) = L(\omega) \cap B(\alpha), \quad \text{......(2.4)}$$

for every $\alpha \in \Sigma$ and $\omega \in \Omega$.

Theorem 1. *With the previous notation, it is sufficient for the required extension to be established that if, for every $\alpha \in \Sigma$, $K[\mu(\alpha)]$ is the kernel of $\mu(\alpha)$ then*

$$K^G[\mu(\alpha)] \cap A(\alpha) = K[\mu(\alpha)], \quad \text{......(2.5)}$$

$$K^G[\mu(\alpha)] \cap B(\beta) = \{e\} \quad \text{......(2.6)}$$

for every $\alpha, \beta \in \Sigma$.

Proof. For every $\omega \in \Omega$, put

$$L[\mu(\alpha)\omega] = L[\mu(\alpha)] = K^G[\mu(\alpha)].$$

If $\omega = \mu(\alpha)\omega'$ and ω_1 are any words in Ω then

$$L(\omega) = L(\omega\omega_1) = K^G[\mu(\alpha)]$$

which simultaneously proves (2.1) and (2.2). Also

$$L[\mu(\alpha)] \cap A(\alpha) = K^G[\mu(\alpha)] \cap A(\alpha) = K[\mu(\alpha)]$$

is the kernel of $\mu(\alpha)$, which proves (2.3).

To prove (2.4) we note that if $\omega = \mu(\gamma)\omega'$ is any word in Ω then

$$\begin{aligned}[L\{\mu(\alpha)\omega\} \cap A(\alpha)]\mu(\alpha) &= [L\{\mu(\alpha)\} \cap A(\alpha)]\mu(\alpha)\\ &= K[\mu(\alpha)]\mu(\alpha) = e,\end{aligned}$$

and

$$\begin{aligned}L(\omega) \cap B(\alpha) &= L[\mu(\gamma)\omega'] \cap B(\alpha)\\ &= K^G[\mu(\gamma)] \cap B(\alpha)\\ &= e, \text{ by (2.6).}\end{aligned}$$

This completes the proof of Theorem 1.

3. Extension in case of E-subgroups

Theorem 2. *Let $A(\alpha)$ be E-subgroups of G. If we define $K[\mu(\alpha)] = e\mu^{-1}(\alpha)$ and inductively*

$$K[\mu(\alpha)\omega] = K^G(\omega)\mu^{-1}(\alpha) \quad \text{......(3.1)}$$

i.e. the greatest subgroup of $A(\alpha)$ mapped into $K^G(\omega)$ by $\mu(\alpha)$, then for the required extension to be established it is sufficient that

$$K[\mu(\alpha)]^m \text{ are normal in } G, \text{ i.e. } \quad \text{......(3.2)}$$

$K[\mu(\alpha)]^{m+1} = K[\mu(\alpha)]^m\mu^{-1}(\alpha)$; for $m = 1, 2, \ldots$, whenever $x[\mu(\alpha)]^{n(\alpha)+1}$ is defined and is equal to e then

$$x[\mu(\alpha)]^{n(\alpha)} = e. \quad \text{......(3.3)}$$

https://doi.org/10.1017/S0013091500014723 Published online by Cambridge University Press

Proof. We can as in (3) prove that

$$K(\omega) \subseteq K(\omega\omega') \quad \text{.................................(3.4)}$$

for any ω, $\omega' \in \Omega$.

Now we prove that

$$K[\mu(\alpha)]^{n(\alpha)+1} = K[\mu(\alpha)]^{n(\alpha)}. \quad \text{........................(3.5)}$$

Let $\quad x \in K[\mu(\alpha)]^{n(\alpha)+1} = K[\mu(\alpha)]^{n(\alpha)}\mu^{-1}(\alpha),$

thus $\quad x\mu(\alpha) \in K[\mu(\alpha)]^{n(\alpha)}.$

Repeating we arrive at

$$x[\mu(\alpha)]^{n(\alpha)} \in K[\mu(\alpha)],$$

and hence

$$x[\mu(\alpha)]^{n(\alpha)+1} = e.$$

This implies by (3.3) that

$$x[\mu(\alpha)]^{n(\alpha)} = e,$$

which in turn gives

$$x \in K[\mu(\alpha)]^{n(\alpha)}.$$

Thus $\quad K[\mu(\alpha)]^{n(\alpha)+1} \subseteq K[\mu(\alpha)]^{n(\alpha)};$

but $\quad K[\mu(\alpha)]^{n(\alpha)} \subseteq K[\mu(\alpha)]^{n(\alpha)+1}$, from (3.4).

These two together prove (3.5).

From (3.2) and (3.5) we get

$$\begin{aligned} K[\mu(\alpha)]^{n(\alpha)+2} &= K[\mu(\alpha)]^{n(\alpha)+1}\mu^{-1}(\alpha) \\ &= K[\mu(\alpha)]^{n(\alpha)}\mu^{-1}(\alpha) \\ &= K[\mu(\alpha)]^{n(\alpha)+1} \\ &= K[\mu(\alpha)]^{n(\alpha)}. \end{aligned}$$

More generally we have

$$K[\mu(\alpha)]^{n(\alpha)+i} = K[\mu(\alpha)]^{n(\alpha)} \text{ for any integer } i>0. \quad \text{..............(3.6)}$$

Now put

$$L(\omega) = K^G(\omega). \quad \text{..................................(3.7)}$$

Thus from (3.4) and (3.6) we get

$$L(\omega) \subseteq L(\omega\omega')$$

for any ω, $\omega' \in \Omega$; and

$$L[\mu(\alpha)]^{n(\alpha)+i} = L[\mu(\alpha)]^{n(\alpha)},$$

for any $\alpha \in \Sigma$ and any integer $i>0$. This proves (2.1) and (2.2). From (3.7) and the fact that $A(\alpha)$ are E-subgroups it follows immediately that

$L[\mu(\alpha)] \cap A(\alpha) = K[\mu(\alpha)]$ is the kernel of $\mu(\alpha)$, which proves (2.3).

The proof that (2.4) also holds is the same as in (3). This completes the proof of Theorem 2.

Special case. If the group G is abelian then every subgroup of G is an E-subgroup and condition (3.2) holds automatically. Thus we have the following result.

Corollary. *If G is abelian then it is sufficient for the required extension to be established that* (3.3) *holds.*

4. Extending all $\mu(\alpha)$ to a single endomorphism

Theorem 3. *For all $\mu(\alpha)$, $\alpha \in \Sigma$ to be extendable to one and the same total endomorphism θ^* of a group $G^* \supseteq G$ such that θ^* is an isomorphism on $G^*(\theta^*)^m$, for some positive integer m, it is necessary and sufficient that if we define θ to map any word $w(a_\tau) \in \{A(\alpha)\}$ onto $w(a_\tau \mu(\tau)) \in \{B(\alpha)\}$ where $a_\tau \in A(\tau)$, τ ranges over some finite set $I \subset \Sigma$, α ranges over Σ then*

θ is a one-valued mapping of $\{A(\alpha)\}$ onto $\{B(\alpha)\}$ which is a homomorphism, (4.1)

there exists in G a sequence of normal subgroups

$$L_1 \subseteq L_2 \subseteq \ldots \subseteq L_m = L_{m+1} = \ldots \qquad (4.2)$$

such that

$$L_1 \cap \{A(\alpha)\} \text{ is the kernel of } \theta,$$

$$[L_{j+1} \cap \{A(\alpha)\}]\theta = L_j \cap \{B(\alpha)\},$$

for $j = 1, 2, \ldots, m$.

Proof. (i). To prove the necessity of (4.1) we assume that the extension is already established, that is we assume the existence of $G^* \supseteq G$ and an endomorphism θ^* which extends $\mu(\alpha)$ for every $\alpha \in \Sigma$ to G^* such that θ^* is an isomorphism on $G^*(\theta^*)^m$.

For any $g^* \in G^*$, $g^*\theta^*$ is uniquely defined. In particular the map $w(a_\tau)\theta^*$ of any word $w(a_\tau) \in \{A(\alpha)\} \subseteq G^*$ is uniquely defined. Since θ^* extends $\mu(\alpha)$ for every $\alpha \in \Sigma$ then

$$w(a_\tau)\theta^* = w(a_\tau \mu(\tau)) = w(a_\tau)\theta$$

and thus the mapping θ is one-valued.

Moreover since θ^* extends θ then for any two words $w(a_\rho)$, $w_1(a_\tau) \in \{A(\alpha)\}$ we have

$$\begin{aligned}[w(a_\rho)w_1(a_\tau)]\theta &= [w(a_\rho)w_1(a_\tau)]\theta^* \\ &= [w(a_\rho)]\theta^*[w_1(a_\tau)]\theta^* \\ &= [w(a_\rho)]\theta \,.\, [w_1(a_\tau)]\theta\end{aligned}$$

which shows that θ is a homomorphism.

The proof that (4.2) is necessary is the same as in (**1**).

(ii). To prove the sufficiency of the conditions we put

$$A_1 = \{A(\alpha)\}, \quad B_1 = \{B(\alpha)\}.$$

Then θ becomes a partial endomorphism of G which maps A_1 onto B_1. Thus because of (4.2) we can extend θ to a total endomorphism θ^* of $G^* \supseteq G$ such that θ^* acts as an isomorphism on $G^*(\theta^*)^m$. Since θ extends $\mu(\alpha)$ for every $\alpha \in \Sigma$, then so does θ^*.

This completes the proof of Theorem 3.

REFERENCES

(1) C. G. Chehata, An embedding theorem for groups, *Proc. Glasgow Math. Assoc.* **4** (1960), 140-143.

(2) C. G. Chehata, Generalisation of an embedding theorem for groups, *Proc. Glasgow Math. Assoc.*, **4** (1960), 171-177.

(3) B. H. Neumann and Hanna Neumann, Extending partial endomorphisms of groups, *Proc. London Math. Soc.* (3) **2** (1952), 337-348.

Faculty of Science
The University
Alexandria, Egypt