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A pure sine-wave oscillator with a fast settling time

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An unforced oscillator which obeys the equation $\ddot{x}+\alpha(x^2+\dot{x}^2-1)\dot{x}+x=0$ has a pure sine-wave limit cycle which is attained rapidly for a range of values of α. In this paper, free and forced oscillation of this equation are examined experimentally and compared with those for the Van der Pol oscillator. Asymptotic solutions for small α confirm the experimental results.

1 Introduction

In 1883 Lord Rayleigh [1] noted that a sustained vibration needs a source of energy that decreases with the amplitude of the vibration, and so, as a simple approximation, he introduced into the equation $\ddot{x}+x=0$ a term $-\alpha\dot{x}$ to describe the supply of energy, and a term $+\alpha\dot{x}^3$ to describe how this supply diminishes with increasing velocity. His complete equation is therefore

$$\ddot{x}+\alpha(\dot{x}^2-1)\,\dot{x}+x=0. \tag{1}$$

If α is small this has an approximate solution

$$x=\sqrt{\frac{4}{3}}\left(\sin t+\frac{\alpha}{24}\cos 3t\right), \tag{2}$$

i.e. a fundamental sine-wave with third harmonic distortion that can be reduced by making α smaller. However, it is easily seen by the method of multiple scales that, when $\alpha \ll 1$, $1/\alpha$ is approximately the *settling time* of the oscillator, that is the time it takes to recover from a perturbation such as being switched on, or tuned. If we want the harmonic content to be 60 db less than the fundamental, α will have to be less than 1/40 and the settling time will be some 40 cycles.

Almost half a century after Rayleigh, Van der Pol [2] discussed a vacuum tube oscillator in terms of the equation

$$\ddot{x}+\alpha(x^2-1)\,\dot{x}+x=0, \tag{3}$$

* Sadly Dr Robinson died while this paper was being processed. He was a mathematically-minded scientist whose interests far transcended those of *EJAM* and he will be greatly missed.

in which Rayleigh's term $\dot{x}^3$ is replaced by $x^2\dot{x}$ and Eq. (3) is essentially the time derivative of Eq. (1). If α is again small the approximate solution is

$$x = 2\left(\sin t - \frac{\alpha}{8}\cos 3t\right), \tag{4}$$

and in this case the third harmonic component is larger than in Eq. (2) for a given α.

Since 1927, many other types of oscillator have been devised in which amplitude control is achieved by a subsidiary circuit that averages either the modulus, or the square, of the oscillation over many cycles, and uses the resulting output to control the active feedback (loop gain) that sustains the oscillation. All these oscillators lead to a similar relation between spectral purity and settling time (see Robinson [3]). Apparently, a pure sine wave can only be achieved at the price of a long recovery time lasting many cycles (see, for example, Robinson [3, 5]). The effects of this relation will be familiar to anyone who had used an audio-frequency signal generator.

In fact, the factors that influence spectral purity have not attracted much theoretical attention. Rayleigh and Van der Pol were more interested in the qualitative behaviour of self-sustained oscillators, though Van der Pol, who was particularly concerned with homodyne radio receivers and receivers with reaction, studied how a marginal oscillator would lock to a small impressed periodic signal at a frequency near resonance. Recent studies have emphasized the possible chaotic behaviour of locked oscillators.

It is unlikely that, until the last decade or so, any real oscillatory or vibrating system, whether electrical or mechanical, was at all precisely described by either Rayleigh's or Van der Pol's equation. Fortunately (see Robinson [4]), the gross features of the behaviour are not very sensitive to the equation's exact form. Eventually, however, the increasing refinement and decreasing cost of electronic components has made it feasible to construct, and investigate, oscillators which are reasonably accurately described by the two classical equations.

We now introduce a new equation obtained by combining Rayleigh's and Van der Pol's equations; it is

$$\ddot{x} + \alpha(x^2 + \dot{x}^2 - 1)\dot{x} + x = 0, \tag{5}$$

and it is immediately obvious that it has exact, pure, sinusoidal solutions, independent of α:

$$x = \sin t \quad \text{or} \quad x = \cos t. \tag{6}$$

These have no higher harmonic components. It is, therefore, a candidate for making an oscillator with a pure spectrum and a short settling time. We may note that the equation is used in Jordan & Smith [6] (Exercises Ch. 3, 4) as an example of an equation with a simple limit cycle; but Jordan & Smith [6] do not emphasise the fact that (unlike the classical Rayleigh and Van der Pol equations) the spectral purity of the solution is independent of α, and so of either the settling time or the rate at which the limit cycle is approached.

The multiple scale methods described in Jordan & Smith [6, Ch. 6] can be used to analyse the convergence of the solution to the limit cycle given by Eq. (6) when α is small. The solution is

$$x = A(T)\sin(t + \phi),$$

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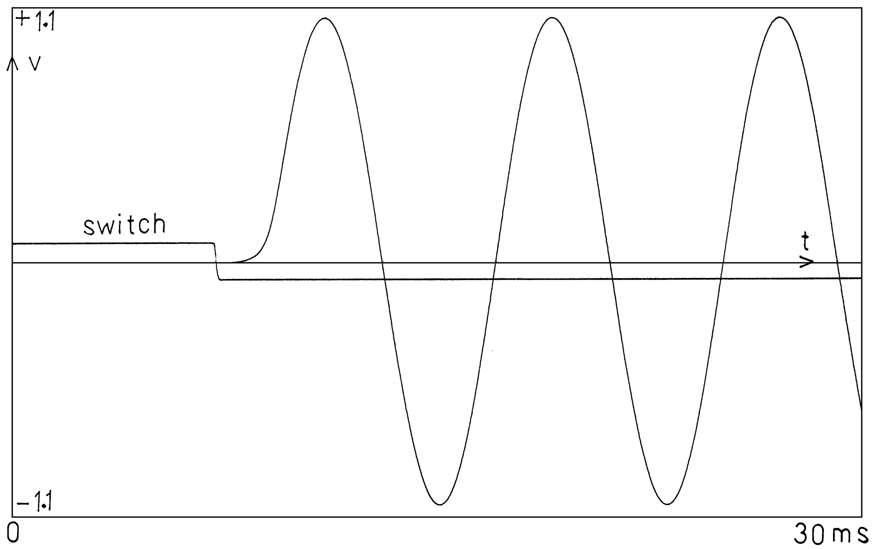

FIGURE 1. The rapid rise of the oscillation to its final form as the logic signal goes negative and the oscillator is turned on.

where $T = \alpha t$ is the slow 'settling' time-scale, $A^2 = 1/(1-K\mathrm{e}^{-T})$ and K and ϕ are constants determined by the initial conditions. This solution demonstrates that the 'settling' time is still of $O(1/\alpha)$ when α is small, and the experimental results now described confirm this, but they also reveal much more interesting behaviour when α is $O(1)$.

2 Free oscillations

Constructing an electronic oscillator obeying Eq. (5) is relatively simple, and Fig. 1 shows how the output of such an oscillator operating at 125 Hz, with $\alpha = 4.5$, behaves when it is suddenly switched on (as the displayed logic signal goes from + to −). We see that the output rises to its final amplitude within little more than $\frac{1}{4}$ of a cycle. Despite this very rapid rise, the total measured noise and harmonic content is over 70 db below the fundamental frequency output. Either a Rayleigh or a Van der Pol oscillator with α as large as 4.5 would only give relaxation oscillations.

Starting from a small value due to noise, or a transient from the switching pulse, the amplitude initially rises exponentially to 1 (corresponding to 10.5 V at the output), with a time constant $1/\alpha$. In real time this is $\tau/\alpha = 0.283$ ms, where $\tau = 1/250\pi$s $= 1.27$ ms, is the actual time constant of the electronic oscillator (see below). In Fig. 1 the time from the switching pulse to the first peak is 3.8 ms and $\exp(\alpha t/\tau) = 8.7 \times 10^5$ so that the oscillation should reach its final amplitude at the first peak to within 0.1%, if the initial value of $r_0 \sim 2.5 \times 10^{-5}$ ($\frac{1}{4}$ mV). (This is a reasonable value since the internal capacitance of the *f.e.t.* switch in the actual circuit, see below, is about 3 pF, the capacity C to ground at the node **b** to which the switch is connected is 1.33×10^5 pF, and the switching pulse is 10 volts.) The oscillator circuit is described in the appendix.

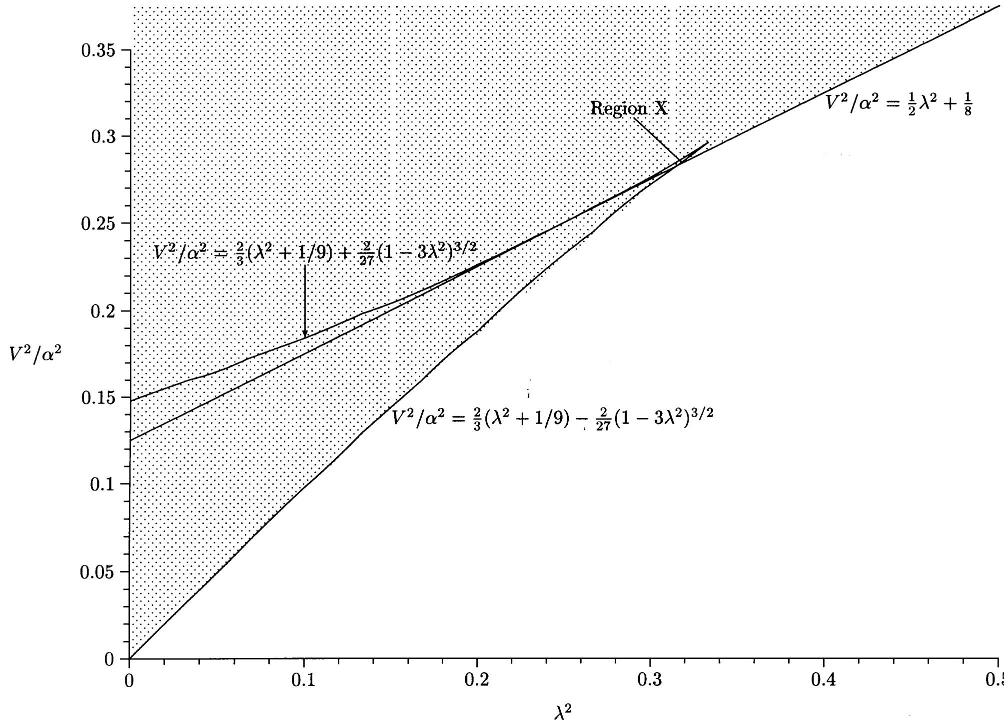

FIGURE 2. Stable equilibrium points for Eq. (9) can be found in the shaded region of the $(\lambda^2, V^2/\alpha^2)$ plane. In the small region X, there are two stable equilibrium points.

3 Forced oscillations

An oscillator, forced by a sinusoidal input $V \sin \omega t$, will be described by

$$\ddot{v} - \alpha(1 - v^2 - \dot{v}^2)\,\dot{v} + v = V \sin \omega t, \tag{7}$$

and an experimental survey indicates that the new oscillator, like the Van der Pol oscillator, will lock to a weak forcing signal near resonance. Experimentally, the width of the locking region increases more or less linearly with the normalized impressed voltage V so that

$$\Delta\omega \sim V/1.4, \tag{8}$$

where $\omega = 1 + \Delta\omega$. When $\alpha, \Delta\omega, V$ are small and of the same order, an asymptotic analysis (see Jordan & Smith [6, Ch. 7, example 21]) shows that the solution can be written approximately in the form $v = a(T)\cos t + b(T)\sin t$, where $T = \alpha t$ and

$$a' = -\frac{1}{2}\frac{V}{\alpha} - \frac{\lambda}{2}b + \frac{a}{2}(1 - a^2 - b^2),$$

$$b' = +\frac{\lambda}{2}a + \frac{b}{2}(1 - a^2 - b^2)$$

and $\lambda = (2\Delta\omega)/\alpha$ is the detuning of the system. These equations are equivalent to those

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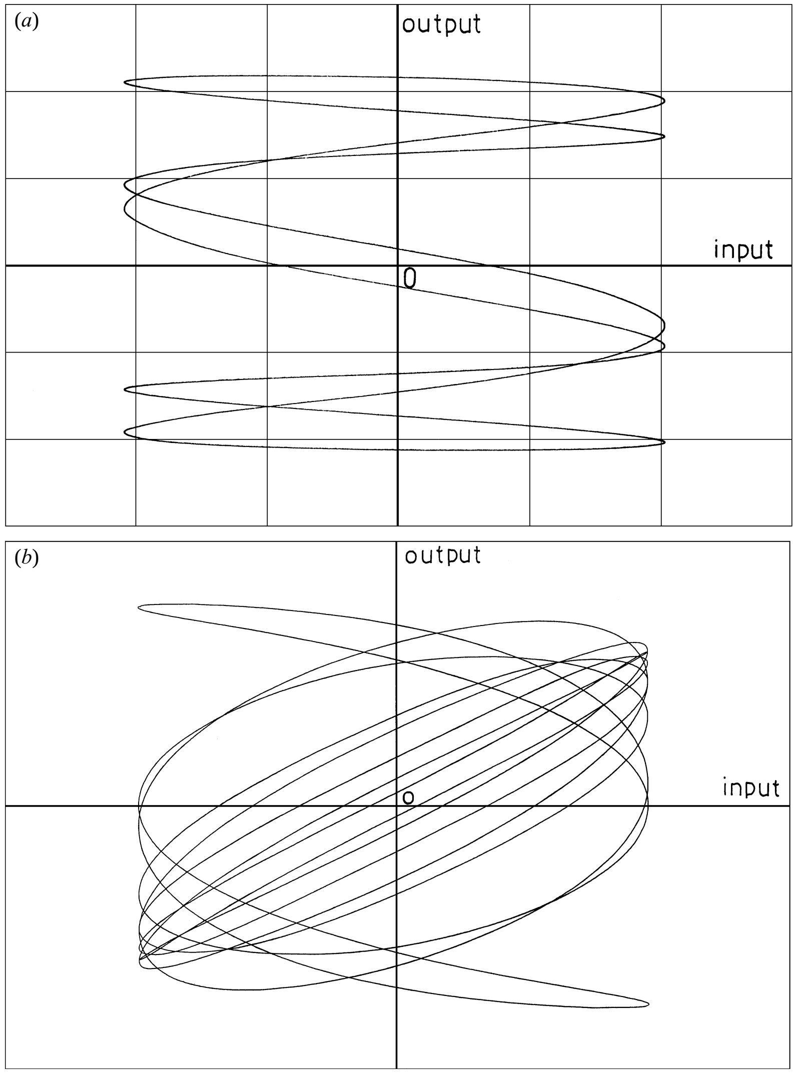

FIGURE 3. (a) A Lissajou figure for a locked subharmonic oscillation for a Van der Pol oscillator with $\alpha = 1$; (b) the Lissajou figure for a period 9 subharmonic obtained with the new oscillator (7) with $\alpha = 4.5$.

derived for the forced Van der Pol equation described in Jordan & Smith [5, Sect. 7.4]. In this case there is a stable solution (a_0, b_0) provided

$$\frac{V^2}{\alpha^2} > \min\left\{\frac{2}{27}(1+9\lambda^2-(1-3\lambda^2)^{3/2}), \frac{\lambda^2}{2}+\frac{1}{8}\right\} \tag{9}$$

and the possible scenarios are shown in Fig. 2. For a very small range of parameters there are two stable points (shown in regions in Fig. 2), but elsewhere in the region described by Eq. (9) the stable point is a global attractor. This result confirms the experimental result (8) for the width of the locking region when λ is sufficiently large. For parameter values outside the range (9) the solution locks to a limit cycle. However, except

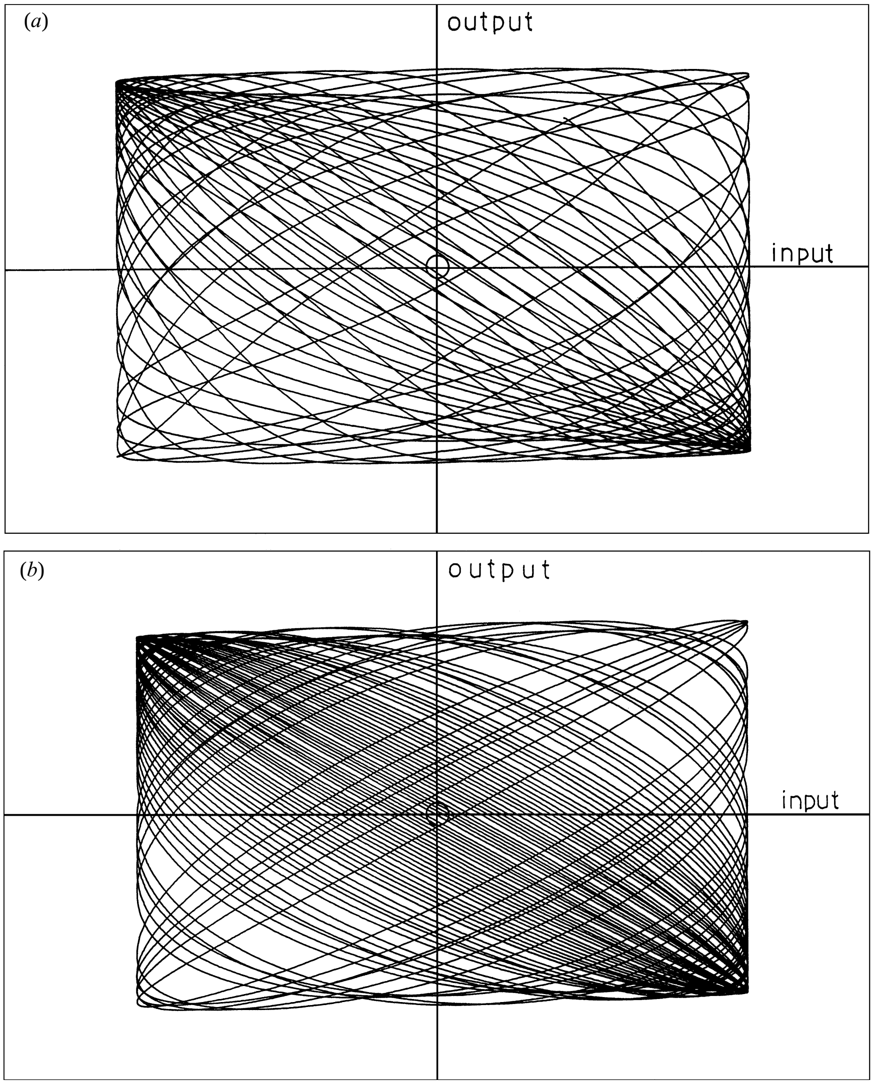

FIGURE 4. (a) An unblocked Lissajou figure for the Van der Pol oscillator with $\alpha = 1$; (b) a Lissajou figure for the new oscillator (7) under similar conditions.

when $\Delta\omega$ is very small, the balance between $\dot{v}^2$ and v^2 in Eq. (7) is now destroyed and, as a result, the locked waveform is no longer sinusoidal. Indeed, the locking behaviour and the observed waveforms are unlike those seen with a Van der Pol oscillator. This is partly because of the presence of both $\dot{v}v^2$ and $\dot{v}^3$ terms, and partly because α is much larger than is usually considered for Van der Pol oscillators.

This anomalous behaviour is illustrated in Figs 3 and 4, which show Lissajou's figures in which the oscillator output is the ordinate and the impressed input signal is the abscissa. The Lissajou figures for both a Van der Pol oscillator, and the new oscillator when they are locked with ω close to 1, are trivial ellipses and, so far as locked oscillations are concerned, that is virtually all that there is to say about the Van der Pol oscillator; however, if α is large enough, the oscillator will lock to a subharmonic signal in narrow regions near an integral multiple of the resonant frequency. Figure 3(a) shows an example, obtained when $\alpha = 1$, which is locked at $\omega/5$ with $V = 0.98$ and $\omega = 4.70$. This is in contrast to the period 9 oscillation in Fig. 3(b), obtained with the new oscillator when V is again 0.98 but ω is only 1.46. The Van der Pol oscillator, even with α as large as unity, produces no such subharmonics when ω is relatively close to unity.

The difference between the unlocked behaviours is less marked. Figure 4(a) is obtained from the Van der Pol oscillator with $\alpha = 1$, when $\omega = 1.11$ and $V = 0.147$, which is 0.76 of the voltage required to produce a lock. Figure 4(b) is from the new oscillator with $\omega = 1.10$ and $V = 0.226$, which is also 0.76 of the voltage required to lock.

4 Conclusion

Clearly, much the most significant feature of this oscillator is that it combines fast recovery, for all but small values of the nonlinearity, with a pure sine-wave output. This is particularly appealing at audio and lower frequencies where purity of the output is otherwise only achieved at the cost of a long settling time, a noticeably inconvenient feature of practical low frequency signal generators. From a more theoretical point of view it is also satisfactory that it is possible to construct an accurate experimental realization of the simple nonlinear Eq (5) for small values of α. However, the theoretical explanation for the fast settling for larger values of α is still lacking. It is hoped that the unusual behaviour that we have observed in the forced case, in particular the existence of sustained subharmonic solutions, may provide some clues for future investigations.

Acknowledgements

The idea of making an oscillator with a fast settling time but no distortion arose several years ago in the conversation with D. J. Jefferies of Surrey University, about an oscillator of a very different structure (see Robinson [5]), which gives two outputs $A\cos\omega t$ and $A\sin\omega t$. These, squared and added, give an amplitude control signal A^2 with no ripple. A numerical calculation, by J. Dean in Jefferies' group, and a practical circuit built by Jefferies, using several 4-quadrant analogue multiplier chips, confirmed this, though it was difficult to maintain adjustment.

Appendix: The oscillator circuit

We now describe the oscillator circuit†. Because it is easier to generate odd functions, rather than even functions and products, we rewrite Eq. (5) in the form:

$$\tau^2 \ddot{v} + \alpha\tau\left(\tau^2 \dot{v}^3 + \frac{1}{3}\frac{dv^3}{dt} - \frac{dv}{dt}\right) + v = 0, \tag{10}$$

where we have put $x = v$ since v is a voltage, and scaled t with τ.

† **Notes on the components.** The diodes are type 1N916 matched to within 2 mV at 100 μA. Most diodes satisfy this, if care is taken to keep them all at the same temperature; but in every batch there are a few rogues that have to be rejected. The op-amps are the ultra-linear type OP77 and, compared with the cheaper type TL071, they lead to 4 db reduction in even harmonic distortion. The diode resistor chains are mounted as compact plug-in units.

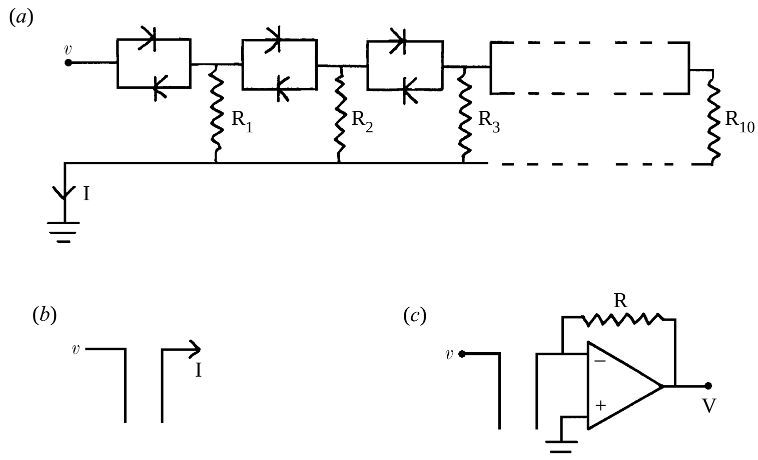

FIGURE 5. (a) The diode resistor chain used to implement the relation $I = \gamma v^3$; (b) the symbol used to represent the chain; (c) the implementation of the relation $V = -kv^3 = -\gamma R v^3$ using the chain and an op-amp.

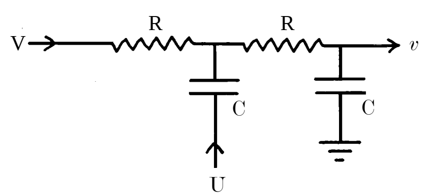

FIGURE 6. The core of the oscillator circuit.

A current I, which is a cubic function $I = \gamma v^3$ of a voltage v between -5.5 V and $+5.5$ V, is generated by the chain of diodes and resistors shown in Fig. 5(a). The resistance values are, starting with R_1, 1 M, 390k, 100k, 56k, 47k, 33k, 27k, 18k, 22k and 15 k. In subsequent diagrams, this chain is denoted by the symbol shown in Fig. 5(b). Figure 5(c) shows it combined with an inverting op-amp (operational amplifier) to give an output voltage $V = -kv^3$, where $k = \gamma R$. Measurements show that, with $R = 20\ k\Omega$, k does not depart from 0.044 by more than 1 %, except for $|v| < 1$ volt where, at $v = 0.5$ volt, it is 40 % larger. This is unimportant, for if either v or $\dot{v}$ is small then $\dot{v}$ or v is near unity, and so an error of 40 % in kv^3, when v corresponds to $x = 0.1$, contributes an error of only 0.002 to $v^2 + \dot{v}^2$.

The core of the circuit is shown in Fig. 6, where the relation between the input voltage V and U and the output voltage v is

$$V = v + 3RC\dot{v} - RC\dot{U} + R^2 C^2 \ddot{v}. \tag{11}$$

With $\tau = RC$, we have only to make $V = -\tau^3 \alpha \dot{v}^3$ and $U = (3+\alpha)v - \frac{1}{3}\alpha v^3$ to regain Eq. (10). The voltage between the nodes a and b is $\tau\dot{v}$ and so we can obtain V and U using two cubing circuits.

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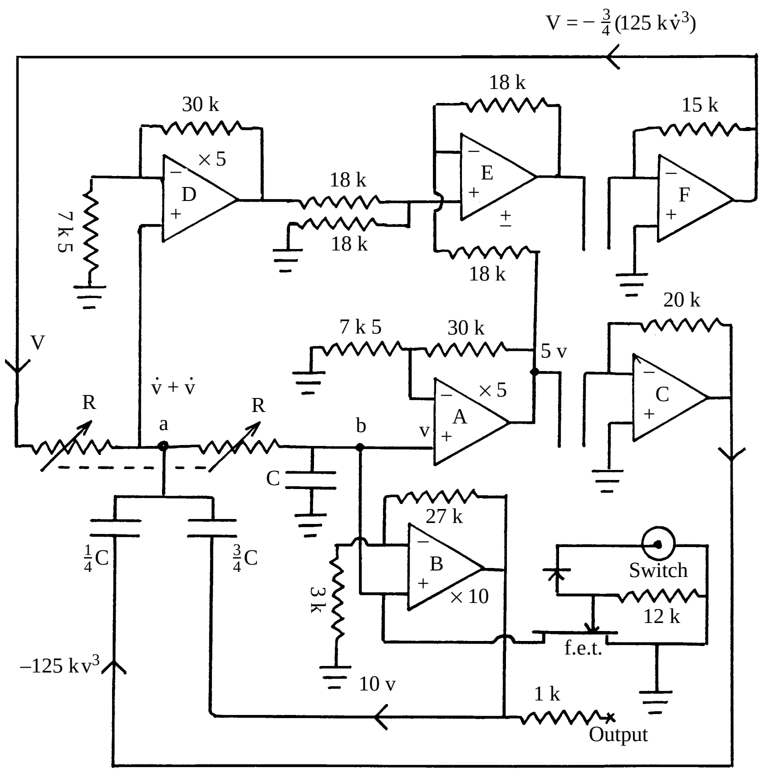

FIGURE 7. The complete oscillator circuit omitting only the ± 15 V supply rails.

Figure 7 shows that complete circuit using six op-amps. For convenience, the linear part of U is injected separately from the linear part, by splitting the capacitor into $C/4$ and $3C/4$. The resistances and capacitances are selected to be equal to their nominal values to within 0.1 %, C is 0.1333 μF. Tuning is accomplished by the accurate, 2-gang, 20 kΩ potentiometer. Henceforth, we adopt a new unit of time RC and eliminate τ.

The op-amps D and A, each of gain $\times 5$, amplify the voltages at the nodes a and b, and then E forms the difference of these voltages (i.e. $5\dot{v}$). This is converted to $-\frac{3}{4}k(5\dot{v})^3$ by the diode chain and F, whose output corresponds to V in Fig. 6. Op-amp C gives an output $-k(5v)^3$ and, because of the split in the capacitor, its contribution to U is $-\frac{1}{4}k(5v)^3$. Op-amp B has a gain $\times 10$ and (because of the split capacitor) contributes 7.5v to U. Thus $\alpha = 7.5 - 3 = 4.5$. It is easy to check that the amplitude of the sinusoidal limit cycle will be $|v| \sim 1.06$ V but, for convenience, the output, about 10.6 V, is taken from the output of B. The switch, which (when off) grounds the input of A, is provided by a field-effect transistor. It is turned on by -10 V pulses.

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