RECURRENT TRANSFORMATION GROUPS

R. A. CHRISTIANSEN

1. Introduction. Let (X, T, π) denote a flow, where X is a compact topological space metrizable by d, and T is a closed non-trivial subgroup of the reals under addition. T is *recurrent* if and only if for each $\epsilon > 0$ and $s > 0$, there exists $t > s$ such that $x \in X$ implies $d(x, xt) < \epsilon$. If T is almost-periodic, then T is both recurrent and distal. In §§ 4 and 5, it is shown that, under more stringent hypotheses, the recurrence of T is neither a necessary nor a sufficient condition for T to be distal. Let S be a closed non-trivial subgroup of T. It is shown in § 3 that T is recurrent if and only if S is recurrent. From this result, we obtain a solution to a problem posed by Nemyckiĭ (**16**, p. 492, Problem 6). In § 3, topological conditions necessary and sufficient for T to be recurrent are also given, thus solving another problem raised by Nemyckiĭ (**16**, p. 492, Problem 5).

2. Preliminaries. The basic references are (**11**) and (**14**). Right-hand $(xf, x\alpha)$ notation for functions and relations will generally be used. R, Z, N, and P will denote the reals, integers, non-negative integers, and positive integers, each with the usual normed structure. We begin with a summary of some known, but somewhat inaccessible, material.

2.1. *Bebutoff spaces.*

LEMMA 1. *Let Y be a topological space and $(X, \mathscr{U})$ a uniform space. Suppose that E is a set of continuous functions from Y to X.*

(a) *The topology $\mathscr{T}_c$ on E of uniform convergence on compacta is the compact open topology.*

(b) *Suppose that Y is locally compact and either Hausdorff or regular. If $y_n \to y$ in Y and $f_n \to f$ in $\mathscr{T}_c$, then $y_n f_n \to yf$ in X.*

(c) *Suppose that f and the net f_n are such that if f_{n_i} is a subnet of f_n and $y_{n_i} \to y$, then $y_{n_i} f_{n_i} \to yf$. Then $f_n \to f$ in $\mathscr{T}_c$.*

Proof. (a) is (**14**, p. 230, Theorem 11). (b) then follows from (**14**, p. 223, Theorem 5). We prove (c). Suppose that for some f_n, f satisfying the premises of (c), $f_n \not\to f$ in $\mathscr{T}_c$. Passing to a subnet, there exist $\alpha \in \mathscr{U}$ and a compact subset C of Y such that

$$(f, f_n) \notin \alpha_C = \{(g, h) \in E \times E;\ (cg, ch) \in \alpha \text{ for all } c \in C\}.$$

Received July 14, 1967. This paper is based on the author's doctoral dissertation, supported by the National Science Foundation and written under Professor James F. Jakobsen at the University of Iowa.

Passing to a subnet, there exist $c_n \to c \in C$ such that $(c_n f, c_n f_n) \notin \alpha$. Since f is continuous, $c_n f \to cf$. $c_n f_n \to cf$ by hypothesis. Hence, $(c_n f, c_n f_n) \in \alpha$ eventually, a contradiction.

Suppose that a locally compact topological group T and a uniform space $(X, \mathscr{U})$ are given. Let F be the set of all continuous functions from T to X. Let $\mathscr{U}_c$ and $\mathscr{T}_c$ be the uniformity and the topology on F of uniform convergence on compacta. Define the shift operator π: $F \times T \to X^T$ by $(f, t)\pi = g$, where $sg = (ts)f$. Briefly, $sft = tsf$. Then $ft \in F$, $fe = f$, and $f(rt) = (fr)t$. By Lemma 1, π is continuous. Hence, $((F, \mathscr{U}_c), T, \pi)$ is a transformation group, called the *Bebutoff space* associated with T and $(X, \mathscr{U})$.

Suppose now that T is a locally compact Hausdorff topological group satisfying the second axiom of countability and $(X, \mathscr{U})$ is a separable, metrizable, and complete uniform space. Then $(F, \mathscr{U}_c)$ is separable metrizable (**3**, p. 41, *Corollaire*), and complete (**14**, p. 231, §§ 12, 13). Since T is locally compact, separable, and metrizable (**14**, p. 186, § 13), there exists (**13**, p. 76, Theorem 2–61) a metric σ for T such that a closed subset of T is compact if and only if it is σ-bounded. For each $s > 0$, let $C_s = \{t \in T; \sigma(t, e) \leqq s\}$. C_s is compact. Let d be a metric for $(X, \mathscr{U})$. If $f, g \in F$ and $s > 0$, define $d(f, g; s) = \sup\{d(tf, tg); t \in C_s\}$. Define ρ: $F \times F \to [0, \infty)$ by $\rho(f, g) = \sup_{s>0} \min\{d(f, g; s), 1/s\}$. Straightforward arguments show that:

(a) ρ is a metric, called a *Bebutoff metric* for F;

(b) $d(ef, eg) \leqq \rho(f, g)$;

(c) If $d(f, g; s) \leqq 1/s$, then $\rho(f, g) \leqq 1/s$;

(d) If $\rho(f, g) < 1/s$, then $d(f, g; s) < 1/s$;

(e) If $d(f, g; s) = 1/s$, then $\rho(f, g) = 1/s$;

(f) $\mathscr{U}_\rho = \mathscr{U}_c$.

If T is a closed subgroup of the reals under addition and σ is the usual metric for T, then σ will be called the *usual Bebutoff metric*.

2.2. *Basic properties of transformation groups*. Suppose that a transformation group (X, T, π) is given. If $t \in T$, then π^t: $X \to X$, defined by $x\pi^t = xt$, is a self-homeomorphism of X. $t \to \pi^t$ defines a homomorphism of T onto the abstract group $G = \{\pi^t; t \in T\}$ of translations of X. T is *effective* if and only if $t \to \pi^t$ is one-to-one. A subset Y of X is *invariant* if and only if $YT \subseteq Y$. If Y is invariant, then so is Y^-. If S is a subgroup of T such that $YS \subseteq Y$, then $(Y, S, \pi | Y \times S)$ is itself a transformation group, denoted by (Y, S, π). A subset Y of X is *minimal* if and only if Y is closed, non-empty, and invariant, but no proper subset of Y is closed, non-empty, and invariant. Y is minimal if and only if $Y = xT^-$ for each $x \in Y$.

Let A be a subset of T. A is *replete* if and only if for each compact subset C of T there exists $t \in T$ such that $tC \subseteq A$. If T is a non-trivial subgroup of the reals under addition, then a semigroup A in T is replete if and only if A contains a ray in T. A is *extensive* if and only if A intersects each replete semigroup of T. A is *left syndetic* if and only if there exists some compact subset

C of T such that $AC = T$. If A is left syndetic, then A is extensive. x is an *almost-periodic point* of X if and only if for each neighbourhood U of x there exists a left syndetic subset A of T such that $xA \subseteq U$. Let X be regular and xT^- be compact. Then x is an almost-periodic point if and only if xT^- is minimal (**11**, p. 31, §§ 4.05, 4.07).

T is *periodic* if and only if there exists a left syndetic subgroup S of T such that $x \in X$ implies $xS = \{x\}$. Suppose that X is uniformizable by $\mathscr{U}$. T is *almost-periodic* if and only if for each $\alpha \in \mathscr{U}$ there exists a left syndetic subset A of T such that $x \in X$ implies $xA \subseteq x\alpha$. T is *recurrent* if and only if for each $\alpha \in \mathscr{U}$ there exists an extensive subset A of T such that $x \in X$ implies $xA \subseteq x\alpha$ (this generalizes the definition given in the introduction). T is periodic implies T is almost-periodic, which implies T is recurrent. T is *distal* if and only if whenever $x \neq y \in X$, there exists $\alpha \in \mathscr{U}$ such that $t \in T$ implies $yt \notin xt\alpha$. Suppose that X is compact and Hausdorff. If T is almost-periodic, then T is distal. If T is distal, then (**4**, p. 402, Theorem 1) T is pointwise almost-periodic.

3. Positive results on recurrent transformation groups.

3.1. *The basic results.* Remarks 1 and 2 are analogues of corresponding results for almost-periodic transformation groups.

REMARK 1. *Let (X, T) be a transformation group uniformizable by $\mathscr{U}$. Let Y be an invariant subset of X. If T is recurrent on Y, then T is recurrent on Y^-.*

Proof. Let $\alpha \in \mathscr{U}$. There exists a symmetric $\beta \in \mathscr{U}$ such that $\beta^3 \subseteq \alpha$. There exists an extensive subset A of T such that $y \in Y$ and $a \in A$ imply $(y, ya) \in \beta$. Let $x \in Y^-$ and $a \in A$. By the continuity of π^a, there exists $\gamma \in \mathscr{U}$ such that $\gamma \subseteq \beta$ and $ya \in xa\beta$ whenever $y \in x\gamma$. There exists $y \in x\gamma \cap Y$. Then $(x, xa) = (x, y)(y, ya)(ya, xa) \in \beta^3 \subseteq \alpha$.

Suppose that $T = Z$ and $X = \{0, 1, \ldots, k\}$ carries the usual discrete metric. (F, Z), the Bebutoff space associated with Z and X, is called a *symbol space*. Since Z is discrete, $F = X^Z$ and $\mathscr{T}_c$ is the product topology. Hence, $\mathscr{T}_c$ is compact.

REMARK 2. *Let (X, Z) be a symbol space, with Y an invariant subset of X. If Z is recurrent on Y, then Z is periodic on Y.*

Proof. Corresponding to $\epsilon = 1$, there exists $k \in P$ such that $f \in Y$ implies $\rho(f, fk) < 1$. Let $f \in Y$ and $m \in Z$. $\rho(fm, (fm)k) < 1$ implies $|mf - mfk| = |0fm - 0f(m + k)| < 1$, which implies $mf = mfk$. Thus $f = fk$.

3.2. *The inheritance problem.*

THEOREM 1. *Suppose that a transformation group (X, T) is given, where X is compact and uniformizable by $\mathscr{U}$ and T is an abelian group such that for some compact neighbourhood C of e, $T = \bigcup\{C^n; n \in P\}$. Let S be a closed syndetic subgroup of T. Then T is recurrent if and only if S is recurrent.*

https://doi.org/10.4153/CJM-1969-064-1 Published online by Cambridge University Press

Proof. The proof is suggested by the proof of the corresponding problem for regional recurrence (**11**, p. 65, § 7.12; p. 67, § 7.21). If S is recurrent, we apply (**11**, p. 59, § 6.19). Suppose that T is recurrent. Let $\alpha \in \mathscr{U}$. By the compactness of X and the local compactness of T, there exist $\alpha_{-1} \in \mathscr{U}$ and a compact symmetric neighbourhood V of e such that if $x \in X$, then $x\alpha_{-1}V \subseteq x\alpha$. Let $A = \{a \in S;\ (x, xa) \in \alpha$ for all $x \in X\}$. We show that A is S-extensive by letting B be a replete semigroup of S and proving that A intersects B. By (**11**, p. 60, § 6.20), there exists a replete semigroup D of T such that $DV \cap S \subseteq B$. There exist $\alpha_0 \in \mathscr{U}$ and $d_0 \in D$ such that $\alpha_0^2 \subseteq \alpha_{-1}$ and $x \in X$ implies $(x, xd_0) \in \alpha_0$. Suppose that for $j = 0, 1, \ldots, i - 1$ we have defined $\alpha_j \in \mathscr{U}$ and $d_j \in D$ such that $\alpha_j^2 \subseteq \alpha_{j-1}$ and $x \in X$ implies $(x, xd_j) \in \alpha_j$. There exist $\alpha_i \in \mathscr{U}$ and $d_i \in D$ such that $\alpha_i^2 \subseteq \alpha_{i-1}$ and $x \in X$ implies $(x, xd_i) \in \alpha_i$. Hence, we obtain sequences $\{\alpha_i\}$ and $\{d_i\}$.

There exists a compact subset C of T such that $SC = T$. Let $d_i = s_i c_i$. By (**11**, p. 60, § 6.22), there exist $n \in N$ and $i_0 < i_1 < \ldots < i_n \in N$ such that $c_{i_0}c_{i_1}\ldots c_{i_n} = s'v \in SV$. By passing to subsequences, we may suppose that $i_j = j$, $0 \leqq j \leqq n$. Let $s = s_0 s_1 \ldots s_n$. Since T is abelian, $d_0 d_1 \ldots d_n v^{-1} = ss' \in DV \cap S \subseteq B$. Let $q = ss'$ and $x \in X$. It suffices to show that $(x, xq) \in \alpha$. Now $(x, xd_n) \in \alpha_n \subseteq \alpha_n^2 \subseteq \alpha_{n-1}$ and $(xd_n, xd_n d_{n-1}) \in \alpha_{n-1}$. Hence $(x, xd_n d_{n-1}) \in \alpha_{n-1}^2 \subseteq \alpha_{n-2}$. Continuing as indicated, we obtain $(x, xd_n d_{n-1}\ldots d_0) \in \alpha_{-1}$. Then $(x, xq) = (x, xd_n d_{n-1}\ldots d_0 v^{-1}) \in \alpha$.

LEMMA 2. *Let T be a locally compact, abelian, Hausdorff topological group and X a complete uniform space. Let F_c be the Bebutoff space associated with T and X. Then $f \in F_c$ is uniformly continuous and totally bounded if and only if fT^- is compact.*

Proof. See (**11**, p. 97, § 11.32).

By *classical Bebutoff space* we mean the Bebutoff space associated with R and R. If $f\colon Z \to R$, then f^*, the *linear extension* of f, is the element of classical Bebutoff space such that if $k \in Z$, then $kf^* = kf$ and $f^*|[k, k+1]$ is a line segment. The restriction to any fixed symbol space of the map $f \to f^*$ is a unimorphism into classical Bebutoff space such that $(fk)^* = f^*k$.

Let us for the moment call a transformation group (X, T) pseudo-minimal if and only if T is recurrent on X and there exists $x \in X$ such that $xT^- = X$. We are now able to solve a problem raised by Nemyckiĭ (**16**, p. 492, Problem 6): Find an example (in classical Bebutoff space) of a compact minimal transformation group which is not pseudo-minimal. Let f be the Morse function (**15**) in the symbol space on $\{0, 1\}$. Let f^* be the linear extension of f. Then $X = (f^*R)^-$ is compact by Lemma 2. Since f is an almost-periodic point of symbol space, f^* is an almost-periodic point of classical Bebutoff space. Hence, X is minimal. Now f is not periodic; hence, by Remark 2, Z is not recurrent on fZ. Thus, Z is not recurrent on f^*Z, whence Z is not recurrent on X. Finally, by Theorem 1, R is not recurrent on X.

3.3. *Topological characterizations of recurrence.* In (**16**, p. 492, Problem 5), Nemyckiĭ raised the problem of finding topological characterizations of recurrent flows. This problem is solved in Theorem 2 by using topologies on the group G of translations π^t of X. As a by-product, Theorem 3 contains topological characterizations of periodic flows. Other characterizations of recurrent flows may be found in (**8**).

LEMMA 3. *Let S be an abstract group and $\mathscr{T}$ a topology for S such that $(S, \mathscr{T})$ is a topological group and for some compact neighbourhood C of e, we have $S = \bigcup\{C^n; n \in P\}$. Let $(S, \mathscr{T}^*)$ be a locally compact Hausdorff topological group. If $\mathscr{T}^* \subseteq \mathscr{T}$, then $\mathscr{T}^* = \mathscr{T}$.*

Proof. Apply (**12**, p. 42, § 5.29) to the identity function from $(S, \mathscr{T})$ onto $(S, \mathscr{T}^*)$.

For the remainder of this section, suppose given a transformation group (X, T, π) with X uniformizable by $\mathscr{U}$. Let $\mathscr{T}$ be the topology for T, and S the underlying abstract group of T. Let $H = \{t \in T; xt = x \text{ for all } x \in X\}$ be the period of T (an invariant subgroup) and $p: S \to G$ defined by $t \to \pi^t$. p is an abstract group epimorphism with kernel H. Let $\mathscr{T}'$ be the quotient topology for G determined by $\mathscr{T}$ and p. $T' = (G, \mathscr{T}')$ is a topological group, the topological isomorph of T/H. Let $\mathscr{T}_c$ and $\mathscr{T}_X$ denote the topologies on G of uniform convergence on compacta and of uniform convergence. Let $\phi: X \times G \to X$ be defined by $(x, \pi^t)\phi = x\pi^t = xt$. If $\mathscr{T}^*$ is a topology on G, then $(X, (G, \mathscr{T}^*), \phi)$ is a transformation group if and only if $(G, \mathscr{T}^*)$ is a topological group and ϕ is continuous.

LEMMA 4. $\mathscr{T}' \supseteq \mathscr{T}_c$.

Proof. We first show that (X, T', ϕ) is an effective transformation group by proving the continuity of ϕ. Let $f_n \to \pi^t$ in T' and $x_n \to x$ in X. Let U be a neighbourhood of xt. There exist open neighbourhoods V of x and W of t such that $VW \subseteq U$. Eventually, $x_n \in V$. Since $WH \in \mathscr{T}$ and $Wpp^{-1} = WH$, we have $Wp \in \mathscr{T}'$. Eventually, f_n is in Wp; thus, there exists $t_n \in W$ such that $f_n = \pi^{t_n}$. Thus, $x_n f_n = x_n t_n \in U$ eventually, and ϕ is continuous. Now let $f_n \to f$ in $\mathscr{T}'$. If $x_{ni} \to x$, then $x_{ni} f_{ni} \to xf$. Hence $f_n \to f$ in $\mathscr{T}_c$ by Lemma 1(c).

THEOREM 2. *Let (X, T) be an effective transformation group such that X is a Hausdorff space uniformizable by $\mathscr{U}$ and T is a closed non-trivial subgroup of R. Consider the following statements:*

(a) *T is recurrent;*
(b) *$\mathscr{T}'$ is not a subcollection of $\mathscr{T}_X$;*
(c) *$\mathscr{T}_c$ is a proper subcollection of $\mathscr{T}'$;*
(d) *$\mathscr{T}_c$ is not locally compact.*

Then (a) *is equivalent to* (b), (b) *implies* (c), *and* (c) *is equivalent to* (d). *If X is compact, then* (a), (b), (c), *and* (d) *are pairwise equivalent.*

Proof. Let C be a compact symmetric non-trivial neighbourhood of 0. Assume (a). We prove (b). For each $\alpha \in \mathscr{U}$ and $n \in P$, there exists $a(\alpha, n) \in T \sim nC$ such that $x \in X$ implies $(x, xa(\alpha, n)) \in \alpha$. Consider the net $\pi^{a(\alpha,n)}$ in G with domain the directed set $\mathscr{U} \times P$. Given $\beta \in \mathscr{U}$, let $\alpha_0 = \beta$ and $n_0 = 1$. Suppose that $\alpha \subseteq \alpha_0$ and $n > n_0$. If $x \in X$, then $(x, xa(\alpha, n)) \in \alpha \subseteq \beta$. Thus, $\pi^{a(\alpha,n)} \in \pi^0\beta_X$, and $\pi^{a(\alpha,n)} \to \pi^0$ in $\mathscr{T}_X$. Since T is effective and $a(\alpha, n) \not\to 0$ in $\mathscr{T}$, we conclude that $\pi^{a(\alpha,n)} \not\to \pi^0$ in $\mathscr{T}'$. Thus, $\mathscr{T}'$ is not a subcollection of $\mathscr{T}_X$.

Assume (b). We prove (a). There exist π^t and a net $\pi^{t_n} \to \pi^t$ in $\mathscr{T}_X$ such that $\pi^{t_n} \not\to \pi^t$ in $\mathscr{T}'$. Suppose that there exists i such that t_n is eventually in iC. Since $t_n \not\to t$, there exist $s \neq t$ and a subnet t_{n_k} such that $t_{n_k} \to s$. For each $x \in X$, $xt_{n_k} \to xs$ and $xt_{n_k} \to xt$. Since X is Hausdorff, $xs = xt$. Since T is effective, $s = t$, a contradiction. For each symmetric $\alpha \in \mathscr{U}$, there exists n_0 such that $n > n_0$ and $x \in X$ imply that $(xt, xt_n) \in \alpha$. Let

$$E = \{\pm(t - t_n)\,;\, n > n_0\}.$$

Thus $n > n_0$, $s \in E$, and $x \in X$ imply $(x, xs) \in \alpha$. If there exists j such that $t_n - t \in jC$ for each $n > n_0$, then choose an i such that $t + jC \subseteq iC$ to deduce a contradiction. Hence E is extensive.

That (b) implies (c) follows from Lemma 4 and $\mathscr{T}_c \subseteq \mathscr{T}_X$. That (d) implies (c) follows from Lemma 4 and the local compactness of $\mathscr{T}'$. If X is compact, then (c) implies (b). Assume (c). We prove (d). Suppose that $\mathscr{T}_c$ is locally compact. Then $(G, \mathscr{T}_c)$ is a locally compact Hausdorff topological space with an abelian abstract group structure such that if $f_n \to f$ in $\mathscr{T}_c$, then $gf_n \to gf$ in $\mathscr{T}_c$. Hence (**5**, Theorem 2), $(G, \mathscr{T}_c)$ is a topological group. Applying Lemmas 3 and 4, $\mathscr{T}' = \mathscr{T}_c$, a contradiction.

THEOREM 3. *Let (X, T) be a transformation group, where X is a Hausdorff space uniformizable by $\mathscr{U}$, and T is a closed non-trivial subgroup of R. The following are then equivalent:*

(a) *T is not effective;*

(b) *T is periodic;*

(c) *$\mathscr{T}'$ is compact;*

(d) *$\mathscr{T}_c$ is compact.*

Proof. That (a) implies (b) is clear. Assuming (b), we will prove (c). There exists a syndetic subset A of T such that $x \in X$ implies $xA = \{x\}$. There exists a compact subset C of T such that $A + C = T$. p is a continuous map from $(C, \mathscr{T}|C)$ onto $(G, \mathscr{T}')$. Hence $\mathscr{T}'$ is compact. That (c) implies (d) follows from Lemma 4. Assuming (d), we will prove (a). $(G. \mathscr{T}_c)$ is a locally compact Hausdorff topological space with an abelian abstract group structure such that if $f_n \to f$ in $\mathscr{T}_c$, then $gf_n \to gf$ in $\mathscr{T}_c$. Applying (**5**, Theorem 2) again, $(G, \mathscr{T}_c)$ is a topological group. Applying Lemmas 3 and 4, $\mathscr{T}' = \mathscr{T}_c$. Since $\mathscr{T}$ is not compact, T cannot be effective.

Let us assume that (X, T) is a transformation group such that X is a compact Hausdorff space and T is a closed non-trivial subgroup of R. Then X

is uniformizable (**14**, p. 198, § 30). Using Theorems 2 and 3, we see that T is recurrent if and only if $\mathscr{T}_c$ is either compact or is not locally compact.

4. T is recurrent does not imply T is distal.

4.1. *Synopsis.* I give an example of a compact minimal subflow X of classical Bebutoff space such that R is recurrent on X but R is not distal on X. The quite lengthy and delicate construction and analysis of X is outlined after some preliminary definitions.

Given a transformation group (X, T), with $x \in X$. x is a *regularly almost-periodic point* if and only if, for any neighbourhood U of x, there exists a left syndetic invariant subgroup A of T such that $xA \subseteq U$. x is an *isochronous point* if and only if, for any neighbourhood U of x, there exist a left syndetic invariant subgroup A of T and an $s \in T$ such that $xsA \subseteq U$. x is a *locally almost-periodic point* if and only if, for any neighbourhood U of x, there exist a neighbourhood V of x and a left syndetic subset A of T such that $VA \subseteq U$. If X is uniformizable, then T is *uniformly equicontinuous* if and only if the group G of translations π^t of X is a uniformly equicontinuous family of functions.

We now outline the construction of X, a compact minimal subset of classical Bebutoff space on which R is pointwise isochronous, pointwise locally almost-periodic, recurrent, but not distal. We will construct $f: Z \to [0, 1]$ such that in the Bebutoff space associated with Z and R, Z is recurrent on fZ, Z is not uniformly equicontinuous on fZ, and f is an isochronous point.

Then let $f^*: R \to [0, 1]$ be the linear extension of f in classical Bebutoff space. f^* is uniformly continuous and bounded; hence, by Lemma 2, $X = (f^*R)^-$ is compact. R is recurrent on f^*R; hence, by Remark 1, R is recurrent on X. R is not uniformly equicontinuous on X, or equivalently, R is not almost-periodic on X (**11**, p. 37, §§ 4.35, 4.38). f^* is an isochronous, hence an almost-periodic point of X, and therefore X is minimal. By (**11**, p. 54, § 5.24), R is pointwise locally almost-periodic. By (**11**, p. 53, §§ 5.20, 5.23), R is pointwise isochronous. By (**9**, p. 710, Theorem 1), R is not distal. Note that by (**11**, p. 53, § 5.23), $X = (gR)^-$ for some regularly almost-periodic point g.

4.2. *Construction preliminaries.* If X and Y are sets, let

$$X \sim Y = \{x; x \in X, x \notin Y\}.$$

A *block* is a subset of R^2 which is a function with domain of $[i, j] \cap Z$ for some $i \leqq j$. If $m \in N$, an *m-block* is a block B containing 3^m elements. With each m-block B we have associated a *centre* c, namely the centre of domain B. A *standard m-block* is an m-block with centre divisible by 3^m. Given $m \in N$ and m-blocks B, B', there exists $j \in Z$ such that domain $B =$ domain $[B' + (j, 0)]$. Let $d(B, B') = \max\{|iB - i[B' + (j, 0)]|; i \in \text{domain } B\}$. d is a pseudo-metric on the collection of all m-blocks. Let $f: Z \to R$ and $m \in N$. The *m-partition* of f is the partition of f into standard m-blocks. That is, the

m-partition of f is $\{B_{m,r}; r \in Z\}$, where

$$B_{m,r} = f|[r3^m - (3^m - 1)/2, r3^m + (3^m - 1)/2]$$

is a standard m-block with centre at $r3^m$. Then symbolically:

$$f = \ldots B_{m,-2}B_{m,-1}B_mB_{m,1}B_{m,2} \ldots,$$

where $B_{m,0} = B_m$. Likewise, we write

$$Z = \ldots A_{m,-2}A_{m,-1}A_mA_{m,1}A_{m,2} \ldots,$$

where $A_{m,0} = A_m$. We adopt in the following the *convention* that for each $k \in N$: $n = n(k) = 2^k$, $m = m(k) = 2^k - 1$, $k' = k + 1$, $n' = n(k')$, $m' = m(k')$, $n_i = n(k_i)$, $m_i = m(k_i)$. Let $k \in N$. Define ${}_kA_m = \{0\}$. Suppose that $0 \leqq k_0 < k$, and ${}_{k_0'}A_m, \ldots, {}_kA_m$ are defined. Define

$${}_{k_0}A_m = \{j3^{m_0}; j \in Z\} \cap A_m \sim \bigcup\{{}_pA_m; k_0' \leqq p \leqq k\}.$$

Since $m(0) = 0$, $\{{}_pA_m; 0 \leqq p \leqq k\}$ is a partition of A_m. For each $r \in Z$ and $0 \leqq p \leqq k$, let ${}_pA_{m,r} = {}_pA_m + (r3^k, 0)$. Then $\{{}_pA_{m,r}; 0 \leqq p \leqq k\}$ is a partition of $A_{m,r}$. If $B_{m,r}$ is a standard m-block with centre at $r3^m$, then $\{{}_pA_{m,r}; 0 \leqq p \leqq k\}$ induces, via the map $i \rightarrow (i, y)$, a partition $\{{}_pB_{m,r}; 0 \leqq p \leqq k\}$, called the *basic partition of the standard m-block* $B_{m,r}$. ${}_pB_{m,r}$ is *symmetric* if and only if $(i, y) \in {}_pB_{m,r}$ whenever $(2 \cdot r3^m - i, y) \in {}_pB_{m,r}$. ${}_pA_{m,r}$ is symmetric. For each $k \in N$ and $r \in Z$, we define functions b: $A_{m,r} \rightarrow A_{m,r}$ and a: $A_{m,r} \rightarrow N$, called the *base* and *accessibility functions*, respectively. Define $r3^mb = r3^m$, $r3^ma = 0$. Suppose that $0 \leqq k_0 < k$ and b and a are defined on $\bigcup\{{}_pA_{m,r}; k_0' \leqq p \leqq k\}$. For each $i \in {}_{k_0}A_{m,r}$, there exists a unique standard m_0'-block $A_{m_0',s}$ such that $i \in A_{m_0',s}$. There exists a unique $j \in Z$ such that $i + j3^{m_0} = s3^{m_0'}$. Define $ib = s3^{m_0'}$ and $ia = |j|$. $b = b_{m,r}$ and $a = a_{m,r}$ are now defined as desired.

LEMMA 5. *If $k \in N$, and $w(k)$ is the number of standard m-blocks in $N \cap A_{m'} \sim A_m$, then:*

(a) $w(k) = (3^n - 1)/2$;

(b) *If $k \geqq 3$, then $w(k) > 2k2^n$.*

Proof. In terms of standard m-blocks,

$$A_{m'} = A_{m,-w(k)} \ldots A_{m,-4}A_{m,-3}A_{m,-2}A_{m,-1}A_mA_{m,1}A_{m,2}A_{m,3}A_{m,4} \ldots A_{m,w(k)}.$$

$\{A_{i+1} \sim A_i; m \leqq i \leqq m' - 1\}$ is a partition of $A_{m'} \sim A_m$. $N \cap A_{i+1} \sim A_i$ contains 3^{i-m} standard m-blocks. Now $m' - 1 - m = (2^{k+1} - 1) - 1 - (2^k - 1) = 2^k - 1 = m$. Thus,

$$w(k) = \sum\{3^{i-m}; m \leqq i \leqq m' - 1\} = 3^0 + 3^1 + \ldots + 3^m$$
$$= (3^{m+1} - 1)/(3 - 1) = (3^n - 1)/2,$$

establishing (a). Now $k \geqq 3$ implies $6k < (3/2)^{2^k}$, hence $k \geqq 3$ implies $2k < (3^{2^k} - 1)/(2 \cdot 2^{2^k})$, from which (b) follows.

4.3. *Construction of f.* The function f: $Z \rightarrow R$ described in § 4.1 will now be constructed and analyzed. We will define $f = \bigcup\{B_m; k \in N\}$, where B_m is a standard m-block with centre at 0, and $B_m \subseteq B_{m'}$. It will be readily seen from the inductive definition of B_m that range $B_m \subseteq [0, 1]$ and if $B_{m_0,r} \subseteq B_m$ and

$0 \leqq p \leqq k_0$, then ${}_pB_{m_0,r}$, hence $B_{m_0,r}$, is symmetric. We will show that if $B_{m_0,r}B_{m_0,r+1} \subseteq B_m$, then $d(B_{m_0,r}, B_{m_0,r+1}) \leqq 1/2^{n_0}$, from which the recurrence of Z on fZ can be easily deduced.

Let $B_0 = \{(0, 0)\}$. Suppose that B_m has been defined satisfying the desired properties. We define $B_{m'}$ after first defining an auxiliary function

$$g: A_{m,1} \ldots A_{m,w(k)} \to R$$

as follows: Let $1 \leqq r \leqq w(k)$ and let $A_{m,r} = \bigcup\{{}_pA_{m,r}; 0 \leqq p \leqq k\}$ be the basic partition of $A_{m,r}$. Let $b = b_{m,r}$ and $a = a_{m,r}$ be the associated base and accessibility functions. Let $i \in {}_kA_{m,r}$. Then $i = r3^m$. If $0 < r \leqq 2^n$, define $ig = r/2^n$. If $2^n < r \leqq 2^{n+1}$, define $ig = (2^{n+1} - r)/2^n$. If $2^{n+1} < r \leqq w(k)$, define $ig = 0$. Suppose that g has been defined on $\bigcup\{{}_pA_{m,r}; k_0' \leqq p \leqq k\}$. If $i \in {}_{k_0}A_{m,r}$, define $ig = ibg - ia/2^{n_0}$. Let $C_{m,r} = g|A_{m,r}$, $D_{m,r} = B_m + (r3^m, 0)$, $B_{m,r} = \max\{C_{m,r}, D_{m,r}\}$, $B_{m,-r} = B_{m,r} + (-2r3^m, 0)$, and $B_{m,0} = B_m$. Define $B_m' = \bigcup\{B_{m,j}; -w(k) \leqq j \leqq w(k)\}$.

We next show that if $C_{m_0,r}$ and $C_{m_0,r+1}$ are standard m_0-blocks contained in g, then $d(C_{m_0,r}, C_{m_0,r+1}) \leqq 1/2^{n_0}$. Suppose first that $k_0 = k$. We will prove, for future use, that $d(C_{m,r}, C_{m,s}) = |r3^mg - s3^mg|$, from which $d(C_{m,r}, C_{m,r+1}) \leqq 1/2^n$ follows. Let $b = b_{m,r}$, $a = a_{m,r}$, $b' = b_{m,s}$ and $a' = a_{m,s}$. For each $i \in A_{m,r}$, let $j = i + (s - r)3^m$ be the corresponding element of $A_{m,s}$. Assume for each $i \in \bigcup\{{}_pA_{m,r}; k_1' \leqq p \leqq k\}$ that $|ig - jg| = |r3^mg - s3^mg|$ and let $i \in {}_{k_1}A_{m,r}$. Then $ia = ja'$, whence $|ig - jg| = |(ibg - ia/2^{n_1}) - (jb'g - ja'/2^{n_1})| = |ibg - jb'g| = |r3^mg - s3^mg|$.

Suppose next that $d(C_{m_1,r}, C_{m_1,r+1}) \leqq 1/2^{n_1}$ for $k_1 = k_0', \ldots, k$. If

$$C_{m_0,r} \subseteq C_{m_0',s}$$

and $C_{m_0,r+1} \subseteq C_{m_0',s+1}$, then let $C_{m_0,t}$ be the first standard m_0-block in $C_{m_0',s}$. By the symmetries of $C_{m_0',s}$ and by the induction assumption on k_0, $d(C_{m_0,r}, C_{m_0,r+1}) \leqq d(C_{m_0,r}, C_{m_0,t}) + d(C_{m_0,t}, C_{m_0,r+1}) \leqq 0 + d(C_{m_0',s}, C_{m_0',s+1}) \leqq 1/2^{n_0'} < 1/2^{n_0}$. If $C_{m_0,r}$ and $C_{m_0,r+1}$ are both subsets of some $C_{m_0',s}$, then they are both subsets of some $C_{m,t}$. Let $b = b_{m,t}$ and $a = a_{m,t}$. Let $i \in A_{m_0,r}$ and $j = i + 3^{m_0}$. Assume first that $i = r3^{m_0}$. If neither i nor j equals $s3^{m_0'}$, then $ib = s3^{m_0'} = jb$ and $|ia - ja| = 1/2^{n_0}$, from which $|ig - jg| = 1/2^{n_0}$. On the other hand, if, without loss of generality, $i = s3^{m_0'}$, then $jg = ig - 1/2^{n_0}$, whence $ig - jg = 1/2^{n_0}$. Assume finally that $|ig - jg| \leqq 1/2^{n_0}$ for

$$i \in \bigcup\{{}_pA_{m_0,r};\ k_1' \leqq p \leqq k_0\},$$

and let $i \in {}_{k_1}A_{m_0,r}$. Then $ia = ja$, whence $|ig - jg| = |igb - jgb| \leqq 1/2^{n_0}$. This completes the proof of the desired inequality.

We now show that $d(B_m, B_{m,1}) \leqq 1/2^n$. Let $i \in A_{m,1}$. If

$$iB_{m,1} \neq (i - 3^m)B_m = iD_{m,1},$$

then $1/2^n \geqq ig = iB_{m,1} > (i - 3^m)B_m \geqq 0$, thus $0 < iB_{m,1} - (i - 3^m)B_m \leqq 1/2^n$. Hence $d(B_m, B_{m,1}) \leqq 1/2^n$. Now let $m_0 < m$, $B_{m_0,r} \subseteq B_m$, and

$$B_{m_0,r+1} \subseteq B_{m,1}.$$

Just as in a previous argument, the symmetries of B_m and the inequality $d(B_m, B_{m,1}) \leqq 1/2^n$ together imply that $d(B_{m_0,r}, B_{m_0,r+1}) \leqq 1/2^{n_0}$.

To show that $d(B_{m_0,r}, B_{m_0,r+1}) \leqq 1/2^{n_0}$ whenever $B_{m_0,r}B_{m_0,r+1} \subseteq B_{m'}$, it thus suffices to let $B_{m_0,r}B_{m_0,r+1} \subseteq B_{m,1} \ldots B_{m,w(k)}$ and to show that $d(B_{m_0,r}, B_{m_0,r+1}) \leqq 1/2^{n_0}$. Using the induction assumption that $d(B_{m_0,r} B_{m_0,s}) \leqq 1/2^{n_0}$ if $B_{m_0,r}$ and $B_{m_0,s}$ are standard m_0-blocks in B_m, one establishes that $d(D_{m_0,r}, D_{m_0,r+1}) \leqq 1/2^{n_0}$. The inequality $d(C_{m_0,r}, C_{m_0,r+1}) \leqq 1/2^{n_0}$ has already been established. Hence $d(B_{m_0,r}, B_{m_0,r+1}) \leqq 1/2^{n_0}$.

By Lemma 5(b), $k \geqq 3$ implies that $w(k) > 3 \cdot 2^n$. Hence, $3 \cdot 2^n 3^m g = 2 \cdot 2^n 3^m g = 0$, which we have seen implies $d(C_{m,3\cdot 2^n}, C_{m,2\cdot 2^n}) = 0$. From this we conclude that $d(B_{m,3\cdot 2^n}, B_{m,2\cdot 2^n}) = 0$. Since $d(B_{m,2\cdot 2^n}, B_{m,2^n}) = 1$, Z is not uniformly equicontinuous on fZ.

We now show that f is an isochronous point. Referring to our synopsis, our analysis will then be complete. For each $\epsilon > 0$, there exists (Lemma 5(b)) $k \in N$ such that $3^m \geqq 2/\epsilon$ and $w(k) > 2^{n+1}$. If $p \in Z$, let $E_p = f|[A_{m,w(k)} + p3^{m'}]$. If $i \in A_{m,w(k)}$, then $ig \leqq w(k)3^m g = 0$. Thus $C_{m,w(k)} \leqq 0$, whence $d(B_m, E_0) = 0$. By various symmetries, $d(E_p, E_{-p}) = 0$. Now to show that f is an isochronous point, it suffices to show that $d(B_m, E_p) = 0$ for each integer p. By our observations just made, it suffices to let $p \in P$ such that $|q| < p$ implies $d(E_q, E_0) = 0$, and to show that $d(E_p, E_0) = 0$. There exists $k_0 > k$ such that $E_p \subseteq B_{k_0'} \sim B_{k_0}$. There exists $r \in (1, w(k_0)]$ such that $E_p \subseteq B_{m_0,r}$. Let i = centre $E_p = w(k)3^m + p3^{m'}$. If $i \notin {}_kA_{m_0,r}$, then

$$3^{m'}|i, 3|w(k), 3|(3^n - 1)/2, 3|1,$$

an impossibility. Let $b = b_{m_0,r}$ and $a = a_{m_0,r}$. Since $ia = w(k) > 2^n$, we conclude that $ig = ibg - ia/2^n < 0$. For each $j \in$ domain E_p, we then have $jg \leqq ig < 0$, which implies that $jf = (j - r3^{m_0})f$. If $E_q = f|[A_{m,w(k)} + (p3^{m'} - r3^{m_0})]$, then we have just seen that $d(E_p, E_q) = 0$. Since $d(E_q, E_0) = 0$ by the induction assumption on p, we have $d(E_p, E_0) = 0$.

4.4. *Miscellaneous examples.* Using the techniques illustrated above, I have constructed various other functions $f\colon Z \to R$ such that if $f^*\colon R \to R$ is the linear extension of f, then f^* (considered as a point of classical Bebutoff space F_c) has the properties given below. Details may be found in my dissertation. We let $X = (f^*R)^-$.

(a) R is recurrent on X, X is compact, and f^* does not have mean value. (f^* is not an almost-periodic point of F_c.)

(b) R is recurrent on X, f^* is neither uniformly continuous nor bounded, hence (Lemma 2) X is not compact. (f^* is not an almost-periodic point of F_c.)

(c) f^* and g^* are uniformly continuous bounded recurrent functions, however $f^* + g^*$ is not a recurrent function. (An example with these properties is also found in (**2**, p. 26).) R is then recurrent on the compact spaces X and $Y = (g^*R)^-$. If R were recurrent on $X \times Y$, then $f^* + g^*$ would be a recurrent function. (f^* and g^* are not almost-periodic points of F_c.)

(d) f^* is an isochronous point of F_c, but f^* is neither uniformly continuous nor bounded. (f^* is not a recurrent function.)

5. T is distal does not imply T is recurrent. In (**7**, p. 478), Furstenberg gives an example of a minimal, distal, non-equicontinuous discrete flow on the torus. We will show that this flow is not recurrent by slightly modifying his proof that it is not equicontinuous. By embedding this discrete flow in a continuous flow, we will show that distal does not imply recurrent, even for continuous flows on compact minimal topological manifolds.

Let S^1 be the set of points in the complex plane having norm 1, and let $X = S^1 \times S^1$ be the torus with the induced coordinatization. Let θ be a fixed real number. For each continuous function $f\colon S^1 \to S^1$, define $h\colon X \to X$ by $(w, z)h = (e^{i\theta}w, wf \cdot z)$. h is one-to-one continuous onto X, hence h is a self-homeomorphism of X. h induces a discrete flow (X, Z, π) such that $\pi^1 = h$. We denote $((w, z), n)\pi$ by $(w, z)n$.

Suppose that $(w, z) \neq (w', z') \in X$. If $w = w'$ and $j \in Z$, then $d((w, z)j, (w', z')j) = d((w, z), (w', z'))$. If $w \neq w'$ and $j \in Z$, then $d((w, z)j, (w', z')j) \geqq d(we^{ij\theta}, w'e^{ij\theta}) = d(w, w')$. From these considerations, we see that Z is distal on X.

Now let θ be a real number which is not a rational multiple of π, let $f\colon S^1 \to S^1$ be the identity function, and let (X, Z, π) be the corresponding discrete distal flow of Furstenberg described above. By (**6**, p. 582, Remark), X is minimal. An inductive argument shows that for each $j \in Z$,

$$(w, z)j = (we^{ij\theta}, w^j z \exp[ij(j-1)\theta/2]).$$

Suppose that Z is recurrent on X. Fix $(w, z) \in X$. There exists $p \in P$ such that $q > p$ implies $d((we^{i\pi/q}, z), (w, z)) < 1/3$. There exists $q > p$ such that $(w', z') \in X$ implies $d((w', z'), (w', z')q) < 1/3$. Thus,

$$d((we^{i\pi/q}, z)q, (w, z)q) < 1.$$

However, the second entries of $(we^{i\pi/q}, z)q$ and $(w, z)q$ are

$$w^q e^{i\pi} z \exp[iq(q-1)\theta/2]$$

and $w^q z \exp[iq(q-1)\theta/2]$, respectively. Hence, we conclude that $d((we^{i\pi/q}, z)q, (w, z)q) \geqq 2$, a contradiction.

For the moment, let (X, Z, π) be any discrete flow. The corresponding cylinder flow, (Y, R, ϕ), is defined as follows. Let $h = \pi^1$. Let Y be the partition of $X \times [0, 1]$ whose elements are $\{(x, t)\}$ if $x \notin X$ and $0 < t < 1$ and $\{(x, 1), (xh, 0)\}$ if $x \in X$ and $t = 1$. Let Y have the quotient topology induced by the projection p. Define $\phi\colon Y \times R \to Y$ as follows: Let $y \in Y$ and $t \in R$. There exist $q \in Z$ and $u \in [0, 1)$ such that $t = q + u$. Let $(x, v) \in y$. If $u + v \in [0, 1]$, define $(y, t)\phi = (xh^q, u + v)p$. If $u + v \in (1, 2)$, define $(y, t)\phi = (xh^{q+1}, u + v - 1)p$. Once one can visualize geometrically this definition, it is readily seen that ϕ is well-defined and indeed does define a continuous flow. It is also easy to see that if X is an n-manifold, then Y is an $(n + 1)$-manifold.

Now let (Y, R, ϕ) be the cylinder flow corresponding to the discrete flow (X, Z, π) of Furstenberg described above. Y is compact metrizable (**13**, p. 126, Theorem 3–23) and a 3-manifold. Since Z is distal on X, R is distal on Y. Since X is minimal, Y is minimal. If R is recurrent on Y, then (Theorem 1) Z is recurrent on Y, whence Z is recurrent on X, a contradiction.

Suppose that a transformation group (X, T), with X a manifold, is given. Let $x \in X$ be an almost-periodic point. Then xT^- is compact (**11**, p. 32, § 4.09). Hence, T is almost-periodic on xT^- if and only if T is equicontinuous on xT^- (**11**, p. 37, §§ 4.35, 4.38). A conjecture by G. D. Birkhoff was shown by Gottschalk (**10**, p. 984) to be equivalent to asserting that if a continuous flow on a manifold X is pointwise almost-periodic, then the flow is equicontinuous on each orbit-closure. Ellis (**4**, p. 405) posed the following question: If X is a compact Hausdorff space minimal under R, and R is distal on X, must R be equicontinuous on X? Both questions have negative answers since examples have been known for several years of compact, minimal, manifold flows which were distal but not equicontinuous (see the Furstenberg example above or (**1**, Chapter 4)). We have shown that a compact, minimal, distal continuous flow on a topological 3-manifold need not even be recurrent.

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Grinnell College,
Grinnell, Iowa