

ARTICLE

Threshold graphs maximise homomorphism densities

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Abstract

Given a fixed graph H and a constant $c \in [0, 1]$, we can ask what graphs G with edge density c asymptotically maximise the homomorphism density of H in G . For all H for which this problem has been solved, the maximum is always asymptotically attained on one of two kinds of graphs: the quasi-star or the quasi-clique. We show that for any H the maximising G is asymptotically a threshold graph, while the quasi-clique and the quasi-star are the simplest threshold graphs, having only two parts. This result gives us a unified framework to derive a number of results on graph homomorphism maximisation, some of which were also found quite recently and independently using several different approaches. We show that there exist graphs H and densities c such that the optimising graph G is neither the quasi-star nor the quasi-clique (Day and Sarkar, *SIAM J. Discrete Math.* 35(1), 294–306, 2021). We also show that for c large enough all graphs H maximise on the quasi-clique (Gerbner et al., *J. Graph Theory* 96(1), 34–43, 2021), and for any $c \in [0, 1]$ the density of $K_{1,2}$ is always maximised on either the quasi-star or the quasi-clique (Ahlswede and Katona, *Acta Math. Hung.* 32(1–2), 97–120, 1978). Finally, we extend our results to uniform hypergraphs.

Keywords: Graph homomorphisms; homomorphism density; threshold graphs

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1. Introduction

We asymptotically study the number of homomorphisms from a fixed graph H to graphs G with a fixed edge density. Specifically, we investigate the properties of graphs G that maximise the *homomorphism density* from H to G :

Definition 1 (Homomorphism Density). Denote the homomorphism density of H in G by

$$t(H, G) = \frac{\text{hom}(H, G)}{|G|^{|H|}},$$

where $|G|$ denotes the number of vertices in G and $\text{hom}(H, G)$ denotes the number of homomorphisms from H to G .

Formally, for a given graph H and $c \in [0, 1]$, we are interested in finding a sequence of graphs that attains the value of

$$\mathcal{M}_H(c) = \limsup_{G \in \mathcal{C}_c} t(H, G),$$

where $\mathcal{C}_c = \{G: t(K_2, G) \leq c\}$ and the lim sup over a class of graphs is defined as

$$\limsup_{G \in \mathcal{C}} f(G) = \lim_{n \rightarrow \infty} \sup\{f(G): G \in \mathcal{C}, |G| \geq n\}.$$



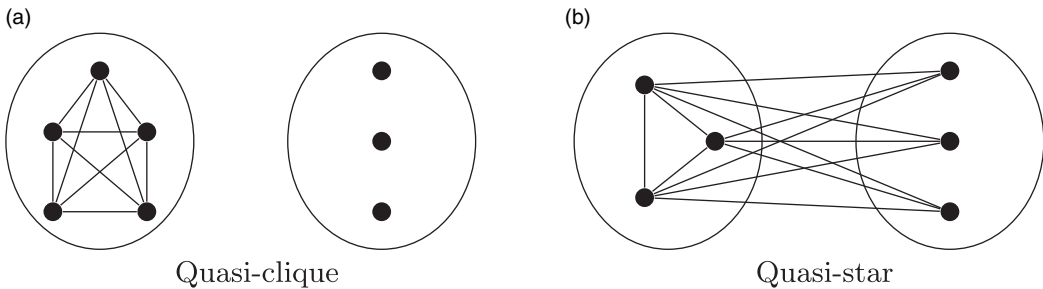


Figure 1. An example of a quasi-star and quasi-clique.

This quantity has been studied for a variety of graphs H . Two families of graphs that frequently maximise $t(H, \cdot)$ are the quasi-clique and quasi-star, where a quasi-clique is an induced clique with isolated vertices and a quasi-star is the complement of a quasi-clique (see Figure 1). We remark that in some papers, the quasi-clique is defined as isolated vertices and an induced clique with one additional vertex that may only be connected to a subset of vertices in the clique [4, 12]. This is important if one wants to determine the non-asymptotic maximiser of $t(H, G)$ for a graph G with say n vertices and m edges. Since we only consider asymptotic results, we will use the simpler definition.

A very general result of Alon [2] implies that if H has a spanning subgraph that is a disjoint union of cycles and isolated edges then $\mathcal{M}_H(c)$ is maximised on the quasi-clique for all c . The study of the behaviour of specific H largely began with Ahlswede and Katona [1], who showed that for any $c \in [0, 1]$ when H is the 2-star, a star with two edges, $\mathcal{M}_H(c)$ is always achieved on either the quasi-star or quasi-clique. This result was later generalised to k -stars, showing that for any $c \in [0, 1]$ the number of homomorphisms from the k -star is maximised when G is the quasi-star or quasi-clique for small k [8], and shortly after for all $k \geq 2$ [13]. The question was also studied in the case where H is the 4-edge path, and again it was shown that the optimising graph is always either the quasi-star and quasi-clique for all densities [12].

A particularly important class of graphs for this problem is the family of threshold graphs, which contains both the quasi-clique and quasi-star.

Definition 2. (Threshold Graph) A graph T is threshold if for any two vertices $u, v \in V(T)$ we have that $N(u) \subseteq \overline{N(v)}$ or $N(v) \subseteq \overline{N(u)}$, where $\overline{N(v)} := \{v\} \cup N(v)$ denotes the closed neighbourhood of v .

We remark that there are other characterisations of threshold graphs [11] which we will discuss in more detail in Section 2. Our main contribution is proving via a local move that the maximiser of $t(H, G)$ for any graph H is always attained on a threshold graph.

Theorem 3. For any graph H and $c \in [0, 1]$, we have that

$$\mathcal{M}_H(c) = \limsup_{G \in \mathcal{C}_c} t(H, G) = \limsup_{G \in \mathcal{C}_c \cap \mathcal{T}} t(H, G),$$

where \mathcal{T} denotes the set of all threshold graphs.

Such a result is of interest as threshold graphs have simpler limit objects than general graphs. Instead of considering the lim sup over all graphs, we can work directly with graphons, which are graph limit objects, and find the graphon that maximises the number of homomorphisms from H [10]. This approach is employed in the results of Nagy [12] and Reiher and Wagner [13]. Threshold graphs may be more convenient to work with as their limits are one-dimensional, as opposed to graphons which are two dimensional [5].

We then extend Theorem 3 to sparse graphs and hypergraphs. To do so, we define

$$\mathcal{M}(H, n, m) = \max\{\text{hom}(H, G) : |V(G)| \leq n, |E(G)| \leq m\},$$

since homomorphism densities are zero for sparse graphs. We remark that the function $\mathcal{M}(H, n, m)$ was determined up to constant factors (depending on H) by Janson, Oleszkiewicz and Ruciński [7] (see Theorem 22). Let us write $f(n) \sim g(n)$ if $\lim_{n \rightarrow \infty} f(n)/g(n) = 1$. We will prove the following theorem, which extends Theorem 3 to the sparse setting.

Theorem 4. *For any graph H and function $m(n) = \omega(n^{3/2})$, we have that*

$$\mathcal{M}(H, n, m) \sim \max\{\text{hom}(H, T) : |V(T)| \leq n, |E(T)| \leq m, T \in \mathcal{T}\},$$

where \mathcal{T} denotes the set of threshold graphs. Moreover if H has no induced C_4 or P_4 , then

$$\mathcal{M}(H, n, m) = \max\{\text{hom}(H, T) : |V(T)| \leq n, |E(T)| \leq m, T \in \mathcal{T}\}$$

for every $m, n \in \mathbb{N}$.

We quickly remark that the bound of $\omega(n^{3/2})$ is likely not tight (cf. Question 50). Additionally, if one wants exact equality for all $n, m \in \mathbb{N}$, some assumptions on H are necessary (cf. Remark 21).

In order to state our result for hypergraphs, we first need to define the class of threshold hypergraphs. We remark that there are several non-equivalent definitions of threshold hypergraphs (see e.g. [14]), but we will use the following definition throughout this paper.

Definition 5 (Threshold Hypergraph). Let G be a k -uniform hypergraph with $k \geq 2$. We then say that G is a threshold hypergraph if for any two vertices x, y we either have that $x \ll_G y$ or $y \ll_G x$, where we say $x \ll_G y$ if for any $e \in E(G)$ with $x \in e$ and $y \notin e$ we have that $(e \setminus \{x\}) \cup y \in E(G)$. We omit the subscript when the underlying hypergraph G is clear from context.

As one might expect, taking $k = 2$ exactly coincides with the definition of threshold graphs. As such, the following theorem generalises Theorem 1.3 to the setting of k -uniform hypergraphs.

Theorem 6. *Let H be a fixed k -uniform hypergraph and $c \in [0, 1]$ then we have that*

$$\mathcal{M}_H(c) = \limsup_{G \in \mathcal{C}_c} t(H, G) = \limsup_{G \in \mathcal{C}_c \cap \mathcal{T}} t(H, G).$$

In a slight abuse of notation, we take \mathcal{T} above to be the set of threshold hypergraphs and \mathcal{C}_c to be the set of hypergraphs of edge density at most c .

We will prove Theorems 3, 4 and 6 in Section 3. The remainder of the paper shows a variety of applications of these ideas. We first give a simple proof in Theorem 42 of a recent result by Gerbner, Nagy, Patkós and Vizer [6] that every graph is maximised on the quasi-clique for c sufficiently close to 1. More precisely, we show that if $\mathcal{M}_{K_{1,|H|-1}}(c)$ is asymptotically maximised on the quasi-clique then so is $\mathcal{M}_H(c)$.

We then contrast this with an example in Theorem 43 of a graph H which for some $c \in [0, 1]$ is optimised on neither the quasi-star nor the quasi-clique, disproving a conjecture of Nagy [12]. To do so, we explicitly find a graph G that has more homomorphisms from H than the quasi-star and quasi-clique when c is small. This result was proved independently by Day and Sarkar [4], who used a similar argument. While the arguments are fundamentally the same, the class of graphs \mathcal{G} that we use with more homomorphisms from H than the quasi-star or quasi-clique is the next simplest threshold graph, fitting well into our results.

Finally, we give a new proof of the result of Ahlswede and Katona [1] that the two-star is maximised on either the quasi-star or quasi-clique.

Theorem 7. *For any $c \in [0, 1]$, we have that $\mathcal{M}_{K_{1,2}}(c)$ is asymptotically attained on the quasi-star or quasi-clique.*

For brevity, we omit the proof of Theorem 7 and refer an interested reader to the Arxiv version of the paper. The proof follows from writing an expression for the number of homomorphisms from the 2-star to a general threshold graph and then explicitly computing the optimum.

2. Preliminaries

2.1 Notation

We typically use H as the fixed graph, G as the target graph, c as the edge density of G , and let n and m denote the number of vertices and edges in G . We let $\text{Hom}(H, G)$ denote the set of homomorphisms from H to G and $\text{hom}(H, G)$ be the cardinality of $\text{Hom}(H, G)$. Moreover, we will be consistent with the notation described in the introduction.

We will say a homomorphism $\varphi : V(H) \rightarrow V(G)$ uses an edge $uv \in E(G)$ if there exists an edge $xy \in H$ such that $u = \varphi(x)$ and $v = \varphi(y)$. Similarly, it uses a vertex $v \in V(G)$ if $v \in \text{im}\varphi$.

We denote the neighbourhood of a vertex u in G by $N_G(u)$, and the closed neighbourhood of a vertex u by $\overline{N}_G(u) := N_G(u) \cup \{u\}$. We frequently drop the subscript G when the graph is clear from context.

Oftentimes when using big-Oh notation, there will be implicit constants depending on the graph H or other parameters k . In this case, we denote this using a subscript, for example, $O_H(f(n))$ or $O_k(f(n))$.

2.2 Threshold graphs

We begin by recalling some basic facts about threshold graphs. There are a number of equivalent definitions of threshold graphs and for a thorough treatment of the subject we refer the reader to [11]. A key second characterisation of threshold graphs will be the following:

Proposition 8. *A graph is a threshold if and only if it can be built, starting from a single vertex graph, by repeatedly adding dominating or isolated vertices.*

We note that this sequence is unique (up to automorphisms). In light of this, we put the set of (unlabelled) n vertex threshold graphs in one-to-one correspondence with binary sequences of length $n - 1$. In such a sequence the i th element is a 1 if the i th vertex that we added is a dominating vertex and a 0 if the i th vertex added is an isolated vertex.

When using this representation we will often refer to the number of parts of a threshold graph. This refers to the number of blocks in the corresponding binary string of the given threshold graph. For instance, 1011100 corresponds to a threshold graph with four parts. Similarly, we note that the quasi-clique and quasi-star are threshold graphs with two parts.

Finally, while we do not use this characterisation directly, we state it as it shows that threshold graphs satisfy the criteria of Corollary 16, below.

Proposition 9. *A graph is threshold if and only if it contains no induced copies of a four vertex path, a four vertex cycle, or the complement of a four vertex cycle.*

2.3 Basic properties of $\mathcal{M}_H(c)$

We note that while in the definition of $\mathcal{M}_H(c)$ we take a lim sup we could have equivalently defined it as the supremum over all graphs in \mathcal{C}_c . This follows from the following useful lemma, whose proof is included in the Arxiv version of the paper.

Lemma 10. *Let G be a graph on n vertices, then there exists a sequence of graphs $G = G_1, G_2, \dots$ such that $|G_{i+1}| > |G_i|$ and $t(H, G) = t(H, G_i)$ for any graph H .*

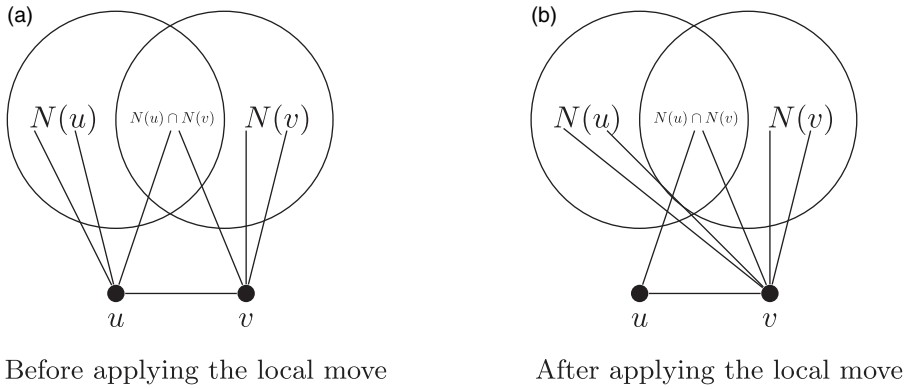


Figure 2. A neighbourhood-ordering local move.

Note that the sequence of graphs does not depend on H . As such, by taking $H = K_2$, we have that each graph G_i in the sequence will also have the same edge density as G .

2.4 Fractional independence number

To prove that there exists a graph not optimised on the quasi-clique or quasi-star, we will need to define the fractional independence number, which is closely related to the number of homomorphisms by a result of Janson, Oleszkiewicz and Ruciński (cf. Theorem 22) [7].

Definition 11 (Fractional Independence Number). Given a graph, $G = (V, E)$, with vertices v_1, \dots, v_n , the fractional independence number denoted $\alpha^*(G)$ is

$$\begin{aligned} &\max_{w_1, \dots, w_n} \sum_i w_i \\ &\text{s.t. } w_i + w_j \leq 1, \quad v_i v_j \in E, \\ &\quad w_i \in [0, 1]. \end{aligned}$$

Since we can take all the $w_i = \frac{1}{2}$, we always have that $\alpha^*(G) \geq \frac{|G|}{2}$. It is well-known that the polytope defined by the constraints above is half-integral, that is, all its vertices are in $\{0, 1/2, 1\}^n$.

Lemma 12. *The fractional independence polytope is half-integral.*

We omit the proof here and refer an interested reader to the Arxiv version of the paper.

3. Threshold graph maximisation results

3.1 A local move and results for dense graphs

Given a graph G there is a natural way of slowly transforming it into a threshold graph (see Figure 2): Take two vertices u and v . Remove the edges between u and $N(u) \setminus \overline{N(v)}$ and add edges from $N(u) \setminus \overline{N(v)}$ to v . This gives us a new graph G' in which we have $N_{G'}(u) \subseteq \overline{N_{G'}(v)}$. Repeating this process takes us from G to a threshold graph. Notationally, we refer to this operation on a graph G as a local move on (u, v) . We will also simply refer to it as moving the neighbours of u to v .

To prove Theorem 3 we start by showing that this local move doesn't significantly decrease the number of homomorphisms from H . Towards this end, we define forbidden paths.

Definition 13 (Forbidden Path). A forbidden path is a path $wxyz$ where wy and xz are not edges.

We show that the homomorphisms lost after applying the local move are those that send a forbidden path in H to a set containing u, v and a vertex in $N(u) \setminus \overline{N(v)}$.

Lemma 14. Let H and G be graphs, u and v be vertices of G , and let G' be the resulting graph after applying a local move on (u, v) . Then $\text{hom}(H, G')$ is at least as large as the number of homomorphisms $\varphi \in \text{Hom}(H, G)$ such that for all $w \in N(u) \setminus \overline{N(v)}$ and all forbidden paths $abcd$ in H , $\{u, v, w\} \not\subseteq \varphi(\{a, b, c, d\})$.

Proof. We proceed by creating an injective function that takes a homomorphism $\varphi(x)$ from H to G satisfying the assumptions in the theorem and maps it to a homomorphism $\varphi'(x)$ from H to G' . For intuition, we attempt to map φ to a φ' such that $\varphi'(x) = \varphi(x)$ whenever possible.

If $\varphi(x) \neq u, v$ then we simply let $\varphi'(x) = \varphi(x)$. If $\varphi(x) = u$, then we again let $\varphi'(x) = \varphi(x)$ so long as there is not an edge $xy \in E(H)$ with $\varphi(y) \in N(u) \setminus \overline{N(v)}$. If there is such an edge $xy \in E(H)$, then we instead set $\varphi'(x) = v$. Finally, if $\varphi(x) = v$ then we set $\varphi'(x) = \varphi(x)$ unless there is an edge $xy \in E(H)$ such that $\varphi(y) = u$ and $\varphi'(y) = v$. If such an edge exists, then we set $\varphi'(x) = u$. Formally,

$$\varphi'(x) = \begin{cases} v & \exists y : xy \in E(H) \text{ and } \varphi(x) = u \text{ and } \varphi(y) \in N(u) \setminus \overline{N(v)} \\ u & \exists y, z : xy, yz \in E(H) \text{ and } \varphi(x) = v \text{ and } \varphi(y) = u \text{ and } \varphi(z) \in N(u) \setminus \overline{N(v)} \\ \varphi(x) & \text{otherwise.} \end{cases}$$

We start with the following claim.

Claim 15. φ' is a homomorphism from H to G' .

Proof. It suffices to check that the edges incident to vertices mapping to u or v are preserved. Let $a, b \in V(H)$ be such that $ab \in E(H)$ and $\varphi(a) = u$. Since φ is a homomorphism, we must have that $\varphi(b) \in N(u)$. We now take cases.

Case I. $\varphi(b) \in N(u) \cap N(v)$

Since $\varphi(b) \neq u, v$, $\varphi'(b) = \varphi(b)$. Moreover, $\varphi'(a)\varphi'(b) \in E(G')$ since a maps to either u or v under φ' and both $u\varphi'(b)$ and $v\varphi'(b)$ are edges in G' .

Case II. $\varphi(b) \in N(u) \setminus \overline{N(v)}$.

If we are in this case, then $\varphi'(a) = v$ as $ab \in E(H)$, $\varphi(a) = u$ and $\varphi(b) \in N(u) \setminus \overline{N(v)}$. Moreover, $\varphi'(b) = \varphi(b)$ as $\varphi(b) \neq u, v$. Since $v\varphi(b) \in E(G')$, we are again done.

Case III. $\varphi(b) = v$ and $\varphi'(a) = u$ (see Figure 3)

We claim that $\varphi'(b) = v$. Suppose not. Then $\exists y, z : by, yz \in E(H)$, $\varphi(y) = u$, and $\varphi(z) \in N(u) \setminus \overline{N(v)}$. Since $yz \in E(H)$, $\varphi'(y) = v$ and thus $y \neq a$. But now note $\varphi(a) = \varphi(y) = u$, so $ay \notin E(H)$. Moreover, $bz \notin E(H)$ since $\varphi(z) \in N(u) \setminus \overline{N(v)}$ and $\varphi(b) = v$. It then follows that $abyz$ is a path in H . But then φ maps the forbidden path $abyz$ to u, v , and a vertex in $N(u) \setminus \overline{N(v)}$, which is a contradiction.

Case IV. $\varphi(b) = v$ and $\varphi'(a) = v$ (see Figure 3)

Note that since $\varphi'(a) = v$ we must have that there is a y such that $ay \in E(H)$ and $\varphi(y) \in N(u) \setminus \overline{N(v)}$. But it then follows that $\varphi'(b) = u$ since $ba, ay \in E(H)$.

Now let a and b be vertices such that $\varphi(a) = v$ and $ab \in E(H)$ where $\varphi(b) \neq u$. Then we have one of two cases

Case V. $\varphi'(a) = v$

Since φ is a homomorphism we have that $\varphi(b) \in N_G(v)$. But then we have that the edge ab is preserved by the homomorphism since $N_G(v) \subseteq N_{G'}(v)$.

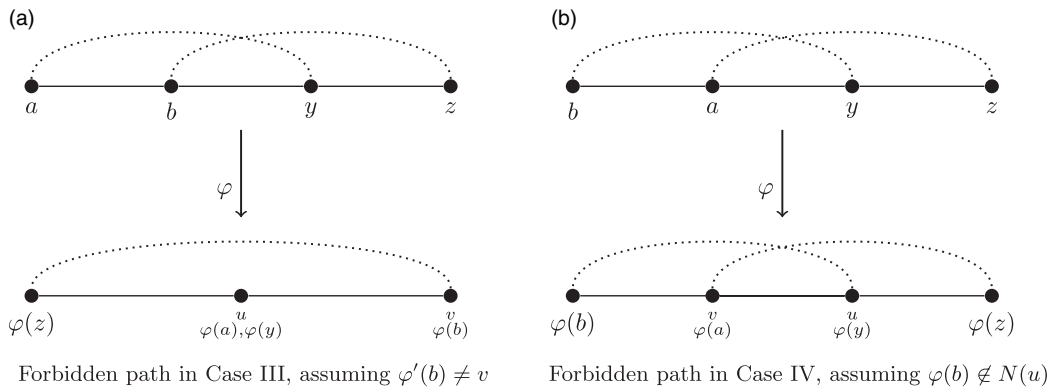


Figure 3. Forbidden paths in proof of Claim 15.

Case VI. $\varphi'(a) = u$

In this case there exists y, z such that $ay, yz \in E(H)$, $\varphi(y) = u$ and $\varphi(z) \in N(u) \setminus \overline{N(v)}$. Now we claim $\varphi(b) \in N(u)$. Suppose not, then $\varphi(b) \in N(v) \setminus \overline{N(u)}$. We now note that $bayz$ is a forbidden path: $by \notin E(H)$ as $\varphi(y) = u$ and $\varphi(b) \notin N(u)$ and $az \notin E(H)$ as $\varphi(a) = v$ and $\varphi(z) \in N(u) \setminus \overline{N(v)}$. But then $bayz$ is forbidden path with u, v and $\varphi(z) \in N(u) \setminus \overline{N(v)}$ in the image of φ , a contradiction.

Since these are all the cases we indeed have that φ' is a valid homomorphism. □

To see that the map is injective, suppose that we had that $\varphi_1, \varphi_2 \mapsto \varphi'$. Clearly $\varphi_1(x) = \varphi_2(x)$ for all x such that $\varphi'(x) \neq u, v$. Now suppose that for some $x \in V(H)$, $\varphi_1(x) = u$ and $\varphi_2(x) = v$. If $\varphi'(x) = v$, then x is adjacent to some y such that $\varphi_1(y) \in N(u) \setminus \overline{N(v)}$. But then φ_2 also maps y to the same vertex, but v is not adjacent to $\varphi_2(y)$ in G , which contradicts the fact that φ_2 is a homomorphism.

So now suppose that $\varphi'(x) = u$. Then there exist y, z such that $xy, yz \in E(H)$ and $\varphi_2(y) = u$ and $\varphi_2(z) \in N(u) \setminus \overline{N(v)}$. But then we must have that $\varphi_1(y) = v$ since $xy \in E(H)$ and $\varphi_1(x) = u$. Similarly, we have that $\varphi'(y) = v$ as $xy \in E(H)$ and $\varphi'(x) = u$. But now y is a vertex such that $\varphi_1(y) = v$ and $\varphi_2(y) = u$ and $\varphi'(y) = v$, which we just showed yields a contradiction.

Hence the map is injective as desired and the statement holds. □

Corollary 16. Let H and G be graphs, $u, v \in V(G)$, and take G' to be the resulting graph after applying a local move (u, v) to G . If H has no induced graphs isomorphic to P_4 or C_4 , then $\text{hom}(H, G') \geq \text{hom}(H, G)$.

Proof. If H does not contain induced subgraphs isomorphic to P_4 and C_4 , then we have that H has no forbidden paths. Hence H has at least as many homomorphisms to G' as it does to G . □

In order to use Lemma 14, we need more quantitative bounds on the number of homomorphisms lost. Towards, this end we prove the following lemma.

Lemma 17. Let H and G be graphs, set $n := |V(G)|$ and let $u, v \in V(G)$ and $S \subseteq V(G) \setminus \{u, v\}$. Then there are at most $O_H(|S|n^{|H|-3})$ homomorphisms $\varphi \in \text{Hom}(H, G)$ such that there exists a forbidden path $abcd$ in H with $u, v, s \in \varphi(\{a, b, c, d\})$ for some $s \in S$.

Proof. Let $abcd$ be a forbidden path in H . Then we must choose one of the four vertices to map to u , one of the remaining three to map to v , and one of the remaining two to map to a vertex in

S. The remaining $|H| - 3$ vertices in the graph could go to any of n vertices. Hence, we have that there are at most

$$4!|S|n^{|H|-3}$$

such homomorphisms. Since there are only a constant number of forbidden paths in H , which we denote by k_H , we have that there are at most

$$4!k_H|S|n^{|H|-3} = O_H(|S|n^{|H|-3})$$

homomorphism mapping a forbidden path to a set including u, v and a vertex from S . □

With this lemma in hand, we note that moving the neighbours of u to v removes at most $O_H(|N(u) \setminus \overline{N(v)}|n^{|H|-3})$ homomorphisms. Thus to argue we didn't lose too many homomorphisms after applying multiple local moves we make the following definition.

Definition 18 (Total Movement). Let $G = G_0, G_1, \dots, G_t$ be a sequence of graphs from applying local moves $(u_1, v_1), \dots, (u_t, v_t)$, that is, each G_i is obtained by applying the local move to the vertices (u_i, v_i) in the graph G_{i-1} . Define the total movement after these t moves as

$$\sum_{i=1}^t \left| N_{G_{i-1}}(u_i) \setminus \overline{N_{G_{i-1}}(v_i)} \right|.$$

We now claim that we can turn G into a threshold graph with small amount of total movement.

Lemma 19. *Let G be a graph. Then using local moves we can turn G into a threshold graph with at most n^2 moves and with total movement at most $|E(G)|$.*

Proof. We prove this by induction on the number of vertices.

For the base case, note that a graph with a single vertex is threshold. Now assume the statement holds for graphs with at most n vertices and let G be a graph on $n + 1$ vertices. Let v_{n+1} be the vertex of maximum degree and v_1, \dots, v_n be the remaining vertices. Consider applying the following n moves: Move the neighbours of v_1 to v_{n+1} , then move the neighbours of v_2 to v_{n+1} , continue in this way until moving the neighbours from v_n to v_{n+1} .

Let G' be the graph after applying all the above moves. Clearly $N_{G'}(v_1) \cup N_{G'}(v_2) \cup \dots \cup N_{G'}(v_n) \subseteq \overline{N_{G'}(v_{n+1})}$. Hence if $v_i v_j \in E(G')$ and $j \neq n + 1$, then we have that $v_{n+1} v_j \in E(G')$. We then conclude that if $v_{n+1} v_j \notin E(G')$ for $j \neq n + 1$, then it must be the case that v_j is isolated.

Let I denote the vertices in G' that are isolated. Applying the inductive hypothesis to $G'[V \setminus (I \cup \{v_{n+1}\})]$, we have that there is a set of at most n^2 moves that transforms $G'[V \setminus (I \cup \{v_{n+1}\})]$ to a threshold graph T with total movement at most $|E(G'[V \setminus (I \cup \{v_{n+1}\})])|$. Applying these moves to G' gives us a graph G'' . But now note that G'' can be described as adding a dominating vertex to T followed by $|I|$ isolated vertices. So G'' is a threshold graph. Note that we made at most $n^2 + n \leq (n + 1)^2$ moves.

To bound the total movement cost, note that any edge incident to v_{n+1} was moved at most once by the initial set of n moves. So by the inductive hypothesis, the total movement cost is at most $E(G'[V \setminus (I \cup \{v_{n+1}\})]) + \deg_{G'}(v_{n+1}) = |E(G)|$, as desired. □

We can now prove a strengthening of Theorem 3.

Theorem 20. *For any graph G with n vertices and any graph H , there exists a threshold graph T on n vertices with $|E(T)| \leq |E(G)|$ such that*

$$\text{hom}(H, T) \geq \text{hom}(H, G) - O_H(n^{|H|-1}).$$

Proof. We use the local moves to transform the true maximum graph into a threshold graph and argue that we haven't lost too many homomorphisms. By Lemma 19, there exists a sequence of local moves on pairs $(u_1, v_1), \dots, (u_k, v_k)$ with total movement at most $|E(G)|$ that transforms G

into a threshold graph T . Let G_i be the graph defined by applying (u_i, v_i) to G_{i-1} . By Lemmas 14 and 17, we have that

$$\text{hom}(H, G_i) \geq \text{hom}(H, G_{i-1}) - O_H(|N_{G_{i-1}}(u_i) \setminus \overline{N_{G_{i-1}}(v_i)}|n^{|H|-3}).$$

We thus conclude that

$$\text{hom}(H, T) \geq \text{hom}(H, G) - \sum_{i=1}^k O_H(|N_{G_{i-1}}(u_i) \setminus \overline{N_{G_{i-1}}(v_i)}|n^{|H|-3}) \geq \text{hom}(H, G) - O_H(n^{|H|-1})$$

where the final inequality uses that the total movement of our sequence of local moves is at most $|E(G)| \leq n^2$. □

Remark 21. Note that such a result will not hold non-asymptotically for $\mathcal{M}(H, n, m)$. Specifically, we note that no threshold graph achieves $\mathcal{M}(C_4, 4, 4)$.

3.2 Extension to sparse graphs

We now extend this result to sparse graphs by strengthening Lemma 17. To do this, we will find it convenient to use the following result.

Theorem 22 (Janson, Oleszkiewicz and Ruciński [7]). Let H be a graph and $m \geq |E(H)|$ and $n \geq |H|$ with $n \leq m \leq \binom{n}{2}$. Then

$$\mathcal{M}(H, n, m) = \Theta_H(m^{|H|-\alpha^*(H)}n^{2\alpha^*(H)-|H|}),$$

where $\alpha^*(H)$ denotes the fractional independence number of the graph.

From this, we easily obtain the following lemma:

Lemma 23. Let H be a graph, k be a non-negative integer, and H' be an induced subgraph of H on $|V(H)| - k$ vertices. Then for every $n \geq |H|$ and $n \leq m \leq \binom{n}{2}$ with $m \geq |E(H)|$

$$\mathcal{M}(H', n, m) = O_H\left(\mathcal{M}(H, n, m) / \left(\frac{m}{n}\right)^k\right).$$

Proof. Let $\Delta = \alpha^*(H) - \alpha^*(H') \geq 0$. By Theorem 22 and some algebraic manipulation we see that

$$\begin{aligned} \mathcal{M}(H', n, m) &= O_H\left(\left(\frac{m}{n}\right)^{|H'|-\alpha^*(H')}n^{\alpha^*(H')}\right) \\ &= O_H\left(\left(\frac{m}{n}\right)^{|H|-k-\alpha^*(H)+\Delta}n^{\alpha^*(H)-\Delta}\right) \\ &= O_H\left(\left(\frac{m}{n}\right)^{-k}\left(\frac{m}{n^2}\right)^\Delta m^{|H|-\alpha^*(H)}n^{2\alpha^*(H)-|H|}\right) \\ &= O_H(\mathcal{M}(H, n, m) / \left(\frac{m}{n}\right)^k). \end{aligned}$$

where the last equality used that $\mathcal{M}(H, n, m) = \Omega_H(m^{|H|-\alpha^*(H)}n^{2\alpha^*(H)-|H|})$ and $m \leq n^2$. □

We remark that the above lemma can be proved without the result of Theorem 22 by taking a graph that approximately achieves $\mathcal{M}(H', n/2, m/4)$ and adding $\Omega(m/n)$ dominating vertices to it to get a graph with many copies of H .

Lemma 24. Let H and G be graphs, where G has $n \geq |H|$ vertices and $m \geq \max(|E(H)|, n)$ edges, and let $u, v \in V(G)$ and $S \subseteq V(G) \setminus \{u, v\}$. Then there are at most $O_H(|S|\mathcal{M}(H, n, m) / \left(\frac{m}{n}\right)^3)$

homomorphisms $\varphi \in \text{Hom}(H, G)$ such there exists a forbidden path $abcd \in H$ in which $\{u, v, s\} \subseteq \varphi(\{a, b, c, d\})$ for some $s \in S$.

Proof. Let $abcd$ be a forbidden path. Then there are clearly $O(|S|)$ ways to map 3 vertices from a, b, c, d into u, v and some vertex from $|S|$. For ease of notation, assume these three vertices are a, b, c . Now consider $H' = H[V(H) \setminus \{a, b, c\}]$. We note that the number of homomorphisms is at most $O_H(|S|\mathcal{M}(H', n, m))$. So it suffices to show that $\mathcal{M}(H', n, m) = O_H\left(\mathcal{M}(H, n, m) / \left(\frac{m}{n}\right)^3\right)$. But this follows from Lemma 23. \square

We can now prove our strengthening of Theorem 20.

Theorem 25. For any graph H and integers n and $m \leq \binom{n}{2}$, there exists a threshold graph T on n vertices and at most m edges such that

$$\text{hom}(H, T) \geq \left(1 - O_H\left(\frac{n^3}{m^2}\right)\right) \mathcal{M}(H, n, m).$$

Proof. The proof follows almost identically to Theorem 3. Let G^* be the graph on n vertices and m edges with

$$\text{hom}(H, G^*) = \mathcal{M}(H, n, m).$$

By Lemma 19, there exists a sequence of local moves on pairs $(u_1, v_1), \dots, (u_k, v_k)$ with total movement at most $|E(G^*)|$ that transforms G^* into a threshold graph T . Let G_i^* be the graph defined by applying (u_i, v_i) to G_{i-1}^* . By Lemmas 14 and 24, we have that

$$\text{hom}(H, G_i^*) \geq \text{hom}(H, G_{i-1}^*) - O_H\left(|N_{G_{i-1}^*}(u_i) \setminus \overline{N_{G_{i-1}^*}(v_i)}| \cdot \mathcal{M}(H, n, m) / \left(\frac{m}{n}\right)^3\right).$$

We thus conclude that

$$\begin{aligned} \text{hom}(H, T) &\geq \text{hom}(H, G^*) - \sum_{i=1}^k O_H\left(|N_{G_{i-1}^*}(u_i) \setminus \overline{N_{G_{i-1}^*}(v_i)}| \cdot \mathcal{M}(H, n, m) / \left(\frac{m}{n}\right)^3\right) \\ &\geq \mathcal{M}(H, n, m) - O_H\left(\frac{n^3}{m^2} \cdot \mathcal{M}(H, n, m)\right) \end{aligned}$$

where the final inequality uses the total movement of our moves is $|E(G)|$. \square

3.3 Extension to hypergraphs

For hypergraphs, we only prove results for asymptotic maximisation in the dense case. As before, we use a sequence of moves to transform our hypergraph. We can view this as a shifting or compression argument, which are common in extremal set theory.

Definition 26 (Hypergraph Move). Let $G = (V, E)$ be a k -uniform hypergraph and u, v be two vertices in G . We say a hypergraph move from u to v in G is as follows: For each hyperedge $e \in E(G)$ containing u and not v such that $(e \setminus \{u\}) \cup \{v\} \notin E(G)$, remove e and add $(e \setminus \{u\}) \cup \{v\}$.

Note that this move generalises our local move for graphs. That said, we will be able to get by with a worse bound on the number of homomorphisms lost from such a hypergraph move:

Lemma 27. Let H and G be k -uniform hypergraphs and let G' be the resulting hypergraph after applying a hypergraph move from u to v in G . Then the number of homomorphisms from H to G' is at least the number of homomorphisms φ from H to G that use at most one of u and v .

Proof. We again construct an injective map from a subset of $\text{hom}(H, G)$ to $\text{hom}(H, G')$. Specifically, given $\varphi \in \text{hom}(H, G)$, we map it to

$$\varphi'(x) = \begin{cases} v & \varphi(x) = u \text{ and } \exists e \in E(H) : x \in e \text{ and } (\varphi(e) \setminus \{u\}) \cup \{v\} \notin E(G) \\ \varphi(x) & \text{otherwise.} \end{cases}$$

We can easily verify that φ' is a homomorphism if φ uses at most one of u and v . To see that the map is injective, suppose $\varphi_1, \varphi_2 \in \text{hom}(H, G)$ both map to a homomorphism $\varphi' \in \text{hom}(H, G')$. Towards a contradiction, assume $\varphi_1 \neq \varphi_2$. By definition, $\varphi'(x) = \varphi_1(x)$ whenever $\varphi_1(x) \neq u$. As an analogous statement holds for φ_2 , we have that $\varphi_1(x) = \varphi_2(x) = \varphi'(x)$ for all x such that $\varphi_1(x) \neq u$ and $\varphi_2(x) \neq u$. So without loss of generality there exists an x such that $\varphi_1(x) = u$ and $\varphi_2(x) = v$. Then $\varphi'(x) = v$ and there exists an $e \in E(H)$ containing x such that $(\varphi_1(e) \setminus \{u\}) \cup \{v\} \notin E(G)$. But now note that $\varphi_2(e) = (\varphi_1(e) \setminus \{u\}) \cup \{v\} \notin E(G)$, so we have reached a contradiction. \square

The key lemma of this section will be the following result to transform hypergraphs into threshold hypergraphs:

Lemma 28. *Let G be a k -uniform hypergraphs with n vertices, then G can be transformed into a threshold hypergraph T by removing $o_k(n^k)$ edges and using $o_k(n^2)$ hypergraph moves.*

This quickly gives Theorem 6.

Proof of Lemma 6, assuming Lemma 28. Let G be a hypergraph with n vertices. By Lemma 28 we can use $o_k(n^2)$ hypergraph moves $(u_1, v_1), \dots, (u_\ell, v_\ell)$ and remove at most $o_k(n^k)$ edges to turn it into a threshold hypergraph T . Now note the number of homomorphisms H to any hypergraph G' that use any two specific vertices $u, v \in V(G')$ is at most $O_H(n^{|H|-2})$. Similarly, there are at most $O_{H,k}(n^{|H|-k})$ homomorphisms that use every vertex in a hyperedge $e \in E(G')$. Thus by Lemma 27 each hypergraph move (u_i, v_i) removes at most $O_H(n^{|H|-2})$ homomorphisms. Moreover, removing a hyperedge e removes at most $O_{H,k}(n^{|H|-k})$ homomorphisms. We conclude that

$$\text{hom}(H, T) \geq \text{hom}(H, G) - o_k(n^2) \cdot O_H(n^{|H|-2}) - o_k(n^k) \cdot O_{H,k}(n^{|H|-k}) = \text{hom}(H, G) - o_{H,k}(n^{|H|}).$$

\square

We now proceed to prove Lemma 28. Like in the case of graphs, we will apply a set of local moves to G , yielding a hypergraph G' , in such a way that there exists a vertex $v \in G'$ with $u \ll_{G'} v$ for all $u \in V(G')$. We then repeat the process on G' to slowly transform it into a threshold hypergraph. One way to do this would of course be to apply a local move from every vertex $u \in V(G)$ to v , however, this would result in a total of $\Omega(n^2)$ moves. To improve on this, we show that we only need to apply moves from a neighbourhood-dominating set to v .

Definition 29 (Neighbourhood-Dominating Set). Given a k -uniform hypergraph $G = (V, E)$, a vertex $v \in V$ and a subset $S \subseteq V$, a neighbourhood-dominating set $D \subseteq V$ of S with respect to v is a set of vertices such that for every vertex $s \in S$ and hyperedge e containing s either $v \in e$ or there exists a $d \in D$ such that $(e \setminus \{s\}) \cup \{d\} \in E(G)$.

Of course, this is only useful if there are neighbourhood-dominating sets of size $o(n)$. To prove that small neighbourhood-dominating sets exist, we will apply the following lemma to a suitably defined graph corresponding to our hypergraph G .

Lemma 30. *Let $G = (A \cup B, E)$ be a bipartite graph with $|A| = n$ and $|B| \leq n^k$ for a positive integer k . Moreover, assume that the minimum degree of a vertex in B is δ , then there exists a set $D \subseteq A$ such that $N_G(D) = B$ and $|D| = O_k(n \log(n)/\delta)$.*

Proof. Assume that $\delta > k \log(n)$ as otherwise we can simply take $D = A$. We will follow the classical proof for dominating sets given in [3]. Take a random set S such that every vertex of A is in S independently with probability p and let N denote the set of vertices in B without a neighbour in S . Build a set D by adding a neighbour of each vertex in N to S . Then observe that $N_G(D) = B$ and in expectation we have

$$\mathbb{E}[|D|] \leq \mathbb{E}[|S| + |N|] \leq pn + n^k(1 - p)^\delta \leq pn + n^k e^{-p\delta}.$$

Taking $p = \frac{1}{\delta} \log(\delta n^{k-1})$ then gives

$$\mathbb{E}[|D|] \leq \frac{kn \log(n)}{\delta} + \frac{n}{\delta} = O_k\left(\frac{n \log(n)}{\delta}\right).$$

□

We now associate some graphs to our hypergraphs:

Definition 31 (Incidence Graph). Given a k -uniform hypergraph $G = (V, E)$, we associate a bipartite incidence graph $I = (A \cup B, E(I))$. Each vertex in A will correspond to a vertex in G and each vertex in B corresponds to a subset $S \in \binom{[n]}{k-1}$ such that $S \subset e \in E(G)$. There is an edge between a vertex $v \in A$ and a subset $S \in B$ if $\{v\} \cup S \in E(G)$.

In particular, we will often be interested in induced subgraphs of the incidence graph with respect to a vertex v .

Definition 32 (Induced Incidence Graphs). Given a k -uniform hypergraph $G = (V, E)$, a set of vertices $S \subseteq V$ and a $v \in V$, we say the the incidence graph induced by S with respect to v is $I[S \cup N_I(S) \setminus \{b \in \binom{[n]}{k-1} : v \in b \text{ or } b \cup v \in E\}]$, where $I = (A \cup B, E(I))$ is the incidence graph of G .

With these definitions in hand, we get the following corollary.

Corollary 33. Let $G = (V, E)$ be a k -uniform hypergraph on n vertices, $S \subseteq V$, $v \in V$ and $I = (A \cup B, E(I))$ be the incidence graph of G induced by S with respect to v . If every vertex $b \in B$ has degree at least $n^{2/3}$, then there exists a neighbourhood-dominating set $D \subseteq S$ of S with respect to v of size at most $O_k(\sqrt{n})$.

Proof. First note that if $|S| < \sqrt{n} + 1$ we can take $D = S$. So we assume $|S| \geq \sqrt{n} + 1$. We then have that $|B| \leq n^{k-1} \leq |A|^{2k-2}$. By Lemma 30, it follows that there exists a set $D \subseteq S$ of size at most $O_k(n \log(n)/n^{2/3}) = O_k(\sqrt{n})$ such that $N_I(D) = B$.

We claim that $D \cup \{v\}$ is a dominating set of S with respect to v . Indeed, let $s \in S$ and e be an edge containing s with $v \notin e$. If $e \setminus \{s\} \cup \{v\} \in E$ then we are done, so suppose this is not the case. It then follows that $e \setminus \{s\} \in B$, so there exists a vertex $d \in D$ that is a neighbour of $e \setminus \{s\}$ in I , which implies $(e \setminus \{s\}) \cup \{d\} \in E(G)$, as desired. □

Our general scheme to transform our hypergraph will now be to remove edges of low degree so the induced incidence graph has high minimum degree and then apply local moves from our dominating sets to v . We will then repeat this process to find a dominating vertex u of $V \setminus \{v\}$.

Formally, we define the following domination procedure from S to v : Let G_0 be a hypergraph, $S \subset V(G_0)$ and $v \in V(G_0)$. Define a sequence of hypergraphs G_0, G_1, \dots, G_k as follows: Let $I_i = (A_i \cup B_i, E(I_i))$ be the incidence graph of G_i induced by S and with respect to v . While there exists a $b \in B_i$ with $\deg(b) \leq n^{2/3}$ remove all edges of the form $b \cup \{s\}$ for some $s \in S$ from G_i to get a hypergraph H_i . Let D_i be the smallest dominating set of S with respect to v in H_i . Apply local moves from each vertex $d \in D_i$ to v , yielding G_{i+1} .

Lemma 34. After running the domination procedure from S to v we have that $s \ll_{G_k} v$ for all $s \in S$.

Proof. Fix $s \in S$ and an edge $e \in E(G_k)$ containing s such that $v \notin e$. We will show that $e \setminus \{s\} \cup \{v\} \in E(G_k)$. First, observe that we never remove hyperedges containing v as no subsets b in the

induced incidence graph contain v or are neighbours of v . Since we only add edges incident to v , $e \in E(G_i)$ for all i . It then follows that there exists d_1, \dots, d_k such that $e \setminus \{s\} \cup \{d_i\} \in E(H_i)$. Note that if $e \setminus \{s\} \cup \{d_i\}$ exists when we apply the local move from d_i to v , then clearly $e \setminus \{s\} \cup \{v\} \in G_{i+1}$. As we never remove edges containing v , $e \setminus \{s\} \cup \{v\} \in G_k$.

Now towards a contradiction, suppose that for each i we applied a local move from $d'_i \in D_i$ to v before the move from d_i to v , and the move from d'_i causes the removal of $e \setminus \{s\} \cup \{d_i\}$. Observe that $d'_i \neq d'_j$ for $i < j$. Indeed suppose $d'_i = d'_j$. Note $d'_i \ll_{G_{i+1}} v$. Since we only add edges incident to v and never remove edges containing v , we have that $d'_i \ll_{H_\ell} v$ for all $\ell > i$. In particular, at the time of making the move from d'_j to v we have that $d'_j \ll v$, so the local move from d'_j to v would remove no edges, a contradiction.

But we now observe that $\{d'_1, \dots, d'_k\} \subseteq e \setminus \{s\}$, a contradiction as $|e \setminus \{s\}| = k - 1$. □

It now remains to show that removing edges of low degree and applying local moves does not ruin the order we have built.

Lemma 35. *Let G be a k -uniform hypergraph and let $u, v, w, x \in V(G)$. Moreover, assume $x \neq u, v$, $u \ll_G x$, $v \ll_G x$ and $w \ll_G x$. Let G' be the resulting hypergraph after applying a hypergraph move from u to v in G . Then $w \ll_{G'} x$.*

Proof. We start with the case that $w = v$.

Claim 36. $v \ll_{G'} x$

Proof. Let $e \in E(G')$ be a hyperedge containing v and not containing x . We will show that $f = (e \setminus \{v\}) \cup \{x\} \in E(G')$. We consider two cases.

Case I. $e \in E(G)$

Since $v \ll_G x$, we must have that $f \in E(G)$. Note that the move from u to v only removes edges containing u , so if $u \notin f$, then $f \in E(G')$. On the other hand, if $u \in f$, we will show that $(f \setminus \{u\}) \cup \{v\} \in E(G)$, which again implies that f is not removed by the local move. Indeed, if $u \in f$, then $u \in e$. Using $u \ll_G x$ then yields $g = (e \setminus \{u\}) \cup \{x\} \in E(G)$. But $g = (f \setminus \{u\}) \cup \{v\}$, so we have $f \in E(G')$ as desired.

Case II. $e \notin E(G)$

Since $e \notin E(G)$, it must have been added during the hypergraph move. In this case, we will show that $f \in E(G)$. Note that this implies that $f \in E(G')$: Since the local move only adds edges not containing u , it follows that $u \notin e$ and thus $u \notin f$. So if $f \in E(G)$, it cannot be removed by the local move, as it only removes edges containing u . To see that $f \in E(G)$, note that in order for the local move to add e , we must have $(e \setminus \{v\}) \cup \{u\} \in E(G)$. Since $u \ll_G x$, $(e \setminus \{v\}) \cup \{x\} = f \in E(G)$, completing the proof. □

Since after applying the hypergraph move we have that $u \ll_{G'} v$, we have also shown that $u \ll_{G'} x$. So it only remains to handle the case of $w \neq u, v$.

Claim 37. $w \ll_{G'} x$ for $w \neq u, v$

Proof. Let $e \in E(G')$ be an edge containing w and not containing x . We will again show $f = (e \setminus \{w\}) \cup \{x\} \in E(G')$ via cases on whether or not e is in $E(G)$.

Case I. $e \in E(G)$

As $w \ll_G x$, $f \in E(G)$. Now, since e wasn't removed it either doesn't contain u , contains both u and v , or $(e \setminus \{u\}) \cup \{v\} \in E(G)$. If $u \notin e$, then $u \notin f$ and $f \in E(G')$, since we only remove edges containing u in the hypergraph move. Similarly, if e contains u and v , then so does f and we again have $f \in E(G')$ as we don't remove edges with both u and v in the hypergraph move. In now remains to handle the case where $(e \setminus \{u\}) \cup \{v\} \in E(G)$.

As $w \ll_G x$, we see $(e \setminus \{u, w\}) \cup \{v, x\} = (f \setminus \{u\}) \cup \{v\} \in E(G)$. But this again implies $f \in E(G')$.

Case II. $e \notin E(G)$

In this case, e must be added during the hypergraph move. Since we only add hyperedges containing v , $v \in e$. Moreover since e was added, $(e \setminus \{v\}) \cup \{u\} \in E(G)$. As $w \ll_G x$, $g = (e \setminus \{v, w\}) \cup \{u, x\} \in E(G)$. Now note that after the hypergraph move we must have that $(g \setminus \{u\}) \cup \{v\} = f \in E(G')$. □

Since we have handled all the cases, that is, $w = v$, $w = u$ and $w \neq u, v$, the proof now follows. □

Lemma 38. Let $G = (V, E)$ be a k -uniform hypergraph, $S \subseteq V$, $v \in S$ and $I = (A \cup B, E(I))$ be the incidence graph induced by S with respect to v . Let $x \in V(G)$ and $y \in V(G) \setminus S$ be such that $x \ll_G y$ and $s \ll_G y$ for all $s \in S$. Suppose that for every subset $b \in B$ with $\deg_I(b) \leq n^{2/3}$, we remove the edges $b \cup \{s\}$ for $s \in S$ from $E(G)$, resulting in a hypergraph G' . Then $x \ll_{G'} y$.

Proof. Let $e \in E(G')$ be an edge containing x and not y . We will show that $f = e \setminus \{x\} \cup \{y\} \in E(G')$. Since $E(G') \subseteq E(G)$, $e \in E(G)$. Combining this with the fact that $x \ll_G y$ implies $f \in E(G)$. Now fix some $s \in f \cap S \subseteq e \cap S$. We will show that $f \setminus \{s\}$ does not lead to the removal of f . To do so, we take cases on why $e \setminus \{s\}$ didn't lead to the removal of e .

Case I. $\deg_I(e \setminus \{s\}) > n^{2/3}$

Note that $s \neq x$ as $x \notin f$. Combining this with the fact that $x \ll_G y$, we observe that for any $t \neq y$, if $e \setminus \{s\} \cup \{t\} \in E(G)$ then $e \setminus \{s, x\} \cup \{t, y\} = (f \setminus \{s\}) \cup \{t\} \in E(G)$. So, $\deg_I(f \setminus \{s\}) \geq \deg_I(e \setminus \{s\}) > n^{2/3}$ and $f \setminus \{s\}$ doesn't lead to the removal of f .

Case II. $v \in e$

If $x \neq v$, then $v \in f$. Observe that by construction, the incidence graph has no edges corresponding to hyperedges in G containing v . So we never remove edges containing v , and $f \in E(G')$. Now assume that $x = v$. Since $s \ll_G y$, we have that $e \setminus \{s\} \cup \{y\} \in E(G)$. But now note $f \setminus \{s\} \cup \{v\} = e \setminus \{s\} \cup \{y\}$. It then follows that $f \setminus \{s\} \notin B$. Thus, we conclude $f \setminus \{s\}$ doesn't lead to the removal of f .

Case III. $e \setminus \{s\} \cup \{v\} \in E(G)$

Since $x \ll_G y$, $e \setminus \{s, x\} \cup \{v, y\} \in E(G)$. But $e \setminus \{s, x\} \cup \{v, y\} = f \setminus \{s\} \cup \{v\}$. Thus $f \setminus \{s\} \notin B$ and again doesn't cause the removal of f .

Since we chose s arbitrarily, $f \in E(G')$, as desired. □

With this, we can now prove Lemma 28.

Proof of Lemma 28. We apply the following algorithm to transform G into a threshold hypergraph:

Initially, let $T = \emptyset$. Repeat n times: Choose a vertex $v \in V \setminus T$. Apply the domination procedure with $G_0 = G$ from $V \setminus T$ to v , resulting in a hypergraphs G_1, G_2, \dots, G_k , intermediate hypergraphs H_0, \dots, H_{k-1} , and dominating sets D_0, \dots, D_{k-1} . Set $T = T \cup \{v\}$ and $G = G_k$.

Claim 39. When the algorithm terminates, G is a threshold hypergraph.

Proof. Suppose that at time r we have that $T = \{t_1, \dots, t_r\}$, where t_i is the i th vertex added to T . We prove by induction on r that $t_1 \gg_G t_2 \gg_G \dots \gg_G t_r \gg s$ for all $s \in V(G) \setminus T$. The base case of $r = 0$ is vacuously true. So now suppose the statement is true for some r .

Note that H_0 is the result of repeatedly deleting edges $b \cup \{s\}$ with $\deg(b) \leq n^{2/3}$ in the incidence graph of G induced by $V \setminus T$ with respect to v . So by Lemma 38, we have that $t_1 \gg_{H_0} t_2 \gg_{H_0} \dots \gg_{H_0} t_r \gg_{H_0} s$ for all $s \in V(G) \setminus T$.

After deleting low degree vertices, the domination procedure applies hypergraph moves from vertices in $V \setminus T$ to v . By Lemma 35, each of these moves preserves the ordering. Thus, $t_1 \gg_{G_1} t_2 \gg_{G_1} \dots \gg_{G_1} t_r \gg_{G_1} s$ for all $s \in V(G) \setminus T$.

We can now repeat this argument k times to get that $t_1 \gg_{G_k} t_2 \gg_{G_k} \dots \gg_{G_k} t_r \gg_{G_k} s$ for all $s \in V(G) \setminus T$. Finally, we are promised by Lemma 34 that $v \gg_{G_k} s$ for all $s \in V(G) \setminus (T \cup v)$. As $t_{r+1} = v$, the inductive hypothesis holds and the proof is complete. \square

Claim 40. *Each iteration applies at most $O_k(\sqrt{n})$ local moves.*

Proof. Note that by Corollary 33, every dominating set D_i has size at most $O_k(\sqrt{n})$. It then follows that each iteration uses at most $O_k(k\sqrt{n}) = O_k(\sqrt{n})$ local moves. \square

Note that the above claim implies that we make $O_k(n^{3/2})$ hypergraph moves in total.

Claim 41. *The algorithm removes $o_k(n^k)$ hyperedges in total.*

Proof. Fix some $b \in \binom{[n]}{k-1}$. We claim that b only causes the removal of edges at most k times. Indeed, after removing all edges of the form $b \cup \{s\}$, we must add some hyperedge containing b for b to cause the removal of more hyperedges. Let e be the first hyperedge containing b added after b 's ℓ th removal. e must have been added by some local move, say from s to v . Now since e is the first such edge we have that $b \not\subseteq e \setminus \{v\} \cup \{s\}$. So $v \in b$. Since we make local moves to any vertex at most once and there are $k - 1$ elements in b , it follows that b 's low degree only causes the remove of hyperedges at most k times. (Note that we are crucially using that b cannot be added and removed multiple times when running the domination procedure with a vertex $v \in b$. Indeed, if b is added due to a local move to v , it cannot be removed until the next iteration as local moves to v never remove hyperedges containing v , and after the addition of an edge e containing b , b will no longer appear in the incidence graph induced by $V \setminus T$ with respect to v .)

We now conclude that each $b \in \binom{[n]}{k-1}$ causes the removal of at most $kn^{2/3}$ edges. So in total we remove at most

$$\binom{n}{k-1} kn^{2/3} = o_k(n^k)$$

hyperedges in total. \square

With these three claims the proof is now complete as we have shown that the algorithm presented transforms the graph into a threshold hypergraph by using $O_k(n^{3/2})$ local moves and removing $o_k(n^k)$ hyperedges. \square

We end this section by remarking that the ideas in this proof simplify considerably in the case of graphs giving an alternative proof of Theorem 3. That said this argument loses slightly more homomorphisms than Theorem 20 and yields a weaker version of Theorem 4.

4. Applications of threshold maximisation

4.1 Maximisation on the clique

Before we begin with the applications, we start by reproving a result of [6] that shows for any graph H the quasi-clique maximises the number of homomorphisms from H for c sufficiently large. This will contrast with the results of Section 4.2, where we show that for small edge density c there are graphs H whose optimisers are threshold graphs that must have strictly more than two parts.

Theorem 42. *For every fixed graph H , there exists $k_H < 1$ such that for all $c > k_H$, $\mathcal{M}_H(c)$ is achieved on the quasi-clique.*

Proof. We will prove the result for a connected graph H . If the graph is not connected, we can apply the result to each of the components and take the maximum of the k_H 's over all the components.

Let T be a spanning tree of H . We note that clearly for any graph G , $t(H, G) \leq t(T, G)$ since T was obtained by removing edges from H . Now using a result from Sidorenko [15] we have that $t(T, G) \leq t(K_{1,|H|-1}, G)$ for all G . By a result of Reiher and Wagner [13], we have that there exists a k_H such that for $c > k_H$, $\mathcal{M}_{K_{1,|H|-1}}(c)$ is achieved on the quasi-clique. But now note that for any quasi-clique \mathcal{K} have that $t(H, \mathcal{K}) = t(K_{1,|H|-1}, \mathcal{K})$, so $\mathcal{M}_H(c)$ is also attained on the quasi-clique. \square

4.2 Graphs requiring three parts

In this section, we show that Theorem 22 implies that some graphs require at least 3 parts to optimise, which disproves a conjecture of Nagy that all graphs require only two parts [12].

Theorem 43. *Let H be a graph with $\alpha^*(H) > \alpha(H)$ and $\alpha^*(H) > \frac{|H|}{2}$, then for c sufficiently small, H is not optimised on the quasi-star or quasi-clique.*

To begin, we show that for all graphs H there exists a threshold graph T on three parts such that $t(H, T)$ matches the upper bound from Theorem 22 up to constant factors depending on H . We will then prove Theorem 44 by showing that threshold graphs with at most two parts have significantly fewer homomorphisms.

Lemma 44. *Let H be a fixed graph and n and m be positive integers such that $2n \leq m \leq \binom{n}{2}$, then there exists a threshold graph T with at most 3 parts, with n vertices and at most m edges, such that*

$$\text{hom}(H, T) = \Omega_H(m^{|H|-\alpha^*(H)} n^{2\alpha^*(H)-|H|}).$$

Proof. We will let G be a threshold graph with three parts:

$$\underbrace{11 \dots 1}_{\alpha \text{ 1's}} \underbrace{00 \dots 0}_{\beta \text{ 0's}} \underbrace{11 \dots 1}_{\gamma \text{ 1's}},$$

where we let $\alpha = \lfloor \sqrt{m} \rfloor$, $\gamma = \lfloor m/(2n) \rfloor$, $\beta = n - \alpha - \gamma$. Then we have that there are at most

$$\frac{(\sqrt{m})^2}{2} + \frac{m}{2n} n = m$$

edges as desired.

To analyse the number of homomorphisms, we note that we lost at most a factor of two from the floors. That is $\alpha \geq \sqrt{m}/2$ and $\gamma \geq (m/4n)$. Clearly, we also have that $\beta \geq n - \sqrt{m} - m/(2n) \geq n/25$.

Now let $f : V(H) \rightarrow \{0, 1/2, 1\}$ be an optimal fractional independence function, that is, $f(v_1), \dots, f(v_n)$ is an optimal solution to the linear program used to define the fractional independence number (see Definition 11). Note that such a function exists since the feasibility polytope for fractional independence is half-integral. Now, we claim that any injective function φ that sends $f^{-1}(1/2)$ to the first block of 1's, $f^{-1}(0)$ to the second block of 1's and $f^{-1}(1)$ to the block of 0's is a homomorphism. This claim will complete the proof since there are at least

$$\left(\frac{\sqrt{m}}{2}\right)^{|f^{-1}(1/2)|} \left(\frac{n}{25}\right)^{|f^{-1}(1)|} \left(\frac{m}{4n}\right)^{|f^{-1}(0)|} = \Omega_H(m^{|H|-\alpha^*(H)} n^{2\alpha^*(H)-|H|})$$

such functions. To see that these are homomorphisms, suppose $uv \in E(H)$. Without loss of generality assume $f(u) \leq f(v)$. Now if $u \in f^{-1}(0)$, then since every vertex in the final block of 1's is dominating we have that $\varphi(u)\varphi(v) \in E(G)$. Otherwise if $f(u) = f(v) = \frac{1}{2}$, then we have that both

$\varphi(u)$ and $\varphi(v)$ are mapped to vertices in the first block of ones. Since any two vertices in this block are connected, we have $\varphi(u)\varphi(v) \in E(G)$ and this is indeed a homomorphism. \square

Corollary 45. *Let H be a fixed graph and $c \in [0, 1]$, then $\mathcal{M}_H(c) = \Omega_H(c^{|H|-\alpha^*(H)})$.*

We will now show that if H satisfies the conditions of Theorem 43, then with only two parts we get far fewer homomorphisms from H as $c \rightarrow 0$ than the optimum.

Proof of Theorem 43. Since isolated vertices do not affect the optimising graph, we assume without loss of generality that H has no isolated vertices.

By Corollary 45, we have that there exists a graph G and constant C_1 such that

$$t(H, G) \geq C_1 c^{|H|-\alpha^*(H)}.$$

This implies that $\mathcal{M}_H(c) \geq C_1 c^{|H|-\alpha^*(H)}$ by Lemma 10. For the sake of contradiction, assume that H is optimised on the quasi-star or quasi-clique. Note that if it's optimised on the quasi-clique then we have that for all graphs G

$$t(H, G) \leq c^{|H|/2}.$$

Since $\alpha^*(H) > |H|/2$, we have that for c sufficiently small $C_1 c^{|H|-\alpha^*(H)} > c^{|H|/2}$, a contradiction.

So we must have that the homomorphism density of H is maximised on the quasi-star. Recall now that every vertex in the quasi-star either has degree d_0 or d_1 for some integers d_0 and d_1 with $d_0 < d_1$. Note that in any homomorphism from H to the quasi-star, the vertices in H mapping to the vertices in the quasi-star of degree d_0 must form an independent set. Hence we have at least $|H| - \alpha(H)$ vertices are mapped to one of the at most cn dominating vertices in the quasi-star of edge density c . By a union bound argument, this implies that there are at most $O_H(c^{n-\alpha(H)} n^{|H|})$ homomorphisms. But now note again that for c sufficiently small we have that $C_2 c^{|H|-\alpha(H)} < C_1 c^{|H|-\alpha^*(H)}$, which is a contradiction.

Thus such a graph H is optimised on neither the quasi-star nor the quasi-clique. \square

Corollary 46. *There exist graphs that are optimised on neither the quasi-star nor the quasi-clique. Moreover these graphs can be taken to be connected and threshold.*

Proof. Take $H = K_3 \sqcup K_{1,2}$. Then $\alpha(H) = 3$ and $\alpha^*(H) = 3.5$. The result then follows by Theorem 43.

If we wish to take the graph to be connected, adding edges from each vertex in the clique to the vertex of degree 2 in the star gives a graph H where we still have $\alpha(H) = 3$ and $\alpha^*(H) = 3.5$. Moreover, such a graph is a threshold graph. We can also take $H = K_\ell \sqcup K_{1,2}$ for any $\ell \geq 3$ if we want arbitrarily large examples. \square

Remark 47. *We note that a graph has $\alpha^*(H) = |H|/2$ if and only if H has a spanning subgraph that is the disjoint union of cycles and edges [2]. (Equivalently, $\alpha^*(H) = |H|/2$ if and only if $t(H, \cdot)$ is optimised on the quasi-clique for all edge densities $c \in [0, 1]$.) For the reverse direction, one simply checks that $\alpha^*(C_\ell) = \ell/2$ for all integers ℓ . The forward direction follows by noting that if $|S| > |N_H(S)|$ for some $S \subseteq V(H)$, then taking*

$$w_v := \begin{cases} 1 & v \in S \setminus N(S) \\ 0 & v \in N(S) \setminus S \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

gives a feasible solution to the fractional independence program of value strictly greater than $|H|/2$. The result now follows from a result of Alon [2], which states that $\max_{S \subseteq V(H)} |S| - |N_H(S)| = 0$ if and only if H has a spanning subgraph that is a disjoint union of cycles and edges.

5. Conclusion

We end with a few open questions. First, perhaps the most natural question to ask

Question 48. For any $c \in [0, 1]$ and graph H is it true that $t(H, \cdot)$ asymptotically maximised on a threshold graph with a bounded number of parts? Is there a bound on the number of parts that is independent of H ?

An intermediate question, easier to resolve than the above, is whether the simpler, one-dimensional limit objects of threshold graphs can help find the optimisers of $t(H, \cdot)$ for particular graphs H . (Recall that much of our current understanding of the optimisers for particular H 's of interest is the result of working with graphons, which are two-dimensional limit objects of graphs [12, 13].)

Question 49. Can the one-dimensional graphons for threshold graphs help us find the maximiser of $t(H, \cdot)$ with edge density at most c for various H ? In particular, can they be used to solve the problem when H is a k -star?

We also remark that our result in Theorem 4 is likely not tight and can probably be extended to sparser graphs, leading us to the next question

Question 50. Does Theorem 4 hold for sparser graphs, that is, for $m = o(n^{3/2})$?

Finally, we note that there are several more restrictive definitions of threshold hypergraphs [14]. For instance, one can consider the following notion.

Definition 51 (T_2 -Threshold Hypergraph). Let $G = (V, E)$ be a k -uniform hypergraph and n denote $|V(G)|$. We say G is a T_2 -threshold hypergraph if there exists a set of positive weights w_1, \dots, w_n and a real number t such that $e \in E(G)$ if and only if

$$\sum_{i \in e} w_i > t.$$

We note that T_2 -threshold hypergraphs are a strict subset of threshold hypergraphs [14] and ask whether our theorems can be extended to this more restrictive class.

Question 52. Does Theorem 6 hold for other definitions of threshold hypergraphs, such as T_2 -threshold hypergraphs?

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