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Singular sets and the Lavrentiev phenomenon

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We show that non-occurrence of the Lavrentiev phenomenon does not imply that the singular set is small. Precisely, given a compact Lebesgue null subset $E \subseteq \mathbb{R}$ and an arbitrary superlinearity, there exists a smooth strictly convex Lagrangian with this superlinear growth such that all minimizers of the associated variational problem have singular set exactly E but still admit approximation in energy by smooth functions.

1. Introduction

For a fixed closed bounded interval $[a, b] \subseteq \mathbb{R}$, we consider the problem of minimizing the functional

$$\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x))\,\mathrm{d}x \tag{1.1}$$

over the class of real-valued absolutely continuous functions $u \in \mathrm{AC}(a, b)$ with fixed boundary conditions, where the function $L = L(x, y, p)\colon \mathbb{R}^3 \to \mathbb{R}$, the *Lagrangian*, is a fixed function of class C^∞. The first general existence results were given by Tonelli [14, 15]; these require the assumptions of superlinearity and convexity of L in the variable p. Assuming the stronger condition that $L_{pp} > 0$, he also proved the following *partial regularity* theorem: minimizers of (1.1) are everywhere differentiable (possibly with infinite derivative) and this derivative is continuous as a map into the extended real line. Thus, the *singular set* of a minimizer, defined as those points where the derivative is infinite, is closed. Since the minimizer is absolutely continuous, we know immediately that it must also be of Lebesgue measure zero. A number of versions of Tonelli's partial regularity theorem, under significantly weaker hypotheses than Tonelli's original statement, can be found in [3–5, 7, 8, 13]; [9] shows that little further improvement is possible.

Tonelli proved conditions guaranteeing that the singular set is empty, i.e. that the minimizer is fully regular. That it can be non-empty given the assumption necessary for partial regularity (i.e. $L_{pp} > 0$) was shown by Ball and Mizel [2] who exhibited examples of minimizers with one-point singular sets. They also constructed, given an arbitrary closed set of measure zero E, a C^∞ Lagrangian depending only on (y, p), superlinear in p and with $L_{pp} > 0$, such that the unique minimizer of (1.1) has singular set precisely E.

Davie [6] completed this work by constructing, for an arbitrary closed null set E, a C^∞ Lagrangian $L = L(x, y, p)$, superlinear in p and with $L_{pp} > 0$, such that any minimizer has singular set exactly E. Davie constructs an admissible function

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$v \in \mathrm{AC}(a,b)$ and a Lagrangian L so that there exists a constant (in his notation) $(8\alpha)^{-1} > 0$ such that $\mathcal{L}(v) < (8\alpha)^{-1}$ but, for any admissible function $u \in \mathrm{AC}(a,b)$, if for some $c \in E$ we have that $u'(c)$ exists and is finite, then $\mathcal{L}(u) \geqslant (8\alpha)^{-1}$. Therefore, any minimizer (and at least one exists) must have infinite derivative on the set E. Thus, the proof rests on the fact that the energy of C^1 functions is bounded away from the infimum of the energy over all $\mathrm{AC}(a,b)$ functions, i.e. that the *Lavrentiev phenomenon* occurs. That such a gap can occur at all was first shown by Lavrentiev [10]. Since the corresponding example of Ball and Mizel described above is autonomous, i.e. has no dependence on the variable x, it follows by a result of Alberti and Serra Cassano [1] that there can be no Lavrentiev gap in this example.

This raises the question of the exact relationship between the singular set and the occurrence of the Lavrentiev phenomenon. If a problem exhibits the Lavrentiev phenomenon, then certainly the singular set of any minimizer over $\mathrm{AC}(a,b)$ must be non-empty, although it should be noted that the first examples of such problems found by Lavrentiev [10] and Manià [11] do not satisfy the $L_{pp} > 0$ condition required for classical partial regularity statements. That a minimizer has a non-empty singular set does not, of course, in general imply the occurrence of a Lavrentiev gap. Quite the reverse is in fact the case: one usually has to go to some effort to prove that a Lavrentiev gap does occur. However, it might be conjectured that if a minimizer has a *large* singular set, for example of Hausdorff dimension 1, then a gap must occur. Thus, the question is, can one prove Davie's result without inducing a Lavrentiev gap? We show, using the methods which Csörnyei *et al*. [5] introduced in the context of universal singular sets, that this is indeed possible, i.e. that the existence of a large singular set does *not* imply occurrence of the Lavrentiev phenomenon. Conversely, knowing that the Lavrentiev phenomenon does not occur does *not* tell us that the minimizer has small singular set, for example in the sense of Hausdorff dimension, nor indeed give us any information about the nature of the singular set not already available.

The methods of Csörnyei *et al*. also naturally allow us to construct a Lagrangian that gives this result and has arbitrary given superlinear growth, and so this result is a generalization of Davie's result even without the further result preventing a Lavrentiev gap.

We prove the following theorem.

THEOREM 1.1. *Let $[a,b]$ be a closed bounded subinterval of the real line and let $E \subseteq [a,b]$ be closed and Lebesgue null. Let $\omega \in C^\infty(\mathbb{R})$ be strictly convex such that $\omega(p) \geqslant \omega(0) = 0$ for all $p \in \mathbb{R}$ and $\omega(p)/|p| \to \infty$ as $|p| \to \infty$ (i.e. ω has superlinear growth).*

There then exists $L \in C^\infty(\mathbb{R}^3)$, $L = L(x,y,p)$, strictly convex in p and such that $L(x,y,p) \geqslant \omega(p)$ for all $(x,y,p) \in \mathbb{R}^3$, and a function $u \in \mathrm{AC}(a,b)$ such that

- *u is the unique minimizer of the functional (1.1) with respect to its own boundary conditions,*
- *the singular set of u is precisely E,*
- *there exist admissible functions $u_k \in C^\infty([a,b])$ (i.e. $u_k(a) = u(a)$ and $u_k(b) = u(b)$) such that $u_k \to u$ uniformly and $\mathcal{L}(u_k) \to \mathcal{L}(u)$.*

For the entire paper we shall assume that $[a, b]$, $\emptyset \neq E \subseteq [a, b]$ and ω are fixed as in theorem 1.1.

NOTATION. We let $\|\cdot\|$ denote the supremum norm on $\mathbb{R}^2$, which is the norm used throughout and for the following definitions. The diameter $\operatorname{diam}(X) \in [0, \infty)$ of a bounded set $X \subseteq \mathbb{R}^2$ is defined by $\operatorname{diam}(X) = \sup_{x,y \in X} \|x - y\|$. For two sets $X, Y \subseteq \mathbb{R}^2$, the notation $X \Subset Y$ is used when the closure $\bar{X}$ of X is compact and contained in Y, and the distance $\operatorname{dist}(X, Y) \in [0, \infty]$ between the two sets is defined by $\operatorname{dist}(X, Y) = \inf_{x \in X,\, y \in Y} \|x - y\|$ and is written $\operatorname{dist}(x, Y)$ when $X = \{x\}$ (this is understood to be $+\infty$ if one of the sets is empty). On the real line, for $r > 0$, we will use $B_r(X)$ for the r-neighbourhood of a subset $X \subseteq \mathbb{R}$.

For a bounded interval $[a, b]$ in $\mathbb{R}$, we shall write $\mathrm{AC}(a, b)$ for the class of absolutely continuous functions on $[a, b]$. For any function $u\colon \mathbb{R} \to \mathbb{R}$ we let $U\colon \mathbb{R} \to \mathbb{R}^2$ be given by $U(x) = (x, u(x))$. The supremum norm of a function on $\mathbb{R}^2$ shall be denoted by $\|\cdot\|_\infty$. Partial derivatives shall be denoted by subscripts, for example, Φ_x and Φ_y for functions $\Phi = \Phi(x, y)\colon \mathbb{R}^2 \to \mathbb{R}$. The Lebesgue measure on the real line shall be denoted by λ.

2. Calibration

Our approach to the construction of minimizers with infinite derivatives is inspired by that in [5]. We use a calibration argument to prove that functions with a specified derivative are minimizers of (1.1), where the Lagrangian L is constructed via a potential defined on $\mathbb{R}^2$. The original context of this method was the study of universal singular sets, specifically the construction of a Lagrangian with universal singular set containing a certain subset S of the plane. Thus, Csörnyei *et al*. constructed the potential to have singular behaviour at these points S. For each point in S a minimizer was constructed with derivative given via the potential (hence, infinite at that point) and graph passing through that point.

We need just one minimizer u but one that has infinite derivative at every point of the set E. Thus, it is more natural to begin by defining u (via its derivative), firstly because this is very easy and secondly because this readily gives us a sequence of smooth admissible functions approximating u with which we shall see that the Lavrentiev phenomenon does not occur. So, we approach the construction of the Lagrangian with the derivative of our intended minimizer already given and with this derivative construct a function ψ on the plane with which we can compare the potential. This is, then, the reverse logic to that used by Csörnyei *et al*. in which minimizers were selected to solve an ordinary differential equation given via the potential. Our function ψ is defined to mimic this idea in the sense that it agrees with the derivative of u on the graph of u'; so our minimizer does satisfy (almost everywhere) the ordinary differential equation $u' = \psi(x, u)$. This is, however, a consequence of our definition of ψ given u, not vice versa.

We first recall [5, lemma 10]. It is stated and used almost as in this original paper, except that later we also need an upper bound of the function for our smooth approximation estimates. We do not repeat the (simple) proof of the other statements.

Lemma 2.1. *There exists a C^∞ function $\gamma\colon \{(p,a,b)\in\mathbb{R}^3\colon b>0\}\to\mathbb{R}$ with the following properties.*

(1) *$p\mapsto\gamma(p,a,b)$ is convex.*

(2) *$\gamma(p,a,b)=0$ for $p\leqslant a-1$.*

(3) *$\gamma(p,a,b)=b(p-a)$ for $p\geqslant a+1$.*

(4) *$\gamma(p,a,b)\geqslant\max\{0,b(p-a)\}$.*

(5) *$\gamma(p,a,b)\leqslant b|p-a+1|$.*

Proof. Recalling the proof from [5], we see that $\gamma(p,a,b)=b\int_\infty^{p-a}\eta$, where the non-decreasing $\eta\in C^\infty(\mathbb{R})$ was chosen such that $\eta(x)=0$ if $x\leqslant-1$, $\eta(x)=1$ if $x\geqslant 1$, and $\int_{-1}^1\eta=1$. The only new statement, lemma 2.1(5), is trivial: if $p\leqslant a-1$ or $p\geqslant a+1$, then the result follows from lemma 2.1(2) or lemma 2.1(3), respectively. If $a-1\leqslant p\leqslant a+1$, then

$$\gamma(p,a,b)=b\int_\infty^{p-a}\eta(x)\,\mathrm{d}x\leqslant b\int_{-1}^{p-a}1\,\mathrm{d}x=b(p-a+1)\leqslant b|p-a+1|.$$

□

The next result is a version of [5, lemma 11]. The main difference, as discussed, is that ψ is given before the potential Φ. We recall that for a function $u\colon[a,b]\to\mathbb{R}$, the function $U\colon[a,b]\to\mathbb{R}^2$ is given by $U(x)=(x,u(x))$.

Lemma 2.2. *Let $S\subseteq\mathbb{R}^2$ be compact, let $\psi\in C^\infty(\mathbb{R}^2\backslash S)$ be such that $\psi(x,y)\to\infty$ as $\operatorname{dist}((x,y),S)\to 0$ and let $\Phi\in C^\infty(\mathbb{R}^2\backslash S)\cap C(\mathbb{R}^2)$ satisfy the following conditions.*

(1) *$-\Phi_x(x,y)\geqslant 4\Phi_y(x,y)>0$ for all $(x,y)\in\mathbb{R}^2\backslash S$.*

(2) *$\Phi_y(x,y)>320\omega'(\psi(x,y))$ for all $(x,y)\in\mathbb{R}^2\backslash S$.*

(3) *$-2(\Phi_x/\Phi_y)(x,y)\leqslant\psi(x,y)\leqslant-160(\Phi_x/\Phi_y)(x,y)$ for all $(x,y)\in\mathbb{R}^2\backslash S$.*

(4) *For all $u\in\mathrm{AC}(a,b)$, the sets $U^{-1}(S)$ and $(\Phi\circ U)(U^{-1}(S))$ are Lebesgue null.*

There then exists a Lagrangian $L\in C^\infty(\mathbb{R}^3)$, strictly convex in p and satisfying $L(x,y,p)\geqslant\omega(p)$ for all $(x,y,p)\in\mathbb{R}^3$, such that, for all $u\in\mathrm{AC}(a,b)$,

$$\mathcal{L}(u)=\int_a^b L(x,u(x),u'(x))\,\mathrm{d}x\geqslant\Phi(U(b))-\Phi(U(a))$$

with equality if and only if $u'(x)=\psi(x,u(x))$ for almost every $x\in[a,b]$. In particular, any such u is the unique minimizer of (1.1) with respect to its boundary conditions.

Proof. This mimics the proof of [5, lemma 11]. Define $\theta,\xi\in C^\infty(\mathbb{R}^2\backslash S)$ by

$$\theta=\Phi_y-\omega'(\psi)\quad\text{and}\quad\xi=\frac{-\Phi_x+\omega(\psi)-\omega'(\psi)\psi}{\theta}.$$

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Fix $(x, y) \in \mathbb{R}^2 \backslash S$. Then note by lemma 2.2(3) and lemma 2.2(1) that $\psi > 0$, so by properties of ω we have that $\omega'(\psi) > 0$. So, using also lemma 2.2(2), we have that

$$\theta > \Phi_y - \tfrac{1}{320}\Phi_y = \tfrac{319}{320}\Phi_y > 0, \tag{2.1}$$

and so ξ is well defined. By convexity of ω, we have that $\omega(p) - \omega'(p)p \leqslant \omega(0) = 0$ for all $p \geqslant 0$. So, using this, lemma 2.2(3) and lemma 2.2(2), we see that

$$-\Phi_x \geqslant -\Phi_x + \omega(\psi) - \omega'(\psi)\psi = \xi\theta \geqslant -\Phi_x - \omega'(\psi)\psi \geqslant -\Phi_x + \frac{\Phi_y}{320} \times \frac{160\Phi_x}{\Phi_y} = -\frac{\Phi_x}{2}. \tag{2.2}$$

So, since $\Phi_y = \theta + \omega'(\psi) > \theta > 0$, we see by lemma 2.2(1) that

$$\xi \geqslant -\Phi_x/(2\theta) \geqslant -\Phi_x/(2\Phi_y) \geqslant 2, \tag{2.3}$$

and so, using lemma 2.2(3), (2.1) and (2.2),

$$\psi \geqslant -2\Phi_x/\Phi_y \geqslant -2 \times 319\Phi_x/(320\theta) \geqslant 3\xi/2 \geqslant \xi + 1.$$

The point of these estimates, and the choice of constants in the assumptions that allows them to be derived, is that

$$0 \leqslant \xi - 1 \tag{2.4}$$

and

$$\psi \geqslant \xi + 1. \tag{2.5}$$

We use the corner-smoothing function γ from lemma 2.1 to define

$$F(x, y, p) = \begin{cases} \gamma(p, \xi(x, y), \theta(x, y)) & (x, y) \in \mathbb{R}^2 \backslash S, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $F \in C^\infty((\mathbb{R}^2 \backslash S) \times \mathbb{R})$. For fixed $p \in \mathbb{R}$, by the growth assumption on ψ, there exists an open set $\Omega \supseteq S$ such that $\psi > 320(p + 2)$ on Ω. By (2.3) and lemma 2.2(3), we see that $\xi \geqslant -\Phi_x/(2\Phi_y) \geqslant \psi/320 \geqslant p + 2$ and so $F = 0$ on $\Omega \times (-\infty, p + 1)$, by lemma 2.1(2) of γ. So in fact, $F \in C^\infty(\mathbb{R}^3)$. Clearly, $F \geqslant 0$ by lemma 2.1(4) and is convex in p by lemma 2.1(1).

Defining $L(x, y, p) = F(x, y, p) + \omega(p)$ gives a Lagrangian $L \in C^\infty(\mathbb{R}^3)$ such that $L \geqslant \omega$ and L is strictly convex in p. For $(x, y) \in \mathbb{R}^2 \backslash S$, we have, by convexity of ω and property (4) of γ (see lemma 2.1), that

$$\begin{aligned} L(x, y, p) &\geqslant \omega(\psi(x, y)) + \omega'(\psi(x, y))(p - \psi(x, y)) + \theta(x, y)(p - \xi(x, y)) \\ &= \Phi_x(x, y) + p\Phi_y(x, y). \end{aligned}$$

Moreover, $p = \psi(x, y)$ implies equality by (2.5) and lemma 2.1(3), and equality in this inequality implies that $p = \psi(x, y)$ by strict convexity of ω. Thus, equality holds in this inequality if and only if $p = \psi(x, y)$.

Let $u \in \mathrm{AC}(a, b)$. Since $\Phi \in C^\infty(\mathbb{R}^2 \backslash S)$, we see that $(\Phi \circ U)\colon [a, b] \to \mathbb{R}$ is differentiable for almost every $x \notin U^{-1}(S)$, which is almost everywhere on $[a, b]$ by

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lemma 2.2(4), with derivative $(\Phi \circ U)'(x) = \Phi_x(U(x)) + u'(x)(\Phi_y(U(x)))$, and for almost every $x \in [a, b]$ the above inequality implies that

$$L(x, u(x), u'(x)) \geqslant \Phi_x(x, u(x)) + u'(x)\Phi_y(x, u(x)) = (\Phi \circ U)'(x) \tag{2.6}$$

with equality if and only if $u'(x) = \psi(x, u(x))$. We note that $(\Phi \circ U)$ has the Lusin property, i.e. it maps null sets to null sets: lemma 2.2(4) implies that any subset of $U^{-1}(S)$ is mapped to a null set and on $[a, b]\backslash U^{-1}(S)$ the function $(\Phi \circ U)$ is locally absolutely continuous since $\Phi \in C^\infty(\mathbb{R}^2\backslash S)$.

Let $\{(a_j, b_j)\}_{j \in J}$ be the (at most countable) sequence of components of $(a, b) \backslash U^{-1}(S)$ such that $\Phi(U(a_j)) < \Phi(U(b_j))$ (if there are no such components, then the result is trivial). Then, using that $(\Phi \circ U)$ is locally absolutely continuous on $(a, b)\backslash U^{-1}(S)$ and the fact from lemma 2.2(4) that $(\Phi \circ U)(U^{-1}(S))$ is null, we see, using (2.6), that

$$\begin{aligned}
\int_a^b L(x, u(x), u'(x))\, \mathrm{d}x &\geqslant \sum_{j \in J} \int_{a_j}^{b_j} L(x, u(x), u'(x))\, \mathrm{d}x \\
&\geqslant \sum_{j \in J} \int_{a_j}^{b_j} \max\{0, (\Phi \circ U)'\}\, \mathrm{d}x \\
&\geqslant \sum_{j \in J} \Phi(U(b_j)) - \Phi(U(a_j)) \\
&\geqslant \Phi(U(b)) - \Phi(U(a)).
\end{aligned}$$

Equality in this relation implies that $L(x, u(x), u'(x)) = (\Phi \circ U)'(x)$ for almost every $x \in \bigcup_{j \in J}(a_j, b_j)$, but also that $\bigcup_{j \in J}(a_j, b_j) = (a, b)\backslash U^{-1}(S)$. Therefore, in fact, $L(x, u(x), u'(x)) = (\Phi \circ U)'(x)$ for almost every $x \in (a, b)\backslash U^{-1}(S)$. By (2.6), this implies that $u'(x) = \psi(x, u(x))$ for almost every $x \in [a, b]$ since $U^{-1}(S)$ is null by lemma 2.2(4).

Conversely, $u'(x) = \psi(x, u(x))$ almost everywhere implies by lemma 2.2(3) that

$$(\Phi \circ U)'(x) = (\Phi_x \circ U)(x) + \psi(x)(\Phi_y \circ U)(x) \geqslant (-\Phi_x \circ U)(x) \geqslant 0$$

almost everywhere. This, combined with the fact that $(\Phi \circ U)$ has the Lusin property, implies that $(\Phi \circ U)$ is absolutely continuous [12, ch. IX, theorem 7.7]. Moreover, (2.6) gives that $L(x, u(x), u'(x)) = (\Phi \circ U)'(x)$ almost everywhere, and hence

$$\int_a^b L(x, u(x), u'(x))\, \mathrm{d}x = \int_a^b (\Phi \circ U)'(x)\, \mathrm{d}x = \Phi(U(b)) - \Phi(U(a)),$$

as required. □

3. Construction of the minimizer

We now begin the construction of our future minimizer u by constructing first its derivative ϕ. The essential property of ϕ is that $\phi(x) \to \infty$ as $\operatorname{dist}(x, E) \to 0$. We naturally define ϕ as the limit of a sequence of non-negative $C^\infty(\mathbb{R})$ functions $\{\phi_k\}_{k=0}^\infty$, where each ϕ_k is bounded above and, on an open set V_k covering E, attains this bound (which tends to ∞ as $k \to \infty$). We construct the ϕ_k so that

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their primitives u_k will be admissible functions in (1.1) (i.e. have the same boundary conditions as u) and converge uniformly to u. In fact, we shall guarantee that $u = u_k$ off V_k. So, since our Lagrangian will be constructed as in lemma 2.2, our estimates showing that there is no Lavrentiev gap reduce just to estimates of the integral over V_k of a function involving the gradient of the potential $\varPhi$. This then requires a certain upper bound for the measure of V_k. We must also remember that our potential $\varPhi$ must have a gradient that satisfies inequalities involving ϕ and, hence, ϕ_k. This $\varPhi$ will be defined, just as in [5], using a sequence of $C^\infty(\mathbb{R}^2)$ functions $\{\varPhi^k\}_{k=0}^\infty$ that have appropriately steep gradients on open sets $\varOmega_k$ around $U(E)$. To guarantee that these $\varPhi^k$ converge, these sets must be small in the directions of these gradients, which is most easily achieved by ensuring they are small in all directions. We choose $\varOmega_k$ so that this measure is controlled by that of V_k; this gives another upper bound for the measure of V_k. Other bounds are required for technical reasons in the proof; we impose just one inequality, which suffices to give all the results.

For $k \geqslant 0$, let $\{h_k\}_{k=0}^\infty$ and $\{A_k\}_{k=0}^\infty$ be strictly increasing sequences of real numbers tending to infinity such that $h_0, A_0 \geqslant 1$. We will eventually need to define explicit values for these sequences to satisfy the exact inequalities required in lemma 2.2, but until we make these definitions, the construction requires only these general assumptions.

Define $V_0 = \mathbb{R}$ and $W_0 = (a-1, b+1)$. For $k \geqslant 1$, we find decreasing sequences of open sets $W_k \Subset V_k \subseteq \mathbb{R}$ covering E of the form

$$V_k = \bigcup_{i=1}^{n_k} (a_k^i, b_k^i) \subseteq \mathbb{R}, \quad \text{where } a_k^1 < b_k^1 \leqslant a_k^2 < b_k^2 \leqslant \cdots \leqslant a_k^{n_k} < b_k^{n_k},$$

and

$$W_k = \bigcup_{i=1}^{n_k} (\tilde{a}_k^i, \tilde{b}_k^i) \subseteq \mathbb{R}, \quad \text{where } a_k^i < \tilde{a}_k^i < \tilde{b}_k^i < b_k^i \text{ for all } 1 \leqslant i \leqslant n_k,$$

for some $n_k \geqslant 1$, such that

$$V_k \subseteq B_{2^{-k}}(E); \tag{3.1}$$

$$V_k \Subset W_{k-1}; \tag{3.2}$$

$$\lambda(V_k) \leqslant (1 + 4(\operatorname{dist}(E, \mathbb{R}\backslash W_{k-1}))^{-1} \times 2^{k+3} A_{k+1}^2 (h_{k+1}+2)(1+\omega(h_{k+1}+2))n_{k-1})^{-1}. \tag{3.3}$$

For $k \geqslant 1$ we define

- $r_k^i = b_k^i - a_k^i > 0$ and $r_k = \min_{1\leqslant i\leqslant n_k} r_k^i > 0$,
- $\tilde{r}_k^i = \tilde{b}_k^i - \tilde{a}_k^i > 0$ and $\tilde{r}_k = \min_{1\leqslant i\leqslant n_k} \tilde{r}_k^i > 0$,
- $\delta_k = \operatorname{dist}(E, \mathbb{R}\backslash V_k)$, where this is strictly positive by compactness of E.

Let $V_k^i = (a_k^i, b_k^i)$ and $W_k^i = (\tilde{a}_k^i, \tilde{b}_k^i)$ for each $1 \leqslant i \leqslant n_k$. We assume that each component of W_k contains a point of E. Then, since $\lambda(V_k) \to 0$ as $k \to \infty$, we see that $\bigcap_{k=1}^\infty V_k = E$. We assume further, as would be natural in the construction of the sets, that, when $E \subseteq (a, b)$, the sets $V_k \subseteq (a, b)$ and otherwise, i.e. when a

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or $b \in E$, the interval(s) covering the endpoint(s) are centred around the relevant endpoint(s) and that all the other intervals lie inside (a,b). Then, in all cases,

$$\tilde{r}_k \leqslant 2\lambda(W_k^j \cap (a,b)) \tag{3.4}$$

for all $1 \leqslant j \leqslant n_k$ and all $k \geqslant 1$.

The sets W_k only play a role later when we have to define the function ψ on the plane that equals u' on the graph of u: it becomes at that point necessary for us to have a gap between the sets V_k, where we shall stipulate the value of u_k', and the sets W_k, where we permit some non-zero addition to u_{k-1}' in the definition of u_k'. Until lemma 4.2, however, little is lost if one does not distinguish between V_k and W_k.

Note that for $1 \leqslant i \leqslant n_k$, $x \in V_k^i$ and $y \notin V_{k-1}$, we in fact, choosing $z \in E \cap V_k^i$, have that $|z-x| \leqslant r_k^i \leqslant \lambda(V_k)$, and so, by (3.3),

$$|x-y| \geqslant |y-z| - |z-x| \geqslant \delta_{k-1} - \lambda(V_k) \geqslant \delta_{k-1}/2.$$

Thus, since this holds for all $1 \leqslant i \leqslant n_k$,

$$\operatorname{dist}(V_k, \mathbb{R}\backslash V_{k-1}) \geqslant \operatorname{dist}(E, \mathbb{R}\backslash V_{k-1})/2. \tag{3.5}$$

Lemma 3.1. *There exists a strictly increasing function $u \in C^\infty([a,b]\backslash E) \cap C([a,b])$ and a sequence $\{u_k\}_{k=0}^\infty$ of strictly increasing functions $u_k \in C^\infty([a,b])$ such that, for all $k \geqslant 0$,*

1. *$u(a) = u_k(a)$ and $u(b) = u_k(b)$,*
2. *$u(x) = u_k(x)$ for all $x \in [a,b]\backslash W_k$ and consequently $u'(x) = u_k'(x)$ for all $x \in [a,b]\backslash \bar{W}_k$,*
3. *$u'(x) \geqslant h_k$ for all $x \in V_k\backslash E$ and $u_k'(x) \geqslant h_l$ for all $x \in V_l$ for all $0 \leqslant l \leqslant k$,*
4. *$u'(x) = u_k'(x)$ for all $x \in [a,b]\backslash V_k$ and $u_k'(x) \leqslant h_k + 2$ for all $x \in [a,b]$,*
5. *$u_k \to u$ uniformly on $[a,b]$.*

Proof. We first exhibit a sequence $\{\phi_k\}_{k=0}^\infty$ of functions $\phi_k \in C^\infty(\mathbb{R})$ such that, for all $k \geqslant 0$,

$$1 \leqslant \phi_k(x) \leqslant h_k + 2 \quad \text{for all } x \in \mathbb{R}, \tag{3.6}$$

$$h_k + 1 \leqslant \phi_k(x) \quad \text{for all } x \in V_k, \tag{3.7}$$

$$\phi_k(x) = \phi_l(x) \quad \text{for all } x \in \mathbb{R}\backslash W_l \text{ for all } 0 \leqslant l \leqslant k, \tag{3.8}$$

$$h_l \leqslant \phi_k(x) \quad \text{for } x \in V_l \text{ for all } 0 \leqslant l \leqslant k, \tag{3.9}$$

$$\int_{W_l^i \cap (a,b)} \phi_k = \int_{W_l^i \cap (a,b)} \phi_l \quad \text{for all } 1 \leqslant i \leqslant n_l \text{ and all } 0 \leqslant l \leqslant k. \tag{3.10}$$

Define $\phi_0(x) = h_0 + 1$ for all $x \in \mathbb{R}$, which clearly satisfies (3.6)–(3.10). Let $k \geqslant 1$ and consider $1 \leqslant j \leqslant n_{k-1}$. Note that (3.3) and (3.4) imply that

$$\begin{aligned}\lambda(W_{k-1}^j \cap V_k \cap (a,b)) \leqslant \lambda(V_k) \leqslant \frac{\operatorname{dist}(E, \mathbb{R}\backslash W_{k-1})}{2(h_k - h_{k-1} + 1)} &\leqslant \frac{\tilde{r}_{k-1}}{2(2(h_k - h_{k-1}) + 1)} \\ &\leqslant \frac{\lambda(W_{k-1}^j \cap (a,b))}{2(h_k - h_{k-1}) + 1},\end{aligned}$$

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and so

$$\frac{\lambda(W_{k-1}^j \cap (a,b))}{\lambda(V_k \cap W_{k-1}^j \cap (a,b))} \geqslant 2(h_k - h_{k-1}) + 1 > h_k - h_{k-1} + 1.$$

Hence, we can choose $\rho_k \in C^\infty(\mathbb{R})$ such that

$$\rho_k(x) = 0 \quad \text{for all } x \in \mathbb{R}\backslash W_{k-1}, \tag{3.11}$$

$$-1 \leqslant \rho_k(x) \leqslant h_k - h_{k-1} \quad \text{for all } x \in \mathbb{R}, \tag{3.12}$$

$$\rho_k(x) = h_k - h_{k-1} \quad \text{for all } x \in V_k, \tag{3.13}$$

$$\int_{W_{k-1}^j \cap (a,b)} \rho_k = 0 \quad \text{for each } 1 \leqslant j \leqslant n_{k-1}. \tag{3.14}$$

For example, fix $1 \leqslant j \leqslant n_{k-1}$ and note that when considering open sets G_k^j and $\tilde{G}_k^j$ such that

$$V_k \cap W_{k-1}^j \Subset G_k^j \Subset \tilde{G}_k^j \Subset W_{k-1}^j,$$

the value $\lambda(\tilde{G}_k^j \cap (a,b))/\lambda(G_k^j \cap (a,b))$ depends continuously on the measures of the two sets $\tilde{G}_k^j \cap (a,b)$ and $G_k^j \cap (a,b)$ and takes values greater than but arbitrarily close to 1, and less than but arbitrarily close to $\lambda(W_{k-1}^j \cap (a,b))/\lambda(W_{k-1}^j \cap V_k \cap (a,b)) > h_k - h_{k-1} + 1$. Thus, we may choose sets $\tilde{G}_k^j$ and G_k^j such that

$$\frac{\lambda(\tilde{G}_k^j \cap (a,b))}{\lambda(G_k^j \cap (a,b))} = h_k - h_{k-1} + 1,$$

that is,

$$(h_k - h_{k-1})\lambda(G_k^j \cap (a,b)) = \lambda((\tilde{G}_k^j \cap (a,b))\backslash(G_k^j \cap (a,b))).$$

Then, defining $\rho_k^j \colon \mathbb{R} \to \mathbb{R}$ by

$$\rho_k^j(x) = \begin{cases} h_k - h_{k-1}, & x \in G_k^j, \\ -1, & x \in \tilde{G}_k^j \backslash G_k^j, \\ 0, & \text{otherwise}, \end{cases}$$

we see that

$$\int_a^b \rho_k^j = \int_{\tilde{G}_{k-1}^j \cap (a,b)} \rho_k^j = (h_k - h_{k-1})\lambda(G_k^j \cap (a,b)) - \lambda((\tilde{G}_k^j \cap (a,b))\backslash(G_k^j \cap (a,b))) = 0.$$

Choosing an appropriate mollification, we can assume that ρ_k^j is of class $C^\infty(\mathbb{R})$, the same equation holds and ρ_k^j satisfies (3.11)–(3.14) with V_k replaced by $W_{k-1}^j \cap V_k$ in (3.13). Then, defining $\rho_k = \sum_{j=1}^{n_{k-1}} \rho_k^j$ gives ρ_k as claimed.

Using this ρ_k, we now suppose ϕ_{k-1} to be defined and set $\phi_k = \phi_{k-1} + \rho_k$. This defines our sequence $\{\phi_k\}_{k=0}^\infty$. We now show by induction on $k \geqslant 0$ that these functions satisfy the requirements (3.6)–(3.10). Let $k \geqslant 1$ and suppose that ϕ_{k-1} has been constructed in this way and satisfies all the conditions.

By (3.11), we see that $\phi_k = \phi_{k-1}$ off W_{k-1}, which gives (3.8) by inductive hypothesis and since $\{W_k\}_{k=0}^\infty$ is a decreasing sequence. Then, for points not in W_{k-1}, we see that the inequality in (3.6) holds by inductive hypothesis (3.6) and since $\{h_k\}_{k=0}^\infty$

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is an increasing sequence. For $x \in W_{k-1}$ we have, by inductive hypothesis (3.7), (3.12) and inductive hypothesis (3.6), that

$$1 \leqslant h_{k-1} \leqslant \phi_{k-1}(x) - 1 \leqslant \phi_k(x) \leqslant \phi_{k-1}(x) + (h_k - h_{k-1}) \leqslant h_k + 2.$$

Hence, the inequality in (3.6) holds everywhere, as required. Note that for $x \in V_k$, by (3.13), inductive hypothesis (3.7) and since $V_k \subseteq V_{k-1}$, we have that

$$\phi_k(x) = \phi_{k-1}(x) + h_k - h_{k-1} \geqslant h_k + 1,$$

as is required for (3.7). This implies (3.9) when $x \in V_k$. Otherwise, choose the greatest index $0 \leqslant l < k$ such that $x \in V_l$. If $l < k-1$, then $x \notin V_{k-1}$, so (3.9) follows by (3.11) and inductive hypothesis (3.9). If $l = k-1$, then $x \in V_{k-1}$, and so, by (3.12) and inductive hypothesis (3.7),

$$\phi_k(x) \geqslant \phi_{k-1}(x) - 1 \geqslant h_{k-1},$$

and hence (3.9) holds in all cases. For the claim (3.10), there is nothing to prove for $l = k$, so let $0 \leqslant l < k$ and fix $0 \leqslant i \leqslant n_l$. Then, using (3.11), (3.14) and the inductive hypothesis, we have that

$$\begin{aligned}\int_{W_l^i \cap (a,b)} \phi_k &= \int_{W_l^i \cap (a,b)} \phi_{k-1} + \int_{W_l^i \cap (a,b)} \rho_k \\ &= \int_{W_l^i \cap (a,b)} \phi_{k-1} + \int_{W_l^i \cap W_{k-1} \cap (a,b)} \rho_k \\ &= \int_{W_l^i \cap (a,b)} \phi_{k-1} \\ &= \int_{W_l^i \cap (a,b)} \phi_l,\end{aligned}$$

using that $W_l^i \cap W_{k-1} \cap (a,b)$ consists of components $W_{k-1}^j \cap (a,b)$ of $W_{k-1} \cap (a,b)$, because $\{W_k\}_{k=0}^{\infty}$ is decreasing.

Using (3.1), we see that for all $x \notin E$ there is $k \geqslant 1$ such that $x \notin W_l$ for all $l \geqslant k$, and thus, by (3.8), letting $\phi(x) = \lim_{k\to\infty} \phi_k(x)$ defines a well-defined function $\phi \in C^{\infty}(\mathbb{R}\backslash E)$ such that

$$\phi(x) = \phi_k(x) \quad \text{for all } x \notin W_k. \tag{3.15}$$

By (3.6), we have that $\phi(x) \geqslant 1$ for all $x \in \mathbb{R}\backslash E$.

Now, $|\phi_k| \leqslant |\phi_0| + \sum_{l=1}^{\infty} |\rho_l|$ for all $k \geqslant 0$ and using (3.11), (3.12) and (3.3) we see that

$$\begin{aligned}\int_a^b |\phi_0| + \int_a^b \sum_{l=1}^{\infty} |\rho_l| &\leqslant (b-a)(h_0+1) + \sum_{l=1}^{\infty} \lambda([a,b] \cap W_{l-1})(h_l - h_{l-1} + 1) \\ &\leqslant (b-a)(h_0 + h_1 + 2) + \sum_{l=1}^{\infty} (h_{l+1}+1)\lambda(V_l) \\ &\leqslant (b-a)(2h_1+2) + \sum_{l=1}^{\infty} 2^{-l} \\ &< \infty.\end{aligned}$$

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So, by the dominated convergence theorem, $\phi \in L^1(a,b)$ and

$$\int_a^b \phi_k \to \int_a^b \phi \quad \text{as } k \to \infty. \tag{3.16}$$

We now define strictly increasing functions $u_k \in C^\infty([a,b])$ for each $k \geqslant 0$ and $u \in C^\infty([a,b]\backslash E) \cap C([a,b])$ by

$$u_k(x) = \int_a^x \phi_k(t)\,\mathrm{d}t \quad \text{and} \quad u(x) = \int_a^x \phi(t)\,\mathrm{d}t,$$

and so $u_k' = \phi_k$ everywhere and $u' = \phi$ off E, in particular, almost everywhere. Lemma 3.1(3) follows immediately from (3.9). Lemma 3.1(4) follows immediately from (3.8) and (3.6). Lemma 3.1(1) follows since, by (3.10) and (3.16),

$$u_k(b) = \int_a^b \phi_k = \int_{W_0\cap(a,b)} \phi_k = \int_{W_0\cap(a,b)} \phi_0 = \int_{W_0\cap(a,b)} \phi = u(b),$$

and since clearly $u_k(a) = 0 = u(a)$ by definition.

Let $k \geqslant 0$ and suppose that $x \in [a,b]\backslash W_k$. Then either we have $x \leqslant \tilde{a}_k^i$ for all $1 \leqslant i \leqslant n_k$ or we have, for some $1 \leqslant i_x \leqslant n_k$, that $\tilde{b}_k^{i_x} \leqslant x$ and $x \leqslant \tilde{a}_k^i$ for all $i_x < i \leqslant n_k$. In the first case we see immediately that, since $[a,x] \cap W_k = \emptyset$, (3.15) implies that

$$u(x) = \int_a^x \phi(t)\,\mathrm{d}t = \int_a^x \phi_k(t)\,\mathrm{d}t = u_k(x).$$

Otherwise we argue by (3.16), (3.10) and (3.8) that

$$\begin{aligned} u(x) = \int_a^x \phi &= \sum_{i=1}^{i_x} \int_{W_k^i\cap(a,b)} \phi + \int_{[a,x]\backslash W_k} \phi \\ &= \sum_{i=1}^{i_x} \int_{W_k^i\cap(a,b)} \phi_k + \int_{[a,x]\backslash W_k} \phi_k \\ &= \int_a^x \phi_k \\ &= u_k(x), \end{aligned}$$

as required for lemma 3.1(2).

Fix $1 \leqslant i \leqslant n_k$ and let $x \in W_k^i$. Since u and u_k are increasing, using lemma 3.1(2), (3.6) and (3.3), we see that

$$\begin{aligned} |u_k(x) - u(x)| &\leqslant u_k(\tilde{b}_k^i) - u(\tilde{a}_k^i) \\ &= u_k(\tilde{b}_k^i) - u_k(\tilde{a}_k^i) = \int_{\tilde{a}_k^i}^{\tilde{b}_k^i} \phi_k \\ &\leqslant (h_k+2)(\tilde{b}_k^i - \tilde{a}_k^i) \leqslant (h_k+2)\lambda(V_k) \\ &\leqslant 2^{-k}. \end{aligned}$$

Since $u = u_k$ off W_k, we then have that $\sup_{x\in[a,b]}|u_k(x) - u(x)| \leqslant 2^{-k}$, and hence u_k converges to u uniformly, as required for lemma 3.1(5). □

4. Construction of the potential

The construction of our potential, Φ, is based on that which constitutes the proof of theorem 10 in [5]. We construct a sequence of $C^\infty(\mathbb{R}^2)$ functions $\{\Phi^k\}_{k=0}^\infty$ that have steep gradients on open sets Ω_k around the graph $U(E)$ of u on E. Because we have fixed the derivative ϕ of our minimizer u with which we have to compare the derivatives of Φ, the sets Ω_k are now given before the construction. This contrasts with the situation of Csörnyei *et al*. where the sets could be chosen small enough at each stage of the construction of the sequence. We have of course carefully chosen Ω_k, or more precisely in fact V_k, so that all the properties required at this stage hold with these fixed sets.

Let $\Omega_0 = \mathbb{R}^2$ and for $k \geqslant 1$ and $1 \leqslant i \leqslant n_k$ define $\Omega_k^i = V_k^i \times u(V_k^i) = V_k^i \times u_k(V_k^i)$ and $\Omega_k = \bigcup_{i=1}^{n_k} \Omega_k^i$. So, Ω_k is an open set satisfying $\Omega_k \Supset U(E)$.

Lemma 4.1. *For this definition of the sequence $\{\Omega_k\}_{k=0}^\infty$, we have that*

$$\bigcap_{k=1}^{\infty} \Omega_k = U(E) \tag{4.1}$$

and for each $k \geqslant 1$,

$$\operatorname{dist}(\Omega_k, \mathbb{R}^2 \backslash \Omega_{k-1}) \geqslant \delta_{k-1}/2, \tag{4.2}$$

$$\sum_{i=1}^{n_k} \operatorname{diam}(\Omega_k^i) < \lambda(V_k)(h_k + 2). \tag{4.3}$$

Proof. The inclusion $U(E) \subseteq \bigcup_{k=1}^\infty \Omega_k$ is clear. Let $(x, y) \notin U(E)$. If $x \notin E$, then there exists $k \geqslant 1$ such that $x \notin V_k$, so $(x, y) \notin \Omega_k$. Otherwise, $x \in E$ but $y \neq u(x)$. Since $x \in E$, for all $k \geqslant 1$ there exists $1 \leqslant i_k \leqslant n_k$ such that $x \in V_k^{i_k}$. Since $|b_k^{i_k} - a_k^{i_k}| < \lambda(V_k) \to 0$, there exists $k \geqslant 1$ such that $|u(b_k^{i_k}) - u(a_k^{i_k})| < |y - u(x)|/2$. If $y \in (u(a_k^{i_k}), u(b_k^{i_k}))$, then

$$|y - u(x)| \leqslant |y - u(a_k^{i_k})| + |u(a_k^{i_k}) - u(x)| \leqslant 2|u(b_k^{i_k}) - u(a_k^{i_k})| < |y - u(x)|,$$

which is a contradiction, so $y \notin (u(a_k^{i_k}), u(b_k^{i_k}))$. Since $x \in (a_k^{i_k}, b_k^{i_k})$ and the components of V_k are pairwise disjoint, this implies that $(x, y) \notin \Omega_k$.

Fix $k \geqslant 1$ and let $(x_1, y_1) \in \Omega_k^i$ for some $1 \leqslant i \leqslant n_k$, but $(x_2, y_2) \notin \Omega_{k-1}$ (the result (4.2) is trivial if $k = 1$, and hence no such point exists). There exists $1 \leqslant j \leqslant n_{k-1}$ such that $x_1 \in V_{k-1}^j$ since the $\{V_k\}_{k=0}^\infty$ are decreasing. First we suppose that $x_2 \notin V_{k-1}^j$. Then, since there must exist at least one point between x_1 and x_2 that does not lie in V_{k-1}, (3.5) implies that

$$\|(x_1, y_1) - (x_2, y_2)\| \geqslant |x_1 - x_2| \geqslant \delta_{k-1}/2.$$

Otherwise, $x_1, x_2 \in V_{k-1}^j$. Notice that by lemma 3.1(3) and (3.5) we have

$$|u(b_{k-1}^j) - u(b_k^i)| \geqslant h_{k-1}\delta_{k-1}/2 \quad \text{and} \quad |u(a_{k-1}^j) - u(a_k^i)| \geqslant h_{k-1}\delta_{k-1}/2.$$

Since $y_1 \in (u(a_k^i), u(b_k^i))$ but $y_2 \notin (u(a_{k-1}^j), u(b_{k-1}^j))$, this implies that

$$\|(x_1, y_1) - (x_2, y_2)\| \geqslant |y_1 - y_2| \geqslant h_{k-1}\delta_{k-1}/2 \geqslant \delta_{k-1}/2,$$

as required for (4.2).

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Finally, for $1 \leqslant i \leqslant n_k$, we easily see, using lemma 3.1(4), that

$$\operatorname{diam}(\Omega_k^i) \leqslant |u_k(b_k^i) - u_k(a_k^i)| \leqslant (h_k + 2)\lambda(V_k^i),$$

and hence, since the $\{V_k^i\}_{i=1}^{n_k}$ are pairwise disjoint,

$$\sum_{i=1}^{n_k} \operatorname{diam}(\Omega_k^i) \leqslant (h_k + 2) \sum_{i=1}^{n_k} \lambda(V_k^i) = (h_k + 2)\lambda(V_k),$$

as required for (4.3). □

The final step before we construct the potential is to lift our derivative ϕ from the real line into the plane, i.e. to construct a function ψ on the plane with which we can compare the potential and which agrees with ϕ where necessary, i.e. on the graph of u.

LEMMA 4.2. *There exists $\psi \in C^\infty(\mathbb{R}^2 \backslash U(E))$ such that the following hold.*

(1) *$\psi(x, y) \leqslant h_k + 2$ for all $(x, y) \notin \bar{\Omega}_k$.*

(2) *$\psi(x, y) \geqslant h_k$ for all $(x, y) \in \bar{\Omega}_k \backslash U(E)$.*

(3) *$\psi(x, u(x)) = u'(x)$ for all $x \in [a, b] \backslash E$.*

Proof. We construct a sequence $\{\psi_k\}_{k=0}^{\infty}$ of functions $\psi_k \in C^\infty(\mathbb{R}^2)$ such that, for $k \geqslant 0$,

$$\psi_k(x, y) \leqslant h_k + 2 \qquad \text{for all } (x, y) \in \mathbb{R}^2, \tag{4.4}$$

$$\psi_k(x, u_k(x)) = u_k'(x) \qquad \text{for all } x \in [a, b], \tag{4.5}$$

$$\psi_k(x, y) = \phi_k(x) \qquad \text{for all } (x, y) \in \Omega_k, \tag{4.6}$$

where ϕ_k is as constructed in the proof of lemma 3.1, and for $k \geqslant 1$,

$$\psi_k(x, y) = \psi_{k-1}(x, y) \quad \text{for all } (x, y) \notin \bar{\Omega}_{k-1}, \tag{4.7}$$

$$\psi_k(x, y) \geqslant h_{k-1} \qquad \text{for all } (x, y) \in \Omega_{k-1}. \tag{4.8}$$

Defining $\psi_0 = h_0 + 1$ satisfies all conditions (4.4)–(4.6). Suppose that ψ_{k-1} has been constructed as required, for $k \geqslant 1$. It is at this point that the positive distance between W_k and $\mathbb{R} \backslash V_k$ becomes useful. We define a new sequence of open sets $\{\tilde{\Omega}_k\}_{k=0}^{\infty}$ in $\mathbb{R}^2$ such that $U(E) \subseteq \tilde{\Omega}_k \subseteq \Omega_k$ and $\tilde{\Omega}_k \supseteq \Omega_{k+1}$ by setting $\tilde{\Omega}_0 = \mathbb{R}^2$, and for $k \geqslant 1$, $\tilde{\Omega}_k^i = W_k^i \times u(W_k^i) = W_k^i \times u_k(W_k^i)$ and $\tilde{\Omega}_k = \bigcup_{i=1}^{n_k} \tilde{\Omega}_k^i$. Choose a function $\pi_k \in C^\infty(\mathbb{R}^2)$ such that $0 \leqslant \pi_k \leqslant 1$ on $\mathbb{R}^2$, $\pi_k = 0$ off Ω_{k-1} and $\pi_k = 1$ on $\tilde{\Omega}_{k-1}$. Using $\phi_k \in C^\infty(\mathbb{R})$ from the proof of lemma 3.1, we define

$$\psi_k(x, y) = \psi_{k-1}(x, y) + \pi_k(x, y)(\phi_k(x) - \psi_{k-1}(x, y)).$$

Condition (4.7) is immediate. Since $\Omega_k \subseteq \tilde{\Omega}_{k-1}$, we see that $\psi_k(x, y) = \phi_k(x)$ for $(x, y) \in \Omega_k$, as required for (4.6).

We note that, by inductive hypothesis (4.4) and (3.6),

$$\psi_k = (1 - \pi_k)\psi_{k-1} + \pi_k \phi_k \leqslant (1 - \pi_k)(h_{k-1} + 2) + \pi_k(h_k + 2) \leqslant h_k + 2$$

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since the $\{h_k\}_{k=0}^{\infty}$ are increasing, as required for (4.4). Now, let $(x,y) \in \Omega_{k-1}$ to check (4.8). Using inductive hypothesis (4.6) and (3.9), we see that

$$\psi_k = (1-\pi_k)\psi_{k-1} + \pi_k\phi_k = (1-\pi_k)\phi_{k-1} + \pi_k\phi_k \geqslant (1-\pi)h_{k-1} + \pi h_{k-1} = h_{k-1},$$

as required.

For (4.5) we need to consider cases. First suppose that $x \notin V_{k-1}$, so $(x, u_k(x)) \notin \Omega_{k-1}$. Then, by lemma 3.1(2) and inductive hypothesis (4.5),

$$\psi_k(x, u_k(x)) = \psi_{k-1}(x, u_k(x)) = \psi_{k-1}(x, u_{k-1}(x)) = u'_{k-1}(x) = u'_k(x),$$

as required. For $x \in W_{k-1}$ we see that then $(x, u_k(x)) \in \tilde{\Omega}_{k-1}$ and so

$$\psi_k(x, u_k(x)) = \psi_{k-1}(x, u_k(x)) + u'_k(x) - \psi_{k-1}(x, u_k(x)) = u'_k(x),$$

as required. The final case is for $x \in V_{k-1}\backslash W_{k-1}$, in which case we argue that by lemma 3.1(2) and inductive hypothesis (4.5),

$$\begin{aligned}
\psi_k(x, u_k(x)) &= \psi_{k-1}(x, u_k(x)) + \pi_k(x, u_k(x))(u'_k(x) - \psi_{k-1}(x, u_k(x))) \\
&= \psi_{k-1}(x, u_k(x)) + \pi_k(x, u_k(x))(u'_{k-1}(x) - \psi_{k-1}(x, u_{k-1}(x))) \\
&= \psi_{k-1}(x, u_{k-1}(x)) \\
&= u'_{k-1}(x) \\
&= u'_k(x),
\end{aligned}$$

as required. Hence, the general result.

Let $(x,y) \in \mathbb{R}^2\backslash U(E)$. By (4.1) there exists $k \geqslant 1$ such that $(x,y) \notin \bar{\Omega}_{k-1}\backslash\bar{\Omega}_k$. Then (4.7) implies that $\lim_{l\to\infty}\psi_l$ exists and equals ψ_k on an open set around (x,y). Hence, ψ_k converges to a function $\psi \in C^\infty(\mathbb{R}^2\backslash U(E))$ such that $\psi = \psi_k$ on $\bar{\Omega}_{k-1}\backslash\bar{\Omega}_k$. Lemma 4.2(1) follows from (4.4) and lemma 4.2(2) follows from (4.8). For lemma 4.2(3), we let $x \in [a,b]\backslash E$, find $k \geqslant 1$ such that $x \in \bar{V}_{k-1}\backslash\bar{V}_k$ and hence that $(x, u(x)) \in \bar{\Omega}_{k-1}\backslash\bar{\Omega}_k$, and use lemma 3.1(2) and (4.5) to see that

$$\psi(x, u(x)) = \psi_k(x, u(x)) = \psi_k(x, u_k(x)) = u'_k(x) = u'(x),$$

as required. □

We now state and prove appropriate versions of lemmas 12 and 13 in [5]. For two vectors $x, y \in \mathbb{R}^2$, we write $[x,y]$ to denote the line segment in $\mathbb{R}^2$ connecting them.

Lemma 4.3. *Let $\tau > 0$, let $e \in \mathbb{R}^2\backslash\{0\}$ and suppose that $\Omega \subseteq \mathbb{R}^2$ is an open set such that $\Omega = \bigcup_{i=1}^{\infty}\Omega_i$ such that $\sum_{i=1}^{\infty}\operatorname{diam}(\Omega_i) < \tau/2\|e\|^2$.*

There then exists $f \in C^\infty(\mathbb{R}^2)$ such that the following hold.

- $0 \leqslant f(x) \leqslant \tau/\|e\|$ *for all* $x \in \mathbb{R}^2$.
- $\operatorname{dist}(\nabla f(x), [0,e]) < \tau$ *for all* $x \in \mathbb{R}^2$.
- $\|\nabla f(x) - e\| < \tau$ *for* $x \in \bar{\Omega}$.

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Proof. We first show that it suffices to prove the result for $e = (1, 0)$. For an arbitrary $e \in \mathbb{R}^2$, find a rotation $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\|e\|^{-1}Re = (1, 0)$. Then

$$\sum_{i=1}^{\infty} \operatorname{diam}(\|e\|R(\Omega_i)) = \|e\| \sum_{i=1}^{\infty} \operatorname{diam}(\Omega_i) < \frac{\tau}{2\|e\|},$$

so $\|e\|R(\Omega)$ satisfies the assumptions for $\tilde{e} := (1, 0)$ and $\tilde{\tau} := \tau/\|e\|$.

So, by assumption there exists $\tilde{f} \in C^\infty(\mathbb{R}^2)$ satisfying the three conclusions for $\tilde{e}$ and $\tilde{\tau}$. Define $f \in C^\infty(\mathbb{R}^2)$ by $f(x) = \tilde{f}(\|e\|Rx)$. Fix $x \in \mathbb{R}^2$. Then,

$$0 \leqslant f(x) = \tilde{f}(\|e\|Rx) \leqslant \frac{\tilde{\tau}}{\|(1, 0)\|} = \frac{\tau}{\|e\|}.$$

By assumption there exists $s \in [0, 1]$ such that

$$\|\nabla\tilde{f}(\|e\|Rx) - s(1, 0)\| < \tilde{\tau}.$$

Then, for this $s \in [0, 1]$, we have

$$\begin{aligned} \|\nabla f(x) - se\| &= \|\|e\|\nabla\tilde{f}(\|e\|Rx)R - s\|e\|\|e\|^{-1}eR^{-1}R\| \\ &= \|\|e\|(\nabla\tilde{f}(\|e\|Rx) - s\|e\|^{-1}eR^{-1})\| \\ &= \|e\|\|\nabla\tilde{f}(\|e\|Rx) - s(1, 0)\| \\ &< \|e\|\tilde{\tau} \\ &= \tau. \end{aligned}$$

Now, let $x \in \bar{\Omega}$. Then, $\|e\|^{-1}Rx \in \|e\|^{-1}R\bar{\Omega}$ so by assumption we have that

$$\|\nabla\tilde{f}(\|e\|^{-1}Rx) - (1, 0)\| < \tilde{\tau}.$$

Thus,

$$\begin{aligned} \|\nabla f(x) - e\| &= \|\|e\|\nabla\tilde{f}(\|e\|Rx)R - \|e\|\|e\|^{-1}eR^{-1}R\| \\ &= \|e\|\|\nabla\tilde{f}(\|e\|Rx) - (1, 0)\| \\ &< \|e\|\tilde{\tau} \\ &= \tau, \end{aligned}$$

as required.

So we can indeed assume without loss of generality that $e = (1, 0)$. By expanding each Ω_k^i slightly so that the inequality $\sum_{i=1}^{\infty} \operatorname{diam}(\Omega_k^i) < \tau/2$ is retained, and using a suitable mollification, it suffices to construct a Lipschitz function $g\colon \mathbb{R}^2 \to \mathbb{R}$ such that the following hold.

- $0 \leqslant g(x) \leqslant \tau/2$ for all $x \in \mathbb{R}^2$.
- $g_x(x) \in [0, 1]$ and $g_y(x) \in [-\tau/2, \tau/2]$ for every $x \in \mathbb{R}^2$.
- $g_x(x) = 1$ for $x \in \Omega$.

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To do this, we first note that for any function $\gamma\colon \mathbb{R} \to \mathbb{R}$,

$$\lambda(\{s \in \mathbb{R}\colon (s,\gamma(s)) \in \Omega\}) \leqslant \sum_{i=1}^{\infty} \lambda(\{s \in \mathbb{R}\colon (s,\gamma(s)) \in \Omega_i\}) \leqslant \sum_{i=1}^{\infty} \operatorname{diam}(\Omega_i) < \frac{\tau}{2}.$$

In particular, defining (just as in [5]) $g\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x,y) = \sup\left\{x - b + \int_{\{s\in\mathbb{R}\colon (s,\gamma(s))\in\Omega\}} \left(1 - \frac{2|\gamma'(s)|}{\tau}\right) \mathrm{d}s\right\},$$

where the supremum is taken over all $b \in \mathbb{R}$ such that $b \geqslant x$ and all $\gamma\colon (-\infty, b] \to \mathbb{R}$ such that $\operatorname{Lip}(\gamma) < 2/\tau$ and $\gamma(b) = y$ satisfies the requirements just as proved in [5]. □

Lemma 4.4. *Let $\varepsilon > 0$, let $e^0, e^1 \in \mathbb{R}^2$ be distinct vectors and let $\Omega \Subset \Omega' \subseteq \mathbb{R}^2$ be open sets such that $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$ such that*

$$\sum_{i=1}^{\infty} \operatorname{diam}(\Omega_i) < \frac{\varepsilon/2\|e^0 - e^1\|^2}{1 + (\|e^0 - e^1\|\delta)^{-1}},$$

where $0 < \delta \leqslant \operatorname{dist}(\Omega, \mathbb{R}^2\backslash\Omega')/2$. Let $g^0 \in C^\infty(\mathbb{R}^2)$.

There then exists $g^1 \in C^\infty(\mathbb{R}^2)$ such that the following hold.

- $\|g^1 - g^0\|_\infty < \varepsilon/\|e^0 - e^1\|$.
- $g^1 = g^0$ *off* Ω'.
- $\operatorname{dist}(\nabla g^1(x), [e^0, e^1]) < \varepsilon + \|\nabla g^0(x) - e^0\|$ *for all* $x \in \mathbb{R}^2$.
- $\|\nabla g^1(x) - e^1\| < \varepsilon + \|\nabla g^0(x) - e^0\|$ *for all* $x \in \bar{\Omega}$.

Proof. Let

$$\tau := \frac{\varepsilon}{1 + (\delta\|e^0 - e^1\|)^{-1}}$$

and apply lemma 4.3 with this τ, the set Ω as given, and the vector $e := e^1 - e^0$. Let $f \in C^\infty(\mathbb{R}^2)$ be the resulting function.

Choose $\chi \in C^\infty(\mathbb{R}^2)$ such that $0 \leqslant \chi \leqslant 1$, $\chi = 1$ on Ω, $\chi = 0$ off Ω' and $\|\nabla\chi\| \leqslant \delta^{-1}$. Define $g^1 = g^0 + \chi f$. Clearly, $g^1 \in C^\infty(\mathbb{R}^2)$. We see immediately from lemma 4.3 that

$$\|g^1 - g^0\|_\infty \leqslant \|f\|_\infty \leqslant \frac{\tau}{\|e^1 - e^0\|} < \frac{\varepsilon}{\|e^1 - e^0\|}.$$

Also note, by the properties of χ and lemma 4.3, we have for $x \in \mathbb{R}^2$ that

$$\begin{aligned}\operatorname{dist}(\nabla g^1(x), [e^0, e^1]) &\leqslant \operatorname{dist}(\nabla g^0(x) - e^0 + (\chi\nabla f)(x) + (f\nabla\chi)(x), [0, e^1 - e^0]) \\ &\leqslant \|\nabla g^0(x) - e^0\| + \operatorname{dist}(\nabla f(x), [0, e]) + \|(f\nabla\chi)(x)\| \\ &\leqslant \|\nabla g^0(x) - e^0\| + \tau + \frac{\tau}{\delta\|e^0 - e^1\|} \\ &= \|\nabla g^0(x) - e^0\| + \varepsilon.\end{aligned}$$

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For $x \in \bar{\Omega}$ we have, since $\chi(x) = 1$, that

$$\begin{aligned}\|\nabla g^1(x) - e^1\| &\leqslant \|\nabla g^0(x) - e^0\| + \|(\chi\nabla f)(x) - (e^1 - e^0)\| + \|(f\nabla\chi)(x)\| \\ &\leqslant \|\nabla g^0(x) - e^0\| + \|\nabla f(x) - e\| + \frac{\tau}{\delta\|e^0 - e^1\|} \\ &\leqslant \|\nabla g^0(x) - e^0\| + \tau\left(1 + \frac{1}{\delta\|e^0 - e^1\|}\right) \\ &= \|\nabla g^0(x) - e^0\| + \varepsilon.\end{aligned}$$

□

We now construct a potential Φ that will satisfy the conditions of lemma 2.2 with the function ψ given by lemma 3.1. We now assign values to our increasing sequences and define two new sequence $\{t_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=0}^{\infty}$ by setting, for $k \geqslant 0$,

- $h_k = 10(3 + 2^{k+1})$,
- $t_k = 3 + 2^k$,
- $B_k = 4 + 320\omega'(h_{k+1} + 2)$ and $A_k = 3t_kB_k$.

Also, for $k \geqslant 0$ define numbers $\eta_k = 1 - 2^{-k-1} > 0$ and $\varepsilon_k = 2^{-k}(4n_k)^{-1} > 0$, and vector $e_k = (-A_k, B_k) \in \mathbb{R}^2$. We inductively construct a sequence $\{\Phi^k\}_{k=0}^{\infty}$ of functions $\Phi^k \in C^{\infty}(\mathbb{R}^2)$ satisfying, for $k \geqslant 0$,

$$\|\nabla\Phi^k(x, y) - e_k\| < \eta_k \quad \text{for all } (x, y) \in \bar{\Omega}_k, \tag{4.9}$$

and for $k \geqslant 1$,

$$\|\Phi^k - \Phi^{k-1}\|_{\infty} < \varepsilon_{k-1}, \tag{4.10}$$

$$\Phi^k = \Phi^{k-1} \quad \text{off } \Omega_{k-1}, \tag{4.11}$$

$$\operatorname{dist}(\nabla\Phi^k(x, y), [e_{k-1}, e_k]) < \eta_k \quad \text{for all } (x, y) \in \bar{\Omega}_{k-1}. \tag{4.12}$$

We define $\Phi^0(x, y) = -A_0x + B_0y$, which clearly satisfies (4.9). Suppose for $k \geqslant 1$ that we have constructed Φ^{k-1} as claimed. To construct Φ^k we apply lemma 4.4 with data $\varepsilon = \varepsilon_{k-1}$, $e^0 = e_{k-1}$, $e^1 = e_k$, $\Omega = \Omega_k$, $\Omega' = \Omega_{k-1}$, $\delta = \delta_{k-1}/4$ and $g^0 = \Phi^{k-1}$. We must check that the assumptions of the lemma hold with these values. First, recall that (4.2) gives that $\delta_{k-1}/4 < \operatorname{dist}(\Omega_k, \mathbb{R}^2\backslash\Omega_{k-1})/2$ indeed. We see by (4.3) and (3.3) that

$$\begin{aligned}\sum_{i=1}^{n_k} \operatorname{diam}(\Omega_k^i) &\leqslant \lambda(V_k)(h_k + 2) \\ &< \frac{2^{-(k-1)}(4n_{k-1})^{-1}/2A_k^2}{1 + 4\delta_{k-1}^{-1}} \\ &< \frac{\varepsilon_{k-1}/2\|e^k - e^{k-1}\|^2}{1 + 4(\delta_{k-1}\|e^k - e^{k-1}\|^2)^{-1}}.\end{aligned}$$

We define Φ^k as the function g^1 given by the lemma.

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Then (4.11) is immediate and since $\|e_k - e_{k-1}\| \geqslant 1$, we see that $\|\Phi^k - \Phi^{k-1}\|_\infty < \varepsilon_{k-1}$, as required for (4.10). For (4.12) we let $(x, y) \in \bar{\Omega}_{k-1}$ and use inductive hypothesis (4.9) and the properties given by lemma 4.4 to see that

$$\begin{aligned}\operatorname{dist}(\nabla\Phi^k(x,y),[e_{k-1},e_k]) &< \varepsilon_{k-1} + \|\nabla\Phi^{k-1}(x,y) - e_{k-1}\| \\ &\leqslant \varepsilon_{k-1} + \eta_{k-1} \\ &\leqslant 2^{-(k-1+2)} + 1 - 2^{-k} \\ &= \eta_k.\end{aligned}$$

Similarly for (4.9), we let $(x, y) \in \bar{\Omega}_k$ and use the lemma and the inductive hypothesis (4.9) again, noting that $\Omega_k \subseteq \Omega_{k-1}$, to see that

$$\|\nabla\Phi^k(x,y) - e_k\|_2 < \varepsilon_{k-1} + \|\nabla\Phi^{k-1}(x,y) - e_{k-1}\|_2 < \eta_k.$$

Hence, we can construct such a sequence $\{\Phi^k\}_{k=0}^\infty$ as claimed. We now check that this gives us the potential we require for lemma 2.2 with $S = U(E)$. By (4.10) and since $\varepsilon_k \leqslant 2^{-(k+2)}$, we see that the Φ^k converge uniformly to some $\Phi \in C(\mathbb{R}^2)$.

Fix $(x, y) \in \mathbb{R}^2 \backslash (U(E))$. By (4.1), there is $k \geqslant 1$ such that $(x, y) \in \bar{\Omega}_{k-1} \backslash \bar{\Omega}_k$, and hence $\Phi \in C^\infty(\mathbb{R}^2 \backslash (U(E))) \cap C(\mathbb{R}^2)$ and $\nabla\Phi = \nabla\Phi^l$ on $\bar{\Omega}_{k-1} \backslash \bar{\Omega}_k$ for all $l \geqslant k$, by (4.11). Moreover, by (4.12),

$$\Phi_y(x,y) = \Phi^k_y(x,y) \geqslant B_{k-1} - \eta_{k-1} \geqslant B_0 - 1 \geqslant 3,$$

as required for the second inequality of lemma 2.2(1). More precisely, by (4.12) there is an $s \in [0, 1]$ such that $\|\nabla\Phi(x, y) - (se_{k-1} + (1 - s)e_k)\|_2 < 1$. Using this we see that

$$\begin{aligned}-\Phi_x(x,y) &\leqslant sA_{k-1} + (1-s)A_k + 1 \\ &\leqslant 3t_k(sB_{k-1} + (1-s)B_k) + 1 \\ &\leqslant 3t_k(\Phi_y(x,y) + 1) + 1 \\ &\leqslant 5t_k\Phi_y(x,y),\end{aligned}$$

and thus $(-\Phi_x/\Phi_y)(x, y) \leqslant 5t_k$. Similarly,

$$\begin{aligned}-\Phi_x(x,y) &\geqslant sA_{k-1} + (1-s)A_k - 1 \\ &\geqslant 3t_{k-1}(sB_{k-1} + (1-s)B_k) - 1 \\ &\geqslant 3t_{k-1}(\Phi_y(x,y) - 1) - 1 \\ &\geqslant t_{k-1}\Phi_y(x,y),\end{aligned}$$

and thus $(-\Phi_x/\Phi_y)(x, y) \geqslant t_{k-1}$. Lemma 2.2(1) follows since $t_{k-1} \geqslant t_0 = 4$. We know from lemma 4.2(2) and lemma 4.2(1) that $h_{k-1} \leqslant \psi(x, y) \leqslant h_k + 2$. Thus, by properties of ω,

$$\Phi_y(x,y) > B_{k-1} - 1 = 3 + 320\omega'(h_k + 2) \geqslant 320\omega'(\psi(x,y)),$$

as required for lemma 2.2(2).

We note that, from the definitions,

$$10t_k = 10(3 + 2^k) = h_{k-1}$$

and

$$h_k + 2 = 10(3 + 2^{k+1}) + 2 \leqslant 10 \times 2^4(3 + 2^{k-1}) = 160t_{k-1}.$$

So we see that

$$-2\Phi_x(x, y)/\Phi_y(x, y) \leqslant 10t_k \leqslant \psi(x, y) \leqslant h_k + 2 \leqslant -160\Phi_x(x, y)/\Phi_y(x, y),$$

and we hence obtain lemma 2.2(3).

We finally check lemma 2.2(4), so let $\tilde{u} \in \mathrm{AC}(a, b)$. The set $\tilde{U}^{-1}(U(E)) \subseteq E$ and is therefore null. Fix $k \geqslant 1$ and note that, by (4.10) and since $\varepsilon_k \leqslant 2^{-(k+2)}$, we have that $\|\Phi - \Phi^k\|_\infty < 2\varepsilon_k$. Fix $1 \leqslant i \leqslant n_k$. The image of Ω_k^i under Φ is connected, and thus

$$\Phi(\Omega_k^i) \subseteq B_{2\varepsilon_k}(\Phi^k(\Omega_k^i)),$$

and hence, since (4.12) implies that Φ^k has Lipschitz constant at most $A_k + B_k + 2$,

$$\lambda(\Phi(\Omega_k^i)) \leqslant \lambda(B_{2\varepsilon_k}(\Phi^k(\Omega_k^i))) \leqslant 4\varepsilon_k + \lambda(\Phi^k(\Omega_k^i)) \leqslant 4\varepsilon_k + (A_k + B_k + 2)\,\mathrm{diam}(\Omega_k^i).$$

So, summing over $1 \leqslant i \leqslant n_k$ gives, by the choice of ε_k, (4.3) and (3.3), and since the $\{\Omega_k^i\}_{i=1}^{n_k}$ are pairwise disjoint,

$$\begin{aligned}\lambda(\Phi(\Omega_k)) &\leqslant \sum_{i=1}^{n_k} \lambda(\Phi(\Omega_k^i)) \\ &\leqslant \sum_{i=1}^{n_k} (4\varepsilon_k + (A_k + B_k + 2)\,\mathrm{diam}(\Omega_k^i)) \\ &\leqslant 2^{-k} + (A_k + B_k + 2)(h_k + 2)\lambda(V_k) \\ &\leqslant 2^{-(k-1)}.\end{aligned}$$

Therefore, since for all $k \geqslant 0$,

$$(\Phi \circ \tilde{U})(\tilde{U}^{-1}(U(E))) = \Phi(\tilde{U}(a, b) \cap U(E)) \subseteq \Phi(\Omega_k),$$

we see that $(\Phi \circ \tilde{U})(\tilde{U}^{-1}(U(E)))$ is indeed a null set.

5. Conclusion

Proof of theorem 1.1. We let $L \in C^\infty(\mathbb{R}^3)$ be the Lagrangian given by lemma 2.2 with u as constructed in lemma 3.1, $S = U(E)$, ψ as given by lemma 4.2 and this potential Φ. The growth condition on ψ follows from lemma 4.2(2). By lemma 4.2(3), we infer from lemma 2.2 that the first statement of the theorem holds for this $u \in \mathrm{AC}(a, b)$.

Since $u' \in C^\infty([a, b]\backslash E)$ satisfies $u'(x) \to \infty$ as $\mathrm{dist}(x, E) \to 0$, we see that $E = \{x \in (a, b) \colon |u'(x)| = \infty\}$. In particular, since our function u is a minimizer with respect to its own boundary conditions, we see that the singular set of u is indeed E.

We now prove the third statement of the theorem. Lemma 3.1 gives us a sequence of admissible functions $u_k \in C^\infty([a, b])$ that converge uniformly to u. We just need to prove that they also converge in energy. Let $\varepsilon > 0$. By lemma 3.1(2) we see that

$$\begin{aligned} 0 \leqslant \mathcal{L}(u_k) - \mathcal{L}(u) &= \int_a^b L(x, u_k(x), u_k'(x)) - L(x, u(x), u'(x))\,\mathrm{d}x \\ &= \int_{V_k} L(x, u_k(x), u_k'(x)) - L(x, u(x), u'(x))\,\mathrm{d}x. \end{aligned}$$

We know from the precise conclusion of lemma 2.2 that $x \mapsto L(x, u(x), u'(x))$ is integrable, so since $\lambda(V_k) \to 0$ as $k \to \infty$ by (3.3), we can choose $k_0 \geqslant 1$ such that $\int_{V_k} L(x, u(x), u'(x))\,\mathrm{d}x < \varepsilon/2$ whenever $k \geqslant k_0$.

Now, for each $k \geqslant 1$ and almost every $x \in [a, b]$, we have that

$$\begin{aligned} L(x, u_k(x), u_k'(x)) &= \omega(u_k'(x)) + F(x, u_k(x), u_k'(x)) \\ &= \omega(u_k'(x)) + \gamma(u_k'(x), \xi(x, u_k(x)), \theta(x, u_k(x))) \end{aligned}$$

by definition of the Lagrangian L in lemma 2.2. Fix such an $x \in [a, b]$. We get the following upper bound for γ by using lemma 2.1(5), lemma 3.1(4), (2.2) (noting that $\xi \geqslant 0$ by (2.3)) and lemma 2.2(1):

$$\begin{aligned} \gamma(u_k'(x), \xi(x, u_k(x)), \theta(x, u_k(x))) &\leqslant \theta(x, u_k(x))|u_k'(x) - \xi(x, u_k(x)) + 1| \\ &\leqslant \Phi_y(x, u_k(x))(u_k'(x) + 1) \\ &\qquad + \theta(x, u_k(x))\xi(x, u_k(x)) \\ &\leqslant \Phi_y(x, u_k(x))((h_k + 2) + 1) - \Phi_x(x, u_k(x)) \\ &\leqslant -\Phi_x(x, u_k(x))(h_k + 7)/4 \\ &\leqslant -\Phi_x(x, u_k(x))h_k/2. \end{aligned}$$

Now, $\omega(u_k'(x)) \leqslant \omega(h_k + 2)$ by properties of ω and lemma 3.1(4), and for sufficiently large $k \geqslant 0$, $h_k + 2 \leqslant \omega(h_k + 2)$ since ω is superlinear and $h_k \to \infty$ as $k \to \infty$. So, again using lemma 3.1(4), and since certainly $-\Phi_x \geqslant 2$, we have, for large $k \geqslant 0$,

$$L(x, u_k(x), u_k'(x)) \leqslant \omega(h_k + 2) - \Phi_x(x, u_k(x))h_k/2 \leqslant -\Phi_x(x, u_k(x))\omega(h_k + 2).$$

Now, if $x \in \bar{V}_{l-1} \backslash \bar{V}_l$, then $(x, u_k(x)) \in \bar{\Omega}_{l-1} \backslash \bar{\Omega}_l$, so $\Phi_x(x, u_k(x)) = \Phi_x^l(x, u_k(x))$, and hence $-\Phi_x(x, u_k(x)) \leqslant A_l + 1$. Thus, for large $l \geqslant 1$, almost everywhere on $\bar{V}_{l-1} \backslash \bar{V}_l$ we have

$$L(x, u_k(x), u_k'(x)) \leqslant (A_l + 1)\omega(h_k + 2).$$

So for sufficiently large $k \geqslant 1$ we have, since $\{h_k\}_{k=0}^{\infty}$ is increasing, by properties of ω and by (3.3), that

$$\begin{aligned} 0 \leqslant \int_{V_k} L(x, u_k(x), u_k'(x))\,\mathrm{d}x &\leqslant \sum_{l=k+1}^{\infty} \int_{\bar{V}_{l-1} \backslash \bar{V}_l} L(x, u_k(x), u_k'(x))\,\mathrm{d}x \\ &\leqslant \sum_{l=k+1}^{\infty} \int_{\bar{V}_{l-1} \backslash \bar{V}_l} (A_l + 1)\omega(h_k + 2)\,\mathrm{d}x \end{aligned}$$

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$$\leqslant \sum_{l=k+1}^{\infty} \omega(h_l+2)(A_l+1)\lambda(V_{l-1})$$
$$\leqslant \sum_{l=k}^{\infty} 2^{-l}$$
$$\leqslant 2^{-k+1}.$$

So choosing $k_1 \geqslant 1$ such that $2^{-k_1+1} \leqslant \varepsilon/2$, we have for large $k \geqslant k_0, k_1$ that

$$0 \leqslant \mathcal{L}(u_k) - \mathcal{L}(u) \leqslant \int_{V_k} L(x, u_k(x), u_k'(x))\,\mathrm{d}x + \int_{V_k} L(x, u(x), u'(x))\,\mathrm{d}x \leqslant \varepsilon,$$

as required. □

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