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FIRST-ORDER RELEVANT REASONERS IN CLASSICAL WORLDS

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Abstract. Sedlár and Vigiani [18] have developed an approach to propositional epistemic logics wherein (i) an agent's beliefs are closed under relevant implication and (ii) the agent is located in a classical possible world (i.e., the non-modal fragment is classical). Here I construct first-order extensions of these logics using the non-Tarskian interpretation of the quantifiers introduced by Mares and Goldblatt [12], and later extended to quantified modal relevant logics by Ferenz [6]. Modular soundness and completeness are proved for constant domain semantics, using non-general frames with Mares–Goldblatt truth conditions. I further detail the relation between the demand that classical possible worlds have Tarskian truth conditions and incompleteness results in quantified relevant logics.

§1. Introduction. This paper presents a framework for first-order epistemic logics that combine classical and relevant first-order modal logics. Modular soundness and completeness are proved for constant domain semantics, using non-general frames with Mares–Goldblatt ([12]) truth conditions. This construction enables one to represent an agent's belief set as closed under relevant implication while the extensional connectives remain completely classical: for the agent, non-modal sentences behave classically, but the sentences in the scope of an epistemic operator need not behave classically. This work builds on the work of Sedlár and Vigiani [18], in which relevant and classical *propositional* modal logics are combined. Semantically, this combination identifies a set of *classical possible worlds*, on which truth and validity are defined, inside a relevant logic's ternary relational model. Here I use and argue for a particular conception of a classical possible world in the first-order setting, while detailing formal problems that can arise on other approaches.

In [18] the representation of epistemic states using relevant logic and relevant situations (parts of possible, possibly inconsistent worlds) have the following features: (i) the situations modeling an agent's belief states can be incomplete and inconsistent, (ii) the agent's beliefs are only closed under relevant consequence (or, in a neighborhood generalization, relevant provable equivalence), (iii) an agent need not believe every theorem (relevant or classical), and (iv) the logics permit a more nuanced and fine-grained representation of an agent's beliefs towards implications. The focus of this paper is on first-order extensions of these logics and some of the additional formal and philosophical problems that arise in these extensions; nevertheless, the

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propositional fragment of the logics developed here include both fusion and left-implication, permitting additional representations of agent's epistemic attitudes.

Sedlár and Vigiani [18] further apply their framework to several problems of logical omniscience. In many presentations of epistemic logics with a classical (two-valued, Boolean) base, certain paradoxes of omniscience are unavoidable or require drastic and sometimes philosophically questionable additions (at least, that is, when taken as a whole). The main problems are that the standard approaches (i) take beliefs as sets of classical possible worlds which entails that all classical theorems are believed by the agent, and (ii) they require belief sets to be closed under classical logic. The former leads to agents believing, for example, 'p or not p', for every p. The latter makes the agents logically perfect, believing all and only the logical consequences of their beliefs. While problem (i) is completely eliminated by the use of relevant situations, problem (ii) is ameliorated by a restriction to closure under provable relevant implication. While this does not solve the problem of logical omniscience as a whole, it is perhaps more reasonable to suppose that an agent's (perhaps implicit) beliefs are closed under relevant implication (see, e.g., [2, 3, 18]).

The combination of classical and relevant first-order epistemic logics presented here proceeds as in [18], by using two modalities. The first modality, $\Box$, is the epistemic modality. Rather, it is a modal operator acting as a placeholder for an epistemic modality. The present work does not assume it is a particular epistemic modality, but rather aims are providing a general framework in which $\Box$ can be used to model different epistemic modalities (by adopting appropriate axioms/frame conditions). The second modality, $\Box_L$, is a formal tool used to bridge the relevant and classical sides of the logic. In particular, $\Box_L \mathcal{A}$ is taken to mean that $\mathcal{A}$ is a theorem of the underlying relevant logic. Roughly, a formula is shown to be a theorem of an underlying relevant logic, and then $\Box_L$ allows us to transfer this theorem into a classical setting.

The paper is divided as follows. Section 2 first introduces a semantic approach for quantified (bi-)modal relevant logics. The approach is an application of the Mares–Goldblatt (MG) style semantics, which was introduced for quantified relevant **R** in [12]. Several authors have generalized the MG approach. The generalization used here is that of Ferenz [6], which extends the MG semantics to a wide range of quantified and quantified modal relevant logics. Then, in Section 3, a Hilbert-style axiomatization is given (which is proved sound and complete in [6]).

In Section 4, we introduce MG-based models for first-order extensions of the work of Sedlár and Vigiani [18], and give an axiomatization. In contrast to [18], we add fusion, a left-implication, and an intensional truth constant to the propositional fragment of the language. In [18], the implication and negation behave classically (i.e., truth-functionally) at possible worlds. Similarly, we enforce this requirement on the new propositional connectives, and consider particular ways of making the quantifiers classical. Sections 5 and 6 respectively contain modular soundness and completeness proofs for the systems developed. In Section 7, a particular classical behavior of quantifiers is given closer inspection. The models developed in earlier sections lack this property. We discuss both its philosophical motivation and the formal problems encountered with its addition.

§2. First-order relevant modal logics.

Here, we present the first-order relevant modal logics which serve as a foundation for the remainder of the paper. The logics and semantics defined here are essentially those found in [5, 6], which combines and furthers the Mares–Goldblatt interpretation of quantifiers (see, e.g., [8, 12]) and Seki's general frame semantics for *regular* relevant model logics (see, e.g., [20, 21]).[1]

For now, we simplify the matter at hand by presenting a constant domain semantics—indeed, a single universal domain—with no existence predicates. Therefore the quantifiers are to be considered *possibilist*, although the *actualist*, variable domain semantics is obtainable, through the modifications detailed in [11]. An agent may claim to have beliefs about possible situations with a different set of objects composing it; that is, objects that either do not exist or have no counterparts in the agent's world. Nonetheless, we will sideline these kinds of philosophical issues for the time being.

The language of the first-order substructural epistemic logic $\mathbf{QBM.C}_{\Box\Box_L}$, hereafter just **QBM.C**, will be built up from a set of symbols that can be divided as follows:

1. a denumerable set of variables $Var = \{x_0, x_1, \dots\}$;
2. an at most denumerable *signature* $\mathbb{S}$ consisting of:
   (a) a set of constant symbols $Con^{\mathbb{S}} = \{c_0, c_1, \dots\}$;
   (b) a non-empty set of predicate symbols $Pred^{\mathbb{S}}$;
3. constant symbol $\boldsymbol{t}$;
4. binary operators $\wedge, \vee, \rightarrow, \circ, \leftarrow$;
5. unary operators $\neg, \Box, \Box_L$;
6. quantifier symbol $\forall$, $\exists$.

Each n-ary predicate symbol will be written as P^n_k (with identifying subscript k often omitted), and the set of n-ary predicates shall be written as $Pred^n \subseteq Pred^{\mathbb{S}}$.

For a signature $\mathbb{S}$, the set of terms of **QBM.C** is $Term^{\mathbb{S}}_{\mathbf{QBM.C}} = Con^{\mathbb{S}} \cup Var$, and τ (often with subscripts) will always denote a term. For the remainder of the paper, it is often the case that a signature will implicitly be taken for granted, or stated explicitly. Hereafter we will drop the "$\mathbb{S}$" in notation.

Officially, we take the set of variables to be ordered, as in $x_1, \dots, x_n, \dots$. Then, given a set of U of individuals, a *variable assignment* is a denumerable sequence of individuals, $f \in U^\omega$, such that the nth element in the sequence (written as fn) is the individual assigned to the n-th variable x_n. Given a variable assignment f, an x-variant of f differs from f in at most the assignment to the variable x. The set of all x-variants of f will be denoted xf. We will write $f[j/n]$ (or $f[j/x_n]$), with $j \in U$ to denote the result of changing the n-th element of f with the individual j.

DEFINITION 2.1 (Language $\mathfrak{L}_{QBM.C}$). *The basic first-order substructural language $\mathfrak{L}_{QBM.C}$, or well-formed formulas (hereby wff) is defined in BNF as follows:*

$$\phi ::= P^n(\tau_1, \dots, \tau_n) | \boldsymbol{t} | \neg\phi | \Box\phi | \Box_L\phi | \phi \wedge \phi | \phi \vee \phi | \phi \rightarrow \phi | \phi \circ \phi | \phi \leftarrow \phi | \forall x\phi | \exists x\phi.$$

In terms of binding strength, for the purposes of omitting parentheses, we assume that unary connectives and quantifiers bind the strongest (and equally so). We assume the left and right arrow bind weaker than fusion, which itself binds weaker than the extensional conjunction and disjunction.

We shall write $\mathcal{A}[\tau/x]$ for the result of replacing every free occurrence of x in $\mathcal{A}$ with the term τ. Similarly, we will use $\mathcal{A}[\tau_0/v_0, \dots, \tau_n/v_n]$ for the result of simultaneously replacing v_0 through v_n with τ_0 through τ_n respectively. A variable is said to be *bound*

[1] Both here and in Seki, the term *regular* is used as introduced in [19].

in a formula $\mathcal{A}$ if it (i) is the instance x in a quantifier $\exists x$ or $\forall x$, or (ii) is an instance of x that occurs within the scope of either $\exists x$ or $\forall x$. Non-bound variables are said to be *free*. A term τ is free for (or freely substitutable for) a variable x in $\mathcal{A}$ if τ does not become bound in the resulting formula $\mathcal{A}[\tau/x]$.

Note the inclusion of $\leftarrow$ and $\circ$. This inclusion is not only formally interesting—e.g., see Section 4, where additional frame conditions for classical possible worlds must be satisfied—but also enables additional expressive power in terms of formulating the beliefs of an agent. With fusion, for example, we are able to represent the difference between an agent merely believing $(\mathcal{A} \rightarrow \mathcal{B}) \wedge \mathcal{A}$ (which does not imply $\mathcal{B}$ in **BM**), and believing $(\mathcal{A} \rightarrow \mathcal{B}) \circ \mathcal{A}$ (which does imply $\mathcal{B}$ in **BM**) The latter is a type of conjunction which "applies" modus ponens in this case. We can thus represent the nuanced difference between believing a set of sentences in a simple way and believing a set of sentences in way that implies belief of the relevant consequences of such a set. One may come to believe that $\mathcal{A} \wedge \mathcal{B}$ is true by believing each conjunct individually, but yet one need not have additionally entertained combining these beliefs to obtain their joint logical consequences. For left-implication, the case only arises when $\circ$ is not commutative (otherwise $\leftarrow$ is just $\rightarrow$). With distinct left-implication, we can represent (epistemic states regarding) implications corresponding to conjoining premises in different orders. I emphasize again that the project here is a general framework: for particular applications one needs to assume the right properties for both these connectives, implication, and the epistemic modality.

The frames and models are defined as in [6], following Mares and Goldblatt's interpretation of quantifiers in the first-order extensions of the relevant logic **R**.

DEFINITION 2.2 (Base First-Order Frames). *A* Base First-Order Frame *is a tuple*

$$F = \langle K, N, R, *, S_{\Box}, S_{\Box_L}, U, Prop, PropFun \rangle,$$

where $N \subseteq K \neq \emptyset$; $R \subseteq K^3$; $* : K \rightarrow K$; $S_{\Box} \subseteq K^2$; $S_{\Box_L} \subseteq K^2$; *U is a non-empty set of individuals; and, defining the "upsets" as* $\mathcal{P}(K)^{\uparrow} = \{X \in \mathcal{P}(K) : \forall a, b, \in K(a \in X \;\&\; a \leq b) \Rightarrow b \in X\}$, *with* $a \leq b =_{df} \exists x \in N(Rxab)$ *we have that* $Prop \subseteq \mathcal{P}(K)^{\uparrow}$, $PropFun \subseteq \{\phi : U^{\omega} \longrightarrow Prop\}$, *and the following conditions hold*:

- (c1) $\leq$ *is reflexive and transitive.*
- (c2) $N \in Prop$.
- (c3) $Rabc$, $a' \leq a$, $b' \leq b$ & $c \leq c'$ *imply* $Ra'b'c'$.
- (c4) $a \leq b$ *implies* $b^* \leq a^*$.
- (c5) $S_{\Box}bc$ *and* $a \leq b$ *imply* $S_{\Box}ac$.
- (c6) $S_{\Box_L}bc$ *and* $a \leq b$ *imply* $S_{\Box_L}ac$.
- (c7) $Prop$ *is closed under* $\cap, \cup, \neg, \Box, \Box_L, \rightarrow, \circ, \leftarrow$ *where*:
  - (a) $\neg X = \{a \in K : \alpha^* \notin X\}$.
  - (b) $\Box X = \{a \in K : \forall b(S_{\Box})ab \Rightarrow b \in X\}$.
  - (c) $\Box_L X = \{a \in K : \forall b(S_{\Box_L})ab \Rightarrow b \in X\}$.
  - (d) $X \rightarrow Y = \{a \in K : \forall b, c \in K(Rabc \;\&\; b \in X \Rightarrow c \in Y)\}$.
  - (e) $X \circ Y = \{a \in K : \exists b, c \in K(Rbca \;\&\; b \in X \;\&\; c \in Y)\}$.
  - (f) $X \leftarrow Y = \{a \in K : \forall b, c \in K(Rbac \;\&\; b \in X \Rightarrow c \in Y)\}$.
- (c8) $\phi_N \in Prop$ (*where* $\phi_N f = N$, *for every* $f \in U^{\omega}$).

(c9) *PropFun is closed under* $\cap, \cup, \neg, \Box, \Box_L, \rightarrow, \circ, \leftarrow$ *where for all* $f \in U^\omega$, *every* $\phi, \psi \in PropFun$:
  (a) $(\oplus\phi)f = \oplus(\phi f)$, *for each* $\oplus \in \{\neg, \Box, \Box_L\}$.
  (b) $(\phi \otimes \psi)f = \phi f \otimes \psi f$, *for each* $\otimes \in \{\cap, \cup, \rightarrow, \circ, \leftarrow\}$.

(c10) *PropFun is closed under* $\forall_n$ *and* $\exists_n$, *for every* $n \in \omega$, *where*:
  (a) $(\forall_n\phi)f = \bigsqcap_{g \in x_n f} \phi g = \bigcup\{X \in Prop \mid X \subseteq \bigcap_{g \in x_n f} \phi g\}$.
  (b) $(\exists_n\phi)f = \bigsqcup_{g \in x_n f} \phi g = \bigcap\{X \in Prop \mid \bigcup_{g \in x_n f} \phi g \subseteq X\}$.

A frame is called *full* when *Prop* is the set of every hereditary subset of K, and *PropFun* contains every function from U^ω to *Prop*.

Definition 2.3 (Basic First-Order Pre-Models for **QBM.C**). *A* basic pre-model *is a tuple* $\mathfrak{M} = \langle F, |{-}| \rangle$ *such that F is a base first-order frame and* $|{-}|$ *is a valuation function that assigns*:

1. *an individual* $|c| \in U$ *to each constant symbol* c;
2. *a function* $|P^n| : U^n \longrightarrow \mathcal{P}(K)$ *to each* n-*ary predicate symbol* P^n; *and*
3. *a propositional function* $|\mathcal{A}| : U^\omega \longrightarrow \mathcal{P}(K)$ *to each formula* $\mathcal{A}$ *such that, when* $\mathcal{A}$ *is atomic, for every* $f \in U^\omega$:[2]

$$|P^n\tau_1, \ldots, \tau_n|f = |P^n|(|\tau_1|f, \ldots |\tau_n|f).$$

Moreover, when $\mathcal{A}$ *is not atomic (or* $\boldsymbol{t}$*), the valuation is extended as follows, for every* $f \in U^\omega$:

$$\begin{aligned} |\neg\mathcal{A}|f &= \neg|\mathcal{A}|f, & |\mathcal{A} \rightarrow \mathcal{B}|f &= |\mathcal{A}|f \rightarrow |\mathcal{B}|f,\\ |\Box\mathcal{A}|f &= \Box|\mathcal{A}|f, & |\mathcal{A} \circ \mathcal{B}|f &= |\mathcal{A}|f \circ |\mathcal{B}|f,\\ |\Box_L\mathcal{A}|f &= \Box_L|\mathcal{A}|f, & |\mathcal{A} \leftarrow \mathcal{B}|f &= |\mathcal{A}|f \leftarrow |\mathcal{B}|f,\\ |\mathcal{A} \wedge \mathcal{B}|f &= |\mathcal{A}|f \cap |\mathcal{B}|f, & |\forall x_n\mathcal{A}|f &= \forall_n|\mathcal{A}|f,\\ |\mathcal{A} \vee \mathcal{B}|f &= |\mathcal{A}|f \cup |\mathcal{B}|f, & |\exists x_n\mathcal{A}|f &= \exists_n|\mathcal{A}|f,\\ |\boldsymbol{t}|f &= \phi_N f (= N). \end{aligned}$$

Definition 2.4 (Basic First-Order Models for **QBM.C**). *A* basic model *for* ***QBM.C*** *is a basic pre-model for* ***QBM.C*** *that assigns elements of PropFun to each atomic formula, including identity statements.*

Although the valuation function is officially extended in terms of propositional functions, we can define a relation $\vDash$ such that $a, f \vDash \mathcal{A}$ exactly when $a \in |\mathcal{A}|f$. See [6] and [12] for a list with the conditions for each formula shape.

A formula is *satisfied* by a variable assignment f in a model $\mathfrak{M}$ when $N \subseteq |\mathcal{A}|f$. A formula is *valid in a model* $\mathfrak{M}$ when it is satisfied by every variable assignment in that model; *valid in a frame* when it is valid in every model based on that frame; *valid in a class of frames* when it is valid in every frame in that class.

Lemma 2.5 (Hereditary Lemma). *For any formula* $\mathcal{A}$, *if* $a \leq b$ *and* $a \in |\mathcal{A}|f$, *then* $b \in |\mathcal{A}|f$.

[2] Here "$|\tau|f$" is defined to be fn when τ is the variable x_n, and $|c|$ when τ is constant symbol c.

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The proof of this lemma is a standard induction on the complexity of formulas, the interesting cases being negation, and the modalities. Negation is given by condition (c4), and modalities by (c5) and (c6).

LEMMA 2.6. *For any formula $\mathcal{A}$, if $f, g \in U^\omega$ agree on each free variable of $\mathcal{A}$, then* $|\mathcal{A}|f = |\mathcal{A}|g$.

LEMMA 2.7 (Semantic entailment). *In a basic model, a formula $\mathcal{A} \to \mathcal{B}$ is satisfied by a variable assignment* f *iff* $|\mathcal{A}|f \subseteq |\mathcal{B}|f$.

The proof of this lemma is as usual, using the hereditary lemma.

§3. Axiomatization.

DEFINITION 3.8. *We write a formula with a variable superscript to indicate that the variable does not occur free in the formula—e.g.,* x *does not occur free in $\mathcal{A}^x$. We use $\Rrightarrow$ as a separator for rules of* proof: *$\mathcal{A}_0, \dots, \mathcal{A}_n \Rrightarrow \mathcal{B}$ should be understood to mean "if $\mathcal{A}_0, \dots, \mathcal{A}_n$ are all theorems, then so is $\mathcal{B}$".*

The following axiom schemes and rules schemes generate the logic ***QBM.C***:

(ID)	$\mathcal{A} \to \mathcal{A}$.
$(\wedge E)$	$\mathcal{A} \wedge \mathcal{B} \to \mathcal{A}$.
$(\wedge E)$	$\mathcal{A} \wedge \mathcal{B} \to \mathcal{B}$.
$(\vee I)$	$\mathcal{A} \to \mathcal{A} \vee \mathcal{B}$.
$(\vee I)$	$\mathcal{B} \to \mathcal{A} \vee \mathcal{B}$.
$(\wedge I)$	$((\mathcal{A} \to \mathcal{B}) \wedge (\mathcal{A} \to \mathcal{C})) \to (\mathcal{A} \to (\mathcal{B} \wedge \mathcal{C}))$.
$(\vee E)$	$((\mathcal{A} \to \mathcal{C}) \wedge (\mathcal{B} \to \mathcal{C})) \to ((\mathcal{A} \vee \mathcal{B}) \to \mathcal{C})$.
$(\wedge\text{-}\vee)$	$\mathcal{A} \wedge (\mathcal{B} \vee \mathcal{C}) \to (\mathcal{A} \wedge \mathcal{B}) \vee (\mathcal{A} \wedge \mathcal{C})$.
$(DM1)$	$\neg(\mathcal{A} \wedge \mathcal{B}) \leftrightarrow (\neg\mathcal{A} \vee \neg\mathcal{B})$.
$(DM2)$	$\neg(\mathcal{A} \vee \mathcal{B}) \leftrightarrow (\neg\mathcal{A} \wedge \neg\mathcal{B})$.
$(\Box\wedge)$	$(\Box\mathcal{A} \wedge \Box\mathcal{B}) \to \Box(\mathcal{A} \wedge \mathcal{B})$.
$(\Box_L\wedge)$	$(\Box_L \mathcal{A} \wedge \Box_L \mathcal{B}) \to \Box_L(\mathcal{A} \wedge \mathcal{B})$.
$(\forall E)$	$\forall x\mathcal{A} \to \mathcal{A}[\tau/x]$, *where τ is free for x in $\mathcal{A}$.*
$(\exists I)$	$\mathcal{A}[\tau/x] \to \exists x\mathcal{A}$, *where τ is free for x in $\mathcal{A}$.*

(MP)	$\mathcal{A}, \mathcal{A} \to \mathcal{B} \Rrightarrow \mathcal{B}$.
(ADJ)	$\mathcal{A}, \mathcal{B} \Rrightarrow \mathcal{A} \wedge \mathcal{B}$.
$(Affix)$	$\mathcal{A} \to \mathcal{B}, \mathcal{C} \to \mathcal{D} \Rrightarrow (\mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{D})$.
$(RCont)$	$\mathcal{A} \to \mathcal{B} \Rrightarrow \neg\mathcal{B} \to \neg\mathcal{A}$.
$(R\Box M)$	$\mathcal{A} \to \mathcal{B} \Rrightarrow \Box\mathcal{A} \to \Box\mathcal{B}$.
$(R\Box_L M)$	$\mathcal{A} \to \mathcal{B} \Rrightarrow \Box_L\mathcal{A} \to \Box_L\mathcal{B}$.
$(R\forall I)$	$\mathcal{A}^x \to \mathcal{B} \Rrightarrow \mathcal{A}^x \to \forall x\mathcal{B}$.
$(R\exists E)$	$\mathcal{A} \to \mathcal{B}^x \Rrightarrow \exists x\mathcal{A} \to \mathcal{B}^x$.
$(R\circ)$	$\mathcal{A} \to (\mathcal{B} \to \mathcal{C}) \Lleftarrow\!\!\Rrightarrow (\mathcal{A} \circ \mathcal{B}) \to \mathcal{C}$.
$(R\leftarrow)$	$\mathcal{A} \to (\mathcal{B} \to \mathcal{C}) \Lleftarrow\!\!\Rrightarrow \mathcal{B} \to (\mathcal{C} \leftarrow \mathcal{A})$.
$(R\boldsymbol{t})$	$\boldsymbol{t} \to \mathcal{A} \Lleftarrow\!\!\Rrightarrow \mathcal{A}$.

LEMMA 3.9. *The following are theorems and derivable/admissible rules of* ***QBM.C***:

(1) $\forall x(\mathcal{A}^x \to \mathcal{B}) \to (\mathcal{A}^x \to \forall x\mathcal{B})$.

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(2) $\forall x(\mathcal{A} \to \mathcal{B}^x) \to (\exists x\mathcal{A} \to \mathcal{B}^x)$.
(RGC1) $\mathcal{A}^x \to \mathcal{B}[c/x] \Rrightarrow \mathcal{A}^x \to \forall x\mathcal{B}$.
(RGC2) $\mathcal{A}[c/x] \Rrightarrow \forall x\mathcal{A}$.

For (1), the proof is left to the reader, and uses fusion. Similarly, (2) follows by using left arrow and existential introduction. (RGC1) and (RGC2) follow from similar arguments to those in [12].

Proposition 3.10. *The following are theorems of **QB.C** (defined below), but not theorems of **QBM.C**:*

$(Dual1)\ \neg\forall x\neg\mathcal{A} \leftrightarrow \exists x\mathcal{A}$, $(Dual3)\ \neg\forall x\mathcal{A} \leftrightarrow \exists x\neg\mathcal{A}$,
$(Dual2)\ \neg\exists x\neg\mathcal{A} \leftrightarrow \forall x\mathcal{A}$, $(Dual4)\ \neg\exists x\mathcal{A} \leftrightarrow \forall x\neg\mathcal{A}$.

Theorem 3.11. *The logic **QBM.C** is sound and complete with respect to the class of all basic frames.*

Proof. The logic **QBM.C** is the logic **QB.C** of [6], but with multiple "Box" modalities, the left-implication, and a weaker negation. However, the proofs of [6] can easily be modified to handle these differences. □

3.1. Extensions. An important class of extensions of the base logic add extensional confinement axioms. In the presence of a weak negation without double negation equivalence, the following axioms are not equivalent:

(EC1) $\forall x(\mathcal{A} \vee \mathcal{B}^x) \to \forall x\mathcal{A} \vee \mathcal{B}^x$.
(EC2) $\mathcal{A}^x \wedge \exists x\mathcal{B} \to \exists x(\mathcal{A}^x \wedge \mathcal{B})$.

However, once we add double negation introduction and elimination, they are interderivable in any logic extending **QBM.C**. These axioms are required to capture a certain classicality of the quantifiers. Furthermore, these axioms are the reasons for employing admissible propositional functions [8, 12].

Definition 3.12. *The logic **BMQ.C** is the base logic **QBM.C**+ $(EC1)$ + $(EC2)$.*

The next lemma can be stated in a more fine-grained manner, separating (EC1) from (EC2), and vice versa, but we state it coarsely as follows.

Lemma 3.13. ***BMQ.C** is sound and complete with respect to the class of all **QBM.C**-models that satisfy $(cEC1)$ and $(cEC2)$, where the latter as defined as follows: For every $\phi \in PropFun$, $X, Y \in Prop$, $n \in \omega$, and $f \in U^\omega$.*

$$(cEC1) \quad X - Y \subseteq \bigcap_{j \in U} \phi(f[j/n]) \text{ only if } X - Y \subseteq (\forall_n \phi)f.$$

$$(cEC2) \quad \bigcup_{j \in U} \phi(f[j/n]) \subseteq X \cup \overline{Y} \text{ only if } |\exists_n \phi|f \subseteq X \cup \overline{Y}.$$

Although **QBM.C** and **BMQ.C** are our foundational logics, we will define a number of common propositional relevant logics and their extensions in the relevant logic literature using the following list of axioms and rules:

(DNE) $\mathcal{A} \leftrightarrow \neg\neg\mathcal{A}$.
(Cont) $(\mathcal{A} \to \mathcal{B}) \to (\neg\mathcal{B} \to \neg\mathcal{A})$.
(RCM) $\mathcal{A} \to \neg\mathcal{A} \Rrightarrow \neg\mathcal{A}$.

(B) $(\mathcal{A} \to \mathcal{B}) \to ((\mathcal{C} \to \mathcal{A}) \to (\mathcal{C} \to \mathcal{B}))$.
(B′) $(\mathcal{A} \to \mathcal{B}) \to ((\mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{C}))$.
(W) $(\mathcal{A} \to (\mathcal{A} \to \mathcal{B})) \to (\mathcal{A} \to \mathcal{B})$.
(C) $(\mathcal{A} \to (\mathcal{B} \to \mathcal{C})) \to (\mathcal{B} \to (\mathcal{A} \to \mathcal{C}))$.

The relevant logic **BM** (or rather **BM** with $\circ, \leftarrow, \boldsymbol{t}$) is defined (in the appropriate non-modal propositional language) as **QBM.C**, but without the axioms and rules containing modalities and quantifiers. Some non-modal propositional extensions of **BM** (with $\circ, \leftarrow, \boldsymbol{t}$) are captured by the following list:

1. **B** = **BM** + (DNE).
2. **DW** = **B** + (Cont).
3. **T** = **DW** + (RCM) + (B) + (B′) + (W).
4. **R** = **T** + (C).

Where $\mathbb{L}$ is a propositional extension of the relevant logic **B**, **Q**$\mathbb{L}$.**C** and $\mathbb{L}$**Q**.**C** denote the extensions of **QBM.C** and **BMQ.C**, respectively, by the same additional axiom and rule schemes.

The modal logic **C** denotes the *least regular modal logic*, over some propositional logic. Namely, the result of adding ($\Box\wedge$) and (R$\Box$M). The extensions in the modal fragment of a regular modal logic are defined using the following, where $\Diamond\mathcal{A} =_{df} \neg\Box\neg\mathcal{A}$:[3]

($K_\Box$) $\Box(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})$.
(RN) $\mathcal{A} \Rrightarrow \Box\mathcal{A}$.
(BD) $\Box(\mathcal{A} \vee \mathcal{B}) \to (\Box\mathcal{A} \vee \Diamond\mathcal{B})$.
(DB) $(\Diamond\mathcal{A} \wedge \Box\mathcal{B}) \to \Diamond(\mathcal{A} \wedge \mathcal{B})$.
(T) $\Box\mathcal{A} \to \mathcal{A}$.
(4) $\Box\mathcal{A} \to \Box\Box\mathcal{A}$.

A more complete list (in the background of relevant logics) can be found in several places, including [20, 21]. It is assumed that the reader is familiar with the naming conventions for modal logics extending **K**, so we only offer the following list:

1. **K** = **C** + ($K_\Box$) + (RN).
2. **S4** = **K** + (T) + (4).

Note, importantly, that here we use these names only to refer to the set of modal axiom schemes, and not to a full propositional modal logic. That is, the name only refers to the set of axiom schemes with modal operators in their shape. For example, (BD) and (DB) are theorems of the **K**-ish extension of classical logic, but not of the **K**-ish extension of relevant logics.

Quantified and modal logics are obtained by taking a quantified relevant logic $\mathbb{L}$ and extending it with a set of modal axiom schemes. Where $\mathbb{M}$ is the name of a set of modal axiom schemes, the logic $\mathbb{L}.\mathbb{M}$ is the result of simply adding the modal axiom schemes to the axiomatization of $\mathbb{L}$. In many cases this axiomatization will lack (BD) and (DB), making the modal fragment lack a certain "classicality". The dot is removed

[3] In general, it is somewhat common to assume that $\Diamond$ and $\Box$ are not necessarily dual in relevant logics, in which case we would take the $\Diamond$ as primitive. However, for our purposes here it is sufficient to only take the defined $\Diamond$ via negation.

when, in the terms of [6], the logic becomes *sufficiently classical*, in that it has both the duality of the modalities and (BD) and (DB) as theorem schemes.[4]

A final note on these axiomatizations is that they make essential use of $\circ$, $\leftarrow$, and $\boldsymbol{t}$. There are quantified relevant logics where the addition these connectives does not result in a conservative extension—e.g., see the logic **QB**⁻ in [24] for a first-order relevant logic which is not conservatively extended by either $\circ$ or $\leftarrow$.

§4. Epistemic models with worlds.

DEFINITION 4.14. *A* bounded frame *is a frame where there are elements* $0, 1 \in K$ *such that for each* $\alpha, \beta \in K$ *and* $S \in \{S_\Box, S_{\Box_L}\}$:

1. $0 \leq \alpha \leq 1$.
2. $1^* = 0$ *and* $0^* = 1$.
3. $S00$.
4. $S1\alpha \Rightarrow \alpha = 1$.
5. $R010$.
6. $R1\alpha\beta \Rightarrow (\alpha = 0$ *or* $\beta = 1)$.
7. $R111$.
8. $R\alpha\beta0 \Rightarrow (\alpha = 0$ *or* $\beta = 0)$.
9. $R100$.
10. $R\alpha1\beta \Rightarrow (\alpha = 0$ *or* $\beta = 1)$.

A bounded (pre-)model *is a (pre-)model based on a bounded frame where* $1 \in |p|f$ *and* $0 \notin |p|f$, *for very atomic sentence* p, *including* $|\boldsymbol{t}|f$, *for every* $f \in U^\omega$.

LEMMA 4.15. *In a bounded model* $\mathfrak{M}$, *for every formula* $\mathcal{A}$ *and every* $f \in U^\omega$:

1. $1 \in |\mathcal{A}|f$.
2. $0 \notin |\mathcal{A}|f$.

Proof. The proof adapts [18], which is by induction on the structural complexity of $\mathcal{A}$.[5] Base cases, including $\boldsymbol{t}$, are given by definition, and most cases can be obtained by straightforward adaptations of the arguments of Sedlár and Vigiani. Thus, we will demonstrate the case for $\circ$, $\leftarrow$, and $\forall x_n$, and leave $\exists x_n$ to the reader.

For the case $\mathcal{A} = \mathcal{B} \circ \mathcal{C}$, we have that $1 \in |\mathcal{B} \circ \mathcal{C}|f$ iff $\exists b, c \in K(Rbc1$ and $b \in |\mathcal{B}|f$ and $c \in |\mathcal{C}|f)$. By 4.14(7), that is $R111$, the result follows by the induction hypothesis. Therefore $1 \in |\mathcal{B} \circ \mathcal{C}|f$, as required. Next, for reductio, suppose that $0 \in |\mathcal{B} \circ \mathcal{C}|f$. Then $\exists b, c \in K(Rbc0$ and $b \in |\mathcal{B}|f$ and $c \in |\mathcal{C}|f)$. Then by 4.14(8), $b = 0$ or $c = 0$. But then either $0 \in |\mathcal{B}|f$ or $0 \in |\mathcal{C}|f$. Either way contradicts the induction hypothesis.

For the case $\mathcal{A} = \mathcal{B} \leftarrow \mathcal{C}$, we have that $1 \in |\mathcal{B} \leftarrow \mathcal{C}|f$ iff $\forall b, c \in K(Rb1c$ and $b \in |\mathcal{B}|f$ imply $c \in |\mathcal{C}|f)$. Take any b, c such that $Rb1c$ and $b \in |\mathcal{B}|f$. By 4.14(10),

4 Naming conventions for quantified modal relevant logics have not been simple. A likely reason for the multitude of decorations and permutations is that many authors have aimed for generality rather than singling out particular logics.

5 Note that (1) actually follows directly from *Prop* being a set of $\leq$-upsets, condition 4.14(1), and *Prop*'s closure under the various operations. The corresponding conditions in the previous definition, however, are satisfied by the canonical model, and will therefore be kept.

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either $b = 0$ or $c = 1$. By the induction hypothesis, the former is impossible. The latter, with the induction hypothesis, entails $1 = c \in |\mathcal{C}|f)$, as required. Next, for reductio, suppose that $0 \in |\mathcal{B} \leftarrow \mathcal{C}|f$. This is iff $\forall b, c \in K(Rb0c$ and $b \in |\mathcal{B}|f$ imply $c \in |\mathcal{C}|f)$. By 4.14(9), $R100$, and furthermore $1 \in |\mathcal{B}|f$ by the induction hypothesis. Therefore $0 \in |\mathcal{C}|f$, which contradicts the induction hypothesis.

The last case shown is $\mathcal{A} = \forall x_n \mathcal{B}$. We have that $1 \in |\forall x_n \mathcal{B}|f$ iff $1 \in \bigsqcap_{g \in xf} |\mathcal{A}|g$. The latter is an element of *Prop* by construction, and is thus an upwardly closed set which contains 1, as required. We have that $0 \notin |\forall x_n \mathcal{B}|f$ iff $0 \notin \bigsqcap_{g \in xf} |\mathcal{A}|g$. By the induction hypothesis, for every $g \in xf$, $0 \notin |\mathcal{A}|g$, so $0 \notin \bigcap_{g \in xf} |\mathcal{A}|g$. So there is no $X \in Prop$ where $0 \in X \subseteq \bigcap_{g \in xf} |\mathcal{A}|g$, as required. □

A possible world is a point in the model where the intensional connectives behave classically: that is, extensionally. Sedlár [17] and Sedlár and Vigiani [18] define worlds such that the negation and implication of the relevant fragment are truth-functional at a world. We will do the same by ensuring that the left-implication and fusion are also classical. However, for the first-order machinery, there are many choices to make in terms of “classicality”.

First, it is likely that a world should satisfy (EC1) and (EC2), as these axioms are decidedly classical. After that, we have options. Should we enforce the Tarskian truth condition, which can be succinctly paraphrased by $\bigcap_{i \in I} X_i = \bigsqcap_{i \in I} X_i$ and $\bigcup_{i \in I} X_i = \bigsqcup_{i \in I} X_i$? If we do, then worlds must be ω-complete: that is, a world-variable assignment pair cannot satisfy all instances of a universally quantified formula $\forall x \mathcal{A}$ without also satisfying $\forall x \mathcal{A}$ itself. Indeed, we suppose (and might even go as far as to claim) that there is good reason to believe that a *robust possible world* is ω-complete. However, issues of completeness lie in the direction of ω-complete possible worlds. So for the time being we do not assume this strong condition on possible worlds. The reader may look forward to Section 7, where this issue is highlighted, and where we develop the relation to Fine’s incompleteness results for constant domain (non-general frame) **RQ**.[6]

Note that (EC1) and (EC2) are not sufficient to get a fully classical behavior of the quantifiers. For example, some formulas needed for prenex normal form are invalid. [14] shows the following “lemons” are (and should be) invalid in any first-order relevant logic worth its weight in salt:

(∃PN1) $(p \to \exists x Ax) \to \exists x(p \to Ax)$.
(∃PN2) $(\forall x Ax \to p) \to \exists x(Ax \to p)$.

These formulas could be upgraded to axiom schemes, nonetheless invalid, simply by replacing ‘p’ with $\mathcal{B}^x$ uniformly. The remainder of the usual formulas required for prenex normal form are valid, including $\forall x(p \to Fx) \leftrightarrow (p \to \forall x Fx)$ [14, p. 279].

[6] There is a lot of precedent towards Tarskian (here, ω-complete) worlds. Nonetheless, there is also some precedent against them. Driven by (the interpretation of) the separation of the Barcan formulas from contracting and expanding domains as in [8], it could be argued that the non-Tarskian approach (using $\bigsqcap, \bigsqcup$) finds more coherence with certain intuitions and practices concerning modal reasoning. Indeed, we don’t often have epistemic access to “that’s all” clauses discussed below in Section 7.

DEFINITION 4.16. *Let* $\mathfrak{M}$ *be a bounded model. An element* $w \in N$ *is a* possible world *if it satisfies conditions* (1)–(9) *below. For every* $s, t, \in K$, *and for every* $\phi \in PropFun$, $X, Y \in Prop$, $n \in \omega$, *and* $f \in U^{\omega}$:

1. $w^* = w$.
2. $Rwww$.
3. $Rwst \Rightarrow (s = 0 \text{ or } w \leq t)$.
4. $Rwst \Rightarrow (t = 1 \text{ or } s \leq w)$.
5. $Rstw \Rightarrow ((s \leq w \,\&\, t \leq w) \text{ or } s = 0 \text{ or } t = 0)$.
6. $Rswt \Rightarrow (s = 0 \text{ or } w \leq t)$.
7. $Rswt \Rightarrow (t = 1 \text{ or } s \leq w)$.
8. *If* $X - Y \subseteq \bigcap_{j \in U} \phi(f[j/n])$ *and* $w \in X - Y$, *then* $w \in (\forall_n \phi) f$.
9. *If* $\bigcup_{j \in U} \phi(f[j/n]) \subseteq X \cup \overline{Y}$ *and* $w \notin X \cup \overline{Y}$, *then* $w \notin (\exists_n \phi) f$.

Note that the possible worlds are all logically normal points with respect to the underlying relevant (subclassical) logic. This holds in the canonical model, but a word or two is in order. At a world the conditional is made truth-functional, and there is no way to falsify (using two truth values) a truth functional conditional corresponding to a relevant conditional, provided the relevant logic is a sublogic of classical logic. Thus, the worlds should make true at least all of the conditional theorems of the relevant logic. And so, by extension, all worlds should be logically normal points in the frame.

DEFINITION 4.17. *A* W-frame *is a structure* $\mathbb{F} = \langle \mathfrak{F}, W \rangle$ *where* $\mathfrak{F}$ *is a bounded frame satisfying* $(cEC1)$ *and* $(cEC2)$, $W \subseteq N$ *is a set of base possible worlds, and the following conditions are satisfied*:

1. $(\forall w \in W)(\forall u \in K)(S_{\Box_L} wu \Rightarrow u \in N)$.
2. $(\forall k \in N)(\exists w \in W) S_{\Box_L} ws$.

DEFINITION 4.18. *A* W-(pre-)model *based on W-frame* $\mathbb{F}$ *is defined as in Definition* 4.14. *Moreover, satisfaction and validity are defined as follows*:

1. *A formula* $\mathcal{A}$ *is* satisfied by variable assignment f in a W-model $\mathfrak{M}$ *iff* $W \subseteq |\mathcal{A}| f$.
2. *A formula* $\mathcal{A}$ *is* valid in a W-model $\mathfrak{M}$ *iff it is satisfied by every* $f \in U^{\omega}$.
3. *A formula* $\mathcal{A}$ *is* valid in a class of W-frames *iff it is valid in each W-model based on a W-frame in the class.*

LEMMA 4.19. $\Box_L \mathcal{A} \to \mathcal{B}$ *is valid in W-model* $\mathfrak{M}$ *iff, for every* $f \in U^{\omega}$, $|\mathcal{A}| f \subseteq |\mathcal{B}| f$.

Proof. The proof is similar to that of Sedlár and Vigiani, except for our use of N in Definition 4.17.

Suppose that $\Box_L \mathcal{A} \to \mathcal{B}$ is valid in W-model $\mathfrak{M}$. Further suppose for reductio that $|\mathcal{A}| f \not\subseteq |\mathcal{B}| f$, for some $f \in U^{\omega}$. By semantic entailment, f does not satisfy $\mathcal{A} \to \mathcal{B}$, which means that there is some $\alpha \in N$ such that $\alpha \notin |\mathcal{A} \to \mathcal{B}| f$. But, by Definition 4.17, there is a world w such that $S_{\Box_L} w\alpha$, which entails that $\alpha \notin |\Box_L(\mathcal{A} \to \mathcal{B}| f)$, which contradicts our starting point.

For the right-to-left direction, suppose that $|\mathcal{A}| f \subseteq |\mathcal{B}| f$ for every $f \in U^{\omega}$. By semantic entailment, $\mathcal{A} \to \mathcal{B}$ is satisfied by every variable assignment. Then for each $\alpha \in N$, $\alpha \in |\mathcal{A} \to \mathcal{B}| f$. For any world w, suppose that $S_{\Box_L} w\beta$. Then $\beta \in N$, by Definition 4.17. As this is the case for every world, we have that $\Box_L \mathcal{A} \to \mathcal{B}$ is valid in the model. □

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LEMMA 4.20. *For any world w in any W-model $\mathfrak{M}$:*

1. $w \in |\neg\mathcal{A}|f$ *iff* $w \notin |\mathcal{A}|f$.
2. $w \in |\mathcal{A} \to \mathcal{B}|f$ *iff* $w \notin |\mathcal{A}|f$ *or* $w \in |\mathcal{B}|f$.
3. $w \in |\mathcal{A} \circ \mathcal{B}|f$ *iff* $w \in |\mathcal{A}|f$ *and* $w \in |\mathcal{B}|f$.
4. $w \in |\mathcal{A} \leftarrow \mathcal{B}|f$ *iff* $w \notin |\mathcal{B}|f$ *or* $w \in |\mathcal{A}|f$.
5. $w \in |\boldsymbol{t}|f$.
6. $w \in |\forall x\mathcal{A}|f$ *iff* $w \in |\neg\exists x\neg\mathcal{A}|f$.
7. $w \in |\exists x\mathcal{A}|f$ *iff* $w \in |\neg\forall x\neg\mathcal{A}|f$.
8. $w \in |\forall x_n(\mathcal{A}^{x_n} \vee \mathcal{B}) \to (\mathcal{A}^{x_n} \vee \forall x\mathcal{B})|f$ *(and therefore it also satisfies* $(EC2)$*).*
9. (a) $w \in |(p \to \exists xAx) \to \exists x(p \to Ax)|f$.
   (b) $w \in |(\forall xAx \to p) \to \exists x(Ax \to p)|f$.

Proof. For (1) and (2), the reader is referred to [18].

For (3), suppose that $w \in |\mathcal{A} \circ \mathcal{B}|f$. Then $\exists b, c \in K(Rbcw)\ \&\ b \in |\mathcal{A}|f\ \&\ c \in |\mathcal{B}|f$. By Lemma 4.15, b and c are not 0. Thus, by applying 4.16(5), we get $b, c \leq w$, and therefore that $w \in |\mathcal{A}|f$ and $w \in |\mathcal{B}|f$, as required. For the other direction, $Rwww$ is sufficient for the result.

For (4), the left-to-right direction follows by $Rwww$. For the other direction, assume that $w \notin |\mathcal{A} \leftarrow \mathcal{B}|f$. Then there are $b, c \in K$ such that $Rbwc$, $b \in |\mathcal{B}|f$ and $c \notin |\mathcal{A}|f$. By Lemma 4.15, $b \neq 0$ and $c \neq 1$. Thus, by applying 4.16(5), $b \leq w \leq c$. But then both $w \in |\mathcal{B}|f$ and $w \notin |\mathcal{A}|f$, as required, on pains of contradiction.

By definition of worlds we have $W \subseteq N$, so (5) follows straightforwardly.

(6) and (7) will follow from the $*$ properties at each world. (As soon as you add double negation, you regain the quantifier dualities.)

For (8) the proof is as in [12], and some of the details are included here. By (2), (8) can be demonstrated by showing that, for every world w, either $w \notin |\forall x_n(\mathcal{A}^{x_n} \vee \mathcal{B})|f$ or $w \in |\mathcal{A}^{x_n} \vee \forall x_n\mathcal{B}|f$, for every f. Assume that $w \notin |\mathcal{A}^{x_n} \vee \forall x_n\mathcal{B}|f$. Then $w \notin |\mathcal{A}^{x_n}|f$ and $w \notin |\forall x_n\mathcal{B}|f$. For reductio, further assume that $w \in |\forall x_n(\mathcal{A}^{x_n} \vee \mathcal{B})|f$. Then, $w \in \bigcap_{g \in x_n f} |(A^{x_n} \vee \mathcal{B})|g$. By the reasoning in [12], there is an $X \in Prop$ such that $w \in X$ and $w \in X - |\mathcal{A}|f$ and $X - |\mathcal{A}|f \subseteq \bigcap_{g \in x_n f} |\mathcal{B}|f$. But by 4.16(8), $w \in (\forall_n|\mathcal{B}|)f = |\forall x_n\mathcal{B}|f$, a contradiction.

For (9), by (2), both cases reduce to the distribution of a universal over conjunction into a single conjunct (where the other conjunct has no free occurrences of the variable in question). □

Unlike in [18], our inclusion of N and requirement that N is "seen in full by W" means that we do not need to consider variants of the usual frame conditions for extensions. In Table 1, we have provided a small list of axioms and rules together with their corresponding frame conditions. (That is, a modular frame-correspondence in the background of **QBM.C**.) The conditions were shortened by adopting the following conventions of notation:

$$R^2abcd =_{df} \exists x(Rabx\ \&\ Rxcd).$$
$$S^2_\Box ab =_{df} \exists x(S_\Box ax\ \&\ S_\Box xb).$$
$$Ra(Rbc)d =_{df} \exists x(Raxd\ \&\ Rbcx).$$
$$S_\Diamond ab =_{df} S_\Box a^*b^*.$$

Table 1. *Frame correspondence for extensions.*

Names	Axioms	Condition
(DNE)	$\neg\neg\mathcal{A} \leftrightarrow \mathcal{A}$	$a^{**} = a$
(Cont)	$(\mathcal{A} \to \mathcal{B}) \to (\neg\mathcal{B} \to \neg\mathcal{A})$	$Rabc \Rightarrow Rac^*b^*$
(RCM)	$\mathcal{A} \to \neg\mathcal{A} \Rrightarrow \neg\mathcal{A}$	Raa^*a
(B)	$(\mathcal{A} \to \mathcal{B}) \to ((\mathcal{C} \to \mathcal{A}) \to (\mathcal{C} \to \mathcal{B}))$	$R^2abcd \Rightarrow Ra(Rbc)d$
(B′)	$(\mathcal{A} \to \mathcal{B}) \to ((\mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{C}))$	$R^2abcd \Rightarrow Rb(Rac)d$
(W)	$(\mathcal{A} \to (\mathcal{A} \to \mathcal{B})) \to (\mathcal{A} \to \mathcal{B})$	$Rabc \Rightarrow R^2abbc$
(C)	$(\mathcal{A} \to (\mathcal{B} \to \mathcal{C})) \to (\mathcal{B} \to (\mathcal{A} \to \mathcal{C}))$	$Rabcd \Rightarrow Racbd$
($K_\Box$)	$\Box(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})$	$Rbcf \;\&\; S_\Box fd \Rightarrow$ $\exists b', c'(S_\Box bb' \;\&\; S_\Box cc' \;\&\; Rb'c'd)$
(RN)	$\mathcal{A} \Rrightarrow \Box\mathcal{A}$	$b \in N \;\&\; S_\Box bc \Rightarrow c \in N$
(BD)	$\Box(\mathcal{A} \vee \mathcal{B}) \to (\Box\mathcal{A} \vee \Diamond\mathcal{B})$	$S_\Box ab \Rightarrow \exists c \leq b(S_\Box ac \;\&\; S_\Diamond ac)$
(DB)	$(\Diamond\mathcal{A} \wedge \Box\mathcal{B}) \to \Diamond(\mathcal{A} \wedge \mathcal{B})$	$S_\Diamond ab \Rightarrow \exists c \leq b(S_\Box ac \;\&\; S_\Diamond ac)$
($T_\Box$)	$\Box\mathcal{A} \to \mathcal{A}$	$S_\Box aa$
($4_\Box$)	$\Box\mathcal{A} \to \Box\Box\mathcal{A}$	$S^2_\Box ab \Rightarrow S_\Box ab$
(PEM)	$\mathcal{A} \vee \neg\mathcal{A}$	$a \in N \Rightarrow a^* \leq a$
(M)	$\mathcal{A} \to (\mathcal{A} \to \mathcal{A})$	$Rabc \Rightarrow (a \leq c \vee b \leq c)$
(RER)	$\mathcal{A} \Rrightarrow (\mathcal{A} \to \mathcal{B}) \to \mathcal{B}$	$\exists x(x \in N \;\&\; Raxa)$

Note that $\Diamond$ is always assumed here to be the negation-dual of $\Box$ by being a notational variant. This is even despite the absence of double negation introduction/elimination.[7] The conditions for (BD) and (DB) are drawn directly from [6], which is a generalization of the condition given by Mares and Meyer [13].

For some further extensions, the reader is directed to [5, 6] for the quantified modal setting, [16] for the proportional relevant setting, and [20, 21] for propositional modal relevant setting.

Lemma 4.21. *For every $\mathbb{L}$-formula $\mathcal{A}$, if there is an $\mathbb{L}$-model $\mathfrak{M}$ in which $\mathcal{A}$ is not valid, then there is a $\mathbb{CL}$-model $\mathfrak{M}'$ in which $\Box_L\mathcal{A}$ is not valid.*

Proof. The proof is as in [18]. Here we add states 0, 1 and world w to the model $\mathfrak{M}$, and show the resulting model is a W-model in which $\Box_L\mathcal{A}$ is not valid.

Suppose we have an $\mathbb{L}$-model $\mathfrak{M}$ based on the frame

$$F = \langle K', N', R', *', S'_\Box, S'_{\Box_L}, U, Prop', PropFun' \rangle$$

which invalidates $\mathcal{A}$. Construct the new model as follows. First, define the W-frame $\mathbb{F}$ as follows:

1. $W = W' \cup \{1, 0, w\}$.
2. $N = N' \cup \{1, w\}$.

[7] That is to say, by definition we can derive $\Diamond\mathcal{A} \leftrightarrow \neg\Box\neg\mathcal{A}$; however, we may not also have $\Box\mathcal{A} \leftrightarrow \neg\Diamond\neg\mathcal{A}(= \neg\neg\Box\neg\neg\mathcal{A})$. Semantically, this means that $S_\Diamond a^*b^*$ need not imply $S_\Box ab$. With (DNE), we can use one accessibility relation. Without (DNE), the $S_\Diamond$ relation is the $S_\Box$ restricted to $*$-points.

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3. $R = R' \cup \{(w, w, w), (1, 1, 1), (1, 0, 0)\}$
$\cup\{(w, a, 1)|a \in W\} \cup \{(w, 0, a)|\forall a \in W\}$
$\cup\{(0, a, b), (a, 0, b), (a, b, 1)|a, b \in W\}$.[8]
4. $* = *' \cup \{(w, w), (0, 1), (1, 0)\}$.
5. $S_\Box = S'_\Box \cup \{(w, w), (a, 1), (0, b)|a, b \in W\}$.
6. $S_{\Box_L} = S'_{\Box_L} \cup \{(w, w), (a, 1), (0, b)|a, b \in W\} \cup \{(w, a)|a \in N\}$.
7. $U = U$.
8. For each $X' \in Prop'$, if $N' \subseteq X'$, add $X = X' \cup \{1, w\}$ to $Prop$, and if $N' \not\subseteq X'$, add $X = X' \cup \{1\}$ to $Prop$.
Given a $X' \in Prop'$, let $h(X')$ be the corresponding $X \in Prop$.
9. For each $\phi' \in PropFun'$, add the propositional function ϕ defined by, for all $f \in U^\omega$, $\phi f = h(\phi' f)$.

First we show that this frame is an $\mathbb{L}$-frame, then that it is a bounded frame, then that it is a W-frame.

As defined, (c1), (c2), (c4), (c5), (c6), and (c8) are straightforward. For (c3), adding $(0, a, b), (a, 0, b), (a, b, 1)$ ensures the tonicity w.r.t. 1 and 0, and indirectly for w, for which $0 \leq w \leq 1$ and $w \leq w$ are the only relevant orderings.

For (c7), first note that every element of $Prop$ is an upset. We show only the case for $X \to Y$, as the rest are similar. Suppose that $X, Y \in Prop$. We show that $X \to Y = h((X \to Y)')$. Suppose that $N' \subseteq (X \to Y)'$. (The other case is similar.) Then $h((X \to Y)') = (X \to Y)' \cup \{1, w\}$. For each $a \in W'$, $a \in (X \to Y)'$ iff $a \in X \to Y$. The right-to-left direction is straightforward. For the left-to-right, assume that $a \notin X \to Y$. Then there is $b, c \in W(Rabc \;\&\; b \in X \;\&\; c \notin Y)$. If $b, c \in W'$, then $a \notin (X \to Y)'$, as required. If at least one of $b, c \in \{1, 0, w\}$, then we know $b \neq 0$, $c \neq 1$. If $b = 1$, then by the tonicity conditions, we obtain the desired result. If $b = w$, then by the definition of the model, $c = 1$, a contradiction. If $c = 0$, the result follows by tonicity. If $c = w$, then by definition $b = 1$, a contradiction.

Thus let's focus on 1 and w. It suffices to show that $w, 1 \in X \to Y$—i.e., that $\forall b, c \in W((Rabc \;\&\; b \in X) \Rightarrow c \in Y)$, for $a = 1$ and $a = w$. Note that the only R additions with 1 in the first place are (1,1,1), (1,0,b), and (1,b,1). In each case, $(R1bc \;\&b \in X) \Rightarrow c \in Y)$, as required. Now consider w. We added $(w, w, w), (w, a, 1), (w, 0, a)$ to obtain R, as these also imply that $(Rwbc \;\&b \in X) \Rightarrow c \in Y)$, again because 1 is in every element of $Prop$ and 0 is in no elements of $Prop$.

For (c9), we may apply similar arguments, relying on (c7). It is straightforward to show that $\forall_n \phi f = h(\forall_n \phi' f)$, from which the (c10) follows.

To show that this is a bounded frame, we need to show (1)–(10) of Definition 4.14. (1)–(5), (7), and (9) are straightforward. We show only (8). Suppose that $Rab0$. We only added (1,0,0), (0, a, 0), and (a,0,0), and in each case (8) is satisfied.

To show that this is a W-frame, we must show that w is a possible world (satisfying (1)–(9) of Definition 4.16), and that conditions (1) and (2) of Definition 4.17.

To show $w \in N$ is a possible world, (1) and (2) are trivial. Of (3)–(7), we only show (6). Suppose that $Rawt$. We only added $(w, w, w), (w, w, 1), (0, w, a)$, and $(a, w, 1)$ and in each case the consequent of (6) is satisfied. Of (8) and (9), we only show (8). Suppose

8 Adding the set $\{(w, w, w), (1, 1, 1), (1, 0, 0)\}$ to the ternary relation ensures that $\leq$ is reflexive on all points, and adding each $(w, a, 1)$ and $(w, 0, a)$ ensures that $\leq$ ordering for the new points is adequate.

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that $w \in X - Y \subseteq \bigcap_{j \in U} \phi(f[j/n])$. For $w \in X - Y$, we have that $w \in X$ and $N' \subseteq X'$, but $w \notin Y$ and so $N' \nsubseteq Y'$. Now, if $w \in \bigcap_{j \in U} \phi(f[j/n])$, then $w \in \phi(f[j/n])$, for each $j \in U$, which means that $N' \subseteq \phi'(f[j/n])$. This entails that $\bigcap_{j \in U} \phi'(f[j/n])$ must be as large as the admissible proposition N'. Thus, $(\forall_n \phi)'f$ either is just N' or contains it properly. By definition, $(\forall_n \phi)f$ must include w, as required.

For (1) and (2) of Definition 4.17, it is easy to see the construction ensures their satisfaction: w relates to all and only the elements of N.

Thus we have indeed constructed a W-frame. To extend this frame to a model, let the valuation function $|–|$ assign to constants the same objects as in the $\mathbb{L}$-model. For n-ary predicates, lets $|P^n|(u_1, \dots, u_n) = h(|P^n|^{\mathfrak{M}}(u_1, \dots, u_n))$. That is, add 1 to the truth set of each atomic formula's truth set, and also add w if the truth set originally contained N. This model is a bounded model: 1 is in every truth set and 0 is in none of the truth sets.

What remains to be shown is that invalidity of $\mathcal{A}$ in $\mathfrak{M}$ implies the invalidity of $\Box_L \mathcal{A}$ is the constructed W-model. We show this in two steps: first, we show that, for every $a \in W'$, $a \in |\mathcal{A}|^{\mathfrak{M}} f$ iff $a \in |\mathcal{A}|f$. Then we use this to show the desired result.

We show that for every $a \in W'$, $a \in |\mathcal{A}|^{\mathfrak{M}} f$ iff $a \in |\mathcal{A}|f$, for every $f \in U^{\omega}$ by induction on the complexity of $\mathcal{A}$. If $\mathcal{A}$ is atomic, either it is either of the form $\boldsymbol{t}$, in which case the result follows trivially, or $P^n(\tau_1, \dots, \tau_n)$. For the latter case, the only disagreement in the models, by definition, is for the elements outside W', and so the base cases are covered.

For the inductive cases, we demonstrate only a few.

Suppose that $\mathcal{A} = \mathcal{B} \circ \mathcal{C}$. If $a \in W'$ is such that $a \in |\mathcal{B} \circ \mathcal{C}|^{\mathfrak{M}} f$, then there are $b, c \in W'$ such that $R'bca$, $b \in |\mathcal{B}|^{\mathfrak{M}} f$, and $c \in |\mathcal{C}|^{\mathfrak{M}} f$. The points b and c bear these relations and inclusions (in the new construction) by the inductive hypothesis, and so $a \in |\mathcal{B} \circ \mathcal{C}|f$. For the converse, suppose that $a \in |\mathcal{B} \circ \mathcal{C}|f$. It suffices to show that $R'abc$ for $b, c \in W'$ such that $b \in |\mathcal{B}|f$, and $c \in |\mathcal{C}|f$. The case to worry about is when we know that, in the constructed model, $Rbca$, when b or c is one of w or 0 or 1. By construction, this is the case only with the added (0,c,a), and (b, 0, a). In both cases, $0 \notin X \in Prop$, in which case $a \in |\mathcal{B} \circ \mathcal{C}|f$ must be witnessed by a different pair, namely a pair of elements in W', as required.

Suppose that $\mathcal{A} = \forall x_n \mathcal{B}$. For every $a \in W'$:

$$
\begin{aligned}
a \in |\forall x_n \mathcal{B}|^{\mathfrak{M}} f \text{ iff } & a \in \prod_{g \in xf} |\mathcal{B}|^{\mathfrak{M}} g && \text{DF} \\
\text{iff } & a \in \prod_{g \in xf} |\mathcal{B}| g && \text{Induction hyp.} \\
\text{iff } & a \in |\forall x_n \mathcal{B}| f && \text{DF.}
\end{aligned}
$$

Note that these "iff" cannot be turned into "=", as the second step would fail, as the induction hypothesis only applies to elements of W'. However, the second step, using the induction hypothesis, does hold here, because $\prod_{g \in xf} |\mathcal{B}|g$ is just $\prod_{g \in xf} |\mathcal{B}|^{\mathfrak{M}} g$ with an extra element or two (of $1, w$).

On to the desired result. Suppose that $\mathcal{A}$ is not valid in the $\mathbb{L}$-model $\mathfrak{M}$. Then there is some $a \in N$ such that $a \notin |\mathcal{A}|^{\mathfrak{M}} f$ for some f. In the constructed model, the last fact shown implies that $a \notin |\mathcal{A}|f$. Moreover, since $S_{\Box_L} wa$, we have that $\Box_L \mathcal{A}$ is not valid in the constructed W-model. □

4.1. Axiomatization. Before giving an axiomatization for the whole system, we need to identify a single system for quantified classical logic. There are many routes to go, and we elect for the route where we have a possibilist interpretation of the quantifiers. Thus the following definition.

DEFINITION 4.22. *The classical first-order logic* ***CQ*** *is defined (with some redundancies) by taking:*

1. *The theorems of classical propositional logic, in the first-order language* $\mathbb{L}_{FO}$.[9]
2. *The axioms* $(\forall E)$, $(\exists I)$, $(R\circ)$, $(R\leftarrow)$, $(R\boldsymbol{t})$, $(EC1)$, $(EC2)$.
3. *The rules* $(R\forall I)$, $(R\exists E)$, (MP).

REMARK 4.23. *Note that the axiom forms of contraposition and double negation elimination are available, so in* $\mathbb{L}_{FO}$ $(Dual1)$–$(Dual4)$ *are derivable.*

LEMMA 4.24. *The following formulas are theorem-schemes and admissible rules of* ***CQ***.

(CQ)	$\forall x \forall y \mathcal{A} \to \forall y \forall x \mathcal{A}$.
$(RGC1)$	$\mathcal{A}^x \to \mathcal{B}[c/x] \Rrightarrow \mathcal{A}^x \to \forall x \mathcal{B}$.
$(RGC2)$	$\mathcal{A}[c/x] \Rrightarrow \forall x \mathcal{A}$.
(UG)	$\mathcal{A} \Rrightarrow \forall x \mathcal{A}$.
(VQ)	$\mathcal{A}^x \to \forall x \mathcal{A}^x$.
(AI)	$\forall y(\forall x \mathcal{A} \to \mathcal{A}[y/x])$.
(RTI)	$\mathcal{A} \Rrightarrow \mathcal{A}[\tau/x]$, *where* τ *is free for* x *in* $\mathcal{A}$.
(EXT)	*All the theorems of Lemma* 3.9.
$(\circ=)$	$\mathcal{A} \circ \mathcal{B} \leftrightarrow \mathcal{A} \wedge \mathcal{B}$.
$(\leftarrow=)$	$(\mathcal{A} \leftarrow \mathcal{B}) \leftrightarrow (\mathcal{B} \to \mathcal{A})$.
$(\boldsymbol{t}=)$	$\boldsymbol{t} \leftrightarrow (\mathcal{A} \vee \neg \mathcal{A})$.

Proof. Using (∀E) and (R∀I) we can derive any instance of (CQ). By rewriting a proof replacing a constant symbol for the quantified variable and then applying the appropriate generalization rule, we obtain (RGC1) and (RGC2). (UG), (VQ), (AI), (RTI), and (EXT) are straightforward. The final three, (∘=), (←=), and ($\boldsymbol{t}$=) show the provable reductions of the connectives not usually taken as primitive in classical logic. □

It is straightforward to show, using the previous lemma, that the logic **CQ** is the logic **QK** + (CQ) + (R∀I) from [8], but without the modal fragment. That is, by reducing the language to exclude modal operators, and deleting the K axiom and necessitation rule. (Note that our naming convention for the rules and axioms differs from Goldblatt's.)

The presentation of **CQ** is a bit odd in that it is in a language that includes ∘, ← and $\boldsymbol{t}$. However, these are definable in first-order classical logic, and such a presentation does not affect the system as a whole.

DEFINITION 4.25. *Let* $\mathbb{L}$ *be an axiom system for a quantified modal relevant logic as given in Section* 3. *The logic* $\mathbb{CL}$ *based on* $\mathbb{L}$ *is defined by:*

[9] In particular, use the implication connective "→" for the material conditional.

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1. *The theorems of* $\boldsymbol{CQ}$, *with* (MP), $(R\forall I)$, $(R\exists E)$, *written in the language* $\mathfrak{L}_{QBM.C}$.
2. *For all axiom schemes* $\mathcal{A}$ *of* $\mathbb{L}$, *and axiom scheme* $\Box_L\mathcal{A}$.
3. *For all rules* $\mathcal{A}_1, \ldots \mathcal{A}_n \Rrightarrow \mathcal{B}$ *of* $\mathbb{L}$, *a rule* $\Box_L\mathcal{A}_1, \ldots \Box_L\mathcal{A}_n \Rrightarrow \Box_L\mathcal{B}$.
4. *The bridge rule* (BR): $\Box_L(\mathcal{A} \to \mathcal{B}) \Rrightarrow \mathcal{A} \to \mathcal{B}$.

The following lemma will be useful in the completeness proof below.

LEMMA 4.26. *The rule* $(RGC2)$*—*$\mathcal{A}[c/x] \Rrightarrow \forall x\mathcal{A}$*—is derivable in any logic* $\mathbb{CL}$.

Proof. The proof has two steps. First, we note that the universal generalization rule is derivable. This follow from the definition of inferential behaviour of $\boldsymbol{t}$, and the rule $(\forall\text{E})$, in $\mathbb{CL}$. Second, we rewrite the terms in a proof of $\mathcal{A}[c/x]$ to obtain a proof of $\mathcal{A}[y/x]$, for a brand new y, which allows us to apply $(\forall\text{E})$ for the desired result. □

We now move on to showing that this axiom system is sound and complete with respect to the W-models defined above. However, before moving on, let's address a worry that the reader may have at this point.

4.2. Believing t and believing theorems. You may notice that, $\vdash_{\mathbb{CL}} \Box\boldsymbol{t} \Rightarrow \Box\mathcal{T}$, for every theorem $\mathcal{T}$ of $\mathbb{L}$. Thus, if one believes $\boldsymbol{t}$ then they believe all $\mathbb{L}$theorems. As noted above, however, $\Box\mathcal{T}$ is not a theorem of $\mathbb{CL}$ for each theorem $\mathcal{T}$ of $\mathbb{L}$. Thus, the constructed systems do not enforce omniscience with respect to the theorems of the underlying relevant logic. Only agents who believe $\boldsymbol{t}$ will will believe every relevant theorem. That being said, let's briefly explain what it means to believe that $\boldsymbol{t}$.

What does it mean to believe the intensional constant proposition $\boldsymbol{t}$? In the ternary relational models, $\boldsymbol{t}$ is essentially the intersection of all theorems (itself a theorem). In effect, it is a finitely expressible proposition that acts as the infinite conjunction of all theorems. To really be in a state of believing $\boldsymbol{t}$, which it should be clear that no one actually is in such a state, is exactly to believe all theorems. Thus, $\Box\boldsymbol{t} \Rightarrow \Box\mathcal{T}$ is harmless. Not because no one believes $\boldsymbol{t}$, but that believing $\boldsymbol{t}$ is just believing all theorems. Thus, one's beliefs are not forced to contain all theorems (unless they already do).[10]

It may be highly plausible that we want to model agents who do in fact believe certain subsets of theorems. For example, an agent might believe all theorems of the form $\mathcal{A} \vee \neg\mathcal{A}$. This can be modeled by restricting $S_\Box$'s second argument place to only include elements a satisfying $a^* \leq a$ (including those outside N, if necessitation is not also desired). An upshot of the approach here is that believing one subset of the theorems need not imply believing every theorem. However, while relevant logics are able to deny that every theorem implies every other theorem, some theorems do in fact relevantly imply other theorems. In the cases where an agent (plausibly) believes a set of theorems that imply another set of theorems, they must believe that other set of theorems as well, in the systems constructed here. This is bad news as far as logical omniscience is concerned, but again only falls under the requirement that beliefs are closed under provable relevant implication.[11]

[10] The author thanks an anonymous reviewer for suggesting that viewing $\boldsymbol{t}$ as the conjunction of all theorems leads to a much better explanation, rather than emphasizing an intensional nature of $\boldsymbol{t}$.

[11] Thanks again to an anonymous reviewer who pointed out that some sets of theorems (of certain logics) are plausible for an agent to believe, and that this highlights some problems concerning logical omniscience. While modeling an agent that believes all instances of $\mathcal{A} \vee \neg\mathcal{A}$ does not collapse the model in such a way that the underlying relevant logic changes (other

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§5. Soundness.

Lemma 5.27 (Soundness of $\mathbb{CL}$). *For all* $\mathbb{L}$ (*containing* ***QBM.C*** *and extended by axiom in Table* 1), *if* $\vdash_{\mathbb{CL}} \mathcal{A}$, *then* $\vDash_{\mathbb{CL}} \mathcal{A}$.

Proof. Proof is by induction on the length of proofs. Two distinct base cases: (i) where $\mathcal{A}$ is a theorem of **CQ**, $\mathcal{A}$ is valid by Lemma 4.20, and (ii) where $\mathcal{A}$ is an axiom of $\mathbb{L}$, $\Box_L \mathcal{A}$ is valid in the class of W-frames, as worlds can only see points in N. For axioms of extensions as in Table 1, the case is just as straightforward.

The inductive step shows that each rule of inference of **CL** preserves validity in **CL**-frames. The cases of (MP), (BR), and the $\Box_L$ variants of the $\mathbb{L}$-rules (MP), (ADJ), (Affix), (RCont), (R$\Box$M), and (R$\Box_L$M), can be given by arguments similar to those of Sedlár and Vigiani. (Note that the presentation of the axiom systems here differs from S&V's presentation by the use of axiom schemes instead of universal substitution, so some slight modifications of their arguments as a whole are required.) The cases for the rules regarding fusion, $\boldsymbol{t}$, and left arrow are covered again by Lemma 4.20. The remaining cases to be shown are for $(R\forall I)$, $(R\exists E)$, $\Box_L$-$(R\forall I)$, $\Box_L$-$(R\exists E)$, and then for extensions.

We will give the cases for the rules with the universal quantifier. Consider $(R\forall I)$. For any world w, suppose that, for every $f \in U^{\omega}$, $w \in |\mathcal{A}^x \to \mathcal{B}|f$. Then either $w \in \neg|\mathcal{A}^x|f$, or $w \in |\mathcal{B}|f$. From the former, immediately $w \in |\mathcal{A}^x \to \forall x\mathcal{B}|f$. From the latter, since it holds for every f, $w \in \bigcap_{g\in xf} |\mathcal{B}|g$, which using Lemma 4.20 entails that $w \in |\mathcal{A}^x \to \forall x\mathcal{B}|f$.

Consider the $\Box_L$ variant of $(R\forall I)$. The proof is similar to that in [12]. Suppose that $\Box_L(\mathcal{A}^x \to \mathcal{B})$ is valid in an arbitrary W-model. Then by Lemma 4.19, for every $g \in U^{\omega}$, $|\mathcal{A}^x|g \subseteq |\mathcal{B}|g$. By Lemma 2.6, $|\mathcal{A}^x|f = |\mathcal{A}^x|g$, for an arbitrary f. In particular, we have $|\mathcal{A}^x|f \subseteq \prod_{g\in xf} |\mathcal{B}|g = |\forall x\mathcal{B}|f$. Since f was arbitrary, we have $|\mathcal{A}^x|f \subseteq |\forall x\mathcal{B}|f$ for each f, as so by Lemma 4.19 we have $\Box_L(\mathcal{A}^x \to \forall x\mathcal{B})$ is valid in the W-model. As the model was arbitrary, the result follows.

For extensions with rule schemes from Table 1, the cases are straightforward, giving the usual soundness arguments for these extensions in relevant logics, and following the reasoning of the previous cases. □

As in [18], it is provable that the $\Box_L$ operator encodes $\mathbb{L}$, as in the following.

Theorem 5.28. *For any* $\mathbb{L}$, $\vdash_{\mathbb{L}} \mathcal{A}$ *iff* $\vdash_{CL} \Box_L\mathcal{A}$.

Proof. For the left-to-right direction, we use induction on the length of $\mathbb{L}$-proofs of a formula $\mathcal{A}$. The base cases are the axioms of $\mathbb{L}$, which is given by the axiomhood of $\Box_L\mathcal{A}$ in $\mathbb{CL}$, as in (2) of Definition 4.25. The induction step is given straightforwardly by (3) of Definition 4.25.

For the converse direction, suppose that $\nvdash_{\mathbb{L}} \mathcal{A}$. Then there is an $\mathbb{L}$-model $\mathfrak{M}$ such $\mathcal{A}$ is not valid in $\mathfrak{M}$. By Lemma 4.21, there is a $\mathbb{CL}$-model $\mathfrak{M}'$ such that $\Box_L\mathcal{A}$ is not valid in $\mathfrak{M}'$. By soundness, $\nvDash_{\mathbb{CL}} \Box_L\mathcal{A}$. □

than adding $\Box(\mathcal{A} \vee \neg\mathcal{A})$ and its *relevant* consequences to the set of theorems). However, it does restrict the graph of the $S_\Box$ relation, even to the point of restricting $S_\Box$ to *worlds* should an agent believe *all* classical theorems (or a set of formulas that *relevantly* implies all classical theorems).

§6. Completeness.

6.1. Theories.

DEFINITION 6.29 (Theories). *Let $\mathbb{L}$ be a quantified modal logic that includes **QBM.C**, and consider also the logic $\mathbb{CL}$ based on it. Where Γ and Δ as sets of $\mathbb{L}$-formulas*:

1. $\Gamma \gg_{\mathbb{L}} \Delta$ *is defined to mean that there are some* $\mathcal{A}_1, \ldots, \mathcal{A}_n \in \Gamma$ *and* $\mathcal{B}_1, \ldots, \mathcal{B}_m \in \Delta$ *such that* $(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_n) \vdash (\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m)$ *is a theorem of* $\mathbb{L}$.
2. $\Gamma \gg_{\mathbb{L}} A$ *is shorthand for* $\Gamma \gg_{\mathbb{L}} \{A\}$.
3. *When* $\Gamma \not\gg_{\mathbb{L}} \Delta$, *we say the pair* (Γ, Δ) *is an* $\mathbb{L}$*-independent pair.*
4. *A set of formulas* Γ *is* an $\mathbb{L}$-theory *when, if* $\Gamma \gg_{\mathbb{L}} \mathcal{A}$, *then* $\mathcal{A} \in \Gamma$.
5. *A theory* Γ *is* prime *if and only if, if* $\mathcal{A} \vee \mathcal{B} \in \Gamma$, *then either* $\mathcal{A} \in \Gamma$ *or* $\mathcal{B} \in \Gamma$.
6. *A theory* Γ *is* $\mathbb{L}$-regular *if and only if it contains every theorem of* $\mathbb{L}$.
7. *A theory* Γ *is non-empty when* $\Gamma \neq \emptyset$, *and non-trivial when* $\Gamma \neq wff$.
8. *A theory* Γ *is maximally consistent when it is non-trivial and prime, whose proper extensions are trivial.*
9. *A theory* Γ *is* ω*-complete when it does not contain every instance of a universally quantified formula without also containing the universally quantified formula.*

DEFINITION 6.30 (Relations on Theories). *We define the following relations on* $\mathbb{L}$ *theories*: *For* $\mathbb{L}$ *theories* α, β, γ:

1. $R'\alpha\beta\gamma$ *iff* $\{\mathcal{A} \circ \mathcal{B} : \mathcal{A} \in \alpha \ \& \ \mathcal{B} \in \beta\} \subseteq \gamma$.
2. $S'_{\Box}\alpha\beta$ *iff* $\{A : \Box\mathcal{A} \in \alpha\} \subseteq \beta$.
3. $S'_{\Box_L}\alpha\beta$ *iff* $\{A : \Box_L\mathcal{A} \in \alpha\} \subseteq \beta$.

LEMMA 6.31 (Extensions and Pair-Extensions). *Let* $\mathbb{L}$ *be a quantified modal relevant logic extending **QBM.C**, and let* Γ, Δ, Σ *be sets of formula.*

1. *If* (Γ, Δ) *is an* $\mathbb{L}$*-independent pair, then there is a prime* $\mathbb{L}$*-theory* $\Gamma' \supseteq \Gamma$ *such that* (Γ', Δ) *is an* $\mathbb{L}$*-independent pair.*
2. *If* (Γ, Δ) *is an* $\mathbb{L}$*-independent pair and* $\Gamma \cup \Delta = wff$, *then* Γ *is prime.*
3. *If* Σ *is prime and* $R'\Gamma\Delta\Sigma$, *then there exist prime* $\mathbb{L}$*-theories* $\Gamma' \supseteq \Gamma$ *and* $\Delta' \supseteq \Delta$ *such that* $R'\Gamma'\Delta'\Sigma$.
4. *If* Σ *is a prime* $\mathbb{L}$*-theory and* Γ *and* $\Delta\mathbb{L}$*-theories,* $R'\Sigma\Gamma\Delta$, *and* $\mathcal{A} \notin \Delta$, *then there is a prime* $\mathbb{L}$*-theories* $\Gamma' \supseteq \Gamma$ *and* $\Delta' \supseteq \Delta$ *such that* $R'\Sigma\Gamma'\Delta'$.
5. *If* Σ *is prime and* $\mathcal{A} \to \mathcal{B} \notin \Sigma$, *then there exist prime theories* Γ *and* Δ *such that* $R'\Sigma\Gamma\Delta$ *where* $\mathcal{A} \in \Gamma$ *and* $\mathcal{B} \notin \Delta$.

Proof. For (i) note that the logic in question is pair extension acceptable (see, e.g., [15, Sections 5.1–5.2] or [1, pp. 123–126]). The remainder of the proof, as for $\mathbb{L}$-theories, is also quite standard in the literature. Specifically, for first-order modal relevant logics, see Ferenz [5, 6]. □

Note that some of the cases in the above lemma carry over to a logic $\mathbb{CL}$, since $\mathbb{CL}$ is also pair extension acceptable. However, in the proofs below only $\mathbb{L}$-extensions are required.

COROLLARY 6.32. *If* $\mathcal{A}$ *is not a theorem of* $(\mathbb{CL})$, *then there is a regular prime* $\mathbb{CL}$*-theory* Γ *where* $\mathcal{A} \notin \Gamma$.

Proof. Suppose that $\mathcal{A}$ is not a theorem of $\mathbb{CL}$. Then $(CL, \mathcal{A})$ is a $\mathbb{CL}$-independent pair, where CL is the set of theorems of CL. By the lemma above, since CL is an $\mathbb{L}$-theory, we obtain a regular prime $\mathbb{CL}$-theory Γ which does not contain $\mathcal{A}$. □

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LEMMA 6.33 (Squeezes). *The following squeeze results hold.*

1. *Suppose that* $S'_{\Box}\Gamma\Delta$ *(or* $S'_{\Box_L}$*) and* $\mathcal{A} \notin \Delta$*, for prime* $\mathbb{L}$*-theory* Γ *and* $\mathbb{L}$*-theory* Δ*. Then there is a prime* Δ' *such that* $S'_{\Box}\Gamma\Delta'$ *(or* $S'_{\Box_L}\Gamma\Delta'$*).*
2. *Suppose that* Γ *is a prime* $\mathbb{L}$*-theory and* $\Box\mathcal{A} \notin \Gamma$ *(or* $\Box_L\mathcal{A} \notin \Gamma$*). Then there is a prime* $\mathbb{L}$*-theory* Δ *such that* $\mathcal{A} \notin \Delta$ *and* $S_{\Box}\Gamma\Delta$ *(or* $S_{\Box_L}\Gamma\Delta$*).*

Proof. The proof is quite standard in modal relevant logics. E.g., the arguments of [21] straightforwardly apply. □

As every $\mathbb{CL}$-theory is also an $\mathbb{L}$-theory, we have the following.

COROLLARY 6.34. *Suppose that* $S'_{\Box}\Gamma\Delta$ *(or* $S'_{\Box_L}$*) and* $\mathcal{A} \notin \Delta$*, for prime* $\mathbb{CL}$*-theory* Γ *and* $\mathbb{L}$*-theory* Δ*. Then there is a prime* Δ' *such that* $S'_{\Box}\Gamma\Delta'$ *(or* $S'_{\Box_L}\Gamma\Delta'$*).*

6.2. Canonical Model.

DEFINITION 6.35 (Canonical Model for **CL**). *Given a logic **CL** based on the quantified modal relevant logic* $\mathbb{L}$ *extending **QBM.C**, the canonical model is defined as follows:*

$$\mathfrak{M}^{\mathbb{C}} = \langle K^{\mathbb{C}}, N^{\mathbb{C}}, W^{\mathbb{C}}, R^{\mathbb{C}}, *^{\mathbb{C}}, S^{\mathbb{C}}_{\Box}, S^{\mathbb{C}}_{\Box_L}, U^{\mathbb{C}}, Prop^{\mathbb{C}}, PropFun^{\mathbb{C}} \rangle,$$

where:

1. $K^{\mathbb{C}}$ *is the set of prime* $\mathbb{L}$*-theories.*
2. $N^{\mathbb{C}}$ *is the set of all regular prime* $\mathbb{L}$*-theories.*
3. $W^{\mathbb{C}}$ *is the set of all regular prime **CL**-theories.*
4. $R^{\mathbb{C}}$ *is the relation* R' *restricted to* $K^{\mathbb{C}}$*.*
5. $*^{\mathbb{C}}$ *is given by* $a^* = \{\mathcal{A} : \neg\mathcal{A} \notin a\}$*.*
6. $S^{\mathbb{C}}_{\Box}$ *is the relation* $S'_{\Box}$ *restricted to* $K^{\mathbb{C}}$*.*
7. $S^{\mathbb{C}}_{\Box_L}$ *is the relation* $S'_{\Box_L}$ *restricted to* $K^{\mathbb{C}}$*.*
8. $U^{\mathbb{C}}$ *is the infinite set of constants Con.*
9. *For every closed formula* $\mathcal{A}$*,* $[\![\mathcal{A}]\!]^{\mathbb{C}} =_{df} \{a \in K^{\mathbb{C}} : \mathcal{A} \in a\}$*.*
10. $Prop^{\mathbb{C}} =_{df} \{[\![\mathcal{A}]\!]^{\mathbb{C}} : \mathcal{A}$ *is a closed formula*$\}$*.*
11. *Given an* $f \in U^{\omega}$*,* fn *is a constant. For any formula* $\mathcal{A}$*, let* $\mathcal{A}^f$ *be the closed formula that results from replacing every free occurrence of a variable* x_n *with the constant* fn*. That is,* $\mathcal{A}^f =_{df} \mathcal{A}[f0/x_0, \dots, fn/x_n, \dots]$*.*
12. *For each formula* $\mathcal{A}$*, the function* $\phi_{\mathcal{A}} : U^{\omega} \longrightarrow Prop_{\mathbb{C}}$ *is given by* $(\phi_{\mathcal{A}})f =_{df} [\![\mathcal{A}^f]\!]^{\mathbb{C}}$*.*
13. $PropFun^{\mathbb{C}}$ *is the set of all functions* $\phi_{\mathcal{A}}$*, for each formula* $\mathcal{A}$*.*
14. *The canonical valuation function is given by:*
    (a) $|c| = c$.
    (b) $|P^n|(c_1, \dots c_n) = [\![P(c_1, \dots c_n)]\!]^{\mathbb{C}}$.
    (c) *The valuation is extended to all wff as before.*

LEMMA 6.36 (Underling quantified modal relevant structure). *The structure*

$$\langle K^{\mathbb{C}}, N^{\mathbb{C}}, R^{\mathbb{C}}, *^{\mathbb{C}}, S^{\mathbb{C}}_{\Box}, S^{\mathbb{C}}_{\Box_L}, U^{\mathbb{C}}, Prop^{\mathbb{C}}, PropFun^{\mathbb{C}} \rangle$$

is an $\mathfrak{L}$*-frame, in the sense of* [6]*, with two box-like modalities. In particular, the following facts are established:*

1. $\leq = \subseteq$.
2. $\otimes\phi_{\mathcal{A}} = \phi_{\otimes\mathcal{A}}$ *(for* $\otimes \in \{\neg, \Box, \Box_L\}$*).*

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3. $\phi_{\mathcal{A}} \otimes \phi_{\mathcal{B}} = \phi_{\mathcal{A}\otimes\mathcal{B}}$(*for* $\otimes \in \{\wedge, \vee, \rightarrow, \circ, \leftarrow\}$).
4. $\forall_n \phi_{\mathcal{A}} = \phi_{\forall x_n \mathcal{A}}$; $\exists_n \phi_{\mathcal{A}} = \phi_{\exists x_n \mathcal{A}}$.
5. *For closed* $\forall x\mathcal{A}$ *and* $\exists x\mathcal{A}$:
   (a) $[\![\forall x\mathcal{A}]\!]^{\mathbb{C}} = \bigsqcap_{c\in con} [\![\mathcal{A}[c/x]]\!]^{\mathbb{C}}$, *and*
   (b) $[\![\exists x\mathcal{A}]\!]^{\mathbb{C}} = \bigsqcup_{c\in con} [\![\mathcal{A}[c/x]]\!]^{\mathbb{C}}$.
6. *And* (*consequently*) *that* (c1)–(c10) *of Definition* 2.2 *hold.*

The proof is an in [6], and will be omitted here.

LEMMA 6.37. *The canonical frame is a bounded frame, in the sense of Definition* 4.14.

Proof. The empty theory and the full theory, which we will call 0 and 1 respectively, are both in the canonical frame. By Lemma 6.36(1), $0 \leq \alpha \leq 1$. Using the arguments of Sedlár and Vigiani we may show (2)–(6), and the arguments for (7)–(10) are straightforwardly similar to (5)–(6) and left to the reader. □

LEMMA 6.38. *Each* $w \in W^{\mathbb{C}}$ *is a world, in the sense of Definition* 4.16.

Proof. Items (1)–(4) can be given by Sedlár and Vigiani's arguments, and (5)–(7) are similar. Of the remaining cases, we show the case for (8), which is a modified and shortened version of the arguments of [12, theorem 10.3], but with the restriction to $w \in W^{\mathbb{C}}$.

Assume that $w \in X - Y \subseteq \bigcap_{j\in U} \phi(f[j/n])$, for some $w \in W^{\mathbb{C}}$. $Y \in Prop$, so $Y = [\![\mathcal{A}]\!]^{\mathbb{C}}$ for a closed $\mathcal{A}$; and $\phi = \phi_{\mathcal{B}}$ for some formula (possible open) $\mathcal{B}$. We thus have, for every $c \in con$,

$$w \in X - [\![\mathcal{A}]\!]^{\mathbb{C}} \subseteq \phi_{\mathcal{B}}(f[c/n])$$

and so (since $\mathcal{A}$ is closed)

$$X \subseteq [\![\mathcal{A} \cup \mathcal{B}^{f[c/n]}]\!]^{\mathbb{C}} = [\![(\mathcal{A} \vee \mathcal{B})^{f\backslash n}[c/x_n]]\!]^{\mathbb{C}},$$

where $\mathcal{A}^{f\backslash n} = \mathcal{A}[f0/x_0, \dots f(n-1)/x_{(n-1)}, x_n/x_n, f(n+1)/x_{(n+1)}, \dots]$, and consequently $\mathcal{A}^{f\backslash n}[c/x_n] = \mathcal{A}^{f[c/n]}$ as well as $\forall x_n(\mathcal{A}^{f\backslash n}) = (\forall x_n \mathcal{A})^f$. Since this entails that $(\mathcal{A} \vee \mathcal{B})^{f\backslash n}[c/x_n] \in w$, for each $c \in con$, by Lemma 6.36(v).(a) we have that $\forall x_n((\mathcal{A} \vee \mathcal{B})^{f\backslash n}) \in w$. By extensional confinement, since $w \in W^{\mathbb{C}}$, we have $\mathcal{A} \vee \forall x_n(\mathcal{B}^{f\backslash n}) \in w$.

Since $w \notin [\![\mathcal{A}]\!]^{\mathbb{C}}$, $\forall x_n(\mathcal{B}^{f\backslash n}) \in w$, and by an earlier equality we have $(\forall x_n \mathcal{B})^f \in w$, and thus $w \in (\forall_n \phi_{\mathcal{B}})f$, as required. □

LEMMA 6.39. *The canonical model satisfies*:

1. $(\forall w \in W)(\forall u \in K)(S_{\Box_L} wu \Rightarrow u \in N)$.
2. $(\forall k \in N)(\exists w \in W) S_{\Box_L} ws$.

Proof. For (1), suppose for some $w \in W^{\mathbb{C}}$ that $S_{\Box_L} wu$ for some $u \in K^{\mathbb{C}}$. Then for every $\mathbb{L}$-theorem $\mathcal{A}$, $\Box_L \mathcal{A} \in w$, and so then $\mathcal{A} \in u$, making u a regular $\mathbb{L}$-theory, and so a member of N, as required.

For (2), suppose that $\alpha \in N$. (Suppose also that $\alpha \neq 1$, in which case the result trivially follows.) We are required to find a world $w \in W^{\mathfrak{C}}$ such that, if $\Box_L \mathcal{A} \in w$, then

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$\mathcal{A} \in \alpha$. Let $\Gamma = \{\mathcal{A} : \vdash_{CL} \mathcal{A}\}$, which is clearly a non-empty, nontrivial CL-theory, and thus an $\mathbb{L}$-regular theory. Further let $\Delta = \{\Box_L \mathcal{A} : \mathcal{A} \notin \alpha\}$. Δ is non-empty, since α is non-trivial.

It follows that (Γ, Δ) is a **CL**-independent pair. Here we use the reasoning of Sedlár and Vigiani. If it were not a pair, then $\vdash_{\mathbb{CL}} \Box_L \mathcal{A}_1 \vee \cdots \vee \Box_L \mathcal{A}_n$. But then $\vdash_{\mathbb{CL}} \Box_L(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n)$, which by Theorem 5.28 entails that $\vdash_{\mathbb{L}} (\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n)$. And so since α is prime and regular, one of $\mathcal{A}_1, \dots \mathcal{A}_n$ is in α, which gives the required contradiction.

Finally, by applying the extension Lemma 6.31(i), we obtain an prime, non-trivial maximally consistent $\mathbb{L}$-theory, indeed $\mathbb{CL}$-theory, $\Gamma' \in W^{\mathbb{C}}$ such that $S^{\mathbb{C}}_{\Box_L} \Gamma' \alpha$, as required. □

Lemma 6.40. *For any extensions obtained using Table 1, the appropriate conditions are satisfied by the canonical model.*

The proof of this lemma is standard and omitted.

Therefore the canonical frame is indeed a W-frame. What's left to show is that it is also a W-model and that truth is membership. Mares and Goldblatt's arguments establish the following lemma.

Lemma 6.41 (Atomic Propositional Functions). *For every* n*-ary predicate symbol P, every $f \in U^{\omega}$, and every set of terms $\tau_1, \dots \tau_n$:*

1. $P(\tau_1, \dots \tau_n)^f = P(|\tau_1|f, \dots, |\tau_n|f)$.
2. $|P(\tau_1, \dots \tau_n)| = \phi_{P(\tau_1,\dots\tau_n)}$.

The above lemma ensures that all atomic formulas are mapped to members of $PropFun^{\mathbb{C}}$, and then Lemma 6.36 extends this fact to all formulas. Thus the canonical model is a model.

Lemma 6.42 (Truth Lemma). *For any formula $\mathcal{A}$, $|\mathcal{A}| = \phi_{\mathcal{A}}$. This is, for every $f \in U^{\omega}$, $|\mathcal{A}|f = [\![\mathcal{A}^f]\!]^{\mathbb{C}}$, which is $a \in |\mathcal{A}|f$ iff $\mathcal{A}^f \in a$.*

The proof of the truth lemma is by induction on the structural complexity of a formula. The arguments of [6] and [12] can be employed, using the facts established in Lemma 6.36.

We thus have both that the canonical model is a model, and that truth is membership. Consequently, we obtain the following.

Theorem 6.43 (Completeness for $\mathbb{CL}$ and extensions). *For any $\mathbb{CL}$ based on an $\mathbb{L}$ obtained from **QBM.C** (and Table 1), we have $\vDash_{\mathbb{CL}}$ implies $\vdash_{\mathbb{CL}}$.*

Proof. Suppose that $\vDash_{\mathbb{CL}} \mathcal{A}$. Then every regular prime $\mathbb{CL}$-theory contains $\mathcal{A}^f$, for each $f \in U^{\omega}$. For every free variable in $\mathcal{A}$, replace it with a new constant not in $\mathcal{A}$. This formula belongs to every regular prime $\mathbb{CL}$-theory, and is therefore a $\mathbb{CL}$-theorem by Corollary 6.32. Repeated but finite applications of (RGC2) (see Lemma 4.26) followed by repeated but finite applications of the axiom ($\forall$E), will produce a proof of $\mathcal{A}$. □

§7. Worlds and ω-completeness.

In a canonical model, an ω-complete situation or world is such that, if it contains every instance $\mathcal{A}^f$ of $\forall x_n \mathcal{A}$, then it also contains $\forall x_n \mathcal{A}$. We also use the phrase ω-complete to describe points in any model such that, for $a \in K$, if $a \in |\mathcal{A}|f$ for every f, then $a \in |\forall x_n \mathcal{A}|f$, for every formula $\mathcal{A}$ and every variable assignment f. For simplicity, in all cases we will use the phrase ω-completeness

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as it is used in the canonical model. This is despite most models not having enough constant symbols for the substitutional view; however, the general idea is the same.

A Russellian and philosophical distinction between ω-complete and ω-incomplete situations is that the ω-complete situations have the *extra information* concerning what all the objects in the domain are. That is, it has a "that's all"-clause. Now, one philosophical intuition one might profess is that *robust possible worlds* decide everything, but what exactly is meant by this? We discuss two possibilities here. First, one can mean that worlds decide anything and everything, including those "meta" properties such as that's-all clauses. The second is that worlds decide every formula expressible in a language. At present, the author believes that neither of these views are correct (and for more than just a rejection of metaphysical intuitions on such things as possible worlds). The first seems to overload worlds with more than we may have reason to suppose. A stronger version of the first, which we shall explore a little here—and which the author also believes is incorrect—is that worlds decide every that's-all clause. The second view appears to limit possible worlds to what we can express in a particular language, which may appeal to certain Ersatzist interpretations. While I am sympathetic to such linguistically based approaches to possible worlds, it doesn't appear that the language of the logics presented here entail some consequences for ω-completeness. In fact, this second view can be seen to be present in the models constructed above, every formula is decided in every world. In short: the first is an ad-hoc delimitation of worlds, and the second is an ad-hoc limitation on possible worlds. Thus, for the remainder of this section, let's consider the modification of the first view, and some formal troubles that arise.

In the completeness proof, for Lemma 6.39, in showing that every member of N has a world that can see it, we encounter an interesting problem in logics with the addition of the Barcan formula (for the $\Box_L$ operator). The $\mathbb{L}$-regular theory α is not necessarily ω-complete as defined. Now, suppose that $\mathcal{A}^{f[j/x_n]} \in \alpha$, for every $j \in Con$, but $\forall x_n \mathcal{A}^f \notin \alpha$. Then Δ, as defined in the lemma, contains $\Box_L \forall x_n \mathcal{A}^f$ but never $\Box_L \mathcal{A}^{f[j/n]}$. So Γ can be extended by every instance $\Box_L \mathcal{A}^{f[j/x_n]}$. Then, if Γ' is ω-complete, then $\forall x_n \Box_L \mathcal{A} \in \Gamma'$. Finally, from the Barcan formula we obtain a contradiction.

Having seen that part of the problem is that the theory α is not ω-complete (and not a possible world, in general), one might respond: "While I have some intuition that robust possible worlds ought to be ω-complete, I either have no intuition regarding situations one way or the other. That is, it appears to do no philosophical harm to the motivations for ω-complete worlds to require all ($\mathbb{L}$-normal) situations to also be ω-complete. Indeed, there may be some intuition that the so-called 'logically normal' points be ω-complete, for that is just part of what 'logically-normal' means."

First, I wish to stave off the argument that logical normality entails ω-completeness in this setting. The so-called logically normal worlds are just the worlds where the logical operators behave according to logic. That is, these points make every theorem true. In other words, similar to the second view above, these points are normal with respect to the behavior of formulas, and not their meta properties such as ω-completeness.

Having prevented a certain misunderstanding, I now turn to the difficulties in carrying out the task of making all situations ω-complete. Then I will investigate the possibility of making only the situations in N ω-complete.

If we make every situation ω-complete, and thus require the Tarskian truth conditions for quantified formulas, then (even after adding the Barcan formula) the defined axiom systems will be incomplete *for non-general frame semantics* for sufficiently strong logics.[12] Fine [7] has shown that Tarskian truth conditions plus non-general frames entails incompleteness for strong relevant logics including **R**, **E**, **T**, and others. Thus, requiring all points to be ω-complete should not be done in conjunction with giving up general frames. However, I conjecture the following:

> **Conjunction 1**: We can obtain completeness for logics with ω-complete points, Tarskian truth conditions, the Barcan Formulas, and general frames.

This conjecture is suggested by some results of [8] and [9]. A couple of first-order modal classical logics are shown to be complete w.r.t. the MG semantics (and in one case with variable domain MG semantics). There logics are also known to be incomplete w.r.t. non-general frame, semantics with a Tarskian truth condition. The results in question, however, show completeness for general frame, Tarskian models. That is, for completeness for these logics, general frames are sufficient.

This conjecture remains to be proved or disproved. Moreover, (we have given) no particular philosophical explanation to settle the more interpretive questions corresponding to this conjecture.

§8. Concluding remarks.

Herein we have constructed first-order epistemic logics based on the propositional epistemic logics of [18]. In addition, we have proven modular soundness and completeness results, and further explored some key formal and philosophical difficulties. The systems defined here, as in [18], avoid some but not all of the problems of logical omniscience; beliefs are still closed under relevant implication. This problem may be further ameliorated by switching to neighbourhood-based semantics, so far as closing beliefs under relevant bi-implication improves matters.

On the first-order side of things, in addition to the ω-completeness, there are also the questions of constructing models with variable domains, identity, epistemic constructions such as *being familiar with (or knowing) an object*, and so forth. Variable domains should be constructible, following [11], by adding an existence predicate, adding a domain function that determines a subdomain of U for each situation, modifying the truth condition of a quantified formula, and appropriately replacing certain first-order axioms.

Identity, however, is not as straightforward. Traditional approaches to identity in relevant logics leads to irrelevant logics—i.e., the validity of implications such as $p \to x = x$. Several MG-based approaches to identity in relevant logics have been recently developed, such as in [4] (based on Kremer's *relevant indiscernibility* interpretation of identity [10]) and [22]. These are starting points for identity adding identity to the epistemic logics here.

It might be supposed that being familiar with an object in some respect is a precondition for knowing that the object has or lacks certain properties. A logic of familiarity is developed, e.g., in [23]. Developing the constructions of this paper to include predicate(s) for familiarity would allow us to further deal with forms of logical

[12] The reader is reminded that the Tarskian truth conditions are the generalized intersections and unions of instances.

omniscience. For an agent to know that $P(\tau)$, we could require the agent to be familiar with the object τ.

The present paper thus serves as a starting point for several philosophical and formal projects in epistemic logic.

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