On the existence of cocycle-invariant Borel probability measures

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Abstract. We show that a natural generalization of compressibility is the sole obstruction to the existence of a cocycle-invariant Borel probability measure.

1. *Introduction*

Suppose that X is a standard Borel space and $T: X \to X$ is a Borel automorphism of X. A Borel measure μ on X is T-*invariant* if $\mu(T(B)) = \mu(B)$ for all Borel sets $B \subseteq X$. The characterization of the class of Borel automorphisms of standard Borel spaces admitting an invariant Borel probability measure is a fundamental problem going back to Hopf (see [**Hop32**]).

A *compression* of an equivalence relation E on X is an injection $\phi: X \to X$ sending each E-class into a proper subset of itself. Building on work by Murray and von Neumann (see [**MVN36**]), Nadkarni has shown that the existence of a Borel compression of the orbit equivalence relation E_T^X induced by T is the sole obstruction to the existence of a T-invariant Borel probability measure (see [**Nad90**]).

Suppose that E is a Borel equivalence relation on X that is *countable*, in the sense that all of its equivalence classes are countable. A Borel measure μ on X is E-*invariant* if it is T-invariant for all Borel automorphisms $T: X \to X$ whose graphs are contained in E. It is easy to see that a Borel measure is T-invariant if and only if it is E_T^X-invariant. Becker and Kechris have pointed out that Nadkarni's argument yields the more general fact that the existence of a Borel compression of E is the sole obstruction to the existence of an E-invariant Borel probability measure (see [**BK96**, Theorem 4.3.1]).

An equivalence relation is *aperiodic* if all of its classes are infinite. A set $Y \subseteq X$ is E-*complete* if it intersects every E-class in at least one point, and a set $Y \subseteq X$ is a *partial*

transversal of E if it intersects every E-class in at most one point. A *transversal* of E is an E-complete partial transversal of E. The Lusin–Novikov uniformization theorem (see, for example, [**Kec95**, Theorem 18.10]) ensures that there is a Borel transversal of E if and only if X is the union of countably many Borel partial transversals of E. We say that E is *smooth* if it satisfies these equivalent conditions. Dougherty, Jackson and Kechris have pointed out that the existence of a Borel compression of E is equivalent to the existence of an aperiodic smooth Borel subequivalence relation of E (see [**DJK94**, Proposition 2.5]), thereby obtaining another characterization of the class of countable Borel equivalence relations on standard Borel spaces admitting an invariant Borel probability measure.

A substantially weaker notion than E-invariance is that of E-*quasi-invariance*, where one asks that $\mu(T(B)) = 0 \iff \mu(B) = 0$ for all Borel sets $B \subseteq X$ and Borel automorphisms $T: X \to X$ whose graphs are contained in E. Given a group Γ, we say that a function $\rho: E \to \Gamma$ is a *cocycle* if $\rho(x, z) = \rho(x, y)\rho(y, z)$ whenever $x \, E \, y \, E \, z$. Given a Borel cocycle $\rho: E \to (0, \infty)$, we say that a Borel measure μ on X is ρ-*invariant* if $\mu(T(B)) = \int_B \rho(T(x), x) \, d\mu(x)$ for all Borel sets $B \subseteq X$ and Borel automorphisms $T: X \to X$ whose graphs are contained in E. Clearly E-invariance is equivalent to invariance with respect to the constant cocycle, whereas the Radon–Nikodym theorem (see, for example, [**Kec95**, §17.A]) and the Feldman–Moore observation that countable Borel equivalence relations on standard Borel spaces are orbit equivalence relations induced by Borel actions of countable groups (see [**FM77**, Theorem 1]) ensure that E-quasi-invariance is equivalent to invariance with respect to some Borel cocycle $\rho: E \to (0, \infty)$ (see, for example, [**KM04**, §8]). A characterization of the class of Borel cocycles $\rho: E \to (0, \infty)$ admitting an invariant Borel probability measure was provided in [**Mil08a**]. Here we investigate more natural generalizations of the characterizations mentioned above.

In §2 we introduce the direct generalizations of *aperiodicity* and *compressibility* to cocycles that come from viewing ρ as endowing each E-class with a notion of relative size. We also introduce the generalization of *smoothness* to cocycles that comes from the Glimm–Effros dichotomy. We note that, unfortunately, even when E is smooth, there are Borel cocycles on E admitting neither a compression nor an invariant Borel probability measure. In order to bypass this obstacle, we introduce the *quotient* of ρ by a finite subequivalence relation of E. Generalizing the observation of Dougherty, Jackson and Kechris, we show that the existence of an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E is equivalent to the existence of a Borel subequivalence relation of E on which ρ is aperiodic and smooth. We also note that, at least when ρ is smooth, the existence of an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E is the sole obstacle to the existence of a ρ-invariant Borel probability measure.

In §3 we introduce *Borel coboundaries*, a natural class of particularly simple Borel cocycles containing the constant cocycles. We note that, unfortunately, there are Borel coboundaries admitting neither an injective Borel compression of the quotient by a finite Borel subequivalence relation of E nor an invariant Borel probability measure. In order to bypass this new obstacle, we then drop the assumption of injectivity, and combine the Becker–Kechris generalization of Nadkarni's theorem, the Dougherty–Jackson–Kechris

characterization of the existence of Borel compressions, and an approximation lemma to generalize Nadkarni's theorem to Borel coboundaries.

THEOREM 1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a Borel coboundary. Then exactly one of the following statements holds.*
(1) *There is a finite-to-one Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E.*
(2) *There is a ρ-invariant Borel probability measure.*

In §4 we no longer restrict our attention to Borel coboundaries. Unfortunately, the direct generalization of Theorem 1 to Borel cocycles remains open. In order to bypass this final obstacle, we consider the weakening of the notion of a compression of the quotient of ρ by a finite subequivalence relation F of E obtained by only taking the quotient in the range, which we refer to as a *compression* of ρ over F. By augmenting the main argument of [**Mil08a**] with an additional approximation lemma, we generalize Nadkarni's theorem to Borel cocycles.

THEOREM 2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a Borel cocycle. Then exactly one of the following statements holds.*
(1) *There is a finite-to-one Borel compression of ρ over a finite Borel subequivalence relation of E.*
(2) *There is a ρ-invariant Borel probability measure.*

2. Smooth cocycles

One can think of a cocycle $\rho \colon E \to (0, \infty)$ as assigning a notion of relative size to each E-class C, with the ρ-size of a point $y \in C$ relative to a point $z \in C$ being $\rho(y, z)$. More generally, the ρ-size of a set $Y \subseteq C$ relative to z is given by $|Y|_z^\rho = \sum_{y \in Y} \rho(y, z)$. We say that Y is ρ-*infinite* if this quantity is infinite. As the definition of cocycle ensures that $|Y|_{z'}^\rho = |Y|_z^\rho \rho(z, z')$ for all $z' \in C$, it follows that the notion of being ρ-infinite does not depend on the choice of $z \in C$. It also follows that the ρ-*size* of Y relative to a non-empty set $Z \subseteq C$, given by $|Y|_Z^\rho = |Y|_z^\rho / |Z|_z^\rho$, does not depend on the choice of $z \in C$.

We say that a cocycle $\rho \colon E \to (0, \infty)$ is *aperiodic* if every E-class is ρ-infinite. Note that the aperiodicity of ρ trivially yields that of E. Conversely, when ρ is bounded, the aperiodicity of E yields that of ρ.

We say that a function $\phi \colon X \to X$ is a *compression* of ρ if the graph of ϕ is contained in E, $|\phi^{-1}(x)|_x^\rho \le 1$ for all $x \in X$, and the set $\{x \in X \mid |\phi^{-1}(x)|_x^\rho < 1\}$ is E-complete. Note that, when ρ is the constant cocycle, a function $\phi \colon X \to X$ is a compression of E if and only if it is a compression of ρ.

PROPOSITION 2.1. *Suppose that X is a standard Borel space and E is an aperiodic smooth countable Borel equivalence relation on X. Then there is an aperiodic Borel cocycle $\rho \colon E \to (0, \infty)$ that does not admit a compression.*

Proof. Fix a strictly decreasing sequence $(r_n)_{n \in \mathbb{N}}$ of positive real numbers for which $\sum_{n \in \mathbb{N}} r_n = \infty$. As E is both aperiodic and smooth, the Lusin–Novikov uniformization

theorem yields a partition $(B_n)_{n \in \mathbb{N}}$ of X into Borel transversals of E. For each $x \in X$, let $n(x)$ denote the unique natural number for which $x \in B_{n(x)}$, and define $\rho \colon E \to (0, \infty)$ by setting $\rho(x, y) = r_{n(x)}/r_{n(y)}$ whenever $x \mathrel{E} y$.

The fact that $\sum_{n \in \mathbb{N}} r_n = \infty$ ensures that ρ is aperiodic. To see that there is no compression of ρ, note that if $\phi \colon X \to X$ is a function such that the graph of ϕ is contained in E and $|\phi^{-1}(x)|_x^\rho \leq 1$ for all $x \in X$, then a straightforward induction on $n(x)$, using the fact that $(r_n)_{n \in \mathbb{N}}$ is strictly decreasing, shows that $\phi(x) = x$ for all $x \in X$. □

A *digraph* on X is an irreflexive set $G \subseteq X \times X$. Given such a digraph, we say that a set $Y \subseteq X$ is *G-independent* if $G \cap (Y \times Y) = \emptyset$. A *$Y$-coloring* of G is a function $c \colon X \to Y$ with the property that $c^{-1}(y)$ is G-independent for all $y \in Y$.

The *vertical sections* of a set $R \subseteq X \times Y$ are the sets of the form $R_x = \{y \in Y \mid (x, y) \in R\}$, where $x \in X$. When G is Borel, it follows from [**KST99**, Proposition 4.5] that there is a Borel $\mathbb{N}$-coloring of G if and only if X is the union of countably many Borel sets $B \subseteq X$ for which the vertical sections of $G \cap (B \times B)$ are finite.

We say that a Borel measure μ on X is *E-ergodic* if every E-invariant Borel set is μ-conull or μ-null. Given a Borel cocycle $\rho \colon E \to \Gamma$ and a set $Z \subseteq \Gamma$, let G_Z^ρ denote the digraph on X with respect to which distinct points x and y are related if and only if they are E-equivalent and $\rho(x, y) \in Z$. The Glimm–Effros dichotomy for countable Borel equivalence relations (see [**Wei84**]) ensures that E is smooth if and only if there is no atomless E-ergodic E-invariant σ-finite Borel measure. In [**Mil08b**], this was generalized to show that if $\rho \colon E \to (0, \infty)$ is a Borel cocycle, then there is an open neighborhood $U \subseteq (0, \infty)$ of 1 for which there is a Borel $\mathbb{N}$-coloring of G_U^ρ if and only if there is no atomless E-ergodic ρ-invariant σ-finite Borel measure. Consequently, we say that a Borel cocycle $\rho \colon E \to (0, \infty)$ is *smooth* if it satisfies these equivalent conditions. Note that the smoothness of E trivially yields that of ρ. Conversely, when ρ is bounded, the smoothness of ρ ensures that X is the union of countably many Borel sets whose intersection with each E-class is finite, thus E is smooth.

We say that a set $Y \subseteq X$ is *ρ-lacunary* if it is G_U^ρ-independent for some open neighborhood $U \subseteq (0, \infty)$ of 1.

PROPOSITION 2.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, Γ is a Polish group, and $\rho \colon E \to \Gamma$ is a Borel cocycle. If there is an open neighborhood $U \subseteq \Gamma$ of 1_Γ for which there is a Borel $\mathbb{N}$-coloring of G_U^ρ, then there is a Borel $\mathbb{N}$-coloring of G_K^ρ for all compact sets $K \subseteq \Gamma$.*

Proof. Given a digraph G on X, we say that a set $Y \subseteq X$ is a *G-clique* if all pairs of distinct points of Y are G-related. It is sufficient to show that if a set $Y \subseteq X$ does not contain an infinite G_U^ρ-clique, then the vertical sections of $G_K^\rho \cap (X \times Y)$ are finite. Towards this end, fix a non-empty open set $V \subseteq \Gamma$ with the property that $V^{-1}V \subseteq U$, as well as a finite sequence $(\gamma_i)_{i < n}$ of elements of Γ for which $K \subseteq \bigcup_{i < n} \gamma_i V$, and note that if $x \in X$, then $(G_K^\rho)_x \subseteq \bigcup_{i < n} (G_{\gamma_i V}^\rho)_x$, so we need only show that each $(G_{\gamma_i V}^\rho)_x$ is a G_U^ρ-clique. But if $i < n$ and $y, z \in (G_{\gamma_i V}^\rho)_x$, then $\rho(y, z) = \rho(y, x)\rho(x, z) \in (\gamma_i V)^{-1}\gamma_i V = V^{-1}V \subseteq U$. □

The following fact ensures that a Borel cocycle $\rho \colon E \to (0, \infty)$ is smooth if and only if there is an E-complete ρ-lacunary Borel set.

PROPOSITION 2.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, Γ is a locally compact Polish group, $\rho\colon E \to \Gamma$ is a Borel cocycle, and $U \subseteq \Gamma$ is a pre-compact open neighborhood of 1_Γ. Then there is a Borel $\mathbb{N}$-coloring of G_U^ρ if and only if there is an E-complete G_U^ρ-independent Borel set.*

Proof. If $c\colon X \to \mathbb{N}$ is a Borel $\mathbb{N}$-coloring of G_U^ρ, then set $A_n = c^{-1}(n)$ and $B_n = A_n \setminus \bigcup_{m<n}[A_m]_E$ for all $n \in \mathbb{N}$. As the Lusin–Novikov uniformization theorem ensures that the latter sets are Borel, it follows that their union is an E-complete G_U^ρ-independent Borel set.

Conversely, suppose that $B \subseteq X$ is an E-complete G_U^ρ-independent Borel set. The Lusin–Novikov uniformization theorem then yields Borel functions $\phi_n\colon B \to X$ such that $E \cap (B \times X) = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(\phi_n)$, from which it follows that there are such functions satisfying the additional constraint that the sets $K_n = \rho(\operatorname{graph}(\phi_n))$ are pre-compact. As Proposition 2.2 yields Borel $\mathbb{N}$-colorings of $G_{K_n U K_n^{-1}}^\rho \cap (B \times B)$, and the Lusin–Novikov uniformization theorem ensures that ϕ_n sends $G_{K_n U K_n^{-1}}^\rho$-independent Borel sets to G_U^ρ-independent Borel sets, there are Borel $\mathbb{N}$-colorings of $G_U^\rho \cap (\phi_n(B) \times \phi_n(B))$, and therefore of G_U^ρ. □

Remark 2.4. Propositions 2.2 and 2.3 easily imply that a Borel cocycle $\rho\colon E \to (0, \infty)$ is smooth if and only if X is the union of countably many ρ-lacunary Borel sets.

We say that a function $\phi\colon X \to X$ is *strictly ρ-increasing* if its graph is contained in E and $|\phi^{-1}(x)|_x^\rho < 1$ for all $x \in X$.

PROPOSITION 2.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho\colon E \to (0, \infty)$ is a smooth Borel cocycle. Then there is an E-invariant Borel set $B \subseteq X$ for which $E \restriction {\sim}B$ is smooth and there is a strictly $(\rho \restriction (E \restriction B))$-increasing Borel automorphism.*

Proof. Fix a partition $(B_n)_{n \in \mathbb{N}}$ of X into ρ-lacunary Borel sets. For each $x \in X$, let $n(x)$ be the unique natural number for which $x \in B_{n(x)}$. Let $\preceq$ be the partial order on X with respect to which $x \preceq y$ if and only if $x \mathrel{E} y$, $n(x) = n(y)$, and $\rho(x, y) \le 1$, and let B be the set of $x \in X$ such that for all $n \in \mathbb{N}$, either $B_n \cap [x]_E = \emptyset$ or $\preceq \restriction (B_n \cap [x]_E)$ is isomorphic to the usual ordering of $\mathbb{Z}$. Then $E \restriction {\sim}B$ is smooth, and the $(\preceq \restriction B)$-successor function is a strictly $(\rho \restriction (E \restriction B))$-increasing Borel automorphism. □

Given a cocycle $\rho\colon E \to (0, \infty)$ and a finite subequivalence relation F of E, define $\rho/F\colon E/F \to (0, \infty)$ by $(\rho/F)([x]_F, [y]_F) = |[x]_F|_{[y]_F}^\rho$. The Lusin–Novikov uniformization theorem ensures that if F is Borel, then X/F is standard Borel, so that E/F is a countable Borel equivalence relation on a standard Borel space. Moreover, if ρ is Borel, then ρ/F is a Borel cocycle on E/F. The Lusin–Novikov uniformization theorem also implies that, when ρ is the constant cocycle, a Borel compression of ρ/F gives rise to a Borel compression of ρ. In spite of Proposition 2.1, such quotients allow us to generalize the fact that aperiodic smooth countable Borel equivalence relations admit Borel compressions.

PROPOSITION 2.6. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho\colon E \to (0, \infty)$ is an aperiodic smooth Borel cocycle. Then there is a finite Borel subequivalence relation F of E for which there is a strictly (ρ/F)-increasing Borel injection.*

Proof. By Proposition 2.5, we can assume that E is smooth. As the aperiodicity of ρ yields that of E, there is a partition $(B_n)_{n\in\mathbb{N}}$ of X into Borel transversals of E. For each $x \in X$, let $n(x)$ be the unique natural number with $x \in B_{n(x)}$, set $n_i(x) = i$ for all $i < 2$, recursively define $n_{i+2}(x)$ to be the least natural number such that the ρ-size of the set $\{y \in [x]_E \mid n_{i+1}(x) \le n(y) < n_{i+2}(x)\}$ relative to the set $\{y \in [x]_E \mid n_i(x) \le n(y) < n_{i+1}(x)\}$ is strictly greater than one for all $i \in \mathbb{N}$, and let $i(x)$ be the unique natural number with the property that $n_{i(x)}(x) \le n(x) < n_{i(x)+1}(x)$. Let F be the subequivalence relation of E with respect to which two E-equivalent points are F-equivalent if and only if $i(x) = i(y)$. Then the function $\phi\colon X/F \to X/F$, given by $\phi([x]_F) = \{y \in [x]_E \mid i(y) = i(x) + 1\}$, is a strictly (ρ/F)-increasing Borel injection. $\square$

The following fact yields an equivalent form of ρ-invariance that will prove useful when considering Borel injections.

PROPOSITION 2.7. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to (0, \infty)$ is a Borel cocycle, and μ is a ρ-invariant Borel measure. Then $\mu(T(B)) = \int_B \rho(T(x), x) \, d\mu(x)$ for all Borel sets $B \subseteq X$ and Borel injections $T\colon B \to X$ whose graphs are contained in E.*

Proof. Fix a countable group $\Gamma = \{\gamma_n \mid n \in \mathbb{N}\}$ of Borel automorphisms of X whose induced orbit equivalence relation is E, recursively define $B_n = \{x \in B \setminus \bigcup_{m<n} B_m \mid T(x) = \gamma_n \cdot x\}$ for all $n \in \mathbb{N}$, and note that

$$\mu(T(B)) = \sum_{n\in\mathbb{N}} \mu(\gamma_n(B_n))$$

$$= \sum_{n\in\mathbb{N}} \int_{B_n} \rho(\gamma_n \cdot x, x) \, d\mu(x)$$

$$= \int_B \rho(T(x), x) \, d\mu(x)$$

by ρ-invariance. $\square$

The following fact yields an equivalent form of ρ-invariance that will prove useful when considering Borel functions.

PROPOSITION 2.8. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to (0, \infty)$ is a Borel cocycle, and μ is a ρ-invariant Borel measure. Then $\mu(\phi^{-1}(B)) = \int_B |\phi^{-1}(x)|_x^\rho \, d\mu(x)$ for all Borel sets $B \subseteq X$ and Borel functions $\phi\colon X \to X$ whose graphs are contained in E.*

Proof. By the Lusin–Novikov uniformization theorem, there are Borel sets $B_n \subseteq B$ and Borel injections $T_n\colon B_n \to X$ with the property that $(\mathrm{graph}(T_n))_{n\in\mathbb{N}}$ partitions

$\text{graph}(\phi^{-1}) \cap (B \times X)$. Then

$$\int_B |\phi^{-1}(x)|_x^\rho \, d\mu(x) = \sum_{n \in \mathbb{N}} \int_{B_n} \rho(T_n(x), x) \, d\mu(x) = \mu(\phi^{-1}(B))$$

by Proposition 2.7.   □

Much as before, we say that a function $\phi \colon X \to X$ is a *compression* of ρ over a finite subequivalence relation F of E if the graph of ϕ is contained in E, $|\phi^{-1}([x]_F)|_{[x]_F}^\rho \le 1$ for all $x \in X$, and the set $\{x \in X \mid |\phi^{-1}([x]_F)|_{[x]_F}^\rho < 1\}$ is E-complete. The Lusin–Novikov uniformization theorem ensures that every Borel compression of the quotient of ρ by a finite Borel subequivalence relation F of E gives rise to a Borel compression of ρ over F. It also implies that, when ρ is the constant cocycle, a Borel compression of ρ over a finite Borel subequivalence relation of E gives rise to a Borel compression of ρ.

PROPOSITION 2.9. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and there is a Borel compression $\phi \colon X \to X$ of ρ over a finite Borel subequivalence relation F of E. Then there is no ρ-invariant Borel probability measure.*

Proof. By the Lusin–Novikov uniformization theorem, there exist a Borel transversal $B \subseteq X$ of F, Borel sets $B_n \subseteq B$, and Borel injections $T_n \colon B_n \to X$ for which $(\text{graph}(T_n))_{n \in \mathbb{N}}$ partitions $F \cap (B \times X)$. If μ is a ρ-invariant Borel measure, then Proposition 2.7 ensures that

$$\mu(X) = \sum_{n \in \mathbb{N}} \mu(T_n(B_n))$$

$$= \sum_{n \in \mathbb{N}} \int_{B_n} \rho(T_n(x), x) \, d\mu(x)$$

$$= \int_B |[x]_F|_x^\rho \, d\mu(x),$$

whereas Propositions 2.7 and 2.8 imply that

$$\mu(X) = \int |\phi^{-1}(x)|_x^\rho \, d\mu(x)$$

$$= \sum_{n \in \mathbb{N}} \int_{T_n(B_n)} |\phi^{-1}(x)|_x^\rho \, d\mu(x)$$

$$= \sum_{n \in \mathbb{N}} \int_{B_n} |(\phi^{-1} \circ T_n)(x)|_{T_n(x)}^\rho \, d((T_n^{-1})_*(\mu))(x)$$

$$= \sum_{n \in \mathbb{N}} \int_{B_n} |(\phi^{-1} \circ T_n)(x)|_x^\rho \, d\mu(x)$$

$$= \int_B |\phi^{-1}([x]_F)|_x^\rho \, d\mu(x).$$

As the set $A = \{x \in B \mid |\phi^{-1}([x]_F)|_x^\rho < |[x]_F|_x^\rho\}$ is E-complete, it follows that if $\mu(X) > 0$, then $\mu(A) > 0$. As $|\phi^{-1}([x]_F)|_x^\rho \le |[x]_F|_x^\rho$ for all $x \in B$, it follows that if $\mu(A) > 0$, then $\mu(X) = \infty$.   □

We next note the useful fact that smoothness is invariant under quotients by finite Borel subequivalence relations of E.

PROPOSITION 2.10. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho: E \to (0, \infty)$ is a Borel cocycle, and F is a finite Borel subequivalence relation of E. Then ρ is smooth if and only if ρ/F is smooth.*

Proof. By partitioning X into countably many F-invariant Borel sets, we can assume that there is a real number $r > 1$ with $|[x]_F|_x^\rho \le r$ for all $x \in X$. As $[Y]_F/F$ is $G_{(1/r,r)}^{\rho/F}$-independent for all $G_{(1/r^2,r^2)}^\rho$-independent sets $Y \subseteq X$, the smoothness of ρ yields that of ρ/F. As every F-invariant set $Y \subseteq X$ for which Y/F is $G_{(1/r^2,r^2)}^{\rho/F}$-independent is itself $(G_{(1/r,r)}^\rho \setminus F)$-independent, the smoothness of ρ/F yields that of ρ. $\qquad \square$

Generalizing the Dougherty–Jackson–Kechris observation that there is a Borel compression of E if and only if there is an aperiodic smooth Borel subequivalence relation of E, we have the following proposition.

PROPOSITION 2.11. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho: E \to (0, \infty)$ is a Borel cocycle. Then the following statements are equivalent.*
(1) *There is an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E.*
(2) *There is a Borel subequivalence relation of E on which ρ is aperiodic and smooth.*
(3) *There exist an E-invariant Borel set $B \subseteq X$ and a Borel subequivalence relation F of E such that $F \upharpoonright {\sim}B$ is smooth, $\rho \upharpoonright (F \upharpoonright {\sim}B)$ is aperiodic, and there is a strictly $(\rho \upharpoonright (F \upharpoonright B))$-increasing Borel automorphism.*

Proof. To see (1) $\implies$ (2), observe that by Proposition 2.10 we can assume that there is an injective Borel compression $\phi: X \to X$ of ρ. Set $A = \{x \in X \mid |\phi^{-1}(x)|_x^\rho < 1\}$, and let F be the orbit equivalence relation generated by ϕ. As the sets $A_r = \{x \in X \mid |\phi^{-1}(x)|_x^\rho < r\}$ are $(\rho \upharpoonright F)$-lacunary for all $r < 1$, it follows that $\rho \upharpoonright (F \upharpoonright A)$ is smooth, thus $\rho \upharpoonright (F \upharpoonright [A]_F)$ is aperiodic and smooth. By the Lusin–Novikov uniformization theorem, there is a Borel extension $\psi: X \to [A]_F$ of the identity function on $[A]_F$ whose graph is contained in E, in which case the restriction of ρ to the pullback of $F \upharpoonright [A]_F$ through ψ is aperiodic and smooth.

To see (2) $\implies$ (3), note that if condition (2) holds, then Proposition 2.5 immediately yields the weakening of condition (3) in which the set B need not be E-invariant. To see that this weakening yields condition (3) itself, note that if $B' \subseteq X$ is a Borel set and F' is a smooth Borel subequivalence relation of $E \upharpoonright B'$ for which $\rho \upharpoonright F'$ is aperiodic, then the Lusin–Novikov uniformization theorem yields a Borel extension $\pi: [B']_E \to B'$ of the identity function on B' whose graph is contained in E, the subequivalence relation F'' of $E \upharpoonright [B']_E$ given by $x \, F'' \, y \iff \pi(x) \, F' \, \pi(y)$ is smooth, and $\rho \upharpoonright F''$ is aperiodic.

It only remains to note that Proposition 2.6 yields (3) $\implies$ (1). $\qquad \square$

We close this section by noting that, at least when ρ is smooth, the existence of an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E is the sole obstacle to the existence of a ρ-invariant Borel probability measure.

PROPOSITION 2.12. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a smooth Borel cocycle. Then exactly one of the following statements holds.*
(1)   *There is an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E.*
(2)   *There is a ρ-invariant Borel probability measure.*

Proof. Proposition 2.9 ensures that conditions (1) and (2) are mutually exclusive. To see that at least one of them holds, note first that if ρ is aperiodic, then Proposition 2.6 yields a finite Borel subequivalence relation F of E for which there is a strictly (ρ/F)-increasing Borel injection. And if there is a ρ-finite equivalence class C of E, then the Borel probability measure μ on X, given by $\mu(B) = |B \cap C|_C^\rho$ for all Borel sets $B \subseteq X$, is ρ-invariant.   □

3. *Coboundaries*
We say that a Borel cocycle $\rho \colon E \to (0, \infty)$ is a *Borel coboundary* if there is a Borel function $f \colon X \to (0, \infty)$ such that $\rho(x, y) = f(x)/f(y)$ for all $(x, y) \in E$. The following observation shows that, even for Borel coboundaries, the equivalent conditions of Proposition 2.11 do not characterize the non-existence of an invariant Borel probability measure.

PROPOSITION 3.1. *Suppose that X is a standard Borel space and E is an aperiodic countable Borel equivalence relation on X admitting an invariant Borel probability measure. Then there is a Borel coboundary $\rho \colon E \to (0, \infty)$ with the property that there is neither an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E nor a ρ-invariant Borel probability measure.*

Proof. Set $B_0 = X$ and let $\iota_0 \colon B_0 \to B_0$ be the identity function. Recursively apply [**KM04**, Proposition 7.4] to obtain Borel sets $B_{n+1} \subseteq \iota_n(B_n)$ and Borel involutions $\iota_{n+1} \colon \iota_n(B_n) \to \iota_n(B_n)$ such that the graph of ι_{n+1} is contained in E and the sets B_{n+1} and $\iota_{n+1}(B_{n+1})$ partition $\iota_n(B_n)$ for all $n \in \mathbb{N}$. For each $x \in X$, let $n(x)$ be the maximal natural number for which $x \in B_{n(x)}$, and set $f(x) = 2^{n(x)}$. Define $\rho \colon E \to (0, \infty)$ by setting $\rho(x, y) = f(x)/f(y)$ for all $(x, y) \in E$.

To see that there is no ρ-invariant Borel probability measure, note that if μ is a ρ-invariant Borel measure, then the fact that $\iota_{n+1}(B_{n+2})$ and $(\iota_{n+1} \circ \iota_{n+2})(B_{n+2})$ partition B_{n+1} for all $n \in \mathbb{N}$ ensures that

$$\mu(B_{n+1}) = \int_{B_{n+2}} \rho(\iota_{n+1}(x), x) + \rho((\iota_{n+1} \circ \iota_{n+2})(x), x) \, d\mu(x) = \mu(B_{n+2})$$

for all $n \in \mathbb{N}$, thus $\mu(X) \in \{0, \infty\}$.

Suppose, towards a contradiction, that there is an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E. Then Proposition 2.11

yields an E-invariant Borel set $A \subseteq X$ and a Borel subequivalence relation F of E such that $F \restriction \sim A$ is smooth, $\rho \restriction (F \restriction \sim A)$ is aperiodic, and there is a strictly $(\rho \restriction (F \restriction A))$-increasing Borel automorphism $\phi \colon A \to A$. Fix an E-invariant Borel probability measure μ. As $\iota_n(B_{n+1})$ and $(\iota_n \circ \iota_{n+1})(B_{n+1})$ partition B_n for all $n \in \mathbb{N}$, it follows that $\mu(B_n) = 2\mu(B_{n+1})$ for all $n \in \mathbb{N}$. As the aperiodicity of $\rho \restriction (F \restriction \sim A)$ yields that of $F \restriction \sim A$, Propositions 2.6 and 2.9 imply that A is μ-conull, thus so too is $A \cap \bigcup_{n \in \mathbb{N}} B_{n+1}$. As the definition of ρ ensures that $\phi(A \cap \bigcup_{n \in \mathbb{N}} B_{n+1}) \subseteq A \cap \bigcup_{n \in \mathbb{N}} B_{n+2}$, and the latter set has μ-measure $1/2$, this contradicts E-invariance. □

The following fact yields an equivalent of ρ-invariance that will prove useful when dealing with finite Borel subequivalence relations.

PROPOSITION 3.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and μ is a ρ-invariant Borel measure on X. Then $\mu(B) = \int |B \cap [x]_F|^\rho_{[x]_F} \, d\mu(x)$ for all Borel sets $B \subseteq X$ and finite Borel subequivalence relations F of E.*

Proof. Fix a Borel transversal $A \subseteq X$ of F, Borel sets $A_n \subseteq A$, and Borel injections $T_n \colon A_n \to X$ with the property that $(\mathrm{graph}(T_n))_{n \in \mathbb{N}}$ partitions $F \cap (A \times X)$, and observe that

$$
\begin{aligned}
\int |B \cap [x]_F|^\rho_{[x]_F} \, d\mu(x) &= \sum_{n \in \mathbb{N}} \int_{T_n(A_n)} |B \cap [x]_F|^\rho_{[x]_F} \, d\mu(x) \\
&= \sum_{n \in \mathbb{N}} \int_{A_n} |B \cap [x]_F|^\rho_{[x]_F} \, d((T_n^{-1})_* \mu)(x) \\
&= \sum_{n \in \mathbb{N}} \int_{A_n} |B \cap [x]_F|^\rho_{[x]_F} \rho(T_n(x), x) \, d\mu(x) \\
&= \int_A |B \cap [x]_F|^\rho_x \, d\mu(x) \\
&= \sum_{n \in \mathbb{N}} \int_{A_n \cap T_n^{-1}(B)} \rho(T_n(x), x) \, d\mu(x) \\
&= \sum_{n \in \mathbb{N}} \mu(T_n(A_n) \cap B) \\
&= \mu(B)
\end{aligned}
$$

by Proposition 2.7. □

Given a Borel set $R \subseteq X \times X$ with countable vertical sections and a Borel function $\rho \colon R \to (0, \infty)$, we say that a Borel measure μ on X is ρ-*invariant* if $\mu(T(B)) = \int_B \rho(T(x), x) \, d\mu(x)$ for all Borel sets $B \subseteq X$ and Borel injections $T \colon B \to X$ whose graphs are contained in R^{-1}. The *composition* of sets $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is given by $R \circ S = \{(x, z) \in X \times Z \mid \exists y \in Y \; x \, R \, y \, S \, z\}$. The Lusin–Novikov uniformization theorem ensures that if R and S are Borel sets with countable vertical sections, then so too is their composition. The following fact will prove useful in verifying ρ-invariance.

PROPOSITION 3.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, R, $S \subseteq E$ are Borel, and $\rho \colon E \to (0, \infty)$ is a Borel cocycle. Then every $(\rho \restriction (R \cup S))$-invariant Borel measure μ is $(\rho \restriction (R \circ S))$-invariant.*

Proof. Note first that if $B \subseteq X$ is a Borel set, $T_S \colon B \to X$ is a Borel injection whose graph is contained in S^{-1}, and $T_R \colon T_S(B) \to X$ is a Borel injection whose graph is contained in R^{-1}, then

$$\mu((T_R \circ T_S)(B)) = \int_{T_S(B)} \rho(T_R(x), x) \, d\mu(x)$$

$$= \int_B \rho((T_R \circ T_S)(x), T_S(x)) \, d((T_S^{-1})_* \mu)(x)$$

$$= \int_B \rho((T_R \circ T_S)(x), x) \, d\mu(x).$$

As the Lusin–Novikov uniformization theorem ensures that every Borel injection whose graph is contained in $(R \circ S)^{-1}$ can be decomposed into a disjoint union of countably many Borel injections of the form $T_R \circ T_S$ as above, the proposition follows. $\qquad \square$

We say that Borel cocycles $\rho \colon E \to (0, \infty)$ and $\sigma \colon E \to (0, \infty)$ are *Borel cohomologous* if their ratio is a Borel coboundary. We say that a Borel function $f \colon X \to (0, \infty)$ *witnesses* that ρ and σ are Borel cohomologous if $f(x)/f(y) = \sigma(x, y)/\rho(x, y)$ for all $(x, y) \in E$.

PROPOSITION 3.4. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $f \colon X \to (0, \infty)$ is a Borel function witnessing that Borel cocycles $\rho, \sigma \colon E \to (0, \infty)$ are Borel cohomologous, and μ is a ρ-invariant Borel measure. Then the Borel measure given by $\nu(B) = \int_B f \, d\mu$ is σ-invariant.*

Proof. Simply observe that if $B \subseteq X$ is a Borel set and $T \colon X \to X$ is a Borel automorphism whose graph is contained in E, then

$$\nu(T(B)) = \int_{T(B)} f \, d\mu$$

$$= \int_B f \circ T \, d((T^{-1})_* \mu)$$

$$= \int_B (f \circ T)(x) \rho(T(x), x) \, d\mu(x)$$

$$= \int_B f(x) \sigma(T(x), x) \, d\mu(x)$$

$$= \int_B \sigma(T(x), x) \, d\nu(x)$$

by ρ-invariance. $\qquad \square$

We say that a Borel set $B \subseteq X$ has *ρ-density* at least ϵ if there is a finite Borel subequivalence relation F of E such that $|B \cap [x]_F|_{[x]_F}^\rho \geq \epsilon$ for all $x \in X$. We say that a Borel set $B \subseteq X$ has *positive ρ-density* if there exists $\epsilon > 0$ for which B has ρ-density at least ϵ.

PROPOSITION 3.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and $B \subseteq X$ is a Borel set with positive ρ-density. Then every $(\rho \upharpoonright (E \upharpoonright B))$-invariant finite Borel measure μ extends to a ρ-invariant finite Borel measure.*

Proof. Fix $\epsilon > 0$ for which B has ρ-density at least ϵ, as well as a finite Borel subequivalence relation F of E such that $|B \cap [x]_F|^{\rho}_{[x]_F} \geq \epsilon$ for all $x \in X$, and let $\overline{\mu}$ be the Borel measure on X given by

$$\overline{\mu}(A) = \int |A \cap [x]_F|^{\rho}_{B \cap [x]_F} \, d\mu(x)$$

for all Borel sets $A \subseteq X$.

As $\overline{\mu}(X) \leq \mu(B)/\epsilon$, it follows that $\overline{\mu}$ is finite, and Proposition 3.2 ensures that $\mu = \overline{\mu} \upharpoonright B$.

LEMMA 3.6. *Suppose that $f \colon X \to [0, \infty)$ is a Borel function. Then $\int f \, d\overline{\mu} = \int \sum_{y \in [x]_F} f(y) |\{y\}|^{\rho}_{B \cap [x]_F} \, d\mu(x)$.*

Proof. It is sufficient to check the special case that f is the characteristic function of a Borel set, which is a direct consequence of the definition of $\overline{\mu}$. $\square$

LEMMA 3.7. *The measure $\overline{\mu}$ is $(\rho \upharpoonright F)$-invariant.*

Proof. Simply observe that if $A \subseteq X$ is a Borel set and $T \colon X \to X$ is a Borel automorphism whose graph is contained in F, then

$$\int_A \rho(T(x), x) \, d\overline{\mu}(x) = \int \sum_{y \in A \cap [x]_F} \rho(T(y), y) |\{y\}|^{\rho}_{B \cap [x]_F} \, d\mu(x)$$

$$= \int |T(A \cap [x]_F)|^{\rho}_{B \cap [x]_F} \, d\mu(x)$$

$$= \overline{\mu}(T(A))$$

by Lemma 3.6. $\square$

As $E = F \circ (E \cap (B \times B)) \circ F$, two applications of Proposition 3.3 ensure that $\overline{\mu}$ is ρ-invariant. $\square$

The primary argument of this section will hinge on the following approximation lemma.

PROPOSITION 3.8. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a Borel cocycle. Then for all Borel sets $A \subseteq X$ and positive real numbers $r < 1$, there exist an E-invariant Borel set $B \subseteq X$, a Borel set $C \subseteq B$, and a finite Borel subequivalence relation F of $E \upharpoonright C$ such that $\rho \upharpoonright (E \upharpoonright {\sim}B)$ is smooth, $r < |A \cap [x]_F|^{\rho}_{[x]_F \setminus A} < 1$ for all $x \in C$, and $A \cap [x]_E \subseteq C$ or $[x]_E \setminus A \subseteq C$ for all $x \in B$.*

Proof. By [**KM04**, Lemma 7.3], there is a maximal Borel set $\mathcal{S}$ of pairwise disjoint non-empty finite sets $S \subseteq X$ with $S \times S \subseteq E$ and $r < |A \cap S|^{\rho}_{S \setminus A} < 1$. Set $D = A \setminus \bigcup \mathcal{S}$ and $D' = ({\sim}A) \setminus \bigcup \mathcal{S}$.

LEMMA 3.9. *Suppose that $(x, x') \in E$. Then there exists a real number $s > 1$ with the property that x has only finitely many $G^\rho_{(1/s,s)}$-neighbors in D or x' has only finitely many $G^\rho_{(1/s,s)}$-neighbors in D'.*

Proof. Fix $n, n' \in \mathbb{N}$ such that $(n/n')\rho(x, x')$ lies strictly between r and 1, and fix $s > 1$ sufficiently small that $(n/n')\rho(x, x')$ lies strictly between rs^2 and $1/s^2$. Suppose, towards a contradiction, that there are sets $S \subseteq D$ and $S' \subseteq D'$ of $G^\rho_{(1/s,s)}$-neighbors of x and x' of cardinalities n and n'. Then $n/s < |S|^\rho_x < ns$ and $n'\rho(x', x)/s < |S'|^\rho_x < n'\rho(x', x)s$, so the ρ-size of S relative to S' lies strictly between $(n/n')\rho(x, x')/s^2$ and $(n/n')\rho(x, x')s^2$. As these bounds lie strictly between r and 1, this contradicts the maximality of $\mathcal{S}$. $\square$

Lemma 3.9 ensures that $[D]_E \cap [D']_E$ is contained in the E-saturation of the union of the sets of the form $\{x \in D \mid |D \cap (G^\rho_{(1/s,s)})_x| < \aleph_0\}$ and $\{x \in D' \mid |D' \cap (G^\rho_{(1/s,s)})_x| < \aleph_0\}$, so $\rho \restriction (E \restriction ([D]_E \cap [D']_E))$ is smooth. Set $B = {\sim}([D]_E \cap [D']_E)$ and $C = B \cap \bigcup \mathcal{S}$, and let F be the equivalence relation on C whose classes are the subsets of C in $\mathcal{S}$. $\square$

We say that a Borel set $B \subseteq X$ has σ-*positive ρ-density* if X is the union of countably many E-invariant Borel sets $A_n \subseteq X$ for which $A_n \cap B$ has positive $(\rho \restriction (E \restriction A_n))$-density.

THEOREM 3.10. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and $A \subseteq X$ is an E-complete Borel set. Then X is the union of an E-invariant Borel set $B \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth, an E-invariant Borel set $C \subseteq X$ for which $A \cap C$ has σ-positive $(\rho \restriction (E \restriction C))$-density, and an E-invariant Borel set $D \subseteq X$ for which there is a finite-to-one Borel compression of the quotient of $\rho \restriction (E \restriction D)$ by a finite Borel subequivalence relation of $E \restriction D$.*

Proof. Fix a positive real number $r < 1$. We will show that, after throwing out countably many E-invariant Borel sets $B \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth, as well as countably many E-invariant Borel sets $C \subseteq X$ for which $A \cap C$ has positive $(\rho \restriction (E \restriction C))$-density, there are increasing sequences of finite Borel subequivalence relations F_n of E and E-complete F_n-invariant Borel sets $A_n \subseteq X$ with the property that $r < |A_n \cap [x]_{F_{n+1}}|^\rho_{(A_{n+1}\setminus A_n) \cap [x]_{F_{n+1}}} < 1$ for all $n \in \mathbb{N}$ and $x \in A_n$.

We begin by setting $A_0 = A$ and letting F_0 be equality. Suppose now that $n \in \mathbb{N}$ and we have already found A_n and F_n. By applying Proposition 3.8 to A_n/F_n, and throwing out an E-invariant Borel set $B \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth, we obtain a finite Borel subequivalence relation $F_{n+1} \supseteq F_n$ of E and an F_{n+1}-invariant Borel set $A_{n+1} \subseteq X$ such that $r < |A_n \cap [x]_{F_{n+1}}|^\rho_{[x]_{F_{n+1}}\setminus A_n} < 1$ for all $x \in A_{n+1}$, and $A_n \cap [x]_E \subseteq A_{n+1}$ or $[x]_E \setminus A_n \subseteq A_{n+1}$ for all $x \in X$. By throwing out an E-invariant Borel set $C \subseteq X$ for which $A \cap C$ has positive $(\rho \restriction (E \restriction C))$-density, we can assume that $A_n \subseteq A_{n+1}$, completing the recursive construction.

Set $B_n = A_n \setminus \bigcup_{m<n} A_m$ and define $\phi_n \colon B_n/F_n \to B_{n+1}/F_{n+1}$ by setting $\phi_n(B_n \cap [x]_{F_n}) = B_{n+1} \cap [x]_{F_{n+1}}$ for all $n \in \mathbb{N}$ and $x \in B_n$. Then the union of $\bigcup_{n\in\mathbb{N}} \phi_n$ and the identity function on ${\sim}\bigcup_{n\in\mathbb{N}} A_n$ is a Borel compression of the quotient of ρ by the union of $\bigcup_{n\in\mathbb{N}} F_n \restriction B_n$ and equality. $\square$

As a corollary, we obtain the desired characterization.

THEOREM 3.11. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a Borel coboundary. Then exactly one of the following statements holds.*
(1) *There is a finite-to-one Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E.*
(2) *There is a ρ-invariant Borel probability measure.*

Proof. Proposition 2.9 ensures that conditions (1) and (2) are mutually exclusive. To see that at least one of them holds, fix a bounded open neighborhood $U \subseteq (0, \infty)$ of 1. As ρ is a Borel coboundary, the Lusin–Novikov uniformization theorem implies that there is an E-complete Borel set $A \subseteq X$ for which $\rho(E \restriction A) \subseteq U$. By Theorem 3.10, after throwing out E-invariant Borel sets $B \subseteq X$ and $D \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth and there is a finite-to-one Borel compression of the quotient of $\rho \restriction (E \restriction D)$ by a finite Borel subequivalence relation of $E \restriction D$, we can assume that A has σ-positive ρ-density.

If there is a $(\rho \restriction (E \restriction A))$-invariant Borel probability measure μ, then by passing to an $(E \restriction A)$-invariant μ-positive Borel set, we can assume that A has positive ρ-density, in which case Proposition 3.5 yields a ρ-invariant Borel probability measure.

If there is no $(\rho \restriction (E \restriction A))$-invariant Borel probability measure, then Proposition 3.4 ensures that there is no $(E \restriction A)$-invariant Borel probability measure, in which case the Becker–Kechris generalization of Nadkarni's theorem and the Dougherty–Jackson–Kechris characterization of the existence of a Borel compression yield an aperiodic smooth Borel subequivalence relation F of $E \restriction A$. Then $\rho \restriction F$ is smooth, and the fact that $\rho \restriction (E \restriction A)$ is bounded ensures that $\rho \restriction F$ is also aperiodic. Fix a Borel extension $\phi \colon X \to A$ of the identity function on A whose graph is contained in E, and observe that ρ is aperiodic and smooth on the pullback of F through ϕ. Proposition 2.6 therefore yields an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E. □

4. *The general case*

Here we generalize Nadkarni's theorem to Borel cocycles. As in §3, our primary argument will hinge on a pair of approximation lemmas. Given a finite set $S \subseteq X$ for which $S \times S \subseteq E$, let μ_S^ρ be the Borel probability measure on X given by $\mu_S^\rho(B) = |B \cap S|_S^\rho$.

PROPOSITION 4.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, $f \colon X \to [0, \infty)$ is Borel, $\delta > 0$, and $\epsilon > \sup_{(x,y) \in E} f(x) - f(y)$. Then there exist an E-invariant Borel set $B \subseteq X$ and a finite Borel subequivalence relation F of $E \restriction B$ for which $\rho \restriction (E \restriction {\sim}B)$ is smooth and $\delta\epsilon > \sup_{(x,y) \in E \restriction B} \int f \, d\mu_{[x]_F}^\rho - \int f \, d\mu_{[y]_F}^\rho$.*

Proof. We can clearly assume that $\delta < 1$, and since one can repeatedly apply the corresponding special case of the proposition over the corresponding quotients, we can also assume that $\delta > 2/3$. For each $x \in X$, let $\overline{f}([x]_E)$ be the average of $\inf f([x]_E)$ and $\sup f([x]_E)$. By [**KM04**, Lemma 7.3], there is a maximal Borel set $\mathcal{S}$ of pairwise disjoint

non-empty finite sets $S \subseteq X$ with $S \times S \subseteq E$ and $\epsilon(\delta - 1/2) > |\int f \, d\mu_S^\rho - \overline{f}([S]_E)|$. Set $C = \{x \in \mathord{\sim}\bigcup \mathcal{S} \mid f(x) < \overline{f}([x]_E)\}$ and $D = \{x \in \mathord{\sim}\bigcup \mathcal{S} \mid f(x) > \overline{f}([x]_E)\}$.

LEMMA 4.2. *Suppose that* $(x, y) \in E$. *Then there exists a real number* $r > 1$ *such that* x *has only finitely many* $G_{(1/r,r)}^\rho$-*neighbors in* C *or* y *has only finitely many* $G_{(1/r,r)}^\rho$-*neighbors in* D.

Proof. As $\delta > 2/3$, a trivial calculation reveals that $-\epsilon(\delta - 1/2)$ is strictly below the average of $-\epsilon/2$ and $\epsilon(\delta - 1/2)$, and that the average of $-\epsilon(\delta - 1/2)$ and $\epsilon/2$ is strictly below $\epsilon(\delta - 1/2)$. In particular, by choosing $m, n \in \mathbb{N}$ for which the ratios $s = m/(m + n\rho(y, x))$ and $t = n\rho(y, x)/(m + n\rho(y, x))$ are sufficiently close to $1/2$, we can therefore ensure that the sums $s(\overline{f}([x]_E) - \epsilon/2) + t(\overline{f}([x]_E) + \epsilon(\delta - 1/2))$ and $s(\overline{f}([x]_E) - \epsilon(\delta - 1/2)) + t(\overline{f}([x]_E) + \epsilon/2)$ both lie strictly between $\overline{f}([x]_E) - \epsilon(\delta - 1/2)$ and $\overline{f}([x]_E) + \epsilon(\delta - 1/2)$. Fix $r > 1$ such that they lie strictly between $(\overline{f}([x]_E) - \epsilon(\delta - 1/2))r^2$ and $(\overline{f}([x]_E) + \epsilon(\delta - 1/2))/r^2$.

Suppose, towards a contradiction, that there exist sets $S \subseteq C$ and $T \subseteq D$ of $G_{(1/r,r)}^\rho$-neighbors of x and y of cardinalities m and n. Then $m/r < |S|_x^\rho < mr$ and $n\rho(y, x)/r < |T|_x^\rho < n\rho(y, x)r$, from which a trivial calculation reveals that $s/r^2 < |S|_x^\rho/|S \cup T|_x^\rho < sr^2$ and $t/r^2 < |T|_x^\rho/|S \cup T|_x^\rho < tr^2$. As $\int f \, d\mu_S^\rho$ lies between $\overline{f}([x]_E) - \epsilon/2$ and $\overline{f}([x]_E) - \epsilon(\delta - 1/2)$, and $\int f \, d\mu_T^\rho$ lies between $\overline{f}([x]_E) + \epsilon(\delta - 1/2)$ and $\overline{f}([x]_E) + \epsilon/2$, it follows that $\int f \, d\mu_{S \cup T}^\rho$ lies between $(s(\overline{f}([x]_E) - \epsilon/2) + t(\overline{f}([x]_E) + \epsilon(\delta - 1/2)))/r^2$ and $(s(\overline{f}([x]_E) - \epsilon(\delta - 1/2)) + t(\overline{f}([x]_E) + \epsilon/2))r^2$, so strictly between $\overline{f}([x]_E) - \epsilon(\delta - 1/2)$ and $\overline{f}([x]_E) + \epsilon(\delta - 1/2)$, contradicting the maximality of $\mathcal{S}$. $\square$

Lemma 4.2 ensures that $[C]_E \cap [D]_E$ is contained in the E-saturation of the union of the sets of the form $\{x \in C \mid |C \cap (G_{(1/r,r)}^\rho)_x| < \aleph_0\}$ and $\{x \in D \mid |D \cap (G_{(1/r,r)}^\rho)_x| < \aleph_0\}$, so $\rho \restriction (E \restriction ([C]_E \cap [D]_E))$ is smooth. Set $B = \mathord{\sim}([C]_E \cap [D]_E)$, and let F be the equivalence relation on B whose classes are the subsets of B in $\mathcal{S}$ together with the singletons contained in $B \backslash \bigcup \mathcal{S}$. $\square$

PROPOSITION 4.3. *Suppose that* X *is a standard Borel space,* E *is a countable Borel equivalence relation on* X, $\rho \colon E \to (0, \infty)$ *is a Borel cocycle,* $f, g \colon X \to [0, \infty)$ *are Borel, and* $r > 1$. *Then there exist an* E-*invariant Borel set* $B \subseteq X$, *a Borel set* $C \subseteq B$, *and a finite Borel subequivalence relation* F *of* $E \restriction B$ *such that* $\rho \restriction (E \restriction \mathord{\sim}B)$ *is smooth and* $\int_C f \, d\mu_{[x]_F}^\rho \le \int_{B \backslash C} g \, d\mu_{[x]_F}^\rho \le r \int_C f \, d\mu_{[x]_F}^\rho$ *for all* $x \in B$.

Proof. As the proposition holds trivially on $f^{-1}(0) \cup g^{-1}(0)$, we can assume that $f, g \colon X \to (0, \infty)$. By [**KM04**, Lemma 7.3], there is a maximal Borel set $\mathcal{S}$ of pairwise disjoint non-empty finite sets $S \subseteq X$ with $S \times S \subseteq E$ and $1 < \int_{S \backslash T} g \, d\mu_S^\rho / \int_T f \, d\mu_S^\rho < r$ for some $T \subseteq S$.

Set $D_{U,V} = (f^{-1}(U) \cap g^{-1}(V)) \backslash \bigcup \mathcal{S}$ for all $U, V \subseteq (0, \infty)$.

LEMMA 4.4. *For all* $x \in X$, *there exists* $s > 1$ *such that* x *has only finitely many* $G_{(1/s,s)}^\rho$-*neighbors in* $D_{(f(x)/s, f(x)s),(g(x)/s, g(x)s)}$.

Proof. Fix $m, n \in \mathbb{N}$ for which $1 < (g(x)/f(x))(n/m) < r$, as well as $s > 1$ sufficiently large that $s^6 < (g(x)/f(x))(n/m) < r/s^6$. Suppose, towards a contradiction, that there is

a set $S \subseteq D_{(f(x)/s, f(x)s), (g(x)/s, g(x)s)}$ of $G^\rho_{(1/s,s)}$-neighbors of x of cardinality $k = m + n$, and fix $T \subseteq S$ of cardinality m. Then $f(x)\mu^\rho_S(T)/s < \int_T f \, d\mu^\rho_S < f(x)\mu^\rho_S(T)s$ and $(m/k)/s^2 < \mu^\rho_S(T) < (m/k)s^2$, so $f(x)(m/k)/s^3 < \int_T f \, d\mu^\rho_S < f(x)(m/k)s^3$. And $g(x)\mu^\rho_S(S\setminus T)/s < \int_{S\setminus T} g \, d\mu^\rho_S < g(x)\mu^\rho_S(S\setminus T)s$ and $(n/k)/s^2 < \mu^\rho_S(S\setminus T) < (n/k)s^2$, so $g(x)(n/k)/s^3 < \int_{S\setminus T} g \, d\mu^\rho_S < g(x)(n/k)s^3$. It follows that $\int_{S\setminus T} g \, d\mu^\rho_S \, / \int_T f \, d\mu^\rho_S$ lies strictly between $(g(x)/f(x))(n/m)/s^6$ and $(g(x)/f(x))(n/m)s^6$, and therefore strictly between 1 and r, contradicting the maximality of $\mathcal{S}$. □

As Lemma 4.2 ensures that $\sim \bigcup \mathcal{S}$ is contained in the union of the sets of the form $\{x \in D_{U,V} \mid |D_{U,V} \cap (G^\rho_{(1/s,s)})_x| < \aleph_0\}$, it follows that $\rho \upharpoonright (E \upharpoonright [\sim \bigcup \mathcal{S}]_E)$ is smooth. Set $B = \sim[\sim \bigcup \mathcal{S}]_E$, let F be the Borel equivalence relation on B whose classes are the subsets of B in $\mathcal{S}$, and appeal to the Lusin–Novikov uniformization theorem to obtain a Borel set $C \subseteq B$ with the property that $1 < \int_{B\setminus C} g \, d\mu^\rho_{[x]_F} \, / \int_C f \, d\mu^\rho_{[x]_F} < r$ for all $x \in B$. □

We are now ready to establish our primary result.

THEOREM 4.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a Borel cocycle. Then exactly one of the following statements holds.*

(1) *There is a finite-to-one Borel compression of ρ over a finite Borel subequivalence relation of E.*

(2) *There is a ρ-invariant Borel probability measure.*

Proof. Proposition 2.9 ensures that conditions (1) and (2) are mutually exclusive. To see that at least one of them holds, fix a countable group Γ of Borel automorphisms of X whose induced orbit equivalence relation is E, and define $\rho_\gamma \colon X \to (0, \infty)$ by $\rho_\gamma(x) = \rho(\gamma \cdot x, x)$ for all $\gamma \in \Gamma$.

By standard change of topology results (see, for example, [**Kec95**, §13]), there exist a Polish topology on $[0, \infty)$ and a zero-dimensional Polish topology on X, compatible with the underlying Borel structures of $[0, \infty)$ and X, with respect to which every interval with rational endpoints is clopen, Γ acts by homeomorphisms, and each ρ_γ is continuous. Fix a compatible complete metric on X, as well as a countable algebra $\mathcal{U}$ of clopen subsets of X, closed under multiplication by elements of Γ, and containing a basis for X as well as the pullback of every interval with rational endpoints under every ρ_γ.

We say that a function $f \colon X \to [0, \infty)$ is $\mathcal{U}$-*simple* if it is a finite linear combination of characteristic functions of sets in $\mathcal{U}$. Note that for all $\epsilon > 0$, $\gamma \in \Gamma$, and $Y \subseteq X$ on which ρ_γ is bounded, there is such a function with the further property that $|f(y) - \rho_\gamma(y)| \le \epsilon$ for all $y \in Y$.

Fix a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive real numbers converging to zero, as well as an increasing sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of finite subsets of $\mathcal{U}$ whose union is $\mathcal{U}$.

By recursively applying Propositions 4.1 and 4.3 to functions of the form $[x]_F \mapsto \mu^\rho_{[x]_F}(A)$ and $[x]_F \mapsto \mu^\rho_{[x]_F}(B) - \mu^\rho_{[x]_F}(A)$, and throwing out countably many E-invariant Borel sets $B \subseteq X$ for which $\rho \upharpoonright (E \upharpoonright B)$ is smooth, we obtain increasing sequences of finite algebras $\mathcal{A}_n \supseteq \mathcal{U}_n$ of Borel subsets of X and finite Borel subequivalence

relations F_n of E such that:

(1) $\forall n \in \mathbb{N} \forall A \in \mathcal{A}_n \forall (x, y) \in E \ \mu^\rho_{[x]_{F_{n+1}}}(A) - \mu^\rho_{[y]_{F_{n+1}}}(A) \leq \epsilon_n$.

(2) $\forall n \in \mathbb{N} \forall A, B \in \mathcal{A}_n (\forall x \in X \ \mu^\rho_{[x]_{F_n}}(A) \leq \mu^\rho_{[x]_{F_n}}(B) \implies$
$\exists C \in \mathcal{A}_{n+1} \forall x \in X \ 0 \leq \mu^\rho_{[x]_{F_{n+1}}}(B \backslash C) - \mu^\rho_{[x]_{F_{n+1}}}(A) \leq \epsilon_n)$.

Set $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ and $F = \bigcup_{n \in \mathbb{N}} F_n$. Condition (1) ensures that we obtain finitely additive probability measures μ_x on $\mathcal{U}$ by setting $\mu_x(U) = \lim_{n \to \infty} \mu^\rho_{[x]_{F_n}}(U)$ for all $U \in \mathcal{U}$ and $x \in X$.

LEMMA 4.6. *Suppose that $(U_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{U}$ whose union is in $\mathcal{U}$ and $B = \{x \in X \mid \sum_{n \in \mathbb{N}} \mu_x(U_n) < \mu_x(\bigcup_{n \in \mathbb{N}} U_n)\}$. Then there is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$.*

Proof. As $\mu_x(\bigcup_{m \geq n} U_m) - \sum_{m \geq n} \mu_x(U_m)$ is independent of n, it follows that for all $x \in B$, there exist $\delta > 0$ and $n \in \mathbb{N}$ with the property that $\delta + 2 \sum_{m \geq n} \mu_x(U_m) \leq \mu_x(\bigcup_{m \geq n} U_m)$. So by partitioning B into countably many E-invariant Borel sets and passing to terminal segments of $(U_n)_{n \in \mathbb{N}}$ on each set, we can assume that $B = \{x \in X \mid \delta + 2 \sum_{n \in \mathbb{N}} \mu_x(U_n) \leq \mu_x(\bigcup_{n \in \mathbb{N}} U_n)\}$ for some $\delta > 0$. Fix a sequence $(\delta_n)_{n \in \mathbb{N}}$ of positive real numbers whose sum is at most δ.

SUBLEMMA 4.7. *There are pairwise disjoint sets $A_n \subseteq \bigcup_{m > n} U_m$ in $\mathcal{A}$ with the property that for all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $\forall x \in B \ 0 \leq \mu^\rho_{[x]_{F_k}}(A_n) - \mu^\rho_{[x]_{F_k}}(U_n) \leq \delta_n$.*

Proof. Suppose that $n \in \mathbb{N}$ and we have already found $(A_m)_{m < n}$. Note that if $x \in B$, then

$$\mu_x(U_n) + \sum_{m \geq n} \delta_m \leq \mu_x\left(\bigcup_{m \in \mathbb{N}} U_m\right) - \left(\mu_x(U_n) + \sum_{m < n} 2\mu_x(U_m) + \delta_m\right)$$

$$\leq \mu_x\left(\bigcup_{m > n} U_m\right) - \sum_{m < n} \mu_x(U_m) + \delta_m,$$

so $\forall x \in B \ \mu^\rho_{[x]_{F_k}}(U_n) \leq \mu^\rho_{[x]_{F_k}}(\bigcup_{m > n} U_m \backslash \bigcup_{m < n} A_m)$ for sufficiently large $k \in \mathbb{N}$, by condition (1). It then follows from condition (2) that there exists $A_n \subseteq \bigcup_{m > n} U_m \backslash \bigcup_{m < n} A_m$ in $\mathcal{A}$ with the property that $\forall x \in B \ 0 \leq \mu^\rho_{[x]_{F_k}}(A_n) - \mu^\rho_{[x]_{F_k}}(U_n) \leq \delta_n$ for sufficiently large $k \in \mathbb{N}$. □

Fix $k_n \in \mathbb{N}$ with the property that $\mu^\rho_{[x]_{F_{k_n}}}(U_n) \leq \mu^\rho_{[x]_{F_{k_n}}}(A_n)$ for all $n \in \mathbb{N}$ and $x \in B$, as well as Borel functions $\phi_n \colon B \cap U_n \to A_n$ whose graphs are contained in F_{k_n} for all $n \in \mathbb{N}$. Then the union of $\bigcup_{n \in \mathbb{N}} \phi_n$ and the identity function on $B \backslash \bigcup_{n \in \mathbb{N}} U_n$ is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over the union of $\bigcup_{n \in \mathbb{N}} F_{k_n} \restriction (A_n \cap B)$ and equality on B. □

Lemma 4.6 ensures that, after throwing out countably many E-invariant Borel sets $B \subseteq X$ for which there is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$, we can assume that for all $\delta > 0$ and $U \in \mathcal{U}$, there is a partition $(U_n)_{n \in \mathbb{N}}$ of U into sets in $\mathcal{U}$ of diameter at most δ such that $\mu_x(U) = \sum_{n \in \mathbb{N}} \mu_x(U_n)$ for all $x \in X$.

LEMMA 4.8. *Each μ_x is a measure on $\mathcal{U}$.*

Proof. Suppose, towards a contradiction, that there are pairwise disjoint sets $U_n \in \mathcal{U}$ with $\bigcup_{n \in \mathbb{N}} U_n \in \mathcal{U}$ but $\mu_x(\bigcup_{n \in \mathbb{N}} U_n) > \sum_{n \in \mathbb{N}} \mu_x(U_n)$, for some $x \in X$. Fix a sequence $(\delta_n)_{n \in \mathbb{N}}$ of positive real numbers converging to zero, and recursively construct a sequence $(V_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ of sets in $\mathcal{U}$, beginning with $V_\emptyset = \bigcup_{n \in \mathbb{N}} U_n$, such that $(V_{t \frown (n)})_{n \in \mathbb{N}}$ is a partition of V_t into sets of diameter at most $\delta_{|t|}$ with the property that $\mu_x(V_t) = \sum_{n \in \mathbb{N}} \mu_x(V_{t \frown (n)})$, for all $t \in \mathbb{N}^{<\mathbb{N}}$. Set $r = \sum_{n \in \mathbb{N}} \mu_x(U_n)$, and recursively construct a sequence $(i_n)_{n \in \mathbb{N}}$ of natural numbers with the property that $\sum_{t \in T_n} \mu_x(V_t) > r$, where $T_n = \prod_{m < n} i_m$, for all $n \in \mathbb{N}$. Set $V_n = \bigcup_{t \in T_n} V_t$ for all $n \in \mathbb{N}$. As $(U_n)_{n \in \mathbb{N}}$ covers the compact set $K = \bigcap_{n \in \mathbb{N}} V_n$, so too does $(U_m)_{m < n}$, for some $n \in \mathbb{N}$. Set $U = \bigcup_{m < n} U_m$, and let T be the tree of all $t \in \bigcup_{m \in \mathbb{N}} T_m$ for which $V_t \nsubseteq U$. Note that T is necessarily well founded, since any branch b through T would give rise to a singleton $\bigcap_{n \in \mathbb{N}} V_{t \restriction n}$ contained in $K \setminus U$. König's lemma therefore yields $m \in \mathbb{N}$ with $T \subseteq \bigcup_{\ell < m} T_\ell$, in which case $V_m \subseteq U$, contradicting the fact that $\mu_x(V_m) > \mu_x(U)$. $\square$

As a consequence, Carathéodory's theorem ensures that there is a unique extension of each μ_x to a Borel probability measure $\overline{\mu}_x$ on X.

LEMMA 4.9. *Suppose that $\gamma \in \Gamma$, $U \in \mathcal{U}$, ρ_γ is bounded on U, and $B = \{x \in X \mid \overline{\mu}_x(\gamma(U)) \neq \int_U \rho_\gamma \, d\overline{\mu}_x\}$. Then there is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$.*

Proof. By the symmetry of our argument, it is enough to establish the analogous lemma for the set $B = \{x \in X \mid \overline{\mu}_x(\gamma(U)) < \int_U \rho_\gamma \, d\overline{\mu}_x\}$. By partitioning B into countably many E-invariant Borel sets, we can assume that $B = \{x \in X \mid \delta + \overline{\mu}_x(\gamma(U)) < \int_U \rho_\gamma \, d\overline{\mu}_x\}$ for some $\delta > 0$.

SUBLEMMA 4.10. *For all $\epsilon > 0$, there exists $n \in \mathbb{N}$ with the property that $|\int_U \rho_\gamma \, d\overline{\mu}_x - \int_U \rho_\gamma \, d\mu_{[x]_{F_n}}^\rho| \leq \epsilon$ for all $x \in X$.*

Proof. Fix a $\mathcal{U}$-simple function $f \colon X \to [0, \infty)$ with the property that $|f(x) - \rho_\gamma(x)| \leq \epsilon/3$ for all $x \in U$. By condition (1), there exists $n \in \mathbb{N}$ such that $|\int_U f \, d\overline{\mu}_x - \int_U f \, d\mu_{[x]_{F_n}}^\rho| \leq \epsilon/3$ for all $x \in X$. But then

$$
\begin{aligned}
\left| \int_U \rho_\gamma \, d\overline{\mu}_x - \int_U \rho_\gamma \, d\mu_{[x]_{F_n}}^\rho \right| &\leq \left| \int_U \rho_\gamma \, d\overline{\mu}_x - \int_U f \, d\overline{\mu}_x \right| \\
&\quad + \left| \int_U f \, d\overline{\mu}_x - \int_U f \, d\mu_{[x]_{F_n}}^\rho \right| \\
&\quad + \left| \int_U f \, d\mu_{[x]_{F_n}}^\rho - \int_U \rho_\gamma \, d\mu_{[x]_{F_n}}^\rho \right| \\
&\leq \epsilon
\end{aligned}
$$

for all $x \in X$. $\square$

Condition (1) and Sublemma 4.10 ensure that there exists $n \in \mathbb{N}$ such that $\mu_{[x]_{F_n}}^\rho(\gamma(U)) < \int_U \rho_\gamma \, d\mu_{[x]_{F_n}}^\rho$ for all $x \in B$. As the former quantity is $|\gamma(U) \cap$

$[x]_{F_n}|_x^\rho / |[x]_{F_n}|_x^\rho$ and the latter is $|\gamma(U \cap [x]_{F_n})|_x^\rho / |[x]_{F_n}|_x^\rho$, it follows that $|\gamma(U) \cap [x]_{F_n}|_x^\rho < |\gamma(U \cap [x]_{F_n})|_x^\rho$ for all $x \in B$, so any function from $B \cap \gamma(U)$ to $B \cap \gamma(U)$, sending $\gamma(U) \cap [x]_{F_n}$ to $\gamma(U \cap [x]_{F_n})$ for all $x \in B \cap \gamma(U)$, is a compression of $\rho \restriction (E \restriction (B \cap \gamma(U)))$ over the equivalence relation $(\gamma \times \gamma)(F_n) \restriction (B \cap \gamma(U))$. The Lusin–Novikov uniformization theorem yields a Borel such function, and every Borel such function trivially extends to a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$. □

Lemma 4.9 ensures that, after throwing out countably many E-invariant Borel sets $B \subseteq X$ for which there is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$, we can assume that $\overline{\mu}_x(\gamma(U)) = \int_U \rho_\gamma \, d\overline{\mu}_x$ for all $\gamma \in \Gamma$, $U \in \mathcal{U}$ on which ρ_γ is bounded, and $x \in X$. As our choice of topologies ensures that every open set $U \subseteq X$ is a disjoint union of sets in $\mathcal{U}$ on which ρ_γ is bounded, we obtain the same conclusion even when $U \subseteq X$ is an arbitrary open set. As every Borel probability measure on a Polish space is regular (see, for example, [**Kec95**, Theorem 17.10]), we obtain the same conclusion even when $U \subseteq X$ is an arbitrary Borel set. And since every Borel automorphism $T \colon X \to X$ whose graph is contained in E is a disjoint union of restrictions of automorphisms in Γ to Borel subsets, it follows that each $\overline{\mu}_x$ is ρ-invariant. □

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