Proceedings of the Edinburgh Mathematical Society (2018) **61**, 869–877
doi:10.1017/S0013091517000475

COHOMOLOGY OF FLAT PRINCIPAL BUNDLES

YANGHYUN BYUN AND JOOHEE KIM

Department of Mathematics, Hanyang University, Seoul, Korea (yhbyun@hanyang.ac.kr; kjh0423@hanyang.ac.kr)

(Received 18 May 2017; first published online 22 May 2018)

Abstract We invoke the classical fact that the algebra of bi-invariant forms on a compact connected Lie group G is naturally isomorphic to the de Rham cohomology $H^*_{\mathrm{dR}}(G)$ itself. Then, we show that when a flat connection A exists on a principal G-bundle P, we may construct a homomorphism $E_A : H^*_{\mathrm{dR}}(G) \to H^*_{\mathrm{dR}}(P)$, which eventually shows that the bundle satisfies a condition for the Leray–Hirsch theorem. A similar argument is shown to apply to its adjoint bundle. As a corollary, we show that that both the flat principal bundle and its adjoint bundle have the real coefficient cohomology isomorphic to that of the trivial bundle.

Keywords: flat principal bundle; de Rham cohomology; adjoint bundle

2010 *Mathematics subject classification:* Primary 53C05; 55R20

1. Introduction

Let G be a compact connected Lie group. We consider a principal G-bundle $\pi : P \to M$, where M is any smooth manifold, which is necessarily neither compact nor connected. Furthermore, we assume that P is flat.

We recall the classical fact that $H^*_{\mathrm{dR}}(G)$ is canonically isomorphic to the algebra of bi-invariant forms on G (see (3.1)). Then, by exploiting a flat connection A on P, we define a homomorphism (see (3.2) below),

$$E_A : H^*_{\mathrm{dR}}(G) \to H^*_{\mathrm{dR}}(P). \tag{1.1}$$

Furthermore, choose any $p \in P$ and let $\iota_p : G \to P$ be the map defined by $\iota_p(g) = pg$. Then we will observe that

$$(\iota_p)^* E_A = 1, \tag{1.2}$$

by (3.9) below, which is the identity on $H^*_{\mathrm{dR}}(G)$. Now, we may apply the Leray–Hirsch theorem (cf., [**3**, 5.11] or [**9**, 4D.1]) to have an isomorphism

$$H^*_{\mathrm{dR}}(M) \otimes H^*_{\mathrm{dR}}(G) \cong H^*_{\mathrm{dR}}(P) \tag{1.3}$$

between $H^*_{\mathrm{dR}}(M)$ modules. The isomorphism is such that $a \otimes b \to (\pi^* a) \wedge (E_A(b))$ for any $a \in H^*_{\mathrm{dR}}(M)$ and any $b \in H^*_{\mathrm{dR}}(G)$. In addition, since E_A is a homomorphism between the algebras, the isomorphism (1.3) is not only between $H^*_{\mathrm{dR}}(M)$-modules but also between $\mathbb{R}$-algebras (see (3.3)). The fact represented by (1.3), in spite of its mundane outlook, does not appear known yet, and the current paper seems the first to point it out.

We will show that a similar argument also applies to the adjoint bundle $\mathrm{Ad}\,P = (P \times G)/G$ in § 4.

It is not difficult to see that connectedness of G is a necessary condition for (1.3). For instance, consider the principal $\mathbb{Z}_2$-bundle $S^n \to P^n$. On the other hand, even if we use in this paper the condition that G should be compact, we do not have at the moment an example which shows that it is indeed a necessary condition.

The current paper is partially motivated by a statement of K. Wehrheim ([**12**, 1.2] and the paragraph below it) which asserts that there is a closed 3-form on the adjoint bundle which is, when restricted to each fibre, a form which she refers to as the Maurer–Cartan 3-form. We will provide the precise definition of the Maurer–Cartan 3-form just above (3.3). Her statement concerns not only trivial bundles but also non-trivial bundles. In particular, the structure group in question is not required to be simply connected. Therefore, one may say that the current work proves her statement under the extra condition that the principal bundle is flat. We will prove it by an explicit construction, applying the map E_A in (1.1) (see Corollaries 3.3 and 4.3 below). In addition, we note that the question remains of how E_A in (1.1) depends on the flat connection A. In fact, E_A depends on A on the form level, as can be seen by (3.2) and (3.5).

Our result represented by (1.1)–(1.3), including the corresponding statements regarding the adjoint bundle, describes the real coefficient cohomology of a flat principal bundle and that of its adjoint bundle completely in terms of the cohomology of the fibre and that of the base, under the condition that the fibre is compact and connected. Part of the current paper appears in another work of ours (see [**4**, § 4]). In the current work, we discuss both the flat principal bundle and its adjoint bundle, while [**4**] dealt with only the adjoint bundle. The overlap emerged in the process of communication with the journal in which [**4**] appeared. Nevertheless, the overlap is much smaller now by the omission of some arguments and proofs in § 4. The omission was advised by an anonymous referee, whom we thank for this and also for many other helpful suggestions.

2. Flat principal bundles

Let G be a Lie group. Let P and M be any smooth finite-dimensional manifolds. A surjective smooth map $\pi : P \to M$ together with a smooth right action $P \times G \to P$ is a principal G-bundle if the following conditions are satisfied.

- The local triviality condition holds. That is, for each $x \in M$, there is an open neighbourhood U of x and a diffeomorphism $\psi : \pi^{-1}U \to U \times G$ which maps $P_y = \pi^{-1}\{y\}$, $y \in U$, onto $\{y\} \times G$ and is also a G-map, when $U \times G$ is considered with the obvious right G-action.

Let $\mathcal{G}$ denote the Lie algebra of G, which we regard as the tangent space at the identity element $e \in G$, i.e., $\mathcal{G} := T_eG$. Furthermore, let $\mathrm{Ad}_g : G \to G$ denote the map defined by $\mathrm{Ad}_g(x) = gxg^{-1} = L_gR_{g^{-1}}(x)$ for any $x \in G$ and for any $g \in G$. Here, L_h and R_h denote the multiplications on G by h from the left and from the right, respectively, for any $h \in G$.

https://doi.org/10.1017/S0013091517000475 Published online by Cambridge University Press

Use the same notation Ad_g to denote the isomorphism $\mathcal{G} \to \mathcal{G}$, which is the derivative of the map $\mathrm{Ad}_g : G \to G$ at $e \in G$. Then a connection A on P is, to begin with, a $\mathcal{G}$-valued 1-form on P which must have the following additional properties.

(1) A is equivariant in the sense that $A(dR_g X) = \mathrm{Ad}_{g^{-1}}(A(X))$ for any tangent vector X of P and for the right multiplication $R_g : P \to P$ by any $g \in G$.

(2) Let $p \in P$ and $\iota_p : G \to P$ be as in the introduction. Then for any $X \in \mathcal{G}$ we have that $A((d\iota_p)_e X) = X$.

The curvature $F_A \in \Omega^2(P; \mathcal{G})$ of a connection A is defined as the twisted derivative,

$$F_A = \mathrm{d}_A A.$$

In turn, the twisted derivative is defined as follows. The horizontal subbundle H of the tangent bundle TP determined by A is the bundle whose fibre H_p at p is given by $H_p = \mathrm{Ker}(A_p : T_pP \to \mathcal{G})$ for any $p \in P$. The vertical subbundle V of TP is defined by $V_p = T_pP_{\pi(p)} \subset T_pP$ for any $p \in P$. Then we have the decomposition $TP = V \oplus H$. Let $\pi_H : TP \to H$ be the associated projection. Now let W be any vector space over $\mathbb{R}$ and $\alpha \in \Omega^k(P; W)$ be any W-valued k-form. Then $\alpha^H \in \Omega^k(P; W)$ is defined as follows:

$$\alpha^H(X_1, \ldots, X_k) = \alpha(\pi_H X_1, \ldots, \pi_H X_k),$$

for any $X_i \in T_pP$, $i = 1, 2, \ldots, k$, and any $p \in P$. Now $\mathrm{d}_A\alpha$ is defined as $(\mathrm{d}\alpha)^H$. For this brief introduction to the principal bundle, the connection and its curvature, we referred mainly to [**1**, pp. 26–37] and [**2**, pp. 332–334].

A connection is flat if its curvature vanishes. A principal bundle is referred to as flat if it admits a flat connection. It is well known that a connection A is flat if and only if the associated horizontal distribution H is integrable (cf., [**6**, pp. 48–49]). In other words, A is flat if and only if for each $p \in P$ there is an integral submanifold of H containing p, which is mapped diffeomorphically onto an open subset of M by π. In fact, this version of flatness will be more useful for the later arguments.

3. Cohomology of flat principal bundles

Let $\pi : P \to M$ be a principal bundle whose structure group G is connected and compact. Let $A \in \Omega^1(P; \mathcal{G})$ be a connection on P. Then let H be the horizontal distribution determined by A and let V denote the vertical distribution of P as in the previous section. Now, we consider the projection onto the vertical distribution, $\pi_A : TP \to V$, given by the decomposition $TP = V \oplus H$.

On the other hand, let $\mathcal{H}^*(G)$ denote the set of all *bi-invariant* real-valued forms on G. That a form $\alpha \in \Omega^*(G; \mathbb{R})$ is bi-invariant means that it satisfies $L_g^*\alpha = R_g^*\alpha = \alpha$ for any $g \in G$, where L_g and R_g respectively denote the left and the right multiplication by g. It is well known that bi-invariant forms are closed, and we have that

$$\mathcal{H}^*(G) \equiv H^*_{\mathrm{dR}}(G) \tag{3.1}$$

under the assumption that G is connected and compact (cf., [**5**, 12.1]).

https://doi.org/10.1017/S0013091517000475 Published online by Cambridge University Press

For any $p \in P$ we have the map $\kappa_p : P_x \to G$, where $x = \pi(p)$, which is defined by $\kappa_p(pg) = g$ for any $g \in G$. Let $h \in G$. Then we have $\kappa_{ph} = L_{h^{-1}}\kappa_p$, which follows from the identities $\kappa_{ph}(pg) = \kappa_{ph}(ph(h^{-1}g)) = h^{-1}g = L_{h^{-1}}\kappa_p(pg)$. The equality $\kappa_{ph} = L_{h^{-1}}\kappa_p$ implies that, for any $\alpha \in \mathcal{H}^*(G)$, the pull-back $(\kappa_p)^*\alpha \in \Omega^*(P_x; \mathbb{R})$ does not depend on the choice of $p \in P_x$.

Now, we define for any $\alpha \in \mathcal{H}^k(G)$ a k-form $\hat{\alpha}$ on P by the rule

$$\hat{\alpha}(X_1, \dots X_k) = ((\kappa_p)^*\alpha)(\pi_A X_1, \dots, \pi_A X_k), \tag{3.2}$$

for any tangent vectors $X_1, \dots, X_k \in T_pP$ and for any $p \in P$.

In fact, we need to prove that $\hat{\alpha}$ is smooth, which is done as follows. Write $\pi^{-1}U = P_U$ and assume there is a local section $s : U \to P_U$. Then, by exploiting s, we may define $\psi : P_U \to U \times G$ as the inverse of the map $U \times G \to P_U$, $(x, g) \to s(x)g$. In fact, ψ is well known to be a local trivialization. On the other hand, let $\varphi : U \times G \to G$ denote the projection. We will use the identity

$$\varphi\psi = \kappa_{s(x)} \tag{3.3}$$

on P_x for any $x \in U$. This holds since we have $\varphi\psi(s(x)g) = g = \kappa_{s(x)}(s(x)g)$ for any $g \in G$. Therefore, we have that

$$\hat{\alpha}(X_1, \dots, X_k) = ((\kappa_{s(\pi(p))})^*\alpha)(\pi_A X_1, \dots, \pi_A X_k) = (\psi^*\varphi^*\alpha)(\pi_A X_1, \dots, \pi_A X_k) \tag{3.4}$$

for any tangent vectors $X_1, \dots, X_k \in T_pP$ and for any $p \in P_U$. Here, the first equality comes from the definition (3.2) together with the fact that $(\kappa_p)^*\alpha = (\kappa_{s(\pi(p))})^*\alpha$ as forms on $P_{\pi(p)}$, since $(\kappa_p)^*\alpha$ does not depend on the choice of p as observed in the above. The second equality of (3.4) comes from (3.3). Since $\pi_A : TP \to V$ is a smooth homomorphism between the smooth bundles covering the identity, (3.4) proves that $\hat{\alpha}$ is smooth. Thus, we have indeed

$$\hat{\alpha} \in \Omega^k(P; \mathbb{R}).$$

Now we write

$$E_A : \mathcal{H}^*(G) \to \Omega^*(P; \mathbb{R}) \tag{3.5}$$

for the map defined by $E_A(\alpha) = \hat{\alpha}$ for any $\alpha \in \mathcal{H}^*(G)$.

For the existence of the map E_A in (3.5), we did not require the flatness of A. As mentioned in the last lines of § 2, a connection A on P is flat if and only if there is a horizontal section $s : U \to P$ such that $\mathrm{d}s(TM|_U) \subset H_{P_U}$ for each member U of an open cover of M, where H is the horizontal distribution of P determined by A. Now we observe the following.

Lemma 3.1. *If A is flat, $E_A(\alpha) \in \Omega^*(P)$ is closed for any $\alpha \in \mathcal{H}^*(G)$.*

Proof. Let $U \subset M$ be an open set for which there is a horizontal section $s : U \to P$ with respect to A. Then we have the local trivialization $\psi : P_U \to U \times G$ determined by s as in the above. Also recall the projection $\varphi : U \times G \to G$. Then we write $\kappa = \varphi\psi : P_U \to G$.

https://doi.org/10.1017/S0013091517000475 Published online by Cambridge University Press

It will be also useful to note the equality

$$p = s(\pi(p))\kappa(p) \tag{3.6}$$

for any $p \in P_U$: note that $\psi(p) = (\pi(p), \kappa(p))$ by definition of κ. Now apply the inverse of ψ, which is given by $(x, g) \to s(x)g$, to both sides of the equation.

Now we will show that

$$\hat{\alpha}|_{P_U} = \kappa^*\alpha$$

which is enough since α is a closed form on G.

Assume that α is homogeneous and its degree is k. Then let $X_1, \ldots, X_k \in T_pP$ and $p \in P_U$. Write $x = \pi(p)$. Then we have by definition that

$$\hat{\alpha}(X_1, \ldots, X_k) = \alpha((\mathrm{d}\kappa_{s(x)})_p\pi_A X_1, \ldots, (\mathrm{d}\kappa_{s(x)})_p\pi_A X_k),$$

where we used the fact that $(\kappa_p)^*\alpha = (\kappa_{s(x)})^*\alpha$ on P_x, in other words, that $(\kappa_p)^*\alpha$ does not depend on the choice of $p \in P_x$. Therefore, the equality $\hat{\alpha}|_{P_U} = \kappa^*\alpha$ follows from the following assertion. □

Claim. *We have the identity*

$$(\mathrm{d}\kappa_{s(x)})_p\pi_A = (\mathrm{d}\kappa)_p : T_pP \to T_{\kappa(p)}G.$$

Proof. Consider a curve $s(\delta(t))\gamma(t)$ on P where δ and γ are some curves on U and on G, respectively, such that $\delta(0) = x$ and $\gamma(0) = \kappa_{s(x)}(p) = \kappa(p)$ (see (3.3)). Note that $s(\delta(0))\gamma(0) = s(x)\kappa_{s(x)}(p) = s(\pi(p))\kappa(p) = p$ by (3.6). Thus $s(\delta(t))\gamma(t)$ is a curve which passes through p at $t = 0$. Therefore, the velocity of the curve at $t = 0$ is a tangent vector of P at p, which we denote by X.

Recall the map $\iota_p : G \to P$ from the introduction. Also write $\kappa(p) = g$. Then the velocity vector X can be written as

$$(\mathrm{d}R_g)_{s(x)}(\mathrm{d}s)_x\dot{\delta}(0) + (\mathrm{d}\iota_{s(x)})_g\dot{\gamma}(0), \tag{3.7}$$

which is by itself a decomposition of X in accordance with that of the tangent space, $T_pP = H_p + V_p$. Note that we have exploited the fact that $s : U \to P$ is horizontal. We also used the invariance of H in the sense that $\mathrm{d}R_hH_q = H_{qh}$ for any $q \in P$ and for any $h \in G$, which holds for general connections.

Thus $\pi_A X$ is the velocity of the curve $s(x)\gamma(t)$ at $t = 0$. Since $\kappa_{s(x)}(s(x)\gamma(t)) = \gamma(t)$, we conclude that $(\mathrm{d}\kappa_{s(x)})_p\pi_A X$ is $\dot{\gamma}(0)$. On the other hand, we observe that $\kappa(s(\delta(t))\gamma(t))$ is $\gamma(t)$, which means $(\mathrm{d}\kappa)_pX = \dot{\gamma}(0)$. Thus both $(\mathrm{d}\kappa_{s(x)})_p\pi_A X$ and $(\mathrm{d}\kappa)_pX$ are $\dot{\gamma}(0)$. Furthermore, the expression (3.7) shows that X can be any vector in T_pP. Thus, the claim has been established. □

Remark. For other examples in which calculations such as (3.7) are used, see, for instance [**1**, pp. 32–33 and p. 38].

Thus, assuming A is flat, we have a map $H^*_{\mathrm{dR}}(G) \to H^*_{\mathrm{dR}}(P)$ induced by E_A, since we have the identity (3.1), $\mathcal{H}^*(G) \equiv H^*_{\mathrm{dR}}(G)$. We will continue using E_A to denote this map. It is clear that $E_A : H^*_{\mathrm{dR}}(G) \to H^*_{\mathrm{dR}}(P)$ respects the wedge product. Assume

https://doi.org/10.1017/S0013091517000475 Published online by Cambridge University Press

that $\alpha \in \mathcal{H}^k(G)$, $\beta \in \mathcal{H}^l(G)$ and $X_1, \ldots, X_k, X_{k+1}, \ldots, X_{k+l} \in T_pP$, $p \in P$. Then it is straightforward to see the equality

$$\widehat{\alpha \wedge \beta}(X_1, \ldots, X_{k+l}) = (\hat{\alpha} \wedge \hat{\beta})(X_1, \ldots X_{k+l}) \tag{3.8}$$

by the definition of wedge product and by that of $\hat{\gamma}$ for a bi-invariant form γ on G. Thus E_A is a homomorphism between algebras.

Now we consider again the inclusion $\iota_p : G \to P$ given by $\iota_p(g) = pg$ for any $g \in G$ and for any $p \in P$. Then we have that $(\iota_p)^*\hat{\alpha} = \alpha$ for any $p \in P$ and for any $\alpha \in \mathcal{H}^*(G)$. Assume again that α is homogeneous and of degree k. Let $X_1, \ldots, X_k \in T_gG$ for a $g \in G$. Then it is straightforward to see that

$$(\iota_p)^*\hat{\alpha}(X_1, \ldots, X_k) = \alpha(\mathrm{d}\kappa_p \pi_A \mathrm{d}\iota_p X_1, \ldots, \mathrm{d}\kappa_p \pi_A \mathrm{d}\iota_p X_k) = \alpha(X_1, \ldots, X_k) \tag{3.9}$$

using the facts that $\pi_A \mathrm{d}\iota_p X = \mathrm{d}\iota_p X$ for any $X \in T_gG$ and $\kappa_p \iota_p$ is the identity map on G when the codomain of ι_p is restricted to $P_{\pi(p)}$. Therefore, we have proved the following theorem.

Theorem 3.2. *Let $P \to M$ be a flat principal bundle with a compact connected structure group G. Then for any flat connection A on P there is a homomorphism between algebras*

$$E_A : H^*_{\mathrm{dR}}(G) \to H^*_{\mathrm{dR}}(P)$$

such that $(\iota_p)^*E_A$ is the identity on $H^*_{\mathrm{dR}}(G)$ for any $p \in P$.

The Maurer–Cartan 3-form Θ on a compact connected Lie group G is defined as follows. Choose a bi-invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathcal{G}$ of G. Let $[\cdot, \cdot]$ denote the Lie bracket on $\mathcal{G}$. Then Θ is defined by

$$\Theta(X, Y, Z) = \langle L_{g^{-1}}X, [L_{g^{-1}}Y, L_{g^{-1}}Z] \rangle$$

for any $X, Y, Z \in T_gG$ and for any $g \in G$. Here, the derivative of $L_h : T_{h^{-1}}G \to \mathcal{G}$ is also denoted by L_h, for any $h \in G$. Then it is straightforward to see that Θ a bi-invariant 3-form which represents a class in $H^3_{dR}(G)$. We have the following result.

Corollary 3.3. *Let $P \to M$ be a flat principal bundle with a compact connected structure group G, and A any flat connection on P, and let Θ denote the Maurer–Cartan 3-form of G associated with an invariant inner product. Then $E_A[\Theta] \in H^3_{\mathrm{dR}}(P)$ is the class such that $\iota_p^* E_A[\Theta] = [\Theta] \in H^3_{\mathrm{dR}}(G) \equiv H^3_{\mathrm{dR}}(P_x)$, where $x = \pi(p)$ and $p \in P$.*

In particular, the above justifies the principal bundle version of Wehrheim's statement [**12**, 1.2] under the additional condition that P is flat.

On the other hand, Theorem 3.2 means that a flat principal bundle with a compact connected structure group G is a fibre bundle to which the Leray–Hirsch theorem (cf., [**3**, 5.11] or [**9**, 4D.1]) applies. As a consequence of the theorem, we have an isomorphism of

https://doi.org/10.1017/S0013091517000475 Published online by Cambridge University Press

$H^*(M)$-modules

$$H^*_{\mathrm{dR}}(P) \cong H^*_{\mathrm{dR}}(M) \otimes H^*_{\mathrm{dR}}(G),$$

where the isomorphism $H^*_{\mathrm{dR}}(M) \otimes H^*_{\mathrm{dR}}(G) \to H^*_{\mathrm{dR}}(P)$. Moreover, $H^*_{\mathrm{dR}}(M) \otimes H^*_{\mathrm{dR}}(G)$ can be given a ring structure given on simple tensors by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{kl}(a_1 \wedge a_2) \otimes (b_1 \wedge b_2),$$

where b_1 and a_2 are homogeneous classes of degrees k and l, respectively. Then, since E_A is a homomorphism between algebras by the observation given by (3.8), we conclude the following.

Theorem 3.4. *Let $P \to M$ and G be as in (3.2) above. Then there is an isomorphism between algebras,*

$$H^*_{\mathrm{dR}}(P) \cong H^*_{\mathrm{dR}}(M) \otimes H^*_{\mathrm{dR}}(G).$$

Remarks. A principal G-bundle P over M is flat if and only if it is induced from the universal covering space $\tilde{M}$ by a homomorphism $\pi_1(M) \to G$ (see [**11**, Lemma 1]). For instance, assume G has a non-trivial discrete centre and $\pi_1(M)$ maps into the centre non-trivially, then there exists a non-trivial flat bundle $P \to M$. The theorem above says that, if G is furthermore connected and compact, $H^*_{\mathrm{dR}}(P) \cong H^*_{\mathrm{dR}}(M) \otimes H^*_{\mathrm{dR}}(G)$.

Non-trivial flat SO(3)-bundles P over a compact connected 3-manifold M were studied intensively in the works by Dostoglou and Salamon [**7**, **8**]. The theorem above also applies in this case.

4. Cohomology of the adjoint bundle

Let us begin with a principal bundle $\pi : P \to M$, not necessarily assuming flatness. Let G act on itself from the right by the rule: $x \cdot g = g^{-1}xg = \mathrm{Ad}_{g^{-1}}(x)$ for any $x, g \in G$. Then $P \times G$ has the diagonal right G-action given by $(p, g)h = (ph, \mathrm{Ad}_{h^{-1}}(g))$, and $\mathrm{Ad}\,P$ is defined as the orbit space. Write $[p, g] = (p, g)G$. Then, let $q : P \times G \to \mathrm{Ad}\,P$ denote the projection defined by $q(p, g) = [p, g]$. For the sake of simplicity, let π also denote the projection $\mathrm{Ad}\,P \to M$ which maps $[p, g]$ to $\pi(p)$. Then $\pi : \mathrm{Ad}\,P \to M$ is referred to as the adjoint bundle of P. Note that the fibre $(\mathrm{Ad}\,P)_x = \pi^{-1}\{x\}$ at $x \in M$ can be written as $\{[p, g] \mid \ g \in G\}$, choosing any $p \in P$ such that $\pi(p) = x$. We will write $(\mathrm{Ad}\,P)_U$ to denote $\pi^{-1}U$ for any subset U of M. The adjoint bundle is a bundle of Lie groups such that each fibre is a Lie group which is isomorphic to G. A gauge transformation $P \to P$ is equivalent to a section $M \to \mathrm{Ad}\,P$ (cf. [**10**]).

Now let A be any connection on P, and H be the horizontal distribution determined by A. Then we define a distribution $\bar{H}$ on $\mathrm{Ad}\,P$ as follows:

$$\bar{H}_{[p,g]} = (\mathrm{d}q)_{(p,g)}(H_p \oplus 0_g) \subset T_{[p,g]}(\mathrm{Ad}\,P)$$

for any $[p, g] \in \mathrm{Ad}\,P$. In fact, the invariance of H with respect to the action of G implies that $\bar{H}$ is well defined. Let $h \in G$ and consider $R_h : P \times G \to P \times G$ defined by $R_h(p,g) = (p,g)h = (ph, \mathrm{Ad}_{h^{-1}}(g))$. Then we have $q = qR_h$, and it follows that

$$(\mathrm{d}q)_{(p,g)}(H_p \oplus 0_g) = (\mathrm{d}q\,\mathrm{d}R_h)_{(p,g)}(H_p \oplus 0_g) = (\mathrm{d}q)_{(ph,h^{-1}gh)}(H_{ph} \oplus 0_{h^{-1}gh}).$$

We observe that $(\mathrm{d}q)_{(p,g)}$ is injective when restricted to $H_p \oplus 0_g$ as follows. Let $\bar{\pi} : P \times G \to M$ denote the map defined by $\bar{\pi}(p,g) = \pi(p)$. Then $(\mathrm{d}\bar{\pi})_{(p,g)}$ is clearly injective on $H_p \oplus 0_g$. Note that $\bar{\pi} = \pi q$, and therefore that $(\mathrm{d}\bar{\pi})_{(p,g)} = (\mathrm{d}\pi)_{[p,g]}(\mathrm{d}q)_{(p,g)}$. Thus $(\mathrm{d}q)_{(p,g)}$ is injective on $H_p \oplus 0_g$.

As in the previous section, we let $\bar{V}$ denote the 'vertical' distribution defined by

$$\bar{V}_{[p,g]} = T_{[p,g]}(\mathrm{Ad}\,P)_{\pi(p)} \subset T_{[p,g]}(\mathrm{Ad}\,P)$$

for any $[p,g] \in \mathrm{Ad}\,P$. We will show that the decomposition of the tangent vector bundle

$$T(\mathrm{Ad}\,P) = \bar{V} \oplus \bar{H} \tag{4.1}$$

indeed makes sense as follows. It suffices to show that $\bar{V}_{[p,g]} \cap \bar{H}_{[p,g]} = 0$ for any $[p,g] \in \mathrm{Ad}\,P$. Let $X \in \bar{H}_{[p,g]}$. Then $X = (\mathrm{d}q)_{(p,g)}(X', 0_g)$ for some $X' \in H_p$. We have that $(\mathrm{d}\pi)_{[p,g]}(X) = (\mathrm{d}\pi)_{[p,g]}(\mathrm{d}q)_{(p,g)}(X', 0_g) = (\mathrm{d}\bar{\pi})_{(p,g)}(X', 0_g) = 0$ if and only if $X' = 0$. Thus $\mathrm{d}\pi(X) = 0$ if and only if $X = 0$. This shows that $\bar{V}_{[p,g]} \cap \bar{H}_{[p,g]} = 0$.

Therefore, there is the projection $\pi_A : T(\mathrm{Ad}\,P) \to \bar{V}$ coming from the decomposition (4.1), where we use the same notation as in the previous section.

Choose any $p \in P$. Recall the map $\kappa_p : P_{\pi(p)} \to G$ from the previous section. This time we let κ_p denote the isomorphism $(\mathrm{Ad}\,P)_{\pi(p)} \to G$ given by $\kappa_p[p,g] = g$ for any $g \in G$, allowing ourselves an obvious abuse of notation. Then, if $h \in G$, we have

$$\kappa_{ph}[p,g] = \kappa_{ph}[ph, \mathrm{Ad}_{h^{-1}}(g)] = \mathrm{Ad}_{h^{-1}}(\kappa_p[p,g])$$

for any $g \in G$. That is, we have $\kappa_{ph} = \mathrm{Ad}_{h^{-1}}\kappa_p = L_{h^{-1}}R_h\kappa_p$. Therefore, if we fix an $x \in M$, for any $\alpha \in \mathcal{H}^*(G)$, $(\kappa_p)^*\alpha \in \Omega^*((\mathrm{Ad}\,P)_x; \mathbb{R})$ does not depend on the choice of $p \in P_x$. Now we define $\hat{\alpha}$, a form on $\mathrm{Ad}\,P$ whose smoothness is yet to be verified, for any $\alpha \in \mathcal{H}^k(G)$ by the same formula as (3.2), using the projection $\pi_A : T(\mathrm{Ad}\,P) \to \bar{V}$ given by (4.1) above.

The smoothness of $\hat{\alpha}$ follows from similar computations to (3.3) and (3.4).

Therefore, we may again write

$$E_A : \mathcal{H}^*(G) \to \Omega^*(\mathrm{Ad}\,P; \mathbb{R})$$

for the map defined by $E_A(\alpha) = \hat{\alpha}$ for any $\alpha \in \mathcal{H}^*(G)$.

We also have the following result, by a proof similar to that of Lemma 3.1.

Lemma 4.1. *If A is flat, $E_A(\alpha) \in \Omega^*(\mathrm{Ad}\,P; \mathbb{R})$ is closed for any $\alpha \in \mathcal{H}^*(G)$.*

The identity (3.8) is valid in the current case, as well to show that E_A is an algebra homomorphism. Also note that now $\iota_p : G \to \mathrm{Ad}\,P$ can be defined for any $p \in P$ by $\iota_p(g) = [p,g]$ for any $g \in G$. Note that $\kappa_p\iota_p$ is the identity on G when the codomain of ι_p is restricted to $(\mathrm{Ad}\,P)_{\pi(p)}$. Therefore, we have that $(\iota_p)^*\hat{\alpha} = \alpha$ for any $\alpha \in \mathcal{H}^*(G)$ by the same observation as (3.9). Again recall the identification (3.1), $\mathcal{H}^*(G) \equiv H^*_{\mathrm{dR}}(G)$. Then we have the following by an argument similar to the proof of Theorem 3.2 above.

https://doi.org/10.1017/S0013091517000475 Published online by Cambridge University Press

Theorem 4.2. *Let $P \to M$, G and A be as in Theorem 3.2. Then there is the homomorphism between algebras*

$$E_A : H^*_{\mathrm{dR}}(G) \to H^*_{\mathrm{dR}}(\mathrm{Ad}\, P)$$

such that $(\iota_p)^* E_A$ is the identity on $H^*_{\mathrm{dR}}(G)$ for any $p \in P$.

In particular, we have a result similar to (3.3), as follows.

Corollary 4.3. *Let $P \to M$ be a flat principal bundle with a compact connected structure group G, and A any flat connection on P. Let Θ denote the Maurer–Cartan 3-form of G associated with an invariant inner product. Then $E_A[\Theta] \in H^3_{dR}(P)$ is the class such that $\iota_p^* E_A[\Theta] = [\Theta] \in H^3_{\mathrm{dR}}(G) \equiv H^3_{\mathrm{dR}}(\mathrm{Ad}\, P_x)$, where $x = \pi(p)$ and $p \in P$.*

This result justifies the statement by Wehrheim (see [**12**, (1.2)] and the paragraph below it) under the additional conditions that G is connected and compact, and P is flat. By the same argument which led to (3.4) above, we conclude as follows.

Theorem 4.4. *Let $P \to M$ and G be as in Theorem 3.2 above. Then there is an isomorphism between algebras,*

$$H^*_{\mathrm{dR}}(\mathrm{Ad}\, P) \cong H^*_{\mathrm{dR}}(M) \otimes H^*_{\mathrm{dR}}(G).$$

References

1. D. Bleecker, *Gauge theory and variational principles* (Addison–Wesley, 1981).
2. B. Booss and D. Bleecker, *Topology and analysis: the Atiyah–Singer index formula and gauge-theoretic physics* (Springer–Verlag, New York, 1985).
3. R. Bott and L. W. Tu, *Differential forms in algebraic topology* (Springer–Verlag, New York, 1982).
4. Y. Byun and J. Kim, Cohomology of flat bundles and a Chern-Simons functional, *J. Geom. Phys.* **111** (2017), 82–93.
5. C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, *Trans. Amer. Math. Soc.* **63** (1948), 85–124.
6. S. Donaldson and P. Kronheimer, *The geometry of four-manifolds* (Oxford University Press, New York, 1990).
7. S. Dostoglou and D. Salamon, Instanton homology and symplectic fixed points, in *Symplectic geometry* (ed. D. Salamon), LMS Lecture Note Series, Volume 192, pp. 57–93 (Cambridge University Press, 1993).
8. S. Dostoglou and D. Salamon, Self-dual instantons and holomorphic curves, *Ann. Math.* **139** (1994), 581–640.
9. A. Hatcher, *Algebraic topology* (Cambridge University Press, 2002).
10. K. Mackenzie, Classification of principal bundles and Lie groupoids with prescribed gauge group bundle, *J. Pure Appl. Algebra* **58** (1989), 181–208.
11. J. Milnor, On the existence of a connection with curvature zero, *Comment. Math. Helv.* **32** (1957), 215–223.
12. K. Wehrheim, Energy identity for anti-self-dual instantons on $\mathbb{C} \times \Sigma$, *Math. Res. Lett.* **13**(1) (2006), 161–166.

https://doi.org/10.1017/S0013091517000475 Published online by Cambridge University Press