EXISTENCE AND UNIQUENESS OF EQUILIBRIUM
IN A REINSURANCE SYNDICATE

BY

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ABSTRACT

In this paper we consider a reinsurance syndicate, assuming that Pareto optimal allocations exist. Under a continuity assumption on preferences, we show that a competitive equilibrium exists and is unique. Our conditions allow for risks that are not bounded, and we show that the most standard models satisfy our set of sufficient conditions, which are thus not restrictive. Our approach is to transform the analysis from an infinite dimensional to a finite dimensional setting.

KEYWORDS

Existence of equilibrium, uniqueness of equilibrium, Pareto optimality, reinsurance model, syndicate theory, risk tolerance, exchange economy, probability distributions, Walras' law.

1. INTRODUCTION

We consider the reinsurance syndicate introduced by Borch (1960-62), a model closely related to the exchange economy studied by Arrow (1953). Bühlmann (1984) shows that, provided that there are Pareto optimal risk exchanges, an equilibrium exists for bounded risks. While this result may be of interest for practical purposes (since the accumulated wealth in the World is obviously bounded), in a modeling context this precludes many probability distributions that are of interest, but which may just happen to have unbounded supports.

Bühlmann's arguments are limited to affine contracts, but we shall extend to arbitrary contracts in this paper. We basically swap his assumption of bounded risks and a Lipschitz condition with a continuity requirement on preferences. The latter we demonstrate is satisfied for the most common exchange economies studied within the "finance context". Under this condition we demonstrate both existence and uniqueness of equilibrium.

When Pareto optimal risk exchanges exist in finite dimensional models, there will be competitive equilibria after a redistribution of the initial endowments

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X_i, $i \in \mathcal{I} := \{1, 2, \cdots, I\}$, here a set of random variables referred to as the initial portfolio allocation of the I members of the reinsurance syndicate (by the Second Welfare Theorem). As it turns out, this is true also in our infinite dimensional setting. However, our focus is elsewhere.

We provide a set of sufficient conditions for the existence of an equilibrium for a *given* set of initial portfolios $X = (X_1, X_2, \cdots, X_I)$, provided Pareto optimal risk exchanges exist. Since the set of sufficient conditions for a Pareto optimal exchange to exist are indeed very weak for the model that we consider (see e.g., DuMouchel (1968)), our approach is not restrictive for this reason. In fact, if there are no Pareto optimal contracts, there can not be a competitive equilibrium either, by the First Welfare Theorem.

Existence of equilibrium in infinite dimensional models is, of course, extensively studied in the mathematical economics literature, and there are many possible approaches. Broadly speaking, the strategy that we take is to search in the Pareto frontier of the set of attainable utilities. This is an approach pioneered by Negishi (1960) and Arrow and Hahn (1971) in the finite dimensional case, and by Bewley (1969), Magill (1981) and Mas-Colell (1986) in the infinite dimensional setting. Bewley (1972) is another early reference to existence in infinite-dimensional spaces, and later this topic has been extensively investigated by many authors, including Mas-Colell and Zame (1991), Araujo and Monteiro (1989), and Dana (1993) among others. Uniqueness of equilibrium is a lesser explored subject in infinite dimensional spaces.

Our approach will be based to a large extent on "risk theory", which requires us to first define what is meant by a reinsurance syndicate. One distinguishing feature of our approach, from the above cited literature on exchange economies, is that we do not restrict "portfolios" to be non-negative. The connection to risk theory essentially enables us to transform the existence problem from the infinite dimensional space of L^2, to the finite dimensional Euclidian space R^I.

In Section 2 we present some of the basic properties of such a market. In Section 3 we discuss existence of equilibrium, and give the basic existence theorem of the paper. Our exposition relies mainly on the results of Section 2, and a fixed point theorem. Here one can also find several examples, and we finally prove uniqueness of equilibrium. Section 4 compares our result to a corresponding theorem emerging from a more general theory of an exchange economy, and Section 5 concludes.

2. The reinsurance Syndicate

Consider a one-period model of a syndicated market with two time points, zero and one. The initial portfolio allocation of the members is denoted by $X = (X_1, X_2, \cdots, X_I)$, i.e., the one whose realizations would result at time one if no reinsurance exchanges took place. At time zero X is a random vector defined on a probability space $(\Omega, \mathcal{F}, P)$. After reinsurance at time zero the random vector $Y = (Y_1, Y_2, \cdots, Y_I)$ — the final portfolio allocation — results, satisfying $\Sigma_{i \in \mathcal{I}} Y_i = \Sigma_{i \in \mathcal{I}} X_i$, since nothing "disappears" or is added in a pure exchange of risks. Each risk is supposed to be member of a space L, which can be finite or infinite dimensional. Typically we will consider L to be an $L^p(\Omega, \mathcal{F}, P)$-space in this paper, $1 \leq p \leq \infty$, with special emphasis on L^2.

One difference between a syndicate and the general exchange economy of Arrow (1953) is that the variables X_i signify economic gains or losses measured in some unit of account, not consumption as in the exchange economy, which implies that negative values are allowed in the syndicate interpretation. For example may we interpret $X_i = w_i - V_i$. i.e., the portfolios are assets less liabilities, and when $V_i(\omega) > w_i$ for some state of nature $\omega \in \Omega$, a negative value for the portfolio results. When this materializes for some member, this person may be interpreted to be bankrupt (but not dead).

Let u_i be be the utility index of member i, all assumed strictly increasing and concave, let $\pi : L \rightarrow R$ be a price functional and consider the problem of each member of the syndicate

$$\sup_{Z_i \in L^2} Eu_i(Z_i) \quad \text{subject to} \quad \pi(Z_i) \leq \pi(X_i), \tag{1}$$

for $i \in \mathcal{I}$; the members maximize expected utility subject to their budget constraints. By a price functional we shall always mean a linear functional, which is defined and finite for each $Z \in L$, and which is also continuous (in the topology of L). This definition demands some further comments, which we will return to shortly. Here we just remark that the requirement that π be defined and finite for each risk in L is certainly desirable, but also a strong one, which we shall see later.

Let us call a treaty Y *feasible* if it satisfies $\Sigma_{i \in \mathcal{I}} Y_i \leq \Sigma_{i \in \mathcal{I}} X_i := X_M$, where by X_M we mean the "market portfolio", which is just the aggregate of the initial portfolios of the members. Our definition of equilibrium is:

Definition 1. *A competitive equilibrium is a collection* $(\pi; Y_1, Y_2, \ldots, Y_I)$ *consisting of a price functional* π *and a feasible allocation* $Y = (Y_1, Y_2, \ldots, Y_I)$ *such that for each i,* Y_i *solves the problem (1).*

Because of strict monotonicity of the utility functions, the constraint in (1) is binding in equilibrium. It is still convenient to formulate the problem with an inequality constraint, since this facilitates the use of Kuhn-Tucker and directional derivatives, and gives us positive signs on the associated Lagrange multipliers.

An important feature of this syndicate is that there are no restrictions on contract formation. As a consequence it can be shown that *the pricing functional* π *must be linear and strictly positive if and only if there does not exist any arbitrage possibilities* (e.g., Aase (2002)).

Recall that an arbitrage possibility simply refers to the existence of some financial instrument in the market that provides a strictly positive amount to its holder with strictly positive probability, at time zero or time one, with no commitments for the holder to make any payments at any time. It is hardly surprising that this possibility can not be allowed in a simple model of a reinsurance market, with rational members. In this part of the paper we shall restrict attention to initial portfolios X_i, all in L^2, and sharing rules Y_i that involve no arbitrage.

Since any (strictly) positive, linear functional on L^2 is also continuous, by the Riesz Representation Theorem there exists a unique strictly positive random variable $\xi \in L^2_+$ (the positive cone of L^2) such that

$$\pi(Z) = E(Z\xi) \quad \text{for all} \quad Z \in L^2.$$

Notice that the system is closed by assuming *rational expectations*. This means that the market clearing price π implied by the members' behavior is assumed to be the same as the price functional π on which the members' decisions are based.

Formally our definition of (strong) Pareto optimality is the following

Definition 2. *A feasible allocation $Y = (Y_1, Y_2, \dots, Y_I)$ is called Pareto optimal if there is no feasible allocation $Z = (Z_1, Z_2, \dots, Z_I)$ with $Eu_i(Z_i) \geq Eu_i(Y_i)$ for all i and with $Eu_j(Z_j) > Eu_j(Y_j)$ for some j.*

The following characterization of Pareto optimal allocations is well known:

Proposition 1. *Assume that each u_i is increasing and concave. Then Y is a Pareto optimal allocation if and only if there exists a nonzero vector of member weights $\lambda \in R^I_+$ such that $Y = (Y_1, Y_2, \dots, Y_I)$ solves the problem*

$$\sup_{(Z_1, \dots, Z_I)} \sum_{i=1}^{I} \lambda_i Eu_i(Z_i) \quad \textit{subject to} \quad \sum_{i=1}^{I} Z_i \leq X_M. \tag{2}$$

If the allocation Y is Pareto optimal, then the problem (2) defines a utility function $u_\lambda(\cdot) : R \to R$ for this λ, such that

$$Eu_\lambda(X_M) = \sum_{i=1}^{I} \lambda_i Eu_i(Y_i). \tag{3}$$

Notice that the existence of the member weights λ is a consequence of the Separating Hyperplane Theorem applied to Euclidian R^I. As it turns out, these member weights determine state prices via the marginal utility $u'_\lambda(X_M)$ of the representative member computed at the aggregate portfolio X_M. Thus, despite of the unfortunate fact that the *interior* of L^2_+ is empty, there is still hope to get price supportability of preferred sets via the construction in Proposition 1. (By "price supportability of preferred sets" we mean, loosely speaking, that a price functional can be constructed that is a serious candidate for an equilibrium price.)

Since the utility functions are all strictly increasing, at the Pareto optimum the constraint in (2) is binding.

Pareto optimal allocations can be further characterized under the above conditions, the following is known as Borch's Theorem (see e.g., Borch (1960-62)):

Proposition 2. *An allocation Y is Pareto optimal if and only if there exist positive member weights $\lambda_1, \lambda_2, \ldots, \lambda_I$ and a real function $u'_\lambda(\cdot) : R \to R$, such that*

$$\lambda_1 u'_1(Y_1) = \lambda_2 u'_2(Y_2) = \cdots = \lambda_I u'_I(Y_I) := u'_\lambda(X_M) \quad a.s. \tag{4}$$

Proposition 2 can be proven from Proposition 1 by the Kuhn-Tucker Theorem and a variational argument (see e.g., Aase (2002)). Karl Borch's characterization of a Pareto optimum $Y = (Y_1, Y_2, \ldots, Y_I)$ simply says that there exist positive constants λ_i such that the marginal utilities at Y of all the members are equal modulo these constants. Condition (4) can be rewritten

$$\frac{u'_i(Y_i(\omega))}{u'_i(Y_i(\omega'))} = \frac{u'_j(Y_j(\omega))}{u'_j(Y_j(\omega'))} \tag{5}$$

for all pairs (i, j) and (ω, ω'). The two sides of the above equality correspond to the marginal rates of substitution in consumption between states ω and ω' respectively, for members i and j. Condition (5) is the classical "efficiency condition" stating that marginal rates of substitution must be equalized across agents. Graphically it means that indifference curves (in a two agent two state world) must be tangent at an "efficient" (i.e., Pareto optimal) allocation.

It also follows from (3) and (4) that the Pareto optimal allocations $Y_i(\omega)$ depend on the state ω only through the aggregate market portfolio$X_M(\omega)$, so we may write $Y_i(\omega) = g_i(X_M(\omega))$ for some real function $g_i : R \to R$. In other words, whenever two states of the world ω and ω' have the same level of aggregate wealth $X_M(\omega') = X_M(\omega)$, then for each member i wealth in state ω must be the same as wealth in state ω': $Y_i(\omega) = Y_i(\omega')$ for $i \in \mathcal{I}$. This is called the *mutuality principle*. For simplicity we shall use the notation $Y_i(\omega) = Y_i(X_M(\omega))$ for all i. (Notice that the symbol $Y_i(\cdot)$ have two distinct meanings, one as a random variable $Y_i : \Omega \to R$, the other as a real fuction $Y_i : R \to R$; the context will reveal which one we mean).

Let now x denote the generic value of X_M. Because of the smoothness assumptions of Proposition 1, which we maintain in this paper, both sides of each of the equations in (4) are real, differentiable functions (the right-hand side because of the implicit function theorem), i.e., the functions $Y_i(\cdot) : B \to R$ and $u'_\lambda(\cdot) : B \to R$, for some subset $B \subseteq R$ of the reals, are all differentiable. Thus taking derivatives gives

$$u''_i(Y_i(x))\, Y'_i(x) = \lambda_i^{-1} u''_\lambda(x), \quad x \in B \subseteq R.$$

Dividing this equation by the first order condition $u'(Y_i(x)) = \lambda_i^{-1} u_\lambda'(x)$, we obtain the following non-linear differential equation for the Pareto optimal allocation function $Y_i(x)$:

$$\frac{dY_i(x)}{dx} = \frac{R_\lambda(x)}{R_i(Y_i(x))}, \quad Y_i(x_0) = b_i, \quad x, x_0 \in B, \tag{6}$$

where the constants b_i represent the relevant boundary conditions, which we return to later, $R_\lambda(x) = -\frac{u_\lambda''(x)}{u_\lambda'(x)}$ is the *absolute risk aversion function* of "the representative member", and $R_i(Y_i(x)) = -\frac{u_i''(Y_i(x))}{u_i'(Y_i(x))}$ is the *absolute risk aversion* of member i in a Pareto optimum $Y_i(x)$, $i \in \mathcal{I}$.

Furthermore, we use the notation $\rho_i(Y_i(x)) = \frac{1}{R_i(Y_i(x))}$ to signify the *absolute risk tolerances* of agent i, $i \in \mathcal{I}$, in any Pareto optimum, and $\rho_\lambda(x) = \frac{1}{R_\lambda(x)}$ is the *absolute risk tolerance function* of the representative member.

Since $\Sigma_{i \in \mathcal{I}} Y_i'(x) = 1$, we get by summation in (6) that

$$\rho_\lambda(x) = \sum_{i \in \mathcal{I}} \rho_i(Y_i(x)), \; x \in B,$$

or

$$\rho_\lambda(X_M) = \sum_{i \in \mathcal{I}} \rho_i(Y_i(X_M)) \quad a.s. \tag{7}$$

as an equality between random variables. Using the definitions of risk tolerance, we may rewrite the differential equations (6) as follows

$$\frac{dY_i(x)}{dx} = \frac{\rho_i(Y_i(x))}{\rho_\lambda(x)}, \quad Y_i(x_0) = b_i, \quad x, x_0 \in B. \tag{8}$$

In other words, provided Pareto optimal sharing rules exist, we have the following results, which we shall utilize later:

Proposition 3. *(a) The risk tolerance of the syndicate $\rho_\lambda(X_M)$ equals the sum of the risk tolerances of the individual members in a Pareto optimum.*

(b) The real, Pareto optimal allocation functions $Y_i(x) : R \to R$, $i \in \mathcal{I}$, satisfy the first order, ordinary nonlinear differential equations (8).

(c) The following relationships hold

$$\frac{\partial}{\partial \lambda_i} u_\lambda'(x) = \frac{1}{\lambda_i} \frac{dY_i(x)}{dx} u_\lambda'(x), \quad x \in B, \quad i \in \mathcal{I}. \tag{9}$$

The result in (a) was found by Borch (1985); see also Bühlmann (1980) for the special case of exponential utility functions, and also Gerber (1978), among others. The result in (c) is contained in Theorem 10 p. 130 in Wilson (1968).

It is well-known that if an equilibrium exists, then the first order necessary and sufficient conditions are given by the equations (4). If this is the case, then the Riesz representation ξ, also called the state price deflator, is given by $\xi = u'_\lambda(X_M)$ a.s. This is our next result:

Assume that $\pi(X_i) > 0$ for each i. It seems reasonable that each member of the syndicate is required to bring to the market an initial portfolio of positive value. In this case we have the following (a proof can be found in Aase (2002)):

Theorem 1. *Suppose that $u'_i > 0$ and $u''_i \leq 0$ for all $i \in \mathcal{I}$, and assume that a competitive equilibrium exists, where $\pi(X_i) > 0$ for each i. The equilibrium is then characterized by the existence of positive constants α_i, $i \in \mathcal{I}$, such that for the equilibrium allocation $Y = (Y_1, Y_2, \ldots, Y_I)$*

$$u'_i(Y_i) = \alpha_i u'_\lambda(X_M), \quad a.s. \quad for\ all \quad i \in \mathcal{I}. \tag{10}$$

Here α_i are the Lagrange multipliers associated with the problem (1), and the relation between these and the member weights λ_i is seen to be $\alpha_i = \lambda_i^{-1}$ for all $i \in \mathcal{I}$.

3. Existence and Uniqueness of Equilibrium

Will there always exist prices such that the budget constraints all hold with equality? We will now analyze this question for the reinsurance syndicate just described. The results that we shall prove in this section are:

Theorem 2. *Suppose $u'_i > 0$, $u''_i \leq 0$, and u'''_i are continuous for all i, and $E\{(u'_\lambda(X_M))^2\} < \infty$. Then a unique equilibrium exists.*

We present a proof of the existence part in Section 3.1, and a proof of the uniqueness part in Section 3.2.

The problem of existence of equilibrium in an infinite dimensional setting has been extensively discussed in the literature. Several difficulties have been identified, among them that the interior of the orthant L^2_+ is empty, so calculus becomes rather difficult. Normally the Separating Hyperplane Theorem guarantees that it will be possible to separate a convex set C from a point $x \notin C$, provided that the interior of C is not empty. Hence, if consumption sets have non-empty interior, then the continuity and convexity of preferences will guarantee that preferred sets can be price supported.

As commented after Proposition 1, despite this difficulty we obtain the member weights by a separation argument, which provides us with state prices via the representative member's marginal utility at X_M. It should thus be possible to use this construction to show existence of equilibrium. As it turns out, from Theorem 2 we see that it is sufficient to make an extra smoothness assumption on preferences. In this section we make this precise by utilizing the results of the previous section to essentially transform the problem from an infinite dimensional to a finite dimensional setting.

To this end we start with the initial portfolios X_i, which are supposed to satisfy $X_i \in L^2$, $i \in \mathcal{I}$. The final portfolios Y_i and the state price deflator ξ are supposed to be in L^2 and L^2_+ respectively, according to this theory, the latter because L^2 is its own dual space, where the positive cone stems from the absence of arbitrage. In fact, ξ is strictly positive a.s. However, both the probability distribution of X and the utility functions are exogenously given, and it is not clear at the outset that any particular choice of these, satisfying $X_i \in L^2$, will have these properties. From (7) and (8) of the previous section, it follows that $|Y_i - b_i| \le |X_M|$ for all i, so if $X_M \in L^2$, then $Y_i \in L^2$ for all $i \in \mathcal{I}$. However it is far from clear that $\xi = u_\lambda'(X_M) \in L^2_+$, which this theory requires to be consistent. That is, will there exist positive state prices $\xi = u_\lambda'(X_M)$ having finite variances such that the budget constraints in (1) are all satisfied with equality? These are the problems we now address.

First we notice a few facts about the existence problem. Since the state prices $u_\lambda'(X_M)$ are determined by the member weights λ, and the budget sets remain unchanged if we multiply all these weights by any positive constant, each member's optimal portfolio $Y_i(X_M) := Y_i^{(\lambda)}$ is accordingly homogeneous of degree zero in λ. Hence we can restrict attention to member weights belonging to the $(I-1)$ dimensional unit simplex

$$S^{I-1} = \left\{ \lambda \in R^I_+ \, : \, \sum_{i=1}^{I} \lambda_i = 1 \right\}.$$

Returning to the question posed at the beginning of this section, recalling that we consider a pure exchange economy with strictly increasing utility functions, an equilibrium will exist if there exists some $\lambda \in S^{I-1}$ such that

$$E\Big(u_\lambda'(X_M)(Y_i^{(\lambda)} - X_i)\Big) = 0, \quad \text{for } i = 1, 2, \cdots, I, \tag{11}$$

where we have chosen to parameterize the optimal allocations $Y_i(X_M)$ by the member weights λ. Equations (11) are just a restatement of the budget constraints in (1) with equality, recalling that for any risk Z, its market price $\pi(Z) = E(Z\xi)$, where $\xi = u_\lambda'(X_M)$ a.s. The existence problem may be resolved if one can identify these budget constraints with a continuous function $f: S^{I-1} \to S^{I-1}$ and then employ Brower's fixed-point theorem.

We find it more convenient to use the constants b_i to span the Pareto optimal frontier instead of the constants λ_i. That this is equivalent follows from the differential equations (6), or equivalently (8), since the solutions to these equations are unique once the boundary conditions have been specified. The idea is perhaps best illustrated by a few examples: In the first one the utility functions are negative exponentials.

EXAMPLE 1: Suppose $u_i(x) = 1 - e^{-\frac{x}{\rho_i}}$, $i \in \mathcal{I}$. It is a consequence of Proposition 2 that the Pareto optimal allocations are affine in the aggregate wealth X_M, i.e.,

$$Y_i^\lambda := Y_i(X_M) = \frac{\rho_i}{\rho} X_M + b_i,$$

where the constants ρ_i are the risk tolerances of the members, $\rho = \sum_{i \in \mathcal{I}} \rho_i$ by the result (7), so that ρ is the risk tolerance of the representative member, or the syndicate, and b_i are zero-sum side-payments, corresponding to $Y_i(x_0) = b_i$ for $x_0 = 0$.

By imposing the normalization $E(u_\lambda'(X_M)) = 1$ (corresponding to a zero risk-free interest rate), the budget constraints of the members correspond to the equations

$$\lambda_i = \frac{e^{\frac{b_i}{\rho_i}}}{E\left\{e^{-\frac{X_M}{\rho}}\right\}}, \quad i \in \mathcal{I}, \tag{12}$$

where the zero-sum side-payments b_i are given by

$$b_i = \frac{E\left\{X_i e^{-X_M/\rho} - \frac{\rho_i}{\rho} X_M e^{-X_M/\rho}\right\}}{E\left\{e^{-X_M/\rho}\right\}}, \quad i \in \mathcal{I}. \tag{13}$$

Since there is a one to one connection between the member weights λ_i and the side-payments b_i, the latter could alternatively be used in the fixed-point argument. The state price deflator $\xi = u_\lambda'(X_M) = ce^{-X_M/\rho}$ for some constant c depending on the ρ_i and the weights λ_i, or equivalently on the ρ_i and the side-payments b_i. □

The second example is that of constant relative risk aversion:

EXAMPLE 2: Preferences represented by power utility means that $u_i(x) = (x^{1-a_i} - 1)/(1 - a_i)$ for $x > 0$, for $a_i \neq 1$ and $u_i(x) = \ln(c_i x + d_i)$ for $c_i x + d_i > 0$ when $a_i = 1$, for some constants c_i and d_i, where the natural logarithm results as a limit when $a_i \to 1$. This example only makes sense in the no-bankruptcy case where $X_i > 0$ a.s. for all i when $a_i \neq 1$.

Let us assume that the supports of the initial portfolios are $(0, \infty)$, and $Y_i(x_0) = b_i$ for some $x_0 > 0$. The parameters $a_i > 0$ are the *relative risk aversions* of the members, here given by positive constants, and we consider the HARA-case where $a_1 = a_2 = \ldots = a_I = a$.

The marginal utilities of the members are given by $u_i'(x) = x^{-a}$, and the Pareto optimal allocations Y_i^λ are found from Proposition 2 to be

$$Y_i(X_M) = \frac{\lambda_i^{1/a}}{\sum_{j \in \mathcal{I}} \lambda_j^{1/a}} X_M, \quad i \in \mathcal{I}. \tag{14}$$

The differential equations (6) for these allocations are

$$\frac{dY_i(x)}{Y_i(x)} = \frac{dx}{x}, \quad Y_i(x_0) = b_i \quad i \in \mathcal{I}, \tag{15}$$

showing that $Y_i(X_M) = \frac{b_i}{x_0} X_M$, where b_i is member i's share of the market portfolio when the latter takes on the value x_0, where $\Sigma_{j \in \mathcal{I}} b_i = x_0$.

Comparing the two versions of the Pareto optimal allocations, we notice that $\frac{b_i}{x_0} = \frac{\lambda_i^{1/a}}{\Sigma_{j \in \mathcal{I}} \lambda_j^{1/a}}$, again giving a one to one correspondence between the boundary conditions b_i of the differential equations (6) and the member weights λ_i. The latter are determined by the budget constraints, implying that

$$\lambda_i = k \left(\frac{E\left(X_i X_M^{-a}\right)}{E\left(X_M^{1-a}\right)} \right)^a, \quad i \in \mathcal{I}, \tag{16}$$

or, λ_i is determined modulo the proportionality constant $k = (\Sigma_{j \in \mathcal{I}} \lambda_j^{1/a})^a$ for each i. Here the state price deflator can be seen to be of the form $\xi = u'_\lambda(X_M) = cX_M^{-a}$ for some constant c depending on a and the weights λ_i, or equivalently on a and the constants b_i. □

For both these examples we have computed the respective equilibria, where it is understood that the expectations, appearing in the expressions for the member weights, exist. This must accordingly follow from any set of sufficient conditions for existence of equilibrium. The reason why the existence of the λ_i, or, equivalently the b_i, is not automatic, is that both the probability distribution of X and the utility functions are given exogenously, as explained in the introduction. Although it is clear that if $X_M \in L^2$, then also $Y_i \in L^2$, it is still not obvious that $\xi = u'_\lambda(X_M)$ is in L^2_+. This has to be checked separately.

While it is a celebrated fact that the first order conditions for an optimal exchange of risks do not depend on the probability distribution of the vector X of the initial endowments[1], clearly the equilibrium allocation $Y^{(\lambda)}$ does depend on this distribution through the budget constraints, and only if this probability distribution allows for the computation of the moments appearing in the expressions for the member weights λ_i, as e.g., in (12) and (16), the relevant equilibrium will stand a chance to exist.

As another example, recall the application of this theory to the risk exchange problem between an insurer and an insurance customer. Here the

[1] They depend on the support of the probability distribution.

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premium represented by a parameter p takes on the role of the weight λ in generating the Pareto efficient frontier.

As these examples illustrate, instead of focusing attention on the member weights λ_i, we might as well consider the constants b_i of the differential equations (6), and try to associate with the budget constraints a fixed-point for these. As noticed above, this observation turns out to be quite general, and is the line of attack we choose to follow.

A natural condition to impose for the constants b_i to exist, might be that all the risks are bounded. Often this is too strong. For example if X is multinormally distributed, and thus possesses unbounded supports, certainly the moments in (13) can still be computed, and are well defined (Aase (2009)). This is also the case for many other distributions with unbounded supports.

However, even in the case with bounded supports it is not clear that the pricing functional π is continuous. To see this, consider Example 2 with $B = (0, 1]$. Here the state prices are represented by the function $u'_\lambda(X_M) = cX_M^{-a}$ for some constant c depending on the member weights λ and a. Suppose that X_M is uniformly distributed on $(0, 1)$. Then all the initial portfolios have bounded supports, but it is seen that $u'_\lambda(X_M)$ is not a member of L^2 if $a > 1/2$, *e.g.*, in the log utility case there would be no equilibrium. Empirical research indicates that the parameter a is in the range between 1 and 20, supposedly close to 2, so for this particular example there is no equilibrium in the interesting parameter range. (Bühlmann (1984) overlooked this possibility, and confined his analysis to situations of the type described by Example 1).

One may wonder if it is at all possible for the *aggregate* portfolio to be *uniformly* distributed. In the above only the properties of this distribution near zero was really utilized. To show that it is possible for an aggregate quantity to be uniformly distributed near zero, consider the following simple example: Assume that $I = 2$ and X_1 and X_2 are independent each with a Beta$(\frac{1}{2}, 1)$-distribution. This means that the probability density function for each portfolio is given by

$$f(x) = \frac{1}{2} x^{-\frac{1}{2}}, \quad \text{for } 0 \leq x \leq 1.$$

Here expectations and variances exist, so let us find the probability density $f_M(s)$ of $X_M = X_1 + X_2$ for $0 \leq s \leq 2$. By the convolution formula it is given by

$$f_M(s) = \int_{-\infty}^{\infty} f(x) f(s-x)\, dx = \frac{1}{4} \int \frac{1}{\sqrt{x}} \frac{1}{\sqrt{s-x}}\, dx.$$

Since $(s - x) \in [0, 1]$ for $x \in [0, 1]$, the last integral is split into two regions as follows:

$$\frac{1}{4} \int \frac{1}{\sqrt{x}} \frac{1}{\sqrt{s-x}}\, dx = \begin{cases} \frac{1}{4} \int_0^s \frac{1}{\sqrt{x}} \frac{1}{\sqrt{s-x}}\, dx, & \text{if } 0 \leq s \leq 1; \\ \frac{1}{4} \int_{s-1}^1 \frac{1}{\sqrt{x}} \frac{1}{\sqrt{s-x}}\, dx, & \text{if } 1 \leq s \leq 2. \end{cases}$$

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Considering the first integral, and noticing that $x(s-x) = -(x-\frac{s}{2})^2 + (\frac{s}{2})^2$, by the substitution $t = x - \frac{s}{2}$ we obtain for $0 \le s \le 1$,

$$\frac{1}{4}\int_{-\frac{s}{2}}^{\frac{s}{2}} \frac{1}{\sqrt{\left(\frac{s}{2}\right)^2 - t^2}}\, dt = \frac{1}{4}\,\text{Arcsin}\left(\frac{t}{s/2}\right)\Big|_{-s/2}^{s/2} = \frac{1}{2}\,\text{Arcsin}(1) = \frac{\pi}{4},$$

and for $1 \le s \le 2$ the second integral reduces to $\frac{1}{2}\,\text{Arcsin}(\frac{s}{2}-1)$. Thus the density $f_M(s)$ is uniform on $[0,1]$, and decaying from $\frac{\pi}{4}$ to 0 for $s =\in [1,2]$:

$$f_M(s) = \begin{cases} \frac{\pi}{4}, & \text{if } 0 \le s \le 1; \\ \frac{1}{2}\,\text{Arcsin}\left(\frac{s}{2} - 1\right) & \text{if } 1 \le s \le 2. \end{cases}$$

For power utility, a common terminology is to call a member *risk tolerant* whenever the relative risk aversion a satisfies $0 < a < 1$. In the above example there is only an equilibrium when the members are *very* risk tolerant ($0 < a < 1/2$).

3.1. A basic fixed point argument

As observed in the previous section, instead of focusing attention on the member weights λ_i (because these determine prices via $u_\lambda'(X_M)$), we restrict attention to the constants b_i of the differential equations (8). The optimal allocations, now parameterized by b instead of λ, are functions of the aggregate risk X_M, so we will use the notation $Y_i^{(b)}$ instead of Y_i^λ for Y_i, i.e., $Y_i^{(b)} := Y_i(X_M)$, where $Y_i(\cdot) : B \to R$. Likewise the state price deflator ξ also depends on b by Proposition 2 and Proposition 3(b), allowing us write $\xi = u_b'(X_M)$ instead of $\xi = u_\lambda'(X_M)$ to emphasize this.

Returning to the first order non-linear differential equations (8) for the optimal allocations $Y_i^{(b)}$, in order to use the standard theory of differential equations of this type, Bühlmann (1984) used the following assumption:

(A1) The risk tolerance functions $\rho_i(y)$ satisfy the Lipschitz condition $|\rho_i(y) - \rho_i(y')| \le C\,|y - y'|$ for all i.

(In Bühlmann (1984), the assumption (A1) was made for the absolute risk aversions $R_i(y)$ instead of the risk tolerances $\rho_i(y)$. In this case we do not obtain that e.g., power, or logarithmic utility functions satisfy Bühlmann's assumption H. Also, it is not clear that the differential equation (8) has a solution under H. But (A1) is what we think he meant).

We now investigate what this requirement means for some familiar examples: For negative exponential utility, the marginal utility is given by $u_i'(x) =$

$\frac{1}{\rho_i}e^{-x/\rho_i}$ and the risk tolerance function $\rho_i(y) = \rho_i$ is a constant for all y, so $|\rho_i(y) - \rho_i(y')| = 0$, and the condition is trivially satisfied.

For power utility $u_i(x) = \frac{1}{(1-a_i)}x^{(1-a_i)}$ with constant relative risk aversion $a_i \neq 1$, the risk tolerance $\rho_i(y) = (1/a_i)y$ and $|\rho_i(y) - \rho_i(y')| = (1/a_i)|y - y'|$, so here the condition is satisfied using $C = \max_i\{\frac{1}{a_i}\}$.

When the relative risk aversion equals one, the logarithmic utility function is appropriate, i.e., $u_i(x) = \ln(c_i x + d_i)$ for constants c_i and d_i. In this case the risk tolerance $\rho_i(y) = y + \frac{d_i}{c_i}$ in which case (A1) holds with $C = 1$.

Our basic assumption is that $X_i \in L^2$ for all $i \in \mathcal{I}$. By Minkowski's inequality also $X_M \in L^2$, but what about the optimal portfolios Y_i? Recall from (7) that $\rho(x) = \sum_{i=1}^{I} \rho_i(Y_i(x))$, and this relationship together with (8) imply that

$$|Y_i(X_M) - Y_i(x_0)| \leq |X_M|, \tag{17}$$

which means that and $Y_i \in L^2$ for all $i \in \mathcal{I}$ as well.

Bühlmann's assumptions of finite supports of the X_i together with assumption (A1) allowed him to use standard, global results of ordinary, non-linear differential equations to guarantee that the optimal allocations are continuous in the constants b_i. In order to relax this condition, observe that the differential equations given by (8) are indeed very "nice", since the non-linear functions

$$F_i(y_i, x) := \frac{\rho_i(y_i)}{\rho(x)}$$

satisfiy $|F_i(y_i, x)| \leq 1$ for all i due to (7). Thus Witner's condition of global existence is satisfied for the differential equations (8). In this case we do indeed have global existence and uniqueness of solutions for these equations, over the entire region $(y_i, x) \in R^2$. In order for the solutions $Y_i(x)$ to be *continuous* functions of the constants b_i, the following is sufficient:

(A2) The functions $F_i(y_i, x)$ and $\frac{d}{dy_i}F(y_i, x)$ are continuous for all (y_i, x).

This assumption will thus replace (A1).We also check (A2) for the standard cases: For the negative exponential utility function we can use the domain B of the X_i to be all of $R = (-\infty, \infty)$, and $F_i(y_i, x) = \frac{\rho_i}{\rho}$ so the condition is trivially satisfied.

For the power utility function the quantity $a_i > 0$ is the relative risk aversion of member i, and the function $F(y_i, x)$ is given by

$$F(y_i, x) = \frac{\frac{1}{a_i}y_i}{\rho(x)},$$

where $\rho(x) > 0$ is a smooth function of x, so again (A2) is satisfied and the domain B of the X_i can be taken to be $B = R_{++} = (0, \infty)$.
For the logarithmic utility function we obtain that

$$F(y_i, x) = \frac{y_i + \frac{d_i}{c_i}}{x + \sum_j \frac{d_j}{c_j}},$$

so $\frac{d}{dy_i} F(y_i, x) = (x + \sum_j \frac{d_j}{c_j})^{-1}$ which is continuous for $x > -\sum_j \frac{d_j}{c_j}$. Here $B = (b, \infty)$ where $b = \max_i \{-d_i / c_i\}$.

We conclude that the assumption (A2) is not restrictive, since it does not rule out any of the most common examples.

A closer examination of Assumption (A2) reveals that the only additional requirement it imposes on the preferences of the members to those of Theorem 1 is that the third derivative of the utility functions must exist and be continuous. Third derivatives of utility is important, since it allows us to check whether the members are prudent or not.

Let us now assume that the moments implied by the budget constraints given in (11) exist. Sufficient for this to be the case is that $E\{(u_b'(X_M))^2\} < \infty$. From Pareto optimality it follows that $\lambda_i u_i'(Y_i) = u_b'(X_M)$, implying that it is also sufficient that $E\{(u_i'(Y_i))^2\} < \infty$ for all i.

Finally notice that the state price deflator $u_b'(X_M)$ is also a continuous function of b for the same reason, since $u_i'(\cdot)$ is a continuous function for each i, and Y_i^b is continuous in b for all i, under (A2).

We are now in position to prove the existence part of the main theorem:

Theorem 3. *Suppose $u_i' > 0$, $u_i'' \leq 0$, and u_i''' are continuous for all i, and $E\{(u_b'(X_M))^2\} < \infty$. Then an equilibrium exists.*

Proof: Consider the mapping $f : R^I \to R^I$ which sends $b = (b_1, b_2, \cdots, b_I)$ into $c = (c_1, c_2, \cdots, c_I)$ by the rule

$$E\Big(u_b'(X_M)\big(X_i - (Y_i^{(b)} - b_i)\big)\Big) = c_i, \quad \text{for } i = 1, 2, \cdots, I. \tag{18}$$

By (17) it follows that $|Y_i^{(b)} - b_i| \leq |X_M|$, so $E(Y_i - b_i)^2 \leq EX_M^2 = C < \infty$, and $EY_i^2 < C_i < \infty$ implies that $b_i \in G$ for some compact rectangle G in R^I. Also

$$|c_i| \leq |E(u_b'(X_M)(X_i - (Y_i^{(b)} - b_i))| \leq$$
$$\left\{E(u_b'(X_M))^2\right\}^{\frac{1}{2}} \left\{\left(E\left(X_i^2\right)\right)^{\frac{1}{2}} + \left(E(Y_i - b_i)^2\right)^{\frac{1}{2}}\right\} < K_i < \infty$$

for any $b \in G$ by first applying the Schwarz inequality and then Minkowski's inequality. This establishes $c \in H$ where H is a rectangle like G. Let J be the

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rectangle in R^I containing both G and H. Denote the hyperplane $\sum_{i=1}^I b_i = x_0$ by F. Note that the intersection $F \cap J$ is non-empty, compact and convex. The mapping $b \to c$ defined by f in (18) maps $F \cap J$ into $F \cap J$ since by Walras' law

$$\sum_{i=1}^{I} c_i = E\left(u_b'(X_M)\left(\sum_{i=1}^{I} X_i - \left(\sum_{i=1}^{I} Y_i^{(b)} - \sum_{i=1}^{I} b_i\right)\right)\right) = \sum_{i=1}^{I} b_i.$$

By our above observation that the optimal allocations $Y_i^{(b)}$ and the state price deflator $u_b'(X_M)$ are all continuous functions of b, and since the linear functional $\pi(Z) = E(u_b'(X_M)Z)$ is continuous in L^2 from our assumption that $\xi = u_b'(X_M) \in L^2$, the mapping f is continuous and hence has a fixed-point by Brower's theorem. Therefore there exist b_i^* such that

$$E\left(u_{b^*}'(X_M)(X_i - (Y_i^{(b^*)} - b_i^*)\right) = b_i^*, \quad \text{for } i = 1, 2, \cdots, I$$

and consequently

$$E\left(u_{b^*}'(X_M)(Y_i^{(b^*)} - X_i)\right) = 0, \quad \text{for } i = 1, 2, \cdots, I$$

This completes the proof. □

Let us consider some illustrations where Theorem 3 is conclusive, but where the assumption of bounded risks is not satisfied.

EXAMPLE 3. Returning to the situation in Example 1 where the utility functions are negative exponential, consider the case where there exists a feasible allocation Z, in which the components Z_i are i.i.d. exponentially distributed with parameter θ. Let $X = DZ$ where D is an $I \times I$-matrix with elements $d_{i,j}$ satisfying $\sum_i d_{i,j} = 1$ for all j, so that $X_M = \sum_{i=1}^I Z_i := Z_M$.

This gives an initial allocation X of dependent portfolios, which seems natural in a realistic model of a reinsurance market. Here it means that the X_i portfolios are mixtures of exponential distributions with a fairly arbitrary dependence structure.

In this case X_M has a Gamma distribution with parameters I and θ. According to Theorem 3 all we have to check for an equilibrium to exist is that $E\{(u_i'(Y_i))^2\} < \infty$ for all i, or equivalently that $E\{(u_\lambda'(X_M))^2\} < \infty$. Since $u_\lambda'(X_M) = Ke^{-X_M/\rho}$ for some constant K, we have to verify that the following integral is finite:

$$E\left(e^{-\frac{2X_M}{\rho}}\right) = \int_0^\infty e^{-2x/\rho} \theta e^{-\theta x} \frac{(\theta x)^{(I-1)}}{(I-1)!} dx.$$

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This is indeed the case, since by the moment generating function of the Gamma distribution, it follows that

$$E\left(e^{-\frac{2X_M}{\rho}}\right) = \left(\frac{\theta}{\theta + \frac{2}{\rho}}\right)^I < 1$$

because both the parameter θ and the risk tolerance ρ of the syndicate are strictly positive.

Instead of the assumption of the exponential distributions, suppose that the Z_i are independent, each with a Pareto distribution, i.e., with probability density function

$$f_{Z_i}(x) = \frac{\alpha_i c_i^{\alpha_i}}{z^{1+\alpha_i}}, \quad c_i \le z < \infty, \quad \alpha_i, c_i \in (0, \infty).$$

This is known as the Pareto distribution of the first kind. In this case EZ_i exists only if $\alpha_i > 1$, and var Z_i exists only if $\alpha_i > 2$, etc. The moment generating functions $\varphi_i(\beta) = Ee^{\beta Z_i}$ of these distributions exist for $\beta \le 0$, since the random variables $e^{\beta Z_i}$ are then bounded. Carrying out the same construction as above, we notice that

$$E\left(e^{-\frac{2}{\rho}X_M}\right) = \prod_{i=1}^{I} E\left(e^{-\frac{2}{\rho}Z_i}\right) < \infty$$

since each of the factors has finite expectation. Accordingly, for these distributions a competitive equilibrium exists by Theorem 3.

Here the X_i are mixtures of Pareto distributions, but we should exert some caution, since our theory is developed for risks belonging to L^2. We are outside this domain regarding the Z_i if $\alpha_i < 2$ for some i, in which case $X_j \notin L^2$ for any j. However, as long as the initial risks are in L^2, an equilibrium exists by Theorem 3.

Finally consider the normal distribution, and assume that each X_i is $\mathcal{N}(\mu_i, \sigma_i)$-distributed and that X is jointly normal, where $\text{cov}(X_i, X_j) = \rho_{ij}\sigma_i\sigma_j$ for $i, j = 1, 2, \cdots, I$. By the moment generating function of the normal distribution we have that

$$E\left(u'_\lambda(X_M)\right)^2 = E\left(e^{-\frac{2}{\rho}X_M}\right) = \exp\left(2\left(\frac{\sigma}{\rho}\right)^2 - 2\frac{\mu}{\rho}\right) < \infty \quad \forall i,$$

where $\mu = \sum_{i=1}^{I}\mu_i$ and $\sigma^2 = \sum_{i=1}^{I}\sigma_i^2 + 2\sum_{i>j}\sigma_i\sigma_j\rho_{ij}$. Thus an equilibrium exists.

Even if the positivity requirements are not met, still all the computations of the equilibrium are well defined, the state price deflator $\xi(X_M)$ is a strictly positive element of L^2_+, prices can readily be computed and an equilibrium exists.

It may admittedly be unclear what negative wealth should mean in a one period model, but aside from this there are no formal difficulties with this case as long as utility is well defined for all possible values of wealth. In the reinsurance syndicate we usually interpret $X_i = w_i - V_i$ where w_i are initial reserves and V_i are claims against the ith reinsurer, or member. In this case negative values of X_i have meaning, in that when this occurs, reinsurer i is simply bankrupt or in financial distress. □

In the above example with the Pareto distributions, if the parameters α_i satisfy $1 \le \alpha_i < 2$ for all i, expectations exist, but not variances. Still $u_\lambda'(X_M) = e^{-\frac{2}{p}X_M} \in L^2_+$, however L^2 is not the relevant dual for L^1, but turns out to be L^∞. We notice that $u_\lambda'(X_M) \in L^\infty_+$ as well, which means that this case will now be covered by our next theorem. Our development in Theorem 3 can be seen to be valid for L^1 replacing L^2, and dual price space L^∞ replacing L^2.

To make this precise, we assume the portfolio space $L = L^p := L^p(\Omega, \mathcal{F}, P)$, where the associated price space is the dual L^q, for $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Again we start by assuming that the moments making up the budget constraints (11) exist. By Hölder's inequality it is sufficient that $E\{u_b'(X_M))^q\} < \infty$. We then have the following result.

Theorem 4. *Suppose $u_i' > 0$, $u_i'' \le 0$, and u_i''' are continuous for all i, and $E\{u_b'(X_M))^q\} < \infty$. Then an equilibrium exists.*

The proof can be carried out along the lines of the proof of Theorem 3, except that we use Hölder's inequality instead of Schwartz', and observe that the price functional $\pi(Z)$ is continuous in L^p since $\xi = u_b'(X_M) \in L^q$.

The space L^∞ is exempted from the above discussion, and deserves some special comments. It is well behaved from the point of view of price supporting preferred sets since the positive cone has a non-empty interior, but neither does L^1 furnish all the continuous linear functionals on L^∞, nor do we know that the strictly positive functionals on L^∞ are continuous. Thus our argument based on no arbitrage can not be used directly to characterize a price functional π for the portfolio space L^∞. As it turns out, so long as the preferences are "continuous enough" i.e., continuous in the Mackey topology, this forces continuous prices to be in L^1. As Bewley (1972) shows, expected utility functions are concave and continuous in the Mackey topology provided $u : [0, \infty) \to (-\infty, \infty)$ is concave. From the results of of Mas-Colell and Zame (1991) we can show the following:

Theorem 5. *Assume, in addition to our assumptions, that $u : [0, \infty) \to (-\infty, \infty)$ is strictly monotone and concave for all i, that the initial allocation $X_M \in intL^\infty_+$ (which is non-empty), and that $\pi(X_i) > 0$ for all i. Then an equilibrium exists, and every equilibrium price belongs to L^1.*

The assumption $\pi(X_i) > 0$ for all i has been discussed previously in connection with Theorem 1. Its significance here is the same as in Section 4.

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This theorem can be related to the development in Bühlmann (1984) in the following sense: The portfolio space is assumed to be $L = L^2$. Recall that we require the price to be defined and finite for every $Z \in L$. Since risks are assumed bounded in Bühlmann (1984), in reality they are in L^∞ (in fact in a subset of this space), which is a subset of L^2. Provided the assumptions of Theorem 5 hold, we can find state prices $\xi \in L^1 \supset L^2$ such that ξ assigns a finite price to every risk in L^∞, and hence to every risk "present in the market". If, on the other hand, $\xi \notin L^2$, then ξ does not assign a finite price to all elements of L^2, so some risks are left unpriced. This shows that the desirable requirement that price is defined and finite for *all* risks is also a strong one.

EXAMPLE 4. Returning to Example 2 with $B = (0,1]$, X_M uniformly (0,1)-distributed, and $\xi = cX_M^{-a}$, this situation does not satisfy the requirements of Theorem 5, but it is easy to see that when the relative risk aversion a satisfies $\frac{1}{2} < a < 1$, then $\xi \in L^1$ but $\xi \notin L^2$, illustrating the above possibility. The optimal allocations $Y_i = \frac{\lambda_i^{\frac{1}{a}}}{\Sigma_j \lambda_j^{\frac{1}{a}}} X_M$ are seen to be in L^∞, which means that there exists an equilibrium in L^∞ for $\frac{1}{2} < a < 1$, demonstrating that the sufficient conditions of Theorem 5 can be too strong. The risk $Z = X_M^{-\frac{1}{3}}$, on the other hand, is seen to not be a member of L^∞, but is in L^2, hence it is not priced. When $0 < a < 1/2$, there is an equilibrium in L^2. □

In this example the aggregate portfolio X_M is rather risky, since it contains much probability mass near zero, where the utility functions of the members are not bounded from below. One may wonder who will hold most of this risk in equilibrium. Since we have assumed that they all have the same preferences, the members are only distinguished by their initial allocations X_i. Here we notice from the expressions for the member weights λ_i that the the member with "stronger reserves" will hold more of the risks than the economically "weaker" members, in that λ_i is proportional to $E(X_i/X_M^a)$. This result appeals to intuition: Those members best fit to carry risk will do so in equilibrium. Notice that no member holds the risk Z in equilibrium. Also, there is only an equilibrium here when the members are risk tolerant.

We now leave this theme and present an example where the relative risk aversions of all the syndicate members are constants, still as in Example 2, but from another perspective:

EXAMPLE 5. Consider the model of Example 2, where $u_i(x) = (x^{1-a_i} - 1)/(1 - a_i)$ for $x > 0$, $a_i \neq 1$. We again restrict attention to the case where $a_1 = a_2 = \ldots = a_I = a$.

Recall that the weights λ_i are determined by the budget constraints, implying that

$$\lambda_i = k\left(\frac{E\left(X_i X_M^{-a}\right)}{E\left(X_M^{1-a}\right)}\right)^a, \quad i \in \mathcal{I},$$

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or, λ_i is determined modulo the proportionality constant $k = (\sum_{j \in \mathcal{I}} \lambda_j^{1/a})^a$ for each i.

Let us again consider a situation where there exists a feasible allocation Z, where the Z_i components are i.i.d. exponentially distributed with parameter θ. Let $X = DZ$ where D is an $I \times I$-matrix with elements $d_{i,j}$ satisfying $\sum_i d_{i,j} = 1$ for all j, so that $X_M = \sum_{i=1}^{I} Z_i := Z_M$.

Regarding existence of equilibrium, according to Theorem 3 it is sufficient to check that $u_\lambda'(X_M) \in L^2$. In this case X_M has a Gamma distribution with parameters I and θ, and all we have to check is if the expectation

$$E\left(X_M^{-2a}\right) = \int_0^\infty x^{-2a} \theta e^{-\theta x} \frac{(\theta x)^{I-1}}{(I-1)!} \, dx$$

is finite. The possible convergence problem is seen to occur around zero, and the standard test tells us that when $(-2a + I - 1) > -1$, or when $I > 2a$, this integral is finite. Thus, for example if $a = 10$, then equilibrium exists in this syndicate if the number of members exceeds 20.

One may wonder if the member weights λ_i can be computed when $I > 2a$. To check this consider the two expectations $E(X_M^{1-a})$ and $E(Z_i X_M^{-a})$. In order to verify that these expectations exist, we have to find the joint distribution of Z_i and X_M. It is given by the probability density

$$f(z_i, x) = \theta^2 e^{-\theta x} \frac{\left(\theta(x - z_i)\right)^{I-2}}{(I-2)!}, \quad z_i \leq x < \infty, \; 0 \leq z_i < \infty.$$

So we have to check if the integral

$$E\left(Z_i X_M^{-a}\right) = \int_0^\infty \int_{z_i}^\infty z_i x^{-a} \theta^2 e^{-\theta x} \frac{\left(\theta(x - z_i)\right)^{I-2}}{(I-2)!} \, dz_i \, dx$$

is finite. The possible convergence problem is again seen to occur around zero, and the standard test requires that $(1 - a + I - 2) > -1$, i.e., when $I > a$ this integral is finite. From this it is obvious that the expectations $E(X_i X_M^{-a})$ also converge in the same region, by linearity of expectation, since $X_i = \sum_j d_{i,j} Z_j$.

Similarly, we have to check the following expectation:

$$E\left(X_M^{1-a}\right) = \int_0^\infty x^{1-a} \theta e^{-\theta x} \frac{(\theta x)^{I-1}}{(I-1)!} \, dx.$$

Near zero the possible problem again occurs, and the standard comparison test gives convergence when $(1 - a + I - 1) > -1$, or when $I > a - 1$. To conclude, when $I > \max\{a, a - 1\} = a$, both expectations exist, showing that the member weights exist in the parameter range $(I > 2a)$ where state prices are known to exist.

Notice that an equilibrium will exist with a fairly low number of participants in the interesting region for the parameter a. For instance, for $a = 1$ corresponding to a logarithmic utility function, an equilibrium exists with only three members in the syndicate. When the relative risk aversion is two, only five members are required, and so on.

Finally consider the case of Pareto distributions for the initial portfolios X_i directly, assuming $\alpha_i > 2$ for all i. The integrals

$$E\left(X_i^{-2a}\right) = \left(c_i^{2a}\left(1 + \frac{2a}{\alpha_i}\right)\right)^{-1} < \infty.$$

Since $\min_{i \in \mathcal{I}} \alpha_i > 0$, there are no problems with convergence, and an equilibrium exists in this case regardless of the values of the relative risk aversion parameter a, ($a > 0$) or its relationship to I, since $E(X_M^{-2a}) \leq \sum_i E(X_i^{-2a})$. In this latter case all the portfolios are bounded away from zero, which helps with the existence problem for power utility, while the exponential distribution has more probability mass near zero, potentially causing problems with existence in certain parameter ranges, as we have seen above. □

The result of this example is in line with the very spirit of a competitive equilibrium, which generally implies that the theory may work better the *more* individuals that participate. Recall that classical economics sought to explain the way markets coordinate the activities of *many* distinct individuals each acting in their own self-interest.

We end this section with a fairly general example, namely the utility functions with affine risk tolerances, also called the class of Harmonic Absolute Risk Aversion (HARA) preferences.

EXAMPLE 6. Assume that the risk tolerance ρ_i of member i is of the form $\rho_i(x_i) = \alpha x_i + c_i$ for $i \in \mathcal{I}$. Here the risk tolerance of the representative member is $\rho_b(x) = \alpha x + c$, where $c = \sum_{i \in \mathcal{I}} c_i$. In this case the optimal sharing rules are all affine and of the form

$$Y_i(x) = \frac{c_i + \alpha b_i}{c + \alpha x_0} x + \frac{c b_i - c_i x_0}{c + \alpha x_0}, \quad i \in \mathcal{I},$$

as solutions of the equations $\frac{dY_i(x)}{dx} = \frac{\alpha Y_i(x) + c_i}{\alpha x + c}$, with boundary conditions $Y_i(x_0) = b_i$, where $\sum_i b_i = x_0$. In equilibrium the constants b_i are determined from the budget constraints as

$$b_i = \frac{(c + \alpha x_0) E(\xi X_i) + c_i\left(x_0 E(\xi) - E(\xi X_M)\right)}{\alpha E(\xi X_M) + c E(\xi)}, \quad i \in \mathcal{I},$$

where the state price deflator is $\xi = u_b'(x) = (\alpha x + c)^{-\frac{1}{\alpha}}$, when $\alpha \neq 0$, $\xi = e^{-\frac{x}{c}}$ when $\alpha = 0$. Notice, α does not depend on i.

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This class of utility functions contains many of the common examples as special cases: When $\alpha = 0$ we have the negative exponential utility function $u(x) = c(1 - e^{-\frac{x}{c}})$ of Example 1, where $x_0 = 0$; when $\alpha = 1$ we get the logarithmic utility function $u(x) = \ln(x + c)$ of Example 2, and $x_0 > 0$; when $\alpha > 0$, $\alpha \neq 1$, we get the power utility $u(x) = \frac{\alpha}{\alpha-1}(c + \alpha x)^{1-\frac{1}{\alpha}}$, where $1/\alpha$ is the relative risk aversion of Example 2; when $\alpha = -1$, utility is quadratic, etc. As usual, equilibrium is only determined modulo some normalizing constant, here represented by x_0.

Using Theorem 3, assumption (A2) holds, so the remaining assumption is $E((\alpha X_M + c)^{-\frac{2}{\alpha}}) < \infty$ when $\alpha \neq 0$ ($E(e^{-\frac{2X_M}{c}}) < \infty$ when $\alpha = 0$) for equilibrium to hold, when the initial portfolios $X_i \in L^2$. □

The HARA utility class has a very interesting property: It is the most general class of preferences for which the following is true: Each member of the pool holds an identical attitude towards aggregate risk. Therefore there is unanimity on the management of the risk followed by the representative member, or the "central planner". It happens that these important properties hold only in this special case of linear risk tolerances with the same "slope", or cautiousness α. To put it differently, first the members agree on a Pareto optimal risk sharing arrangement. Second, each member, using his own utility function and this sharing rule, will reach the same decision regarding risk taking as the representative member: Defining the "implicit" utility function v_i of member i by $v_i(x) = u_i(Y_i(x))$, it is the case that $-v_i'(x) / v_i''(x) = \rho_b(x)$ for all x and i (Wilson (1968)).

3.2. Uniqueness of Equilibrium

The question of uniqueness of equilibrium is largely unexplored in the infinite dimensional setting. However, given our smoothness assumptions one would expect equilibrium to be unique, provided one exists. In this section we show that this conjecture holds.

Approaches that take preferences and endowments as primitives seem to encounter many difficulties in addition to the usual difficulty of doing calculus in infinite dimensional spaces. As mentioned before the natural domain of prices is a subset of the dual space of L^2, the positive orthant L^2_+, but this set has empty interior, which is very inconvenient for doing calculus. Excess demand functions are not defined in general, and are not smooth even when they are defined. Araujo (1987) argues that excess demand functions can be smooth only if the "commodity" space is a Hilbert space, which is noticed to be the case in our model (when $p = q = 2$).

Inspired by our approach in Theorem 3, where we basically transformed the infinite dimensional problem into a finite dimensional one represented by the member weights λ, or equivalently, the constants b, we attempt the same line of reasoning regarding the uniqueness question.

Going back to the first order non-linear differential equations in (8), to each point $(x_0, b_1, b_2, \cdots, b_I)$ there is only one solution $Y = (Y_1, Y_2, \ldots, Y_I)$ to these equations under the assumption (A2). However, there could be several fixed-points and thus one possible equilibrium associated with each of them.

Arguing in terms of the member weights λ instead of the b's, let us define the individual demands of the I members by $Z_i^{(\lambda)} = (Y_i^{(\lambda)} - X_i)$ and the excess demand $Z^{(\lambda)} = \Sigma_{i \in \mathcal{I}} Z_i^{(\lambda)}$. Below we show that these are well defined and smooth functions of the member weights λ_i, $i \in \mathcal{I}$.

One reason why we consider the member weights here instead of the constants b, is due to Proposition 3(c), equation (9), where it was shown that the state price $\xi(\lambda)$ is an increasing function of the weights λ_i. As a consequence, by increasing λ_i, member i's optimal portfolio function must increase to maintain equilibrium, since, loosely speaking, this can be associated with a strengthening of member i's initial portfolio X_i, while all the other members' optimal portfolio functions will decrease. This will be formalized below.

The excess demand is zero at the possible equilibrium points λ^*, corresponding to the points b^* of Theorem 3. If the excess demand curve as a function of each member weight λ_i is downward sloping for all i at all equilibria where Theorem 3 holds, there can only be one equilibrium. It is enough that $Z^{(\lambda)}$ is downward sloping in $(I-1)$ of the λ's because of the normalization of the weights. Because of the smoothness of the excess demand function in λ, this will be a sufficient condition for uniqueness.

By investigating the marginal effect on the excess demand Z^{λ^*} from a marginal increase in λ_i^*, making sure that the resulting λ is still on the simplex S^{I-1}, we may use this procedure to check for uniqueness. As real functions the demands $Z_i^{\lambda} : R \to R$ can be expressed as $Z_i^{\lambda} = Y_i^{\lambda}(x) - x_i$ where $\Sigma_i x_i = x$. Thus, in the language of calculus, we must therefore consider the quantities

$$Z^{\lambda^*} - \alpha\Big(\sum_{i \in \mathcal{I}} \lambda_i - 1\Big),$$

where α is the Lagrange multiplyer associated with the constraint of remaining on the simplex. Since any marginal change in one of the member weights will necessarily bring the resulting vector of weights outside the simplex unless the other weights are correspondingly lowered, $\alpha > 0$. Thus, we calculate the following

$$\frac{\partial Z^{\lambda^*}}{\partial \lambda_i} - \alpha \quad \text{for} \quad i = 1, 2, \cdots, (I-1)$$

at any equilibrium point λ^*, and check wether all these have the same sign for all $x \in B$.

In order to calculate the quantities $\frac{\partial Z^{\lambda^*}}{\partial \lambda_i}$, we must find $\frac{dY_j^{\lambda^*}(x)}{\partial \lambda_i}$ for all $i, j \in \mathcal{I}$. It follows by differentiation of the first order conditions

$$\lambda_i u_i'(Y_i^{\lambda}(x)) = u_\lambda'(x) \quad \text{for any } i$$

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that

$$\frac{dY_i^{\lambda}(x)}{d\lambda_i} = \frac{1}{\lambda_i u_i''(Y_i^{\lambda}(x))}\left(\frac{\partial}{\partial\lambda_i}u_i'\lambda(x) - u_i'(Y_i^{\lambda}(x))\right) \quad \text{for } i = j,$$

for all $x \in B$, and using equation (9) and the first order conditions, we obtain

$$\frac{dY_i^{\lambda}(x)}{d\lambda_i} = \frac{1}{\lambda_i}\rho_i\left(Y_i^{\lambda}(x)\right)\left(1 - \frac{dY_i^{\lambda}(x)}{dx}\right) \quad \text{for } i = j, \tag{19}$$

for all $x \in B$. Similarly we get

$$\frac{dY_j^{\lambda}(x)}{d\lambda_i} = -\frac{1}{\lambda_i}\rho_j\left(Y_j^{\lambda}(x)\right)\frac{dY_i^{\lambda}(x)}{dx} \quad \text{for } j \neq i, \tag{20}$$

for all $x \in B$. Notice that $\frac{dY_i^{\lambda}(x)}{dx} \in (0,1)$ by equation (8), in other words, an increase in the market portfolio leads to an increase in all the members' portfolios Y_i, and no member assumes the entire increase because they are all risk averse. It follows that $\frac{dZ_i^{\lambda}(x)}{d\lambda_i} > 0$ for all i and $\frac{dZ_j^{\lambda}(x)}{d\lambda_i} < 0$ for all $j \neq i$, demonstrating what was loosely explained above.

We are now in position to compute the required marginal changes in excess demand within the simplex. It is

$$\frac{\partial Z^{\lambda^*}}{\partial\lambda_i} - \alpha = \sum_{j\in\mathcal{I}}\frac{\partial Z_j^{\lambda^*}}{\partial\lambda_i} - \alpha = \frac{\partial Y_i^{\lambda^*}}{\partial\lambda_i} + \sum_{j\neq i}\frac{\partial Y_j^{\lambda^*}}{\partial\lambda_i} - \alpha =$$

$$\frac{1}{\lambda_i}\rho_i(Y_i^{\lambda^*}(x))\left(1 - \frac{dY_i^{\lambda^*}(x)}{dx}\right) - \sum_{j\neq i}\frac{1}{\lambda_i}\rho_j(Y_j^{\lambda^*}(x))\frac{dY_i^{\lambda^*}(x)}{dx} - \alpha,$$

for all $x \in B$, where we have used (19) and (20). Continuing, we get

$$\frac{\partial Z^{\lambda^*}}{\partial\lambda_i} - \alpha = \frac{1}{\lambda_i}(\rho_i(Y_i^{\lambda^*}(x)) - \frac{dY_i^{\lambda^*}(x)}{dx}\rho_{\lambda^*}(x)) - \alpha$$

for all $x \in B$, where we have used that

$$\rho_{\lambda^*}(x) = \sum_{i\in\mathcal{I}}\rho_i(Y_i(x)), \quad x \in B,$$

according to Proposition 3(a). Finally using (8) we observe that

$$\frac{\partial Z^{\lambda^*}}{\partial\lambda_i} - \alpha = -\alpha < 0 \quad \text{for all } x \in B \text{ and } i \in \mathcal{I}.$$

The conclusion is formulated in the following uniqueness part of the main theorem:

Theorem 6. *Under the assumptions of Theorem 3, or Theorem 4, or Theorem 5, the existing equilibrium in the reinsurance syndicate is unique.*

Thus, our conjecture is confirmed. Notice that in the examples we have presented we were able to find the equilibrium by direct calculation, and the weights λ_i were uniquely determined (modulo multiplication by a positive constant) from the budget constraints. Thus, these equilibria are all unique.

4. Comparison with a more general theory

As mentioned in the introduction, an exchange economy deals with consumption, which can not be negative by definition. An allocation $V \geq 0$ a.s. thus means that $V_i \geq 0$ a.s. for each $i \in \mathcal{I}$. A pair (Y, π) is a *quasi-equilibrium* if $\pi(X_M) \neq 0$, and for each i, $\pi(Z_i) \geq \pi(X_i)$ whenever $Eu_i(Z_i) > Eu_i(X_i)$.

Drawing on the results of a more general theory of an exchange economy, as in e.g., in Mas-Colell and Zame (1991) and Araujo and Monteiro (1989), based on proper preference relations (Mas-Colell (1986)), Aase (1993) formulated the following existence theorem for quasi-equilibrium in an exchange economy in L^2:

Theorem 7. *Assume $u_i(\cdot)$ continuously differentiable for all i. Suppose that $X_M > 0$ a.s., and there is an allocation $V \geq 0$ a.s. with $\sum_{i=1}^{I} V_i = X_M$ a.s., such that $E\{(u_i'(V_i))^2\} < \infty$ for all i, then there exists a quasi-equilibrium.*

If every member i brings something of value to the market, in that $E(\xi \cdot X_i) > 0$ for all i, we have that a quasi-equilibrium is also an equilibrium, which then exists under the above stipulated conditions. This assumption we also met in Theorem 1, as well as in Theorem 5, in the latter case for the same reason as here.

We notice that these requirements put joint restrictions on both preferences and probability distributions that are rather similar to the ones of Theorem 3. Although we have stronger requirements on the utility functions u_i, our requirement on X_M is weaker. In addition we also have demonstrated uniqueness of equilibrium. An example may illustrate the differences between the two theories:

Example 7. Consider the case of power utility of Example 5, where $u_i(x) = (x^{1-a_i} - 1)/(1 - a_i)$ for $x > 0$, $a_i \neq 1$. In this example the exponentially distributed Z_i's satisfy the assumptions of the allocation V in Theorem 7, and $X_M > 0$ a.s. since X_M has a Gamma distribution. Provided $E(\xi \cdot X_i) > 0$ for all i, an equilibrium will exist if

$$E(Z_i^{-2a_i}) = \int_0^\infty x^{-2a_i} \theta_i e^{-\theta_i x} dx < \infty,$$

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which holds true when $a_i < 1/2$, i.e., for very risk tolerant members. As we demonstrated in Example 5, in the case where $a_1 = a_2 = \dots = a_I := a$, an equilibrium exists for $I > 2a$. Thus our previous result is stronger, or perhaps more relevant, since empirical studies suggest that the interesting values of a_i may be in the range between one and 4, say.

Here it is simple to verify existence also when the parameters a_i are unequal, and provided $E(\xi \cdot X_i) > 0$ for all i, an equilibrium will exist in the region $a_i < 1/2$ for all i according to the above theorem[2].

In the case where all the $d_{j,i}$ are equal (to $\frac{1}{I}$), the initial portfolios all have the same Gamma (θI, I)-distribution, in which case the allocation X satisfies the requirements of the allocation V of Theorem 7. In this case we get existence in the region $I > 2 \max_i\{a_i\}$, which is quite similar to the result of Example 4, □

We see that the two theories give comparable results, albeit they guarantee existence in slightly different regions depending upon circumstances. In general it seems easier to reach an equilibrium the more risk tolerant the members are.

5. Summary

Classical economics sought to explain the way markets coordinate the activities of many distinct individuals each acting in their own self-interest. An elegant synthesis of two hundred years of classical thought was achieved by the general equilibrium theory. The essential message of this theory is that when there are markets and associated prices for all goods and services in the economy, no externalities or public goods and no informational asymmetries or market power, then competitive markets allocate resources efficiently.

In this paper the idea of general equilibrium has been applied to a reinsurance syndicate, where many of the idealized conditions of the general theory may actually hold. The most critical assumption seems to be that of no informational asymmetries. Reinsurers like to stress that their transactions are carried out under conditions of "utmost good faith" – *uberrima fides*. This means that the reinsurers usually accept, without question, the direct insurer's estimate of the risk and settlement of claims. The mere existence of rating agencies in this industry is an indication that there may be both adverse selection, and also elements of moral hazard in these markets. Nevertheless, the above theory may still give a good picture of what goes on in syndicated markets.

In models of such markets properties of competitive equilibria have only academic interest so long as it is not clear under what conditions they exist. Existence is thus an issue of great importance.

[2] The explicit computation of the state price deflator ξ is not straightforward when the parameters are no longer equal across the agents. In this case sharing rules are certainly not linear.

The advantage with the existence and uniqueness theorems of this paper is that they rest largely on results in risk theory, or the theory of syndicates, which implies that we may essentially restrict attention to the member weights in Euclidian I-dimensional space, thus reducing the dimensionality of the problems. In contrast, Theorem 7 requires a rather demanding, infinite dimensional equilibrium theory, where the topological structure of infinite dimensional spaces plays an important role.

To sum up, the specific contributions of this paper are: (i) We extend the analysis in Bühlmann (1984) from merely affine contracts to any contracts, and from bounded risks to unbounded ones, (ii) we demonstrate uniqueness of equilibrium, using fairly deep results in Risk Theory, and (iii) we demonstrate that the existence results based on risk theoretic tools are not inferior to, or uniformly dominated by, existence results based on the theory of proper preference relations of an exchange economy.

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