Bull. Aust. Math. Soc. **102** (2020), 430–438
doi:10.1017/S0004972720000246

FINITE GROUPS WITH COMPLEMENTED 2-MINIMAL p-SUBGROUPS

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(Received 28 January 2020; accepted 15 February 2020; first published online 11 March 2020)

Abstract

For a given prime p, we investigate the finite groups all of whose 2-minimal p-subgroups are complemented.

2010 *Mathematics subject classification*: primary 20D10.

Keywords and phrases: finite group, complemented subgroup.

1. Introduction

In this paper, G always denotes a finite group and p always denotes a prime. A subgroup H of G is said to be complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. Given a finite group, its structure can be determined if certain types of subgroups have complements. For instance, a classical theorem of P. Hall says that G is solvable if and only if all its Sylow subgroups have complements. When all subgroups of G are complemented, Hall proved that G is necessarily supersolvable [4]. Gorchakov [2] showed that Hall's requirement of the complementability of all subgroups can be reduced to the complementability of all minimal subgroups.

We say that G is a $\mathfrak{C}_{p^d}$-group if all its subgroups of order p^d are complemented. In particular, G is a $\mathfrak{C}_{p^2}$-group if every 2-minimal p-subgroup of G is complemented. A $\mathfrak{C}_{p^d}$-group G is said to be nontrivial if $|G|_p \geq p^d$. Recently, Monakhov and Kniahina [6] studied $\mathfrak{C}_p$-groups and established criteria for their solvability and supersolvabilty. In [11], we gave a classification for nontrivial $\mathfrak{C}_p$-groups. In this paper, we study the $\mathfrak{C}_{p^2}$-groups.

Let $\Phi(G)$, $\mathrm{Soc}(G)$, $F(G)$ and $F^*(G)$ denote the Frattini subgroup, the socle, the Fitting subgroup and the generalised Fitting subgroup, respectively, of G. Note that $F^*(G) = F(G)E(G)$ with $[F(G), E(G)] = 1$, where $E(G)$ is the subgroup generated by all components of G. Let V be a finite-dimensional $\mathbb{F}_p$-space (where $\mathbb{F}_p$ is the

Project supported by the NSF of China (no. 11671324).

field of p elements). A Borel subgroup of $\mathrm{GL}(V)$ is the normaliser of a Sylow p-subgroup of $\mathrm{GL}(V)$. The notation $\mathcal{S}_n$ and $\mathcal{A}_n$ stands for the symmetric group and the alternating group of degree n, respectively. We use $M_p(1,1,1)$ to denote an extraspecial p-subgroup of order p^3 with exponent an odd prime p, D_8 to denote the dihedral group of order eight, $E(p^n)$ to denote an elementary abelian group of order p^n and $\mathbb{Z}_n$ to denote a cyclic group of order n.

According to [1], if $S \cong \mathrm{PSL}(n,q)$, then $\mathrm{Out}(S) = \langle d, f\rangle \rtimes \langle g\rangle$, where d, f and g are a diagonal, a field and a graph automorphism of S, respectively. We use $\mathrm{Out}_{df}(S)$ to denote the normal subgroup $\langle d, f\rangle$.

The main result of our paper is the following theorem.

Theorem 1.1. *Let T be a finite group, $G = T/O_{p'}(T)$ and $P \in \mathrm{Syl}_p(G)$. Then T is a nontrivial $\mathfrak{C}_{p^2}$-group if and only if one of the following statements is true.*

(1) *$G = H \ltimes P$, where $H \lesssim \mathbb{Z}_{p-1}$ and $P \cong \mathbb{Z}_{p^2}$ such that $C_H(P) = 1$.*
(2) *$G = B \ltimes V$, where $V \cong E(p^2)$ with $C_G(V) = V$ and B is a subgroup of a Borel subgroup of $\mathrm{GL}(V)$ with $|B|_p = p$.*
(3) *$F^*(G) \cong \mathrm{PSL}(n,q)$, $G/F^*(G) \lesssim \mathrm{Out}_{df}(F^*(G))$ and $P \cong \mathbb{Z}_{p^2}$, where $p^2 = \frac{q^n-1}{q-1} \geq 9$.*
(4) *$F^*(G) = O^{p'}(G) = S_1 \times S_2$ is minimal normal in G, where S_1 and S_2 are isomorphic nonabelian simple groups and $N_G(S_1) = N_G(S_2)$ are both $\mathfrak{C}_p$-groups.*
(5) *$G = H \ltimes (N_1 \times \cdots \times N_t)$, where $H \in \mathrm{Hall}_{p'}(G)$ and all N_i are H-isomorphic irreducible $\mathbb{F}_p[H]$-modules of dimension two. Furthermore, if $t > 1$, then H is cyclic.*
(6) *G is a nontrivial $\mathfrak{C}_p$-group.*

Remark 1.2. If $t = 1$ in (5), then H is not necessarily cyclic. For example, let $G = H \ltimes N$ be a Frobenius group with kernel $N \cong E(11^2)$ and complement $H \cong \mathrm{SL}(2,5)$. Then G is a $\mathfrak{C}_{11^2}$-group with N the unique minimal subgroup of G and $H \in \mathrm{Hall}_{11'}(G)$.

2. Preliminaries

We begin with the following facts which will be used freely in the proof.

Lemma 2.1 [7]. *Let G be a group and N a minimal normal subgroup of G such that N is a direct product of t isomorphic nonabelian simple groups, say $N = S_1 \times \cdots \times S_t$. Assume that M is a maximal subgroup of G such that $G = MN$. Then one of the following statements holds.*

(1) *$M \cap N = M_1 \times \cdots \times M_t$, where $|S_i : M_i| = |S_1 : M_1| > 1$ for all i.*
(2) *$M \cap N = E_1 \times \cdots \times E_k$ is minimal normal in M, where $E_1 \cong \cdots \cong E_k \cong S_1$, $k \mid t$.*

Lemma 2.2. *Let G be a group and N a minimal normal subgroup of G such that N is a direct product of t isomorphic simple groups. Suppose that G has a proper subgroup H so that $G = HN$ and $|G : H| = p^k > 1$. Then $t \leq k$.*

https://doi.org/10.1017/S0004972720000246 Published online by Cambridge University Press

Proof. Let $N = S_1 \times \cdots \times S_t$, where the S_i are isomorphic simple groups, and let M be a maximal subgroup of G which contains H. If N is abelian, then $G = HN = H \ltimes N$ and hence $t \leq k$. Suppose that N is nonabelian. Then $G = MN$ is such that

$$|N : M \cap N| = |G : M| := p^s \mid p^k.$$

It follows from Lemma 2.1 that $N \cap M = M_1 \times \cdots \times M_t$, where $|S_1 : M_1|^t = p^s$. Thus, $t \leq s \leq k$. □

Lemma 2.3. *Let G be a group and E a normal subgroup of G. Suppose that $Q \in \mathrm{Syl}_p(E)$.*

(1) *Then $G = EN_G(Q)$ and $G/E \cong N_G(Q)/N_E(Q)$ with $|G|_p = |N_G(Q)|_p$.*
(2) *If D is a minimal normal subgroup of G with $D \cap E = 1$, then D is minimal normal in $N_G(Q)$ and DE/E is minimal normal in G/E.*

Proof. (1) The statement follows from the Frattini argument.

(2) Since D is a minimal normal subgroup of G with $D \cap E = 1$, it follows from $G = EN_G(Q)$ that D is minimal normal in $N_G(Q)$. Let $X/E \trianglelefteq G/E$ so that $E < X \leq DE$ and hence $X = X \cap DE = (X \cap D)E$. Since $1 < X \cap D \trianglelefteq G$, we have $X \cap D = D$ and therefore $X = DE$, as desired. □

Lemma 2.4. *Let $\Gamma = G \ltimes V$, where V is an abelian minimal normal subgroup of Γ so that $p \mid |V|$. Suppose that $O_{p'}(\Gamma/V) > 1$ and $O_{p'}(\Gamma) = 1$. Then all complements of V in Γ are conjugate.*

Proof. Write $L/V = O_{p'}(\Gamma/V)$ so that $G \cap L \trianglelefteq G$ and $N_\Gamma(G \cap L) = GN_V(G \cap L)$. Since V is an abelian minimal normal subgroup of Γ, we conclude that $N_V(G \cap L) \in \{1, V\}$. It follows from $O_{p'}(\Gamma) = 1$ that $N_V(G \cap L) = 1$ and $G = N_\Gamma(G \cap L)$. Let H be another complement for V in Γ so that $H = N_\Gamma(H \cap L)$. Since $H \cap L$ and $G \cap L$ are both complements of V in L, it follows that $(H \cap L)^x = G \cap L$ for some $x \in V$ by the Schur–Zassenhaus theorem. Thus, $H^x = (N_\Gamma(H \cap L))^x = N_\Gamma(G \cap L) = G$, as desired. □

Now we present some lemmas related to nonabelian simple groups.

Lemma 2.5 [8]. *Let S be a nonabelian simple group and $S \leq G \leq \mathrm{Aut}(S)$. Suppose that $p \mid (|S|, |G : S|)$. Then G has nonabelian Sylow p-subgroups.*

Lemma 2.6 [11]. *Let $S \leq G \leq \mathrm{Aut}(S)$, where S is a nonabelian simple group with $p \mid |S|$. Then G is a $\mathfrak{C}_p$-group if and only if one of the following holds:*

(1) $S = \mathcal{A}_p$;
(2) $S = G = \mathrm{PSL}(2, 11)$, $p = 11$;
(3) $S = M_{11}$, $p = 11$;
(4) $S = M_{23}$, $p = 23$;
(5) $S = \mathrm{PSL}(n, q)$, $p = (q^n - 1)/(q - 1)$ *and* $G/S \leq \mathrm{Out}_{df}(S)$.

https://doi.org/10.1017/S0004972720000246 Published online by Cambridge University Press

3. The proof of the main theorem

Lemma 3.1. *For a $\mathfrak{C}_{p^d}$-group G, the following statements hold.*

(1) *If $H \leq G$, then H is also a $\mathfrak{C}_{p^d}$-group.*
(2) *If N is minimal normal in G with $p \mid |N|$, then N is a direct product of at most d copies of a simple group S.*
(3) *If N is a normal subgroup of G with $|N|_p = p^e \leq p^d$, then G/N is a $\mathfrak{C}_{p^{d-e}}$-group. Furthermore, if $e = 0$, then G/N being a $\mathfrak{C}_{p^{d-e}}$-group implies that G is a $\mathfrak{C}_{p^d}$-group.*
(4) *If $|G|_p \geq p^d$ and $N \lhd\lhd G$ with $|N|_p = p^e$, then N is a $\mathfrak{C}_{p^m}$-group, where $m = \min\{d, e\}$.*
(5) *G is a $\mathfrak{C}_{p^{md}}$-group for every positive integer m.*

Proof. (1) If $P \leq H$ is of order p^d and K is a complement of P in G, then $K \cap H$ is a complement of P in H.

(2) Suppose that $|N|_p \geq p^d$. Let $P \leq N$ be of order p^d and let K be a complement of P in G. Then $G = KP = KN$ with $|G : K| = p^d$. Thus, (2) follows from Lemma 2.2.

(3) Suppose that N is a p-group. Let $Q/N \leq G/N$ be of order p^{d-e}. Then Q has order p^d and has a complement K in G. Now KN/N is a complement of Q/N in G/N.

Suppose that N is not a p-group. Let $P \in \mathrm{Syl}_p(N)$. We proceed by induction on e. For $e = 0$, the statement can be checked by a standard argument. By Lemma 2.3(1), we know that $G/N \cong N_G(P)/N_N(P)$ and $|G|_p = |N_G(P)|_p$. Therefore, $N_G(P)$ is a $\mathfrak{C}_{p^d}$-group by (1). According to the previous statement, $N_G(P)/P$ is a $\mathfrak{C}_{p^{d-e}}$-group. Since $|N_N(P)/P|_p = p^0$, it follows that $G/N \cong N_G(P)/N_N(P)$ is a $\mathfrak{C}_{p^{d-e}}$-group by induction.

Suppose that $e = 0$ and hence N is a p'-group. Let $Q \leq G$ be of order p^d. Then $QN/N \leq G/N$ also has order p^d. Note that G/N is a $\mathfrak{C}_{p^d}$-group and QN/N has a complement K/N in G/N. Now K is a complement of Q in G, as desired.

(4) We may assume by (1) and induction that $m = e < d$ and $N \trianglelefteq G$. Let $P \in \mathrm{Syl}_p(N)$ and Q be a p-subgroup of order p^d with $P < Q$. Observe that $Q \in \mathrm{Syl}_p(QN)$ has a complement K in QN by (1). It follows that K is also a complement of P in N.

(5) The statement follows directly from [6, Lemma 1(1)]. □

Remark 3.2. If $e > 0$ in (3), then the fact that G/N is a $\mathfrak{C}_{p^{d-e}}$-group does not imply that G is a $\mathfrak{C}_{p^d}$-group. For example, if G is an extraspecial p-group of order p^5 and $N = \Phi(G)$, then G is not a $\mathfrak{C}_{p^2}$-group (see Lemma 3.3), even if G/N is a $\mathfrak{C}_p$-group.

Lemma 3.3. *Let G be a group of order p^n with $n \geq 3$. Then G is a $\mathfrak{C}_{p^2}$-group if and only if G is isomorphic to $E(p^n)$, D_8 or $M_p(1, 1, 1)$.*

Proof. We only prove the necessity. Write $|\Phi(G)| = p^s$. Assume that G is not elementary abelian. Then $s \geq 1$ and $2 \cdot 2 \geq n + s$ by [9, Proposition F]. Hence, $n = 3$ and G is an extraspecial p-group of order p^3. Let $\Omega_1(G)$ be the subgroup of G generated by all elements of order p. Since G is a $\mathfrak{C}_{p^2}$-group, as shown in [9, Proposition F], $\Omega_1(G) = G$. It follows that $G \cong D_8$ when $p = 2$. Now suppose that $p > 2$. Since G has class two, all elements in $\Omega_1(G)$ have order p. This implies that $G \cong M_p(1, 1, 1)$. □

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Lemma 3.4. *Let* $S \leq G \leq \mathrm{Aut}(S)$, *where* S *is a nonabelian simple group with* $p \mid |S|$, *and let* $P \in \mathrm{Syl}_p(G)$. *Then* G *is a nontrivial* $\mathfrak{C}_{p^2}$*-group if and only if* $S \cong \mathrm{PSL}(n, q)$, $G/S \leq \mathrm{Out}_{df}(S)$ *and* $P \cong \mathbb{Z}_{p^2}$, *where* $p^2 = (q^n - 1)/(q - 1) \geq 9$.

Proof. ($\Leftarrow$) Suppose that $S \cong \mathrm{PSL}(n, q)$. Since $|G|_p = p^2$, to see that G is a $\mathfrak{C}_{p^2}$-group, it suffices to show that G admits a Hall p'-subgroup. Let $H \leq S$ be a stabiliser of a line (or a hyperplane) so that H is a maximal subgroup of S. From $|S|_p = p^2$, it follows that $H \in \mathrm{Hall}_{p'}(S)$. Conversely, by [3, Theorem 1], every Hall p'-subgroup of S is also a stabiliser of a line or a hyperplane. Now we claim that $|G : N_G(H)| = |S : N_S(H)|$. If $n = 2$, the claim follows because all Hall p'-subgroups of S are conjugate in S by [11, Lemma 2.3]. If $n \geq 3$, since $\{H^s \mid s \in S\}$ is fixed by both diagonal and field automorphisms of S by [11, Lemma 2.4], then $|\{H^g \mid g \in G\}| = |\{H^s \mid s \in S\}|$. This yields $|G : N_G(H)| = |S : N_S(H)|$. Now we conclude that $|G : N_G(H)| = |S : N_S(H)| = |S : H| = p^2$, where the second equality holds because H is a maximal subgroup of S. Therefore, $N_G(H)$ is a Hall p'-subgroup of G.

($\Rightarrow$) Suppose that G is a nontrivial $\mathfrak{C}_{p^2}$-group. If $|S|_p = p$, then Lemma 2.5 implies that $|\mathrm{Aut}(S)|_p = p$, which is a contradiction. Hence, $|S|_p \geq p^2$ and S is a nontrivial $\mathfrak{C}_{p^2}$-group by Lemma 3.1(1). Note that if $S \cong \mathcal{A}_{p^2}$, where $p^2 > 4$, then $Q \in \mathrm{Syl}_p(\mathcal{A}_{p^2})$ is also a $\mathfrak{C}_{p^2}$-group. However, Q is not elementary abelian with $|Q| \geq p^4$ and Lemma 3.3 yields a contradiction. Since S is a $\mathfrak{C}_{p^2}$-group and thus admits a subgroup with index p^2, we conclude by [3, Theorem 1] that $S \cong \mathrm{PSL}(n, q)$, where $p^2 = (q^n - 1)/(q - 1)$ and n is a prime.

Note that if $p = 2$, then S has a subgroup of index four and this implies that $S \lesssim S_4$ is solvable. Hence, $p \geq 3$. Suppose that $n = p$. Then $p^2 = (q^n - 1)/(q - 1) \geq 2^n - 1 = 2^p - 1$ and hence $p = 3$ and $9 = p^2 = q^2 + q + 1$, which is a contradiction. Therefore, $n \neq p$ and hence $(n, (q^n - 1)/(q - 1)) = 1$. It follows from [5, Kap. II, Theorem 7.3] that $\mathrm{PSL}(n, q)$ admits a cyclic subgroup (Singer cycle) of order $(q^n - 1)/(q - 1) = p^2$. Since $P \in \mathrm{Syl}_p(G)$ and P is a $\mathfrak{C}_{p^2}$-group which has an element of order p^2, it follows from Lemma 3.3 that $P \cong \mathbb{Z}_{p^2}$ since $p \geq 3$.

Since G is a $\mathfrak{C}_{p^2}$-group, P admits a complement K in G. We may assume that $n \geq 3$ provided that $\mathrm{PSL}(2, q)$ has only the trivial graph automorphism. It follows from $G = PK = SK$ that $|S : K \cap S| = p^2$. Applying [3, Theorem 1] to S and $K \cap S$ shows that $K \cap S$ is a stabiliser of a line or a hyperplane. Suppose that G/S is not isomorphic to a subgroup of $\mathrm{Out}_{df}(S)$. Then, by [1], there is $\theta \in G/S$ such that $\theta = dgf$, where d, g $(\neq 1)$ and f are respectively a diagonal, a graph and a field automorphism of S. Since $S = \mathrm{PSL}(n, q)$ with $n \geq 3$, we have $o(g) = 2$. By [11, Lemma 2.4],

$$|G : N_G(K \cap S)| = 2|S : N_S(K \cap S)| = 2|S : K \cap S| = 2p^2.$$

But $K \leq N_G(K \cap S)$ and $|G : K| = p^2$, which is a contradiction. Therefore, $G/S \leq \mathrm{Out}_{df}(S)$. □

Lemma 3.5. *Let* $G = A \times B$ *be a* $\mathfrak{C}_{p^2}$*-group, where* $|A|_p \geq p$ *and* $|B|_p \geq p$. *Then* B *is a* $\mathfrak{C}_p$*-group. If in addition* B *is simple, then* $|B|_p = p$.

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Proof. By induction, we may assume that $|A| = p$. Now Lemma 3.1(3) implies that $B \cong G/A$ is a $\mathfrak{C}_p$-group. Assume that B is simple. Since B has a subgroup of index p, we have $B \lesssim \mathcal{S}_p$ and therefore $|B|_p = p$. □

Hypothesis 3.6. *Let G be a nontrivial $\mathfrak{C}_{p^2}$-group with $O_{p'}(G) = 1$ and let $P \in \mathrm{Syl}_p(G)$.*

Lemma 3.7. *Assume that G satisfies Hypothesis 3.6. Then all components of G are simple. In particular, $E(G) \leq \mathrm{Soc}(G)$ and $F^*(G) = F(G) \times E(G)$.*

Proof. Let S be a component of G. By Lemma 3.1(4), S is a $\mathfrak{C}_{p^d}$-group with $d = \min\{2, \log_p |S|_p\}$. Also, since $O_{p'}(G) = 1$, we know that $Z(S)$ is a p-group with $|S|_p \geq p$. Assume that $Z(S) > 1$. Note that $\Phi(S) = Z(S)$ for a quasisimple group S and $\Phi(S)$ is not complemented in S. It follows that $|Z(S)| = |\Phi(S)| = p$ and $|S/Z(S)|_p \geq p$. Since $S/Z(S)$ is a $\mathfrak{C}_p$-group by Lemma 3.1(3), by Lemma 2.6, $S/Z(S)$ is isomorphic to $\mathcal{A}_p, \mathrm{PSL}(2, 11), M_{11}, M_{23}$ or $\mathrm{PSL}(n, q)$ with $p = (q^n - 1)/(q - 1)$. Checking the Schur multipliers of those groups in [1], we get a contradiction. Hence, S is simple. □

Lemma 3.8. *Assume that G satisfies Hypothesis 3.6 and admits different minimal normal subgroups. Then one of the following statements holds.*

(1) *All minimal normal subgroups E of G are simple with $|E|_p = p$.*
(2) *All minimal normal subgroups E of G are isomorphic to $E(p^2)$.*

Proof. Let D and E be different minimal normal subgroups of G. Since G satisfies Hypothesis 3.6, $|D|_p, |E|_p \in \{p,\ p^2\}$ by Lemmas 3.1(2), 3.4 and 3.5.

(1) Suppose that $|E|_p = p$. By Lemma 3.1(3), G/E is a $\mathfrak{C}_p$-group. If $L/E = O_{p'}(G/E)$, then $D \cap L = 1$. It follows from Lemmas 2.3(2) and 3.1(3) that G/L is a $\mathfrak{C}_p$-group with a minimal normal subgroup DL/L. Since $|DL/L| = |D/D \cap L| = |D|$, we conclude from [11, Theorem 1.1] that $|D|_p = |DL/L|_p = p$, as desired.

(2) Suppose that $|E|_p = |D|_p = p^2$. Write $P \in \mathrm{Syl}_p(G)$ so that $|P| \geq |DE|_p = p^4$. It follows from Lemma 3.3 that P is elementary abelian and hence $E = S_1 \times S_2$, where S_i are simple by Lemma 3.4. Suppose that $Q \in \mathrm{Syl}_p(E)$, $H = N_G(Q)$, $\overline{H} = H/O_{p'}(H)$ and $\overline{U}(\leq \overline{Q})$ is a minimal normal subgroup of $\overline{H}$. By Lemmas 2.3 and 3.1(3), $\overline{H}$ satisfies Hypothesis 3.6, so it has different minimal normal subgroups $\overline{U}(\lesssim E(p^2))$ and $\overline{D}$ (with $|\overline{D}|_p = p^2$). It follows by (1) that $\overline{U} = \overline{Q} \cong E(p^2)$. Thus, by induction, we may assume that one of D and E is abelian, say E. Let $x \in D$ and $y \in E$ both be elements of order p so that $G = \langle x, y\rangle M$, where $\langle x, y\rangle \cap M = 1$, and $|G : M| = p^2$ and $G = (D \times E)M$. Write $N = D \times E$ so that $N \cap M$ is a complement of $\langle x, y\rangle$ in N. Considering $N_N(M \cap D)$, we have $M \cap N \leq N_N(M \cap D) = N_D(M \cap D) \times E$. Since E is abelian, $E \cap M \trianglelefteq G$. Now $y \notin M$, so $E \cap (M \cap N) \leq E \cap M = 1$. However, $|N : M \cap N| = |MN : M| = p^2$. It follows that $N = (M \cap N)E \leq N_D(M \cap D) \times E \leq N$. Hence, $M \cap D = 1$ provided that $M \cap D \trianglelefteq G$ and $x \notin M$. Thus, $|D| = |DM|/|M| = |G|/|M| = p^2$. Thus, D and E are both isomorphic to $E(p^2)$. □

Lemma 3.9. *Assume that G and P satisfy Hypothesis 3.6 with $\Phi(G) > 1$. Then P is not elementary abelian. Moreover, each of the following statements is true.*

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(1) *$G = H \ltimes P$, where $H \lesssim \mathbb{Z}_{p-1}$ and $P \cong \mathbb{Z}_{p^2}$ such that $C_H(P) = 1$.*
(2) *$G = B \ltimes V$, where $V \cong E(p^2)$ with $C_G(V) = V$ and B is a subgroup of a Borel subgroup of $\mathrm{GL}(V)$ with $|B|_p = p$.*

Proof. Let $\Phi = \Phi(G)$. Since G is a nontrivial $\mathfrak{C}_{p^2}$-group with $\Phi > 1$, it follows from $O_{p'}(G) = 1$ that $|\Phi| = p$. By Lemma 3.1(3), $\overline{G} := G/\Phi$ is a nontrivial $\mathfrak{C}_p$-group. Let $\overline{N} := N/\Phi$ be a minimal normal subgroup of $\overline{G}$ and note that $G/C_G(\Phi) \lesssim \mathrm{Aut}(\Phi) \cong \mathbb{Z}_{p-1}$ and $O_{p'}(\overline{G}) = 1$. We conclude from [11, Theorem 1.1] that $\overline{N} \leq F^*(\overline{G}) = O^{p'}(\overline{G}) \leq \overline{C_G(\Phi)}$ and $|\overline{N}|_p = p$. Then $N \leq C_G(\Phi)$ and hence $\Phi \leq Z(N)$. Since N is not quasisimple by Lemma 3.7, $\overline{N}$ is either abelian or $N = S \times \Phi$ with S $(\trianglelefteq G)$ a nonabelian simple group. Suppose that $N = S \times \Phi$ with S a nonabelian simple group. Let U be a subgroup of S with order p and K a complement of ΦU in G. (We remark that K exists because G is a $\mathfrak{C}_{p^2}$-group.) Therefore, $G = \Phi S K = SK$, which is a contradiction. Since every minimal normal subgroup of $\overline{G}$ is abelian, $\overline{P} = F^*(\overline{G}) = O^{p'}(\overline{G}) \trianglelefteq \overline{G}$ by [11, Theorem 1.1] and hence $P \trianglelefteq G$. Thus, $\Phi(P) = P \cap \Phi = \Phi$ by [10, Lemma 2.4] and therefore P is not elementary abelian. By Lemma 3.3, P is isomorphic to one of the groups $\mathbb{Z}_{p^2}$, D_8 and $M_p(1,1,1)$. Also, $G = H \ltimes P$ with $H \in \mathrm{Hall}_{p'}(G)$ by the Schur–Zassenhaus theorem.

(1) Suppose that $P \cong \mathbb{Z}_{p^2}$. Since $\overline{P} = F^*(\overline{G})$, it follows from $\overline{C_G(P)} \leq C_{\overline{G}}(\overline{P}) \leq \overline{P}$ that $C_G(P) = P$. Thus,
$$G/P = G/C_G(P) \lesssim \mathrm{Aut}(P) \cong \mathbb{Z}_p \times \mathbb{Z}_{p-1}.$$
Hence, $G = H \ltimes P$, where $H \lesssim \mathbb{Z}_{p-1}$ and $P \cong \mathbb{Z}_{p^2}$ such that $C_H(P) = 1$.

(2) Suppose that $P \cong D_8$ or $M_p(1,1,1)$. Since $|N| = |\overline{N}||\Phi| = p^2$, it follows that $C_G(N)$ is a $\mathfrak{C}_{p^2}$-group by Lemma 3.1(1) and hence $C_G(N) = N \times L$, where L is a Hall p'-subgroup of $C_G(N)$. Therefore, $L \trianglelefteq G$. However, $O_{p'}(G) = 1$, so $L = 1$ and then $C_G(N) = N$. It follows that $G = B \ltimes N$ and $B \cong G/C_G(N) \lesssim \mathrm{Aut}(N)$, since G is a $\mathfrak{C}_{p^2}$-group. If $N \cong \mathbb{Z}_{p^2}$, then $G \cong D_8$ with $p = 2$. Suppose that $N \cong E(p^2)$. Then $B \lesssim \mathrm{GL}(2,p)$. As $P \cap B \trianglelefteq B$ and $|\mathrm{GL}(2,p)|_p = |P \cap B| = p$, we see that B is isomorphic to a subgroup of a Borel subgroup of $\mathrm{GL}(2,p)$. Note that D_8 is also of this type. Our result follows. □

Lemma 3.10. *Assume that G and P satisfy Hypothesis 3.6 with $\Phi(G) = 1$. Then P is abelian.*

Proof. Let G be a counterexample of minimal order. Then P is a nonabelian subgroup of order p^3 by Lemma 3.3. Since the hypothesis is inherited by normal subgroups with order divisible by p^2, P is not normal in G. Furthermore, if $N \triangleleft G$, then $|N|_p \leq p^2$. Hence, $G = O^{p'}(G)$ does not have a nontrivial direct factor provided that $O_{p'}(G) = 1$. Let E be a minimal normal subgroup of G. Suppose that $|E|_p = p$. Then $p \nmid |G/EC_G(E)|$ by Lemma 2.5 and hence $G = EC_G(E)$. If E is nonabelian, then $G = E \times C_G(E)$, which is a contradiction. If E is abelian, then $G = C_G(E) = E \times M$, where M is a maximal subgroup of G since $\Phi(G) = 1$, which is a contradiction too. Hence, $|E|_p = p^2$. By Lemma 3.4, $F^*(G) = \mathrm{Soc}(G)$ as $\Phi(G) = 1$ and hence $|F^*(G)|_p \leq p^2$. It follows that $E = F^*(G)$. If E is nonabelian simple, then $|G|_p = p^2$ by Lemma 3.4, which is a

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contradiction. Suppose that $E = S_1 \times S_2$, where the S_i are isomorphic nonabelian simple groups. Then S_i are $\mathfrak{C}_p$-groups by Lemma 3.1(4) and hence $p > 2$. Notice that $G/E \lesssim \mathrm{Out}(E) \cong \mathbb{Z}_2 \ltimes (\mathrm{Out}(S_1) \times \mathrm{Out}(S_2))$. It follows from Lemma 2.5 that $p \nmid |G/E|$, which is a contradiction. Hence, $E \cong E(p^2)$. Since G is a $\mathfrak{C}_{p^2}$-group with $F^*(G) = E$, we know that $G = E \rtimes H$ and $H \lesssim \mathrm{GL}(2, p)$. Since H has more than two Sylow p-subgroups, $H \cong \mathrm{SL}(2, p)$ provided that $G = O^{p'}(G)$. Let $U = (P \cap H)Z(P)$ and K be a complement of U in G. We conclude from $E \not\leq K$ and $|G : K| = |E| = p^2$ that $G = E \rtimes K$. Since $O_{p'}(G) = 1$ and $O_{p'}(G/E) > 1$, it follows from Lemma 2.4 that $H^x = K$ for some $x \in E$. Then $G = UK = UH^x = U^{x^{-1}}H = UH$, where the last equality holds since $U \lhd P$. However, $U \cap H = P \cap H > 1$, which is a contradiction. Thus, P is abelian, as desired. □

Proof of Theorem 1.1. Let P be a Sylow p-subgroup of G. Note that T is a $\mathfrak{C}_{p^2}$-group if and only if $G = T/O_{p'}(T)$ is a $\mathfrak{C}_{p^2}$-group by Lemma 3.1(3).

(⇒) Suppose first that $\Phi(G) > 1$. Then (1) and (2) follow by Lemma 3.9.

Now suppose that $\Phi(G) = 1$. Then $F^*(G) = F(G) \times E(G) = \mathrm{Soc}(G)$ by Lemma 3.7 and P is abelian by Lemma 3.10. If G is simple, then, by Lemma 3.4, $G \cong \mathrm{PSL}(n, q)$, where $|G|_p = (q^n - 1)/(q - 1) = p^2 \geq 9$.

Suppose G contains the unique minimal normal subgroup E. Then $E = F^*(G)$ and $C_G(E) \leq E$. If E is abelian, then $E = P$ and therefore $G = H \ltimes E$, where $H \in \mathrm{Hall}_{p'}(G)$ and $E \cong E(p^2)$. Suppose that E is nonabelian. If E is simple, then our result follows from Lemma 3.4. Otherwise, by Lemmas 3.1(2) and 3.5, $E = S_1 \times S_2$, where the S_i are isomorphic nonabelian simple groups with $|S_i|_p = p$ provided that $O_{p'}(G) = 1$. Since $E = S_1 \times S_2$ is minimal normal in G, we have that $M := N_G(S_1) = N_G(S_2)$ is normal in G with index two. Since p is the largest prime divisor of $|S_i|$, we have $O^{p'}(G) = O^{p'}(M)$. Now M is a $\mathfrak{C}_{p^2}$-group with $O_{p'}(M) = 1$ and, from Lemma 3.1(3), M/S_1 and M/S_2 are both $\mathfrak{C}_p$-groups. Therefore, $M \lesssim M/S_1 \times M/S_2$ is also a $\mathfrak{C}_p$-group by [11, Lemma 3.2]. Applying [11, Theorem 1.1], $F^*(M) = O^{p'}(M)$. Since $F^*(M) = M \cap F^*(G) = F^*(G)$ because $M \trianglelefteq G$, it follows that $F^*(G) = O^{p'}(G)$.

Suppose that G contains at least two minimal normal subgroups. Then we can write $F^*(G) = N_1 \times \cdots \times N_t$, where $t > 1$, and so either $|N_i|_p = p$ for all i or $N_i \cong E(p^2)$ for all i by Lemma 3.8(2). Suppose that $|N_i|_p = p$ for all i. Then, by Lemma 3.1(3), we know that G/N_1 and G/N_2 are nontrivial $\mathfrak{C}_p$-groups. We deduce from $G \lesssim G/N_1 \times G/N_2$ that G is a nontrivial $\mathfrak{C}_p$-group by [11, Lemma 3.2]. Thus, (6) holds. Suppose that $N_i \cong E(p^2)$. Since $|P| \geq p^4$, we see that P is elementary abelian by Lemma 3.3 and hence $P = F^*(G)$. It follows that $G = H \ltimes P$, where $H \in \mathrm{Hall}_{p'}(G)$. Then, by [10, Corollary 2.10] and [10, Theorem B′], (5) holds.

(⇐) By Lemma 3.1(5), Lemma 3.4, [10, Corollary 2.10] and [10, Theorem B′], it suffices to verify when (2) or (4) holds.

Suppose that (2) holds. Then $B = (P \cap B) \rtimes K$, where $|K| \mid (p-1)^2$ since B is a subgroup of a Borel subgroup of $\mathrm{GL}(V)$. Furthermore, since $P \cap B \trianglelefteq B$, we know that $P = (P \cap B)V \trianglelefteq G$. Considering the coprime action of K on V by conjugation, $V = Q \times Z(P)$, where Q and $Z(P)$ are both K-invariant. Choose a subgroup U of

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order p^2. We may assume that $U \neq V$. Then $Q \nleq U$ and hence $G = U \cdot (QK)$. Now $U \cap (QK) = 1$ follows from $P \cap K = 1$. Thus, G is a $\mathfrak{C}_{p^2}$-group, as desired.

Suppose that (4) holds. Since $|N_G(S_1)|_p = |G|_p = p^2$, it follows that $N_G(S_1)$ is a $\mathfrak{C}_{p^2}$-group by Lemma 3.1(5). Hence, $P \leq F^*(G) \leq N_G(S_1)$. Therefore, $N_G(S_1) = P \cdot H$, where $P \cap H = 1$. Since $N_G(S_1) = F^*(G)H$, there is a maximal subgroup M of G such that $H < M$. It follows that $G = F^*(G)M$ and $|G : M| \mid |G : H| = 2p^2$. We conclude from Lemma 2.1 that $|G : M| = p^2$ and thus G is a $\mathfrak{C}_{p^2}$-group. □

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https://doi.org/10.1017/S0004972720000246 Published online by Cambridge University Press