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ON THE LIMIT CYCLES OF POLYNOMIAL DIFFERENTIAL SYSTEMS WITH HOMOGENEOUS NONLINEARITIES

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Abstract We consider three classes of polynomial differential equations of the form $\dot{x} = -y + P_n(x, y)$, $\dot{y} = x + Q_n(x, y)$, where P_n and Q_n are homogeneous polynomials of degree n, having a non-Hamiltonian centre at the origin. By using a method different from the classical ones, we study the limit cycles that bifurcate from the periodic orbits of such centres when we perturb them inside the class of all polynomial differential systems of the above form. A more detailed study is made for the particular cases of degree $n = 2$ and $n = 3$.

Keywords: limit cycles; centres; bifurcation

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1. Introduction

The main problem in the qualitative theory of real planar differential systems is the determination of limit cycles. Limit cycles of planar vector fields were defined by Poincaré [**30**] and started to be studied intensively at the end of the 1920s by van der Pol [**34**], Liénard [**24**] and Andronov [**1**].

One of the classical ways to produce limit cycles is by perturbing a system which has a centre in such a way that limit cycles bifurcate in the perturbed system from some of the periodic orbits of the unperturbed system (see, for example, [**31**]).

In this paper we consider three subclasses of real planar polynomial differential systems of the form

$$\dot{x} = -y + P_n(x, y), \qquad \dot{y} = x + Q_n(x, y), \tag{1.1}$$

where P_n and Q_n are homogeneous polynomials of degree n having a centre at the origin, and we study the limit cycles which bifurcate from their periodic orbits when we perturb such subclasses inside the class of all system (1.1) (see Theorem 2.3). In the future we want to study the perturbation of the centres of system (1.1) with $n = 2$ inside the class of all polynomial systems of arbitrary degree.

There are essentially three methods for studying the bifurcated limit cycles from a centre. The first is based on the Poincaré return map (see, for example, [**3, 9**]); the second on the Poincaré–Melnikov integral or abelian integral, which are equivalent in the plane (see [**3**], § 6 of Chapter 4 of [**20**], and § 5 of Chapter 6 of [**2**]); and the third is based on Theorem 9 of [**16**] (see § 6 of this paper and [**17**]). These methods have been used by several authors for studying bifurcated limit cycles from a centre of a Hamiltonian system, or of a system which can be reduced after a change of variables to a Hamiltonian one. In general, these methods are difficult to apply for studying the limit cycles that bifurcate from the periodic orbits of a centre when the system is integrable but not Hamiltonian. As far as we know there are few papers that study non-Hamiltonian centres (see, for example, [**9, 12, 18**]).

The method that we have used in the proof of Theorem 2.3 is different and easier to apply to the three classes of non-Hamiltonian centres studied here. The difference consists in reducing the problem to a one-dimensional differential equation where explicit integral expressions (easier than the usual abelian integrals for these classes of problems) for the solution of the first variational equation can be computed. For more details on first variational equations (see, for example, [**10**]). This method is inspired by ideas that appeared in a paper of Lins Neto [**27**] when he studied a class of Abel differential equations.

When $n = 2$, system (1.1) are quadratic polynomial differential systems (or simply *quadratic systems* in what follows). Quadratic systems have been intensively studied for the last 30 years, and more than a thousand papers have been published on them (see, for example, the bibliographical survey of Reyn [**32**]).

There are 32 topologically different phase portraits of quadratic systems having a centre (see Vulpe [**35**]). The three subclasses of system (1.1) studied here for $n = 2$ are classes 29, 15 and 20 in Vulpe's classification.

We note that the classes of systems 29 and 15 studied here contain the full classes 29 and 15 of Vulpe. But our class of systems 20 is only a subclass of all quadratic systems having phase portrait topologically equivalent to class 20 of Vulpe. In Theorem 2.4 we study how many limit cycles bifurcated from the periodic orbits of these quadratic centres.

Many authors have studied the limit cycles which bifurcate from periodic orbits of a centre for a quadratic system (see, for example, [**21, 23, 26, 28, 29, 33, 36**]).

In Theorems 2.5 and 2.6 we study the number of limit cycles which bifurcate from the periodic orbits of two classes of these centres when the degree of the system is $n = 3$.

The main results of the paper, i.e. Theorems 2.3, 2.4, 2.5 and 2.6, are stated in § 2. The other sections are dedicated to their proofs.

2. Statement of the results

In order to be more precise we need some preliminary notation and results. Thus, in polar coordinates (r, θ) defined by $x = r\cos\theta$, $y = r\sin\theta$, system (1.1) becomes

$$\dot{r} = f(\theta)r^n, \qquad \dot{\theta} = 1 + g(\theta)r^{n-1}, \tag{2.1}$$

where

$$f(\theta) = \cos\theta P_n(\cos\theta, \sin\theta) + \sin\theta Q_n(\cos\theta, \sin\theta),$$
$$g(\theta) = \cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\cos\theta, \sin\theta).$$

We remark that f and g are homogeneous polynomials of degree $n+1$ in the variables $\cos\theta$ and $\sin\theta$. In the region

$$R = \{(r, \theta) : 1 + g(\theta) r^{n-1} > 0\},$$

the differential system (2.1) is equivalent to the differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{f(\theta) r^n}{1 + g(\theta) r^{n-1}}. \tag{2.2}$$

It is known that the periodic orbits surrounding the origin of system (2.1) do not intersect the curve $\dot{\theta} = 0$ (see the Appendix of [**5**]). Therefore, these periodic orbits are contained in the region R, and consequently they are also periodic orbits of equation (2.2).

The transformation $(r, \theta) \mapsto (\rho, \theta)$ with

$$\rho = \frac{r^{n-1}}{1 + g(\theta) r^{n-1}}$$

is a diffeomorphism from the region R into its image. As far as we know, the first to use this transformation was Cherkas in [**7**]. If we write equation (2.2) in the variable ρ we obtain

$$\frac{\mathrm{d}\rho}{\mathrm{d}\theta} = -(n-1) f(\theta) g(\theta) \rho^3 + [(n-1) f(\theta) - g'(\theta)] \rho^2, \tag{2.3}$$

which is a particular case of an Abel differential equation. These differential equations appeared in the studies of Abel on the theory of elliptic functions. For more details on Abel differential equations see [**22**]. In fact we have proved the following result.

Proposition 2.1. *The function $r = r(\theta)$ is a periodic solution of system (2.1) surrounding the origin if and only if $\rho(\theta) = r(\theta)^{n-1}/(1 + g(\theta) r(\theta)^{n-1})$ is a periodic solution of the Abel differential equation (2.3).*

The class $\boldsymbol{H}$ is defined by (1.1). We say that a system of the class $\boldsymbol{H}$ belongs to the subclass $\boldsymbol{F}$ if its function $f(\theta)$ is identically zero. We note that the orbits of those systems have the polar coordinate r constant. We say that a system of the class $\boldsymbol{H}$ belongs to the subclass $\boldsymbol{G}$ if its function $g(\theta)$ is identically zero and $\int_0^{2\pi} f(\theta)\, \mathrm{d}\theta = 0$. We note that if the degree n of system (1.1) is even, then the condition $\int_0^{2\pi} f(\theta)\, \mathrm{d}\theta = 0$ is always satisfied. Also we remark that $g(\theta) \equiv 0$ means that the infinity of system (1.1) in the Poincaré compactification is fulfilled of singular points. For more details on the Poincaré compactification, see, for example, [**19**]. Additionally, $g(\theta) \equiv 0$ implies that $\dot{\theta} = 1$, and consequently the centre is isochronous.

We say that a system of the class $\boldsymbol{H}$ belongs to the subclass $\boldsymbol{C}$ if the function $(n-1)f(\theta)-g'(\theta)$ is identically zero.

An easy computation shows that the subclass $\boldsymbol{F}$ has dimension $2(n+1)-(n+2)=n$ inside the class $\boldsymbol{H}$. The subclass $\boldsymbol{G}$ has dimension n if n is even, or $n-1$ if n is odd inside the class $\boldsymbol{H}$. Finally, the subclass $\boldsymbol{C}$ has dimension n inside the class $\boldsymbol{H}$.

Using the expression of equation (2.3) it is easy to prove the following results.

Proposition 2.2.

(a) *Systems of class $\boldsymbol{F}$ have a centre at the origin, and in coordinates (ρ,θ) their periodic solutions contained in the region R are given by*

$$\rho(\theta,z)=\frac{z}{1+(g(\theta)-g(0))z}, \tag{2.4}$$

where $z\in(0,\min_\theta|g(\theta)-g(0)|^{-1})$.

(b) *Systems of class $\boldsymbol{G}$ have a centre at the origin, and in coordinates (ρ,θ) their periodic solutions contained in the region R are given by*

$$\rho(\theta,z)=\frac{z}{1-F(\theta)z}, \tag{2.5}$$

where $z\in(0,\min_\theta|F(\theta)|^{-1})$ and $F(\theta)=(n-1)\int_0^\theta f(t)\,\mathrm{d}t$.

(c) *Systems of class $\boldsymbol{C}$ have a centre at the origin, and in coordinates (ρ,θ) their periodic solutions contained in the region R are given by*

$$\rho(\theta,z)=\frac{z}{\sqrt{1+2(n-1)G(\theta)z^2}}, \tag{2.6}$$

where $z\in(0,\min_\theta|G(\theta)|^{-1})$ and $G(\theta)=\int_0^\theta f(t)g(t)\,\mathrm{d}t$.

We note that the function $F(\theta)$ cannot be identically zero, otherwise $f(\theta)\equiv 0$ and, since $g(\theta)\equiv 0$ in class $\boldsymbol{G}$, we would have that $P_n(x,y)\equiv 0$ and $Q_n(x,y)\equiv 0$, in contradiction with the assumptions, because P_n and Q_n must be homogeneous polynomials of degree n. We also note that the function $G(\theta)$ cannot be identically zero; otherwise $f(\theta)g(\theta)\equiv 0$, consequently either $f(\theta)\equiv 0$ or $g(\theta)\equiv 0$, and, since $(n-1)f(\theta)-g'(\theta)\equiv 0$ in class $\boldsymbol{C}$, it follows that $f(\theta)\equiv 0$ and $g(\theta)\equiv 0$, and we would arrive at the above contradiction.

Three of our main results are as follows.

Theorem 2.3. *The following statements hold.*

(1) *We assume that system (1.1) belongs to class $\boldsymbol{F}$. Then for ϵ sufficiently small the system*

$$\dot{x}=-y+P_n(x,y)+\epsilon\bar{P}_n(x,y),\qquad \dot{y}=x+Q_n(x,y)+\epsilon\bar{Q}_n(x,y), \tag{2.7}$$

of class $\boldsymbol{H}$ has a hyperbolic limit cycle surrounding the origin for each root z of the function

$$U(z)=z^2(1-g(0)z)\int_0^{2\pi}\frac{\bar{f}(\theta)\,\mathrm{d}\theta}{1+(g(\theta)-g(0))z},$$

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such that $U'(z) \neq 0$ and $z \in (0, \min_\theta |g(\theta) - g(0)|^{-1})$. Here,

$$\bar{f}(\theta) = \cos\theta \bar{P}_n(\cos\theta, \sin\theta) + \sin\theta \bar{Q}_n(\cos\theta, \sin\theta).$$

Moreover, the limit cycle associated to such a root z tends to the periodic solution (2.4) when $\epsilon \to 0$.

(2) *We assume that system (1.1) belongs to class $\boldsymbol{G}$. Then, for ϵ sufficiently small, system (2.7) of class $\boldsymbol{H}$ has a hyperbolic limit cycle surrounding the origin for each root z of the function*

$$V(z) = -z^2\biggl(z\int_0^{2\pi} \frac{f(\theta)\bar{g}(\theta)\,\mathrm{d}\theta}{1 - F(\theta)z} - \int_0^{2\pi} \bar{f}(\theta)\,\mathrm{d}\theta\biggr),$$

such that $V'(z) \neq 0$ and $z \in (0, \min_\theta |F(\theta)|^{-1})$. Here

$$\bar{g}(\theta) = \cos\theta \bar{Q}_n(\cos\theta, \sin\theta) - \sin\theta \bar{P}_n(\cos\theta, \sin\theta).$$

Moreover, the limit cycle associated to such a root z tends to the periodic solution (2.5) when $\epsilon \to 0$.

(3) *We assume that system (1.1) belongs to class $\boldsymbol{C}$. Then, for ϵ sufficiently small, system (2.7) of class $\boldsymbol{H}$ has a hyperbolic limit cycle surrounding the origin for each root z of the function*

$$\begin{aligned} W(z) = z^2\biggl(\int_0^{2\pi} ((n-1)\bar{f}(\theta) - \bar{g}'(\theta))\sqrt{1 + 2(n-1)G(\theta)z^2}\,\mathrm{d}\theta \\ - (n-1)z\int_0^{2\pi} (f(\theta)\bar{g}(\theta) + \bar{f}(\theta)g(\theta))\,\mathrm{d}\theta\biggr), \end{aligned}$$

such that $W'(z) \neq 0$ and $z \in (0, \min_\theta |G(\theta)|^{-1})$. Moreover, the limit cycle associated to such a root z tends to the periodic solution (2.6) when $\epsilon \to 0$.

Theorem 2.3 will be proved in § 3.

We note that all integrals which appear in the determination of the functions $U(z)$ and $V(z)$ are integrals of rational functions in the variables $\sin\theta$ and $\cos\theta$.

Some other results about limit cycles of system (1.1) for arbitrary $n \geqslant 2$ have been given in [**4–7**, **13**, **25**]. The same systems with $P_n(x, y) = (ax + by)R_{n-1}(x, y)$, $Q_n(x, y) = (cx + dy)R_{n-1}(x, y)$, where R_{n-1} is a homogeneous polynomial of degree $n-1$ have been studied in [**8**, **14**, **15**].

Now we apply Theorem 2.3 to subclasses $\boldsymbol{F}$, $\boldsymbol{G}$ and $\boldsymbol{C}$ of quadratic systems, i.e. for $n = 2$.

Theorem 2.4. *The following statements hold.*

(1) *Every quadratic system of class $\boldsymbol{F}$ can be written in the form*

$$\left.\begin{aligned} \dot{x} &= -y - axy - by^2, \\ \dot{y} &= x + ax^2 + bxy. \end{aligned}\right\} \tag{2.8}$$

The number of limit cycles of the perturbed system

$$\left.\begin{aligned} \dot{x} &= -y - axy - by^2 + \epsilon(Lx^2 + Mxy + Ny^2), \\ \dot{y} &= x + ax^2 + bxy + \epsilon(Ax^2 + Bxy + Cy^2) \end{aligned}\right\} \tag{2.9}$$

that bifurcate from periodic orbits of the centre of system (2.8), is at most 1 if $U(z)$ is not identically zero, where $U(z)$ is defined in Theorem 2.3 (1).

(2) *Every quadratic system of class **G** can be written in the form*

$$\left.\begin{aligned} \dot{x} &= -y + bx^2 + cxy, \\ \dot{y} &= x + bxy + cy^2. \end{aligned}\right\} \tag{2.10}$$

The number of limit cycles of the perturbed system

$$\left.\begin{aligned} \dot{x} &= -y + bx^2 + cxy + \epsilon(Lx^2 + Mxy + Ny^2), \\ \dot{y} &= x + bxy + cy^2 + \epsilon(Ax^2 + Bxy + Cy^2) \end{aligned}\right\} \tag{2.11}$$

that bifurcate from periodic orbits of the centre of system (2.10), is at most 1 if $V(z)$ is not identically zero, where $V(z)$ is defined in Theorem 2.3 (2).

(3) *Every quadratic system of class **C** can be written in the form*

$$\left.\begin{aligned} \dot{x} &= -y + lx^2 - 2axy - ly^2, \\ \dot{y} &= x + ax^2 + 2lxy - ay^2. \end{aligned}\right\} \tag{2.12}$$

There are no limit cycles of the perturbed system

$$\left.\begin{aligned} \dot{x} &= -y + lx^2 - 2axy - ly^2 + \epsilon(Lx^2 + Mxy + Ny^2), \\ \dot{y} &= x + ax^2 + 2lxy - ay^2 + \epsilon(Ax^2 + Bxy + Cy^2) \end{aligned}\right\} \tag{2.13}$$

that bifurcate from periodic orbits of the centre of system (2.12) if $(A+C)l + (L+N)a \neq 0$.

Statement (1) of Theorem 2.4 will be proved in § 4. Statements (2) and (3) are due to Chicone and Jacobs [**9**]. Both can be easily proved by using Theorem 2.3, but since their proofs are similar to the proof of statement (1) we omit them.

Now we apply Theorem 2.3 to subclasses ***F*** and ***G*** of cubic systems, i.e. for $n = 3$.

Theorem 2.5. *Every cubic system of class **F** can be written in the form*

$$\left.\begin{aligned} \dot{x} &= -y + ax^2y + bxy^2 + cy^3, \\ \dot{y} &= x - ax^3 - bx^2y - cxy^2. \end{aligned}\right\} \tag{2.14}$$

Then the maximum number of limit cycles of the perturbed system

$$\left.\begin{aligned} \dot{x} &= -y + ax^2y + bxy^2 + cy^3 + \epsilon(a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3), \\ \dot{y} &= x - ax^3 - bx^2y - cxy^2 + \epsilon(b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3) \end{aligned}\right\} \tag{2.15}$$

that bifurcate from the periodic orbits of the centre of system (2.14) is two, if $U(z)$ is not identically zero, where $U(z)$ is defined in Theorem 2.3 (1).

The proof of Theorem 2.5 is given in § 5.

Theorem 2.6. *Every cubic system of class G after a linear change of variables and a rescaling of time (if necessary) can be written in the form*

$$\left.\begin{aligned} \dot{x} &= -y + yx^2, \\ \dot{y} &= x + xy^2. \end{aligned}\right\} \tag{2.16}$$

Then the maximum number of limit cycles of the perturbed system

$$\left.\begin{aligned} \dot{x} &= -y + yx^2 + \epsilon(a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3), \\ \dot{y} &= x + xy^2 + \epsilon(b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3) \end{aligned}\right\} \tag{2.17}$$

that bifurcate from the periodic orbits of the centre of system (2.16) is two, if $V(z)$ is not identically zero, where $V(z)$ is defined in Theorem 2.3 (2).

The proof of Theorem 2.6 is given in § 6.

We remark that this study that we have made of limit cycles which bifurcate from these three kinds of centres for polynomial differential systems with homogeneous nonlinearities can be extended to differential systems defined by the sum of two quasi-homogeneous vector fields (see [**11**]).

3. Proof of Theorem 2.3

We only prove statement (1) of Theorem 2.3, the other two statements can be proved in a similar way.

We assume that system (1.1) belongs to class $\boldsymbol{F}$, i.e. $f(\theta) \equiv 0$. Then equation (2.3) for system (2.7) becomes

$$\left.\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}\theta} &= -(n-1)\epsilon\bar{f}(g+\epsilon g)\rho^3 + [(n-1)\epsilon\bar{f} - g' - \epsilon\bar{g}']\rho^2 \\ &= -g'\rho^2 + \epsilon(-(n-1)\bar{f}g\rho^3 + [(n-1)\bar{\rho} - \bar{g}']\rho^2) - \epsilon^2(n-1)\bar{f}\bar{g}\rho^3. \end{aligned}\right\} \tag{3.1}$$

Therefore, for ϵ sufficiently small, the solution $\phi(\theta, z, \epsilon)$ of system (3.1) such that $\phi(0, z, \epsilon) = z$ (by the theorem of analytic dependence on initial conditions and parameters for the solutions of an analytic differential equation) can be written in the form

$$\phi(\theta, z, \epsilon) = \varphi(\theta, z) + \epsilon u(\theta, z) + \epsilon^2 R(\theta, z, \epsilon), \tag{3.2}$$

where

$$\left.\begin{aligned} \phi(\theta, z, 0) = \varphi(\theta, z) &= \frac{z}{1 + (g(\theta) - g(0))z}, \\ u(\theta, z) &= \frac{\partial\phi}{\partial\epsilon}(\theta, z, 0), \\ u(0, z) &= 0. \end{aligned}\right\} \tag{3.3}$$

We note that $\varphi(\theta, z)$ is the solution of (3.1) for $\epsilon = 0$.

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We define $U(z) = u(2\pi, z)$. Now we assume that $U(z)$ has a root $z = z_0$ for which $U'(z_0) \neq 0$. Then, by the Implicit Function Theorem, for some $\epsilon_0 > 0$ and all ϵ with $|\epsilon| < \epsilon_0$, there exists a point $\bar{z} = \bar{z}(\epsilon)$ such that $\bar{z}(0) = z_0$ and $U(\bar{z}) + \epsilon R(2\pi, \bar{z}, \epsilon) = 0$. Therefore, from (3.2) we obtain

$$\phi(2\pi, \bar{z}, \epsilon) - \bar{z} = \epsilon[U(\bar{z}) + \epsilon R(2\pi, \bar{z}, \epsilon)] = 0.$$

Hence, the solution $\phi(\theta, \bar{z}, \epsilon)$ is periodic of period 2π. Moreover, since $U'(z_0) \neq 0$, this periodic solution is a hyperbolic limit cycle.

To end the proof of Theorem 2.3 we must compute the function $U(z)$. First we compute $u(\theta, z)$ in terms of the coefficients of ρ^2 and ρ^3 in equation (3.1). For the solution $\phi(\theta, z, \epsilon)$ of system (3.1), we have

$$\dot{\phi} = -g'\phi^2 + \epsilon(-(n-1)\bar{f}g\phi^3 + [(n-1)\bar{f} - \bar{g}']\phi^2) - \epsilon^2(n-1)\bar{f}\bar{g}\phi^3.$$

Therefore,

$$\frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial\epsilon}\right) = \frac{\partial\dot{\phi}}{\partial\epsilon} = -g'2\phi\frac{\partial\phi}{\partial\epsilon} - (n-1)\bar{f}g\phi^3 + [(n-1)\bar{f} - \bar{g}']\phi^2 + O(\epsilon). \tag{3.4}$$

Since $\dot{\varphi} = -g'\varphi^2$, from (3.3) and (3.4), it follows for $\epsilon = 0$ that

$$\frac{\partial u}{\partial\theta} = \dot{u} = 2\frac{\dot{\varphi}}{\varphi^2}\varphi u - (n-1)\bar{f}g\varphi^3 + [(n-1)\bar{f} - \bar{g}']\varphi^2.$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{u}{\varphi^2}\right) = \frac{\dot{u}\varphi - 2u\dot{\varphi}}{\varphi^3} = -(n-1)\bar{f}g\varphi + (n-1)\bar{f} - \bar{g}'.$$

Integrating both sides in the interval $[0, 2\pi]$, we obtain

$$\frac{U(z)}{z^2} = -(n-1)\left(\int_0^{2\pi} \bar{f}(\theta)g(\theta)\varphi(\theta, z)\,\mathrm{d}\theta - \int_0^{2\pi} \bar{f}(\theta)\,\mathrm{d}\theta\right).$$

Therefore,

$$U(z) = -(n-1)z^2\left(z\int_0^{2\pi} \frac{\bar{f}(\theta)g(\theta)\,\mathrm{d}\theta}{1 + (g(\theta) - g(0))z} - \int_0^{2\pi} \bar{f}(\theta)\,\mathrm{d}\theta\right).$$

Since

$$\frac{\bar{f}g}{1 + (g - g(0))z} = \frac{\bar{f}}{z} - \frac{\bar{f}(1 - g(0)z)}{z(1 + (g - g(0))z)},$$

we get

$$U(z) = (n-1)z^2(1 - g(0)z)\int_0^{2\pi} \frac{\bar{f}(\theta)\,\mathrm{d}\theta}{1 + (g(\theta) - g(0))z}.$$

This completes the proof of Theorem 2.3 (1). □

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4. Proof of Theorem 2.4

We only prove statement (1) of Theorem 2.4, the other two statements can be proved in a similar way.

We note that $a^2 + b^2 \neq 0$, otherwise system (2.8) would not be quadratic. From Theorem 2.3, for ϵ sufficiently small, there is a limit cycle surrounding the origin of system (2.9) for each root z of the function

$$U(z) = z^2(1 - g(0)z) \int_0^{2\pi} \frac{\bar{f}(\theta)\,\mathrm{d}\theta}{1 + (g(\theta) - g(0))z},$$

such that $U'(z) \neq 0$ and $z \in (0, \min_\theta |g(\theta) - g(0)|^{-1})$, where this minimum is $(a + \sqrt{a^2+b^2})^{-1}$ if $a \geqslant 0$, and $|a - \sqrt{a^2+b^2}|^{-1}$ if $a < 0$. An easy computation shows that

$$\begin{aligned} g(\theta) &= a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta - \theta_0), \\ \bar{f}(\theta) &= L\cos^3\theta + (A+M)\cos^2\theta\sin\theta + (B+N)\cos\theta\sin^2\theta + C\sin^3\theta, \end{aligned}$$

where θ_0 satisfies $\cos\theta_0 = a/\sqrt{a^2+b^2}$ and $\sin\theta_0 = b/\sqrt{a^2+b^2}$.

Clearly, for obtaining $U(z)$ we only need to compute the following four integrals

$$\int_0^{2\pi} \frac{\cos^i\theta\sin^j\theta\,\mathrm{d}\theta}{1 - az + z\sqrt{a^2+b^2}\cos(\theta-\theta_0)},$$

for $i + j = 3$, where i and j are non-negative integers. Or, equivalently, after the change of variables $\varphi = \theta - \theta_0$ we must compute the integrals

$$\frac{1}{(a^2+b^2)^{3/2}} \int_0^{2\pi} \frac{(a\cos\varphi - b\sin\varphi)^i(a\sin\varphi + b\cos\varphi)^j\,\mathrm{d}\varphi}{p + q\cos\varphi},$$

where $p = 1 - az$, $q = z\sqrt{a^2+b^2}$. These integrals will follow from the computation of the following ones

$$I_{ij} = \int_0^{2\pi} \frac{\cos^i\varphi\sin^j\varphi\,\mathrm{d}\varphi}{p + q\cos\varphi},$$

whose values are

$$\begin{aligned} I_{30} &= \frac{\pi}{q^3\sqrt{p^2-q^2}}[(2p^2+q^2)\sqrt{p^2-q^2} - 2p^3], \\ I_{12} &= \frac{\pi}{q^4\sqrt{p^2-q^2}}[p(2p^2-q^2)\sqrt{p^2-q^2} + 2p^2(q^2-p^2)], \\ I_{21} &= I_{03} = 0. \end{aligned}$$

Now, a tedious computation shows that

$$U(z) = z^2(1 - az)(\alpha I_{30} + \beta I_{12}),$$

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where

$$\alpha = \frac{1}{(a^2+b^2)^{3/2}}[a^3L + a^2b(A+M) + ab^2(B+N) + b^3C],$$

$$\beta = \frac{1}{(a^2+b^2)^{3/2}}[a^3(B+N) + a^2b(3C-2A-2M) + ab^2(3L-2B-2N) + b^3(A+M)].$$

We need to look for the roots z of $U(z)$, or, equivalently, for the roots u of

$$\begin{aligned} U_1(u) &= [\alpha q(2p^2+q^2) + \beta p(2p^2-q^2)]\sqrt{p^2-q^2} - 2\alpha p^3 q + 2\beta p^2(q^2-p^2) \\ &= p^4([\alpha u(2+u^2) + \beta(2-u^2)]\sqrt{1-u^2} - 2\alpha u + 2\beta(u^2-1)), \end{aligned}$$

where $0 < u = q/p < 1$, because $0 < z < \min_\theta |g(\theta) - g(0)|^{-1}$.

All roots of $U_1(u)$ are also roots of the expression

$$\begin{aligned} &[\alpha u(2+u^2) + \beta(2-u^2)]^2(1-u^2) - [2\alpha u - 2\beta(u^2-1)]^2 \\ &\qquad = u^4[\beta^2 - 2\alpha\beta u - (3\alpha^2+\beta^2)u^2 + 2\alpha\beta u^3 - \alpha^2 u^4] = u^4 h(u). \end{aligned}$$

Since $h(\pm 1) = -4\alpha^2$, $h(0) = \beta^2$ and $h'(u)$ has a unique real zero, it follows that $h(u)$ has at most one real root in the interval $(0,1)$. Therefore, since $z = u[au + \sqrt{a^2+b^2}]^{-1}$, we obtain that $U(z)$ has at most one root in the interval $(0, \min_\theta |g(\theta) - g(0)|^{-1})$. Hence Theorem 2.4 is proved. □

5. Proof of Theorem 2.5

By using the condition $f(\theta) \equiv 0$, it is easy to find expression (2.14) for any cubic system of class $\boldsymbol{F}$. By a rotation

$$x = u\cos\varphi + v\sin\varphi, \qquad y = -u\sin\varphi + v\cos\varphi,$$

system (2.14) keeps the same form

$$\dot{u} = -v + \alpha u^2 v + \beta u v^2 + \gamma v^3, \qquad \dot{v} = u - \alpha u^3 - \beta u^2 v - \gamma u v^2,$$

with

$$\beta = -b\sin^2\varphi - 2(a+c)\sin\varphi\cos\varphi + b\cos^2\varphi.$$

Hence there is an angle φ such that $\beta = 0$, and without loss of generality we can suppose that $b = 0$ in system (2.14).

Case 1: $c \neq 0$. Making the scaling $(x,y) \to (x/\sqrt{|c|}, y/\sqrt{|c|})$, we can transform system (2.14) into the same form but with $c = \pm 1$. We will only consider the case $c = 1$, the case $c = -1$ can be treated in a similar way. Therefore, instead of system (2.14) we consider the system

$$\left.\begin{aligned} \dot{x} &= -y + ax^2y + y^3, \\ \dot{y} &= x - ax^3 - xy^2. \end{aligned}\right\} \tag{5.1}$$

An easy computation shows that

$$
\begin{aligned}
g(\theta) &= -a\cos^2\theta - \sin^2\theta = -\tfrac{1}{2}[(a-1)\cos 2\theta + (a+1)],\\
\bar{f}(\theta) &= a_0\cos^4\theta + (b_0+a_1)\cos^3\theta\sin\theta + (b_1+a_2)\cos^2\theta\sin^2\theta\\
&\qquad\qquad + (b_2+a_3)\cos\theta\sin^3\theta + b_3\sin^4\theta\\
&= \bar{a}\cos^2 2\theta + \bar{b}\cos 2\theta\sin 2\theta + \bar{c}\sin^2 2\theta + \bar{d}\cos 2\theta + \bar{e}\sin 2\theta + \bar{a},
\end{aligned}
$$

where

$$\bar{a} = \tfrac{1}{4}(a_0+b_3), \qquad \bar{c} = \tfrac{1}{4}(b_1+a_2), \qquad \bar{d} = \tfrac{1}{2}(a_0-b_3). \tag{5.2}$$

By Theorem 2.3, we need to study the number of zeros of the function

$$U(z) = z^2(1-g(0)z)\int_0^{2\pi}\frac{\bar{f}(\theta)\,\mathrm{d}\theta}{1+(g(\theta)-g(0))z},$$

for $z \in (0, \min_\theta|g(\theta)-g(0)|^{-1})$. Hence we need to compute

$$I_{ij} = \int_0^{4\pi}\frac{\cos^i\varphi\sin^j\varphi\,\mathrm{d}\varphi}{p+q\cos\varphi},$$

for $0 \leqslant i+j \leqslant 2$, where $p = 1+(a-1)z/2$ and $q = -(a-1)z/2$. If $a = 1$, then the I_{ij} are independent of z, hence $U(z)$ has no zeros for $z > 0$. We suppose $a \neq 1$. Since

$$g(\theta) - g(0) = \tfrac{1}{2}(a-1)(1-\cos 2\theta),$$

we obtain that $z \in (0, \min_\theta|g(\theta)-g(0)|^{-1}) = (0, 1/|a-1|)$, and $p > |q| > 0$ for $z \in (0, 1/|a-1|)$. Some easy computations show that

$$
\begin{aligned}
I_{00} &= \frac{4\pi}{\sqrt{p^2-q^2}},\\
I_{10} &= \frac{4\pi(p-\sqrt{p^2-q^2})}{q\sqrt{p^2-q^2}},\\
I_{01} &= 0,\\
I_{20} &= \frac{4\pi p(p-\sqrt{p^2-q^2})}{q^2\sqrt{p^2-q^2}},\\
I_{11} &= 0,\\
I_{02} &= \frac{4\pi(q^2-p^2+p\sqrt{p^2-q^2})}{q^2\sqrt{p^2-q^2}}.
\end{aligned}
$$

Hence $U(z) = 0$ for $z \in (0, 1/|a-1|)$ is equivalent to $U_1(z) = 0$, where

$$
\begin{aligned}
U_1(z) &= \bar{a}I_{20} + \bar{c}I_{02} + \bar{d}I_{10} + \bar{a}I_{00}\\
&= \frac{4\pi}{q^2\sqrt{p^2-q^2}}[\bar{a}(p^2+q^2) + \bar{c}(q^2-p^2) + \bar{d}pq - (\bar{a}p - \bar{c}p + \bar{d}q)\sqrt{p^2-q^2}].
\end{aligned}
$$

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Note that $p > |q| > 0$. If $u = q/p$, then $0 < |u| < 1$, and $U_1(z) = 0$ if and only if $h(u) = 0$, where

$$\begin{aligned} h(u) &= \bar{a}(1+u^2) + \bar{c}(u^2-1) + \bar{d}u - (\bar{a} - \bar{c} + \bar{d}u)\sqrt{1-u^2} \\ &= (1-\sqrt{1-u^2})h_1(u), \end{aligned}$$

where

$$h_1(u) = 2\bar{a} + (\bar{a}+\bar{c})\sqrt{1-u^2} + \bar{d}u. \tag{5.3}$$

Since $h_1''(u) = -(\bar{a}+\bar{c})/(1-u^2)^{3/2}$, the function $h_1(u)$ (and consequently the function $h(u)$) has at most two zeros. On the other hand, from (5.2) and (5.3) we know that the coefficients of $h_1(u)$ are arbitrary if the coefficients a_i and b_j of system (2.15) are arbitrary. Therefore, the two zeros of $h_1(u)$ could be reached for $0 < |u| < 1$.

Case 2: $c = 0$. In this case, system (2.14) can be written as

$$\dot{x} = -y + ax^2y, \qquad \dot{y} = x - ax^3. \tag{5.4}$$

Since $a \neq 0$, we can make a scaling in order to transform (5.4) into the same form but with $a = \pm 1$.

Since $g(\theta) = -a\cos^2\theta = -a(\cos 2\theta + 1)/2$ and consequently $1 + (g(\theta) - g(0))z = 1 + az(1 - \cos 2\theta)/2$, now the cases $a = \pm 1$ correspond to the cases $a = 2$ and $a = 0$ for $c \neq 0$, respectively. This completes the proof of Theorem 2.5. □

6. Proof of Theorem 2.6

The cubic systems of class $\boldsymbol{G}$ have the form

$$\left.\begin{aligned} \dot{x} &= -y + ax^3 + bx^2y - axy^2, \\ \dot{y} &= x + ax^2y + bxy^2 - ay^3. \end{aligned}\right\} \tag{6.1}$$

By a rotation of the coordinates we can assume that $a = 0$. Doing the rescaling $(x, y) \to (x/\sqrt{|b|}, y/\sqrt{|b|})$, b becomes ± 1. If $b = -1$, exchanging x by y and t by $-t$, we get system (2.16).

For system (2.16) we have that $f(\theta) = \sin 2\theta/2$. Therefore,

$$F(\theta) = 2\int_0^\theta f(s)\,\mathrm{d}s = (1 - \cos 2\theta)/2.$$

Moreover,

$$\int_0^{2\pi} \bar{f}(\theta)\,\mathrm{d}\theta = c \quad \text{and} \quad \bar{g}(\theta) = \sum_{i=0}^{4} c_i \cos^i\theta \sin^{4-i}\theta.$$

It is easy to check that the linear map which passes from the coefficients a_i and b_j of system (2.17) to the coefficients c and c_i is surjective.

Now we have

$$\begin{aligned}
V(z) &= -z^2\left(z\int_0^{2\pi}\frac{f(\theta)\bar{g}(\theta)\,\mathrm{d}\theta}{1-F(\theta)z}-c\right)\\
&= -z^2\left(z\int_0^{2\pi}\frac{\frac{1}{2}\sin 2\theta\sum_{i=0}^{4}c_i\cos^i\theta\sin^{4-i}\theta}{p+q\cos 2\theta}\,\mathrm{d}\theta-c\right)\\
&= -z^2\left(z\int_0^{2\pi}\frac{c_1\cos^2\theta\sin^4\theta+c_3\cos^4\theta\sin^2\theta}{p+q\cos 2\theta}\,\mathrm{d}\theta-c\right)\\
&= -z^2[z(c_1I_{24}+c_3I_{42})-c],
\end{aligned}$$

where $p=1-z/2$, $q=z/2$ and

$$I_{ij}=\int_0^{2\pi}\frac{\cos^i\theta\sin^j\theta\,\mathrm{d}\theta}{p+q\cos 2\theta}.$$

A tedious computation shows that

$$\begin{aligned}
I_{24} &= \frac{\pi}{8z}(1+2u+2u^2-2(1+u)\sqrt{u^2-1}),\\
I_{42} &= -\frac{\pi}{8z}(1-2u+2u^2+2(1-u)\sqrt{u^2-1}),
\end{aligned}$$

where $u=p/q$. Therefore, $V(z)=-z^2\varphi(u)$ with

$$\varphi(u)=d_1(u+1)(u-\sqrt{u^2-1})-d_3(u-1)(u-\sqrt{u^2-1})-d_2,$$

where $d_1=\pi c_1/4$, $d_2=c+\pi(c_3-c_1)/8$ and $d_3=\pi c_3/4$. Therefore, we have that $\varphi(u)=0$ if and only if

$$d_1(u+1)-d_3(u-1)=\frac{d_2}{u-\sqrt{u^2-1}}=d_2(u+\sqrt{u^2-1}),$$

or equivalently

$$(d_1-d_2-d_3)u+d_1+d_3=d_2\sqrt{u^2-1}.$$

Since $(\sqrt{u^2-1})''=-(u^2-1)^{-3/2}\neq 0$, we obtain that the number of zeros $u>1$ of $\varphi(u)=0$ is at most two. Due to the fact that the linear map which passes from the coefficients a_i and b_j of system (2.17) to the coefficients d_i is surjective, it follows that there are examples with two simple zeros. This completes the proof of Theorem 2.6.

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