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QUANTUM INCREASING SEQUENCES GENERATE QUANTUM PERMUTATION GROUPS

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Abstract. We answer a question of Skalski and Sołtan (2016) about inner faithfulness of the Curran's map of extending a quantum increasing sequence to a quantum permutation. Roughly speaking, we find a inductive setting in which the inner faithfulness of Curran's map can be boiled down to inner faithfulness of similar map for smaller algebras and then rely on inductive generation result for quantum permutation groups of Brannan, Chirvasitu and Freslon (2018).

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1. Introduction. Let $\mathbb{G}$ be a compact quantum group (in the sense of Woronowicz, but throughout the note we will not need any of the functional analytic features of the associated Hopf-C^*-algebra), let $\mathcal{O}(\mathbb{G})$ be its associated coordinate ring and assume $\beta\colon\ \mathcal{O}(\mathbb{G}) \to \mathcal{B}$ is a *-representation of $\mathcal{O}(\mathbb{G})$ as a *-algebra in some *-algebra. Via abstract Gelfand–Naimark duality, such a map corresponds to a map $\hat{\beta}\colon\ \mathbb{X} \to \mathbb{G}$ and it is natural to ask what is the smallest quantum subgroup containing $\hat{\beta}(\mathbb{X})$, or – in other words – what the quantum subgroup generated by $\hat{\beta}(\mathbb{X}) \subset \mathbb{G}$ is. The answer to this type of questions was studied earlier in [**1**, **4**, **8**, **9**, **19**] in the case of compact quantum groups and later extended to locally compact quantum groups in [**12**, **14**].

The concept of a subgroup is central to treating quantum groups from the group-theoretic perspective and many efforts were made to provide accurate descriptions of various aspects of this concept, as well as providing some nontrivial examples, see for example, [**3**, **6**, **11**, **18**]. This note deals with subgroups of the quantum permutation groups, introduced by Wang in [**21**]. It was observed in [**15**] that quantum permutations can be used to study distributional symmetries of infinite sequences of non-commutative random variables that are identically distributed and free modulo the tail algebra, thus extending the classical de Finetti's theorem to the quantum/free realm.

Another extension of de Finetti's theorem was given by Ryll-Nardzewski: he observed that instead of symmetry of joint distributions under permutations of random variables, it is enough to consider subsequences and compare these type of joint distributions to obtain the same conclusion. What this theorem really boils down to is the fact that one can canonically treat the set $I_{k,n}$ of increasing length-k sequences (of indices) as a subset of all permutations S_n, and this subset is big enough to generate the whole symmetric group: $\langle I_{k,n}\rangle = S_n$, unless $k = 0$ or $k = n$.

This viewpoint was utilised in [**10**] by Curran to extend a theorem of Ryll-Nardzewski to the quantum case: he introduced the space of quantum increasing sequences $I^+_{k,n}$ and described how to canonically extend a quantum increasing sequence to a quantum permutation in S_n^+. The analytic properties of the C^*-algebra $C(I_{k,n}^+)$ were strong enough to provide an extension of Ryll-Nardzewski to the quantum/free case. However, these results did not say anything about the subgroup of the quantum permutation group that is generated by quantum increasing sequences.

If the analogy with the classical world is complete, one would expect that in fact $\overline{\langle I_{k,n}^+\rangle} = S_n^+$ for all n and $k \neq 0, n$. This was ruled out already in [19], where it was observed that $\overline{\langle I_{k,n}^+\rangle} = S_n$ whenever $k = 1, n-1$. However, if only $\overline{\langle I_{k,n}^+\rangle} = S_n^+$ for at least one $k \in \{2, \dots, n-2\}$, this would explain the results of Curran in a more group-theoretic manner. In general, [19, Question 7.3] asks for the complete description of all $\overline{\langle I_{k,n}^+\rangle}$ and emphasises the case $n = 4$ and $k = 2$ as the first non-trivial case to study. We gave a positive answer in this case in [13] and explained that in general $S_n \subset \overline{\langle I_{k,n}^+\rangle} \subset S_n^+$, the left inclusion being strict whenever $2 \leq k \leq n-2$. Banica's conjecture asks whether there exists a compact quantum group $\mathbb{G}$ satisfying $S_n \subset \mathbb{G} \subset S_n^+$ such that both inclusions are strict: conjecturally only one of them should be. The conjecture found its positive answer for $n = 4$ at [3], and only recently for $n = 5$ by providing a deep connection to the subfactor theory, see [2]. The latter was further utilised to argue a inductive-type generation result for quantum permutation groups in [7].

In this manuscript, we use those results to show that $\overline{\langle I_{k,n}^+\rangle} = S_n^+$ for all $n \geq 4$ and $2 \leq k \leq n-2$, answering a question of Skalski and Sołtan from [19]. To do so, we extend the criterion we gave in [13] to cover a wider class of subsets of the generating sets. Informally, we show that $I_{k,n} \subset I_{k,n}^+$ and that $I_{k,n-1}^+, I_{k-1,n-1}^+ \subset I_{k,n}^+$. Consequently, $\overline{\langle I_{k-1,n-1}^+, I_{k,n}\rangle}, \overline{\langle I_{k,n-1}^+, I_{k,n}\rangle} \subset \overline{\langle I_{k,n}^+\rangle}$. This enables us to put ourselves in the inductive generation framework: knowing that $\overline{\langle I_{k,n-1}^+\rangle} = \overline{\langle I_{k-1,n-1}^+\rangle} = S_{n-1}^+$ and $\langle I_{k,n-1}\rangle = S_n$, we deduce that $\overline{\langle S_{n-1}^+, S_n\rangle} \subset \overline{\langle I_{k,n}^+\rangle}$. From [7], one knows that $\overline{\langle S_{n-1}^+, S_n\rangle} = S_n^+$. The latter relies heavily on the solution of Banica's conjecture for $n = 5$.

The precise meaning of the aforementioned ideas is described in the course of the paper. Section 2 serves mainly as preliminaries needed to settle the notation for compact quantum groups (Section 2.1), Hopf images (Section 2.2) and quantum permutation groups together with quantum increasing sequences (Section 2.3). The proof of the main Theorem is outlined in Section 3.

2. Preliminaries.

2.1. Compact quantum groups. This part is devoted to specifying notation and some viewpoints in the theory of compact quantum groups. For details, we refer to [22, 20, 17].

DEFINITION 2.1. A *compact quantum group* is defined by its *continuous functions algebra* $C(\mathbb{G})$ and a *coproduct* $\Delta_{\mathbb{G}}\colon C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$, where $C(\mathbb{G})$ is a unital C^*-algebra and $\Delta_{\mathbb{G}}$ is a unital *-homomorphism satisfying the Podleś conditions:

$$\operatorname{span}_{\mathbb{C}} \left\{(\mathbb{1} \otimes a)\Delta_{\mathbb{G}}(a') : a, a' \in C(\mathbb{G})\right\}^{-\|\cdot\|} = C(\mathbb{G}) \otimes C(\mathbb{G}),$$
$$\operatorname{span}_{\mathbb{C}} \left\{(a \otimes \mathbb{1})\Delta_{\mathbb{G}}(a') : a, a' \in C(\mathbb{G})\right\}^{-\|\cdot\|} = C(\mathbb{G}) \otimes C(\mathbb{G}).$$

The tensor product on the right-hand side of $\Delta_{\mathbb{G}}$ is the C^*-algebraic minimal tensor product.

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DEFINITION 2.2. An n-dimensional *representation* of $\mathbb{G}$ is a corepresentation of $C(\mathbb{G})$, that is, an element $u \in \mathsf{B}(\mathbb{C}^n) \otimes C(\mathbb{G})$ such that

$$\Delta_{\mathbb{G}}(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj},$$

where $u_{ij} = (\langle e_i| \cdot |e_j\rangle \otimes \mathrm{id})u$ and $(e_i)_{1\le i\le n} \subset \mathbb{C}^n$ denote the standard orthonormal basis of $\mathbb{C}^n$.

Developing the representation theory of $\mathbb{G}$, one argues the existence of a dense Hopf-*-algebra $\mathcal{O}(\mathbb{G}) \subseteq C(\mathbb{G})$. It has numerous C^*-completions ($C(\mathbb{G})$ among them), but in particular it has the universal C^*-completion $C^u(\mathbb{G})$. The difference is like between $C^*_{max}(\mathbb{F}_2)$ and $C^*_r(\mathbb{F}_2)$ – even though the C^*-algebras are different, the group object under the hood is the same. We will adopt this viewpoint when studying compact quantum groups and will not dig into the issues of different completions (for further discussion, see e.g. [16]). We will just assume that the compact quantum group $\mathbb{G}$ is studied by $C^u(\mathbb{G})$, as one can replace the original completion with the universal one.

DEFINITION 2.3. A *compact matrix quantum group* is a pair $(C^u(\mathbb{G}), u)$ such that $u \in \mathsf{B}(\mathbb{C}^n) \otimes C^u(\mathbb{G})$ is a representation of $\mathbb{G}$ and $\{u_{ij} : 1 \le i, j \le n\}$ generates $\mathcal{O}(\mathbb{G})$ as *-algebra. Such an element u is called a *fundamental corepresentation* of $C^u(\mathbb{G})$.

DEFINITION 2.4. Given two compact quantum groups $\mathbb{H}$, $\mathbb{G}$, we say that $\mathbb{H}$ *is embedded into* $\mathbb{G}$ (or shorter: $\mathbb{H} \subset \mathbb{G}$ is a closed quantum subgroup), if there exists a surjective unital *-homomorphism $\pi : C^u(\mathbb{G}) \to C^u(\mathbb{H})$ intertwining the respective coproducts:

$$(\pi \otimes \pi) \circ \Delta_{\mathbb{G}} = \Delta_{\mathbb{H}} \circ \pi.$$

Note that the same compact quantum group $\mathbb{H}$ can be embedded into $\mathbb{G}$ in a not necessarily unique way, thus when we write $\mathbb{H} \subset \mathbb{G}$ we mean an inclusion specified by a particular π.

2.2. Hopf image. The Hopf image, studied in detail in the case of compact quantum groups in [4, 19] and in the case of locally compact quantum groups in [14], is concerned with the following situation. Let $\mathbb{G}$ be a compact quantum group. Let B be a unital C^*-algebra and let $\beta : C^u(\mathbb{G}) \to B$ be a unital *-homomorphism. We think of it as the Gelfand dual of a map $\widehat{\beta} : \mathbb{X} \to \mathbb{G}$ from a quantum space into a quantum group and ask what is the closed quantum subgroup of $\mathbb{G}$ generated by $\widehat{\beta}(\mathbb{X}) \subset \mathbb{G}$. We will abuse the notation and write $C^u(\mathbb{X})$ instead of B.

Formally speaking, we consider the following category, which we denote by $\mathcal{C}_\beta$. Objects of $\mathcal{C}_\beta$ are triples $(\pi, \mathbb{H}, \tilde{\beta})$ consisting of: a closed quantum subgroup $\mathbb{H}$ of $\mathbb{G}$ such that $\pi : C^u(\mathbb{G}) \to C^u(\mathbb{H})$ is the associated *-homomorphism intertwining the coproducts and $\tilde{\beta} \circ \pi = \beta$, that is, the map β factors through the C^*-algebra $C^u(\mathbb{H})$ of functions on the subgroup $\mathbb{H}$ (where $\mathbb{H}$ is embedded into $\mathbb{G}$ via π) and $\tilde{\beta} \circ \pi = \beta$ is the factorisation. For two objects $\mathbb{h} = (\pi, \mathbb{H}, \tilde{\beta})$, $\mathbb{k} = (\pi', \mathbb{K}, \beta') \in \mathrm{Ob}(\mathcal{C}_\beta)$, a morphism $\varphi \in \mathrm{Mor}_{\mathcal{C}_\beta}(\mathbb{h}, \mathbb{k})$ is a unital *-homomorphism $\varphi : C^u(\mathbb{K}) \to C^u(\mathbb{H})$ (note that the direction of arrows in $\mathcal{C}_\beta$ matches the direction of maps on the level of quantum groups, which is opposite to the ones on the level of C^*-algebras of functions), which intertwines the respective coproducts and such that the diagram in Figure 1 commutes.

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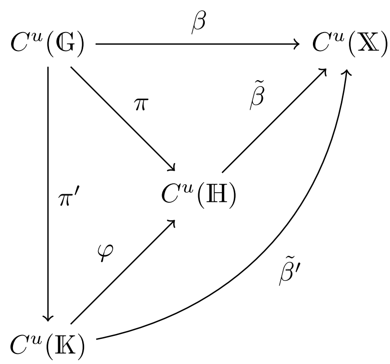


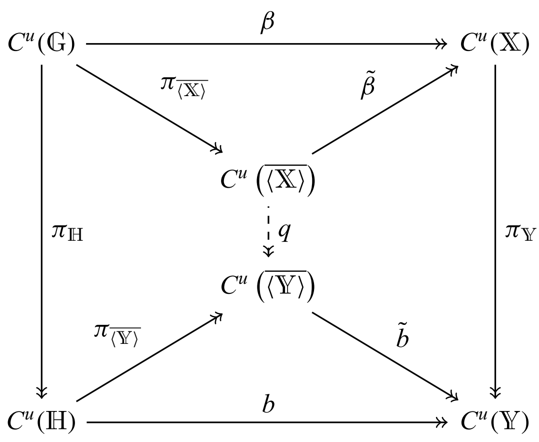
Figure 1. Morphisms in $\mathcal{C}_\beta$.

The object we are interested in is the initial object of the category $\mathcal{C}_\beta$. This object is, classically, the closed subgroup generated by $\mathbb{X}$ and thus we will denote this initial object by $\overline{\langle\mathbb{X}\rangle}$. We will use the universal property as follows. Let $\beta\colon C^u(\mathbb{G}) \to C^u(\mathbb{X})$, $b\colon C^u(\mathbb{H}) \to C^u(\mathbb{Y})$ be two morphisms and let

$$C^u(\mathbb{G}) \xrightarrow{\pi_{\overline{\langle\mathbb{X}\rangle}}} C^u\left(\overline{\langle\mathbb{X}\rangle}\right) \xrightarrow{\tilde{\beta}} C^u(\mathbb{X}) \quad \text{and} \quad C^u(\mathbb{H}) \xrightarrow{\pi_{\overline{\langle\mathbb{Y}\rangle}}} C^u\left(\overline{\langle\mathbb{Y}\rangle}\right) \xrightarrow{\tilde{b}} C^u(\mathbb{Y})$$

be the Hopf image factorisations of the morphisms β, b, respectively. Furthermore, assume that $\mathbb{H} \subset \mathbb{G}$ and $\mathbb{Y} \subset \mathbb{X}$, that is, there exists maps $\pi_\mathbb{H}$ and $\pi_\mathbb{Y}$ such that the diagram on the right commutes.

The universal property of the Hopf image $\overline{\langle\mathbb{Y}\rangle}$ yields the existence of the map q in the diagram on the right. We summarise this observation as:



FACT 2.5. *If* $\mathbb{Y} \subset \mathbb{X}$, $\mathbb{Y} \subset \mathbb{H}$, $\mathbb{X} \subset \mathbb{G}$ *and* $\mathbb{H} \subset \mathbb{G}$ *is a closed quantum subgroup, then* $\overline{\langle\mathbb{Y}\rangle} \subset \overline{\langle\mathbb{X}\rangle}$.

Note that the role of $\mathbb{H}$ could be equally well played by $\overline{\langle\mathbb{Y}\rangle}$ in the above diagram (leading to minor simplifications). The reason we present it in this way is to emphasise that the maps $\pi_\mathbb{H}$, $\pi_\mathbb{Y}$, β, b, represented by arrows located at the circumference of the square diagram, are the ones one can work with directly (by means of formulas), and the maps

$\pi_{\overline{\langle X\rangle}}$, $\pi_{\overline{\langle Y\rangle}}$, $\tilde{\beta}$, $\tilde{b}$, q, represented by arrows located in the interior of the square diagram, are the ones whose existence follows from the universal property and the formulas need not be grasped.

2.3. The Quantum Permutation Group S_n^+ and Quantum Increasing Sequences $I_{k,n}^+$. Quantum permutation groups S_n^+ were introduced in [21] (cf. [19, Section 3]). Consider the universal C^*-algebra generated by n^2-elements u_{ij}, $1 \le i, j \le n$ subject to the following relations:

(1) the generators u_{ij} are all projections.
(2) $\sum_{i=1}^n u_{ij} = \mathbb{1} = \sum_{j=1}^n u_{ij}$.

Denote it by $C^u(S_n^+)$. The matrix $u = [u_{ij}]_{1\le i,j\le n}$ is a fundamental corepresentation of $C^u(S_n^+)$, this yields the quantum group structure. Moreover, $S_n^+ = S_n$ for $n \le 3$ and $S_n^+ \supsetneq S_n$ for $n \ge 4$.

The algebra of continuous functions on the set of quantum increasing sequences was defined by Curran in [10, Definition 2.1]. Let $k \le n \in \mathbb{N}$ and let $C^u(I_{k,n}^+)$ be the universal C^*-algebra generated by p_{ij}, $1 \le i \le n$, $1 \le j \le k$ subject to the following relations:

(1) the generators p_{ij} are all projections.
(2) each column of the rectangular matrix $p = [p_{ij}]_{1\le i\le n, 1\le j\le k}$ forms a partition of unity: $\sum_{i=1}^n p_{ij} = \mathbb{1}$ for each $1 \le j \le k$.
(3) increasing sequence condition: $p_{ij}p_{i'j'} = 0$ whenever $j < j'$ and $i \ge i'$.

This definition stems from the liberation philosophy ([5]): let $I_{k,n}$ be the set of length-k increasing $\{1, \dots, n\} = [n]$-valued sequences. To an increasing sequence $\underline{i} = (i_1 < \dots < i_k)$, one associates a matrix $A(\underline{i}) \in M_{n\times k}(\{0,1\})$ defined by $A(\underline{i})_{s,t} = \delta_{s,i_t}$. The algebra $C(\{A(\underline{i}) : \underline{i} \in I_{k,n}\})$ of functions on these matrices is generated by the coordinate functions $x_{i,j}$ subject to the relations introduced above and the commutation relation (cf. the discussion after [10, Remark 2.2]).

Curran defined also a *-homomorphism $\beta_{k,n}\colon C(S_n^+) \to C(I_{k,n}^+)$ ([10, Proposition 2.5]) by:

- $u_{ij} \mapsto p_{ij}$ for $1 \le i \le n$, $1 \le j \le k$,
- $u_{i\,k+m} \mapsto 0$ for $1 \le m \le n-k$ and $i < m$ or $i > m+k$,
- $u_{m+p\,k+m} \mapsto \displaystyle\sum_{i=0}^{m+p-1} p_{ip} - p_{i+1\,p+1}$ for $1 \le m \le n-k$ and $0 \le p \le k$,

where we set $p_{00} = \mathbb{1}$, $p_{i0} = p_{0i} = p_{i\,k+1} = 0$ for $i \ge 1$.

This *-homomorphism is well defined thanks to the relations obtained in [10, Proposition 2.4] and the universal property of $C^u(S_n^+)$. The classical counterpart of the map $\beta_{k,n}$, denoted $b_{k,n}\colon C(S_n) \to C(I_{k,n})$, is precisely the 'completing an increasing sequence to a permutation' map. More precisely, one draws the diagram of an increasing sequence $\underline{i} = (i_1 < \dots < i_k)$ in the following way: there are k dots in upper row and n dots in the bottom row, l-th dot in the upper row is connected to the i_l-th dot in the bottom row. Then one draws additional $n-k$ dots in the upper row next to previously drawn k dots and connects them as follows: the $(k+j)$-th dot is connected to the j-th leftmost non-connected dot in the bottom row. This results in a diagram of a permutation on n letters, called $b_{k,n}(\underline{i})$, see the example in Figure 2.

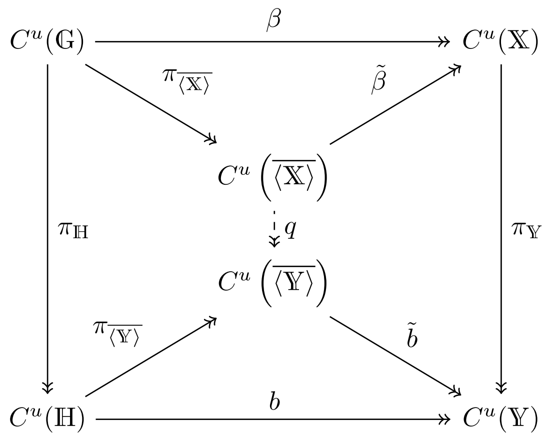


Figure 2. The sequence $(2<3<5<6<8) \in I_{5,9}$, drawn with full circles and segments, is completed, with the aid of empty circles and dashed segments, to the permutation (1, 2, 3, 5, 8, 7, 4, 6).

LEMMA 2.6. *Consider $I_{k,n} \subseteq S_n$ via the above map. Then:*

(1) $\langle I_{k,n} \rangle = S_n$ *for all n and all $k \neq 0, n$,*
(2) $S_n \subset \overline{\langle I^+_{k,n} \rangle}$ *for $k, n \neq 0, n$.*

Proof. The first item is routine. Consider the following two increasing sequences: $a_l = l$ for $l < k$, and $a_k = k + p$ with $p \in \{1, n - k\}$. These yield the transposition $(k, k + 1)$ and the cycle $(k, k + 1, \ldots, n)$, respectively. Further, the sequence $a_l = l + 1$ for all $l \leq k$ yield the cycle $(1, 2, \ldots, k + 1)$. Conjugating the transposition $(k, k + 1)$ by powers of $(k, k + 1, \ldots, n)$ and $(1, 2, \ldots, k + 1)$, one gets every adjacent transposition. It is known that S_n is generated by them.

The second item follows easily from Fact 2.5. □

3. Quantum Increasing Sequences generate Quantum Permutation Groups. The main result is based on a technical observation Lemma 3.3, whereas Lemma 3.1 is a formal preparation to it. We will first formulate both statements and proceed with the proof further on.

LEMMA 3.1. *Fix $n \in \mathbb{N}$ and $1 \leq k \leq n - 1$. Let $(p_{ij})_{1 \leq i \leq n, 1 \leq j \leq k}$, $(\tilde{p}_{ij})_{1 \leq i \leq n-1, 1 \leq j \leq k}$ and $(\dot{p}_{ij})_{1 \leq i \leq n-1, 1 \leq j \leq k-1}$ be the standard generators of $C^u(I^+_{k,n})$, $C^u(I^+_{k,n-1})$ and $C^u(I^+_{k-1,n-1})$, respectively. The maps $\tilde{\eta}_{k,n}\colon C^u(I^+_{k,n}) \to C^u(I^+_{k,n-1})$ and $\dot{\eta}_{k,n}\colon C^u(I^+_{k,n}) \to C^u(I^+_{k-1,n-1})$ determined by:*

$$\dot{\eta}_{k,n}(p_{ij}) = \begin{cases} \mathbb{1} & i = j = 1 \\ 0 & i = 1 \text{ or } j = 1 \text{ and } i \neq j \\ \dot{p}_{i-1\,j-1} & 1 < i \leq n,\ 1 < j \leq k \end{cases}$$

$$\tilde{\eta}_{k,n}(p_{ij}) = \begin{cases} \tilde{p}_{ij} & i < n \\ 0 & i = n \end{cases}$$

*are well-defined surjective *-homomorphisms.*

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REMARK 3.2. The classical counterparts of the above maps are as follows: $\dot{e}_{k,n}\colon I_{k-1,n-1} \to I_{k,n}$ simply shifts an length-$(k-1)$ increasing $[n-1]$-valued sequence right by identifying $[n-1]$ with $\{2, \ldots, n\} \subset [n]$ and adds 1 as the first element. More precisely: $\dot{e}_{k,n}(a_1, \ldots, a_{k-1}) = (1, a_1+1, \ldots, a_{k-1}+1)$. The map $\tilde{e}_{k,n}\colon I_{k,n-1} \to I_{k,n}$ is the canonical embedding induced by the identification $[n-1] = \{1, \ldots, n-1\} \subset [n]$, hence acts as formal identity.

LEMMA 3.3. *Fix* $n \in \mathbb{N}$ *and* $1 \le k \le n-1$ *and let* $\tilde{\eta}_{k,n}, \dot{\eta}_{k,n}$ *be the maps from Lemma 3.1. Let* $(u_{ij})_{1\le i,j\le n}$ *and* $(u'_{ij})_{1\le i,j\le n-1}$ *be the standard generators of* $C^u(S_n^+)$ *and* $C^u(S_{n-1}^+)$*, respectively, and let* $q_n, \bar{q}_n\colon C^u(S_n^+) \to C^u(S_{n-1}^+)$ *be the maps*

$$q_n(u_{ij}) = \begin{cases} u'_{ij} & i,j<n \\ \mathbb{1} & i=j=n \\ 0 & \textit{otherwise} \end{cases} \qquad \bar{q}_n(u_{ij}) = \begin{cases} u'_{i-1\,j-1} & i,j>1 \\ \mathbb{1} & i=j=1 \\ 0 & \textit{otherwise} \end{cases}$$

(1) The diagram below is commutative:

$$\begin{array}{ccc} C^u(S_n^+) & \xrightarrow{\beta_{k,n}} & C^u(I_{k,n}^+) \\ \downarrow{\scriptstyle q_n} & & \downarrow{\scriptstyle \tilde{\eta}_{k,n}} \\ C^u(S_{n-1}^+) & \xrightarrow[\beta_{k,n-1}]{} & C^u(I_{k,n-1}^+) \end{array}$$

(2) The diagram below is commutative:

$$\begin{array}{ccc} C^u(S_n^+) & \xrightarrow{\beta_{k,n}} & C^u(I_{k,n}^+) \\ \downarrow{\scriptstyle \bar{q}_n} & & \downarrow{\scriptstyle \dot{\eta}_{k,n}} \\ C^u(S_{n-1}^+) & \xrightarrow[\beta_{k-1,n-1}]{} & C^u(I_{k-1,n-1}^+) \end{array}$$

REMARK 3.4. The classical counterparts of maps q_n and $\overline{q}_n$ are obtained by identifying $[n-1]$ with $\{1, \ldots, n-1\} \subset [n]$ and $\{2, \ldots, n\} \subset [n]$, respectively. Alternatively, $\widehat{q}_n\,[S_{n-1}] \subset S_n$ is a subgroup operating on first $n-1$ indices and $\widehat{\overline{q}}_n\,[S_{n-1}] \subset S_n$ is a subgroup operating on last $n-1$ indices.

Proof of Lemma 3.1. In both cases, the hypothesis is a consequence of the universal property of $C^u(I_{k,n}^+)$, as long as relations satisfied by p_{ij} are satisfied by their images.

Clearly, $\tilde{\eta}_{k,n}(p_{ij})$ are all projections, which add up to $\mathbb{1}$ in each column. The increasing sequence condition amounts to verifying that $\forall j < j'\ \forall i \ge i'$

$$\tilde{\eta}_{k,n}(p_{ij})\tilde{\eta}_{k,n}(p_{i'j'}) = 0. \tag{3.1}$$

If $i = n$, then $\tilde{\eta}_{k,n}(p_{ij}) = 0$ from the definition, hence (3.1). Otherwise, $n > i \ge i'$ and consequently

$$\tilde{\eta}_{k,n}(p_{ij})\tilde{\eta}_{k,n}(p_{i'j'}) = \tilde{p}_{ij}\tilde{p}_{i'j'} = 0.$$

thanks to increasing sequence condition satisfied by $\tilde{p}_{ij}$'s in $C^u(I_{k,n-1}^+)$.

Likewise, the projections $\dot{\eta}_{k,n}(p_{ij})$ add up to $\mathbb{1}$ in each column, and $\dot{\eta}_{k,n}(p_{ij})\dot{\eta}_{k,n}(p_{i'j'})$ satisfy (3.1) for $n \ge i \ge i' > 1$ and $1 < j < j'$ due to relations in $C^u(I_{k-1,n-1}^+)$. If $i' = 1$, then $1 \le j < j'$ and consequently $\dot{\eta}_{k,n}(p_{i'j'}) = 0$, hence (3.1). Similarly, if $j = 1$ and $i \ge i' > 1$, then $\dot{\eta}_{k,n}(p_{ij}) = 0$ and consequently (3.1) is satisfied. □

Proof of Lemma 3.3. Thanks to Lemma 3.1, it is enough to check if the paths of the diagram agree on generators $(u_{ij})_{1\le i,j\le n}$ of $C^u(S_n^+)$. Note that the classical counterparts of the two diagrams, in view of the interpretations given in Remarks 3.2 and 3.4, are clearly valid.

Let $\mathrm{span}_{\mathbb{C}}\{p_{i,j}, \mathbb{1} \mid 1\le i\le n, 1\le j\le k\}\subset C^u(I_{k,n}^+)$ and $\mathrm{span}_{\mathbb{C}}\{u_{i,j}, \mathbb{1} \mid 1\le i,j\le n\}\subset C^u(S_n^+)$ be called the *affine* part of the respective C^*-algebras. Note that the abelianization maps $C^u(I_{k,n}^+)\to C(I_{k,n})$ and $C^u(S_n^+)\to C(S_n)$, restricted to the affine parts, are *injective*.

Lastly, all the maps involved restrict and co-restrict to the affine parts. Due to commutativity of the classical counterparts of the diagram, the two possible compositions agree on generators. □

REMARK 3.5. For the sake of clarity, let us unpack the idea of the Proof of Lemma 3.3 in one case. Fix $1\le i,j\le n$, the goal is to verify if $X := \tilde{\eta}_{k,n}(\beta_{k,n}(u_{ij})) - \beta_{k,n-1}(q_n(u_{ij})) = 0$. Apply abelianization to the left-hand side:

$$\begin{aligned}\mathrm{Ab}(X) &= \mathrm{Ab}\left(\tilde{\eta}_{k,n}(\beta_{k,n}(u_{ij}))\right) - \mathrm{Ab}\left(\beta_{k,n-1}(q_n(u_{ij}))\right)\\ &= \tilde{e}_{k,n}(\mathrm{Ab}(\beta_{k,n}(u_{ij}))) - b_{k,n-1}(\mathrm{Ab}(q_n(u_{ij})))\\ &= \tilde{e}_{k,n}(b_{k,n}(u_{ij}^{S_n})) - b_{k,n-1}(q_n(u_{ij}^{S_n})) = 0\end{aligned}$$

where the last equality is due to commutativity of the diagram in the classical case, and other equalities are consequences of the definitions of the maps involved. Thus, $\mathrm{Ab}(X)=0$ and X belongs to the affine part, on which Ab is injective, hence $X=0$, as desired.

THEOREM 3.6. *If $n\ge 4$ and $2\le k\le n-2$, then $\overline{\langle I_{k,n}^+\rangle} = S_n^+$.*

Proof. We will proceed inductively. Case $n=4$, $k=2$ was solved in [**13**], let us assume $n\ge 5$ and assume $\overline{\langle I_{k,n-1}^+\rangle} = S_{n-1}^+$ for all $2\le k\le n-3$.

Firstly, $S_n\subseteq \overline{\langle I_{k,n}^+\rangle}$ from Lemma 2.6. Then

(1) if $k\le n-3$, we can use Fact 2.5 with $\mathbb{X}=I_{k,n}^+$, $\mathbb{Y}=I_{k,n-1}^+$, $\pi_{\mathbb{H}}=q_n$, $\beta=\beta_{k,n}$, $\pi_{\mathbb{Y}}=\tilde{\eta}_{k,n}$ and $b=\beta_{k,n-1}$ thanks to Lemma 3.3,

(2) if $k\ge 3$, we can use Fact 2.5 with $\mathbb{X}=I_{k,n}^+$, $\mathbb{Y}=I_{k-1,n-1}^+$, $\pi_{\mathbb{H}}=\bar{q}_n$, $\beta=\beta_{k,n}$, $\pi_{\mathbb{Y}}=\dot{\eta}_{k,n}$ and $b=\beta_{k-1,n-1}$ thanks to Lemma 3.3.

In both cases, we conclude that $S_{n-1}^+\subset \overline{\langle I_{k,n}^+\rangle}$. Together with the first observation, this yields $\overline{\langle S_n, S_{n-1}^+\rangle}\subset \overline{\langle I_{k,n}^+\rangle}\subset S_n^+$, but from [7, Theorem 3.3] it follows that all inclusions are equalities. □

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