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# ON THE OPTIMAL $L^2$ EXTENSION THEOREM AND A QUESTION OF OHSAWA

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**Abstract.** In this paper, we present a version of Guan-Zhou's optimal $L^2$ extension theorem and its application. As a main application, we show that under a natural condition, the question posed by Ohsawa in his series paper VIII on the extension of $L^2$ holomorphic functions holds. We also give an explicit counterexample which shows that the question fails in general.

## §1. Introduction

### 1.1 Background

The Ohsawa–Takegoshi $L^2$ extension theorem concerning the extension problem with $L^2$ estimates of holomorphic sections of a Hermitian holomorphic bundle over complex manifolds under certain geometric conditions is one of the most important results in several complex variables and complex geometry. After it was established in [15], many versions of the Ohsawa–Takegoshi type extension theorems have been obtained such as the generalizations given in a series of papers on the extension of $L^2$ holomorphic functions by Ohsawa himself [15–17], Manivel–Demailly $L^2$ extension theorem [3, 13], the $L^2$ extension theorems by Siu [21] and Berndtsson [1], the $L^2$ extension theorem with gain by McNeal and Varolin [14] and an $L^2$ extension theorem of Demailly et al. [4], and so on. We remark here that recently Guan and Zhou have established in a unified way an optimal version of $L^2$ extension theorem [6, 7, 8] which gives all optimal estimates of the above theorems.

Also various results have been established by using the Ohsawa–Takegoshi extension theorem as well as its optimal version, such as the invariance of plurigenera on projective manifolds [22, 23], Demailly's strong openness conjecture [9], the Suita conjecture, L-conjecture, Berndtsson's theorem [2] on the log-plurisubharmonicity of the relative Bergman kernel (see [8]), and so on. We remark here that the last one can be directly derived by the optimal version of $L^2$ extension theorem. Also as an application of the above mentioned Guan-Zhou's unified result about the Ohsawa–Takegoshi extension theorem with optimal estimates, a complete solution of the Suita conjecture [8] has been obtained by Guan-Zhou.

In [17], Ohsawa posed the following question about $L^2$ extension.

Question. Given a subharmonic function $\psi$ on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} e^{-\psi}\left(=\int_{\mathbb{C}} e^{-\psi}dxdy\right)<\infty,$$

and any subharmonic function $\varphi$ on $\mathbb{C}$, does there exist a holomorphic function $f$ on $\mathbb{C}$ such that

$$f(0)=1$$

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and

$$\int_{\mathbb{C}} |f(z)|^2 e^{-\varphi(z)-\psi(z)} \leq e^{-\varphi(0)} \int_{\mathbb{C}} e^{-\psi} \ ?$$

In the present paper, we give an answer to this question by using Guan-Zhou's optimal $L^2$ extension theorem.

### 1.2 Notions and notations

In order to state the optimal $L^2$ extension theorem which we'll use, we recall some symbols and notions in [15] and [16]. For the fundamentals on $L^2$ method, the reader is referred to [11, 12].

Let $M$ be a complex $n$-dimensional manifold, and $S$ be a closed complex subvariety of $M$. Let $dV_M$ be a continuous volume form on $M$. We consider a class of upper-semi-continuous function $\Psi$ from $M$ to the interval $[-\infty, A)$, where $A \in (-\infty, +\infty]$, such that

(1) $\Psi^{-1}(-\infty) \supset S$, and $\Psi^{-1}(-\infty)$ is a closed subset of $M$;

(2) If $S$ is $l$-dimensional around a point $x \in S_{reg}$ ( $S_{reg}$ is the regular part of $S$), there exists a local coordinate $(z_1, \ldots, z_n)$ on a neighborhood $U$ of $x$ such that $z_{l+1} = \cdots = z_n = 0$ on $S \cap U$ and

$$\sup_{U \setminus S} |\Psi(z) - (n-l) \log \sum_{l+1}^{n} |z_j|^2| < \infty.$$

The set of such polar functions $\Psi$ will be denoted by $\#_A(S)$ and the subset of plurisubharmonic functions $\Psi$ in $\#_A(S)$ will be denoted by $\Delta_A(S)$.

For each $\Psi \in \#_A(S)$, one can associate a positive measure $dV_M[\Psi]$ on $S_{reg}$ as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$\int_{S_l} f d\mu \geq \limsup_{t \to \infty} \frac{2(n-l)}{\sigma_{2n-2l-1}} \int_M f e^{-\Psi} \mathbb{I}_{\{-1-t<\Psi<-t\}} dV_M$$

for any nonnegative continuous function $f$ with supp $f \Subset M$, where $\mathbb{I}_{\{-1-t<\Psi<-t\}}$ is the characteristic function of the set $\{-1-t<\Psi<-t\}$. Here we denote by $S_l$ the $l$-dimensional component of $S_{reg}$, denote by $\sigma_m$ the volume of the unit sphere in $\mathbb{R}^{m+1}$.

Let $\omega$ be a Kähler metric on $M \setminus (X \cup S)$, where $X$ is a closed subset of $M$ such that $S_{sing} \subset X$ ($S_{sing}$ is the singular part of $S$). We can also define a measure $dV_\omega[\Psi]$ on $S \setminus X$ as the minimum element of the partially ordered set of positive measures $d\mu'$ satisfying

$$\int_{S_l} f d\mu' \geq \limsup_{t \to \infty} \frac{2(n-l)}{\sigma_{2n-2l-1}} \int_{M \setminus (X \cup S)} f e^{-\Psi} \mathbb{I}_{\{-1-t<\Psi<-t\}} dV_\omega$$

for any nonnegative continuous function $f$ with supp $(f) \Subset M \setminus X$. As

$$\{-1-t<\Psi<-t\} \cap \mathrm{supp}(f) \Subset M \setminus (X \cup S),$$

the right-hand side of the above inequality is well-defined.

Let $u$ be a continuous section of $K_M \otimes E$, where $E$ is a holomorphic vector bundle equipped with a continuous metric $h$ on $M$. We define

$$|u|_h^2|_V := \frac{c_n h(e,e) v \wedge \bar{v}}{dV_M},$$

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and

$$|u|^2_{h,\omega}|_V := \frac{c_n h(e,e) v \wedge \bar{v}}{dV_\omega},$$

where $u|_V = v \otimes e$ for an open set $V \subset M \setminus (X \cup S)$, $v$ is a continuous section of $K_M|_V$, $e$ is a continuous section of $E|_V$ and $c_n = i^{n^2}$ such that $c_n v \wedge \bar{v}$ is positive (especially, we define

$$|u|^2|_V := \frac{c_n u \wedge \bar{u}}{dV_M},$$

when $u$ is a continuous section of $K_M$). It is clear that $|u|^2_h$ is independent of the choices of $V$. In fact, one may see the relationship between $dV_\omega[\Psi]$ and $dV_M[\Psi]$ (respectively, $dV_\omega$ and $dV_M$). It is clear that

$$\int_{M\setminus(X\cup S)} f|u|^2_{h,\omega} dV_\omega[\Psi] = \int_{M\setminus(X\cup S)} f|u|^2_h dV_M[\Psi],$$
$$\left(\text{resp.} \int_{M\setminus(X\cup S)} f|u|^2_{h,\omega} dV_\omega = \int_{M\setminus(X\cup S)} f|u|^2_h dV_M,\right),$$

where $f$ is a continuous function with compact support on $M \setminus X$.

It is clear that $|u|^2_h$ is independent of the choices of $U$, while $|u|^2_h dV_M$ is independent of the choices of $dV_M$ (respectively, $|u|^2_h dV_M[\Psi]$ is independent of the choices of $dV_M$).

Definition 1.1. Let $M$ be an $n$-dimensional complex manifold with a continuous volume form $dV_M$, and let $S$ be a closed complex subvariety of $M$. We call a pair $(M, S)$ an almost Stein pair if $M$ and $S$ satisfy the following conditions:

There exists a closed subset $X \subset M$ such that:

(a) $X$ is locally negligible with respect to $L^2$ holomorphic functions, that is, for any local coordinate neighborhood $U \subset M$ and for any $L^2$ holomorphic function $f$ on $U \setminus X$, there exists an $L^2$ holomorphic function $\tilde{f}$ on $U$ such that $\tilde{f}|_{U\setminus X} = f$ with the same $L^2$ norm.

(b) $M \setminus X$ is a Stein manifold which intersects every component of $S$, such that $S_{sing} \subset X$.

Remark 1.2. In fact, when $S$ is smooth, the conditions of an almost Stein pair are the same as in [15] and [16]. The almost Stein pair $(M, S)$ includes all the following well-known examples:

(1) $M$ is a Stein manifold (including any open Riemann surface), and $S$ is any closed complex submanifold of $M$;
(2) $M$ is a complex projective algebraic manifold (including any compact Riemann surface), and $S$ is any closed complex submanifold of $M$;
(3) $M$ is a projective family (see [23]), and $S$ is any closed complex submanifold of $M$.

Remark 1.3. Actually, all $L^2$ holomorphic sections of the holomorphic vector bundles can be extended holomorphically from $M \setminus X$ to $M$. In fact, let $(M, S)$ be an almost Stein pair, $h$ be a singular metric on holomorphic line bundle $L$ on $M$ (respectively, continuous metric on holomorphic vector bundle $E$ on $M$ with rank $r$), where $h$ has a locally positive lower bound.

Let $F$ be a holomorphic section of $K_{M\setminus X}\otimes L|_{M\setminus X}$ (respectively, $K_{M\setminus X}\otimes E|_{M\setminus X}$), which satisfies $\int_{M\setminus X}|F|^2_h<\infty$. As $h$ has a locally positive lower bound and $M$ satisfies (a) of the above condition (ab), there is a holomorphic section $\tilde{F}$ of $K_M\otimes L$ on $M$ (resp. $K_M\otimes E$), such that $\tilde{F}|_{M\setminus X}=F$.

### 1.3 The optimal $L^2$ extension theorem

In [8], Guan and Zhou found an Ohsawa–Takegoshi type $L^2$ extension theorem with an optimal $L^2$ estimate, which is stated as follows.

Let $c_A(t)$ be a positive function in $C^\infty((-A,+\infty))$ $(A\in(-\infty,+\infty])$, satisfying

$$\int_{-A}^{\infty}c_A(t)e^{-t}dt<\infty$$

and

$$\Big(\int_{-A}^{t}c_A(t_1)e^{-t_1}dt_1\Big)^2>c_A(t)e^{-t}\int_{-A}^{t}\int_{-A}^{t_2}c_A(t_1)e^{-t_1}dt_1dt_2, \tag{1.1}$$

for any $t\in(-A,+\infty)$, that is,

$$F(t)=\log\left(\int_{-A}^{t}\int_{-A}^{t_2}c_A(t_1)e^{-t_1}dt_1dt_2\right)$$

is strictly concave for any $t\in(-A,+\infty)$.

An easy example of such functions $c_A(t)$ is when $c_A(t)e^{-t}$ is decreasing with respect to $t$.

Remark 1.4 (See [8]). Another important class of functions $c_A(t)$ satisfying (1.1) contains the functions satisfying the following,

(1) $\frac{d}{dt}c_A(t)e^{-t}>0$, for $t\in(-A,a)$;
(2) $\frac{d}{dt}c_A(t)e^{-t}\le 0$, for $t\in[a,\infty)$;
(3) $\frac{d^2}{dt^2}\log(c_A(t)e^{-t})<0$, for $t\in(-A,a)$,

where $a\ge -A$ is a constant.

Theorem 1.5. *Let $(M,S)$ be an almost Stein pair, and $\Psi$ be a plurisubharmonic function in $\Delta_A(S)\cap C^\infty(M\setminus(S\cup X))$ ($X$ is as in the definition of almost Stein pair). Let $h$ be a smooth metric on a holomorphic vector bundle $E$ on $M$ with rank $r$, such that $he^{-\Psi}$ is semi-positive in the sense of Nakano on $M\setminus(S\cup X)$ (when $E$ is a line bundle, $h$ can be chosen as a semipositive singular metric). Then there exists a uniform constant $\mathbf{C}=1$, which is optimal, such that, for any holomorphic section $f$ of $K_M\otimes E|_S$ on $S$ satisfying condition that*

$$\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2_h dV_M[\Psi]<\infty,$$

*there exists a holomorphic section $F$ of $K_M\otimes E$ on $M$ satisfying $F=f$ on $S$ and*

$$\int_M c_A(-\Psi)|F|^2_h dV_M\le \mathbf{C}\int_{-A}^{\infty}c_A(t)e^{-t}dt\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2_h dV_M[\Psi].$$

As we have mentioned, various results have been derived from the above theorem. A survey about the optimal $L^2$ extension and its applications is given in [24].

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Especially, when we take $M$ to be a pseudoconvex domain $D$ in $\mathbb{C}^n$ with coordinates $(z_1, z_2, \ldots, z_n)$, $H$ be the intersection of the set $\{z_{n-l+1} = \cdots = z_n = 0\}$ with $D$, let $dV_D$ and $dV_H$ be the Lebesgue measures on $D$ and $H$, respectively, $\Psi(z) = l\log(|z_{n-l+1}|^2 + \cdots + |z_n|^2)$, then $dV_D[l\log(|z_{n-l+1}|^2 + \cdots + |z_n|^2)] = dV_H$. For any positive function $c_A(t)$ that satisfies (1.1), Theorem 1.5 degenerates to the following.

THEOREM 1.6. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain, let $l$ be an integer between 1 and $n$, let $\varphi$ be a plurisubharmonic function on $D$, and let $H = D \cap \{z_n = \cdots = z_{n-l+1} = 0\}$. Then for any holomorphic function $f$ on $H$ satisfying*

$$\int_H |f|^2 e^{-\varphi} dV_H < \infty,$$

*there exists a holomorphic function $F$ on $D$ satisfying $F = f$ on $H$ and*

$$\int_D c_A(-l\log(|z_n|^2 + \cdots + |z_{n-l+1}|^2))|F|^2 e^{-\varphi} dV_D \leq \frac{\pi^l}{l!} \int_{-A}^{\infty} c_A(t)e^{-t}dt \int_H |f|^2 e^{-\varphi} dV_H.$$

By using Theorem 1.6, we obtain the following

THEOREM 1.7. *If $D \subset \mathbb{C}^n$ be a pseudoconvex domain, $l$ be an integer between 1 and $n$, $H = D \cap \{z_{n-l+1} = \cdots = z_n = 0\}$, $\psi(z) = \rho((\sum_{j=n-l+1}^{n} |z_j|^2)^{l/2})$ where $\rho$ is a radial subharmonic function on $\mathbb{C}$. Then for every plurisubharmonic function $\varphi$ on $D$ and every holomorphic function $f$ on $H$ such that*

$$\int_H |f|^2 e^{-\varphi} dV_H < \infty,$$

*there exists a holomorphic function $F$ on $D$ such that*

$$F = f$$

*on $H$ and*

$$\int_D |F|^2 e^{-\varphi-\psi} dV_D \leqslant 2\sigma_l \int_0^{e^{\frac{A}{2}}} e^{-\rho(t)} t dt \int_H |f|^2 e^{-\varphi} dV_H, \tag{1.2}$$

*where $A = \sup_{z \in D} l\log(\sum_{j=n-l+1}^{n} |z_j|^2)$.*

REMARK 1.8. Since $\rho$ is radial, we may regard $\rho$ as a function defined on $\mathbb{R}_{\geq 0}$ and thus $\rho((\sum_{j=n-l+1}^{n} |z_j|^2)^{l/2})$ and $\rho(t)$ in the right-hand side of (1.2) make sense.

When $n = 1$, as a corollary of Theorem 1.7, we conclude that the question of Ohsawa holds under a natural condition.

COROLLARY 1.9. *If $\psi$ is a subharmonic function on $\mathbb{C}$ such that $i\partial\bar{\partial}\psi$ is rotational invariant and satisfies*

$$\int_{\mathbb{C}} e^{-\psi} < \infty,$$

*then for every subharmonic function $\varphi$ on $\mathbb{C}$, there exists a holomorphic function $f$ on $\mathbb{C}$ such that*

$$f(0) = 1$$

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*and*

$$\int_{\mathbb{C}} |f|^2 e^{-\varphi-\psi} \leqslant e^{-\varphi(0)} \int_{\mathbb{C}} e^{-\psi}.$$

However, without the condition that $i\partial\bar{\partial}\psi$ is rotational invariant, the above question of Ohsawa fails in general. Recently, Guan gave a counterexample in [5] by showing that there exist subharmonic functions $\psi$ and $\varphi$ on $\mathbb{C}$ such that the Bergman space $A^2(\mathbb{C}, e^{-\varphi-\psi})$ with respect to $\varphi+\psi$ contains only zero. In the last section, we will give another counterexample with nontrivial $A^2(\mathbb{C}, e^{-\varphi-\psi})$.

## §2. Proof of main results

In order to prove our main theorem and corollaries, we recall some well-known results.

Proposition 2.1. *Let $D \subset \mathbb{C}^n$ be a domain and set $D_j = \{z \in D : |z| < j$ and $\delta_D(z) > \frac{1}{j}\}$ ($\delta_D(z)$ denotes the distance between $z$ and $\partial D$), $j = 1, 2, \ldots$. Suppose $u \not\equiv -\infty$ is plurisubharmonic on $D$. Assume that $u(z)$ is radial. Then there is a sequence $\{u_j\}_{j=1}^{\infty} \subset C^{\infty}(D)$ with the following properties:*

(1) *$u_j$ is radial and strictly plurisubharmonic on $D_j$.*
(2) *$u_j \geq u_{j+1}$ on $D_j$ and $\lim_{j\to\infty} u_j(z) = u(z)$ for any $z \in D$.*

*Proof.* This is the classical regularization theorem of plurisubharmonic functions on domains in $\mathbb{C}^n$ which can be done by taking convolution (see [19]). Note that if we choose the mollifier in the convolution to be radial, that is, depends only on $|z|$, it is easy to verify the convolution leads to a plurisubharmonic function satisfying the requirements. □

Proposition 2.2 (See [8]). *Let $D$ be a domain in $\mathbb{C}^n$ and $\{F_j\}_{j=1}^{\infty}$ be a sequence of holomorphic functions on $D$. Assume that for any compact subset $K$ of $D$, there exists a constant $C_K > 0$, such that*

$$\int_K |F_j|^2 dV \leq C_K$$

*holds for any $j = 1, 2, \ldots$. Then we have a subsequence of $\{F_j\}_{j=1}^{\infty}$, which is convergent to a holomorphic function uniformly on any compact subset of $D$.*

Proposition 2.3 (Riesz's decomposition, see [20]). *Let $\psi \not\equiv -\infty$ be a subharmonic function on a domain $D$ in $\mathbb{C}$. Then for any given relatively compact open subset $U$ of $D$, we can write $\psi$ as*

$$\psi(z) = \frac{1}{2\pi} \int_U \log|z-\zeta| i\partial\bar{\partial}\psi(\zeta) + h(z)$$

*on $U$, where $h$ is a harmonic function on $U$.*

Now we turn to prove Theorem 1.7.

*Proof.* First, we prove the result with the assumption that $\rho$ is smooth and strictly subharmonic.

Let $\Psi(z) = l \log \sum_{j=n-l+1}^{n} |z_j|^2$ and let $c_A(t) = e^{-\rho(e^{-t/2})}$. As we have mentioned in Remark 1.8, we regard $\rho$ as a function defined on $\mathbb{R}_{\geq 0}$. It is clear that $c_A(-\Psi(z)) = e^{-\rho((\sum_{j=n-l+1}^{n} |z_j|^2)^{l/2})} = e^{-\psi(z)}$.

Let $T = e^{-\frac{t}{2}}$, then we have

$$\int_{-A}^{\infty} c_A(t)e^{-t}dt = -2\int_{-A}^{\infty} e^{-\rho(e^{-\frac{t}{2}})}e^{-\frac{t}{2}}de^{-\frac{t}{2}} = 2\int_0^{e^{\frac{A}{2}}} e^{-\rho(T)}TdT.$$

On the other hand, our choice of $c_A(t) = e^{-\rho(e^{-t/2})}$ belongs to the class mentioned in Remark 1.4. Indeed, we have

$$\begin{aligned}\frac{d}{dt}c_A(t)e^{-t} =& \frac{d}{dT}(e^{-\rho(T)}T^2)\frac{dT}{dt}\\ =&(2Te^{-\rho(T)} - T^2\rho'(T)e^{-\rho(T)})(-\frac{1}{2}T),\\ =& -\frac{1}{2}T^2(2 - T\rho'(T))e^{-\rho(T)}\end{aligned}$$

and

$$\log(c_A(t)e^{-t}) = \log(e^{-\rho(T)}T^2) = \log T^2 - \rho(T).$$

Then we have

$$\frac{d}{dT}\log(c_A(t)e^{-t}) = \frac{2}{T} - \rho'(T),$$

and

$$\frac{d^2}{dT^2}\log(c_A(t)e^{-t}) = \frac{d}{dT}(\frac{2}{T} - \rho'(T)) = -\frac{2}{T^2} - \rho''(T).$$

Therefore,

$$\begin{aligned}\frac{d^2}{dt^2}\log(c_A(t)e^{-t}) =& \frac{d}{dt}(\frac{d}{dT}\log(e^{-\rho(T)}T^2)\frac{dT}{dt})\\ =& \frac{d}{dT}\frac{d}{dT}\log(e^{-\rho(T)}T^2)(\frac{dT}{dt})^2 + \frac{d}{dT}\log(e^{-\rho(T)}T^2)\frac{d^2T}{dt^2}\\ =&(-\frac{2}{T^2} - \rho''(T))\frac{1}{4}T^2 + (\frac{2}{T} - \rho'(T))\frac{1}{4}T\\ =& -\frac{1}{4}T(\rho'(T) + \rho''(T)T).\end{aligned}$$

Note that

$$\rho'(T) + \rho''(T)T > 0$$

is exactly the condition of a radial smooth function on $\mathbb{C}$ being a strictly subharmonic function. It is then obvious that $\frac{d^2}{dt^2}\log(c_A(t)e^{-t}) < 0$ and thus $c_A(t)$ satisfies the condition (3) in Remark 1.4.

Now we denote

$$g(T) = \rho'(T) + \rho''(T)T$$

and we get a nonhomogeneous linear ordinary differential equation. Solving this differential equation, we get

$$\rho(T) = (\int_0^T g(x)dx)\log T - \int_0^T g(x)\log x dx.$$

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Notice that

$$T\rho'(T) = \int_0^T g(x)dx$$

is an increasing function. Thus, the set such that $\frac{d}{dt}c_A(t)e^{-t} > 0$ or $\frac{d}{dt}c_A(t)e^{-t} \le 0$ is indeed an interval. Therefore by Theorem 1.6, we have proved the theorem with the assumption that $\rho$ is smooth and strictly subharmonic.

For the general case, let $\{D_j\}_{j=1}^{\infty}$ be a sequence of pseudoconvex domains satisfying $D_j \Subset D_{j+1}$ for all $j$ and $\bigcup_{j=1}^{\infty} D_j = D$. By Proposition 2.1, we can find a sequence of smooth radial strictly subharmonic functions $\{\rho_j\}_{j=1}^{\infty}$ such that for any $k \ge j$, $\rho_k((\sum_{j=n-l+1}^{n}|z_j|^2)^{l/2})$ decreases to $\rho((\sum_{j=n-l+1}^{n}|z_j|^2)^{l/2})$ on $D_j$.

Let

$$\psi_k(z) = \rho_k((\sum_{j=n-l+1}^{n}|z_j|^2)^{l/2}),$$

then on $D_j$, for every $k \ge j$ we have a holomorphic function $F_{j,k}$ on $D_j$ such that

$$\begin{aligned}\int_{D_j}|F_{j,k}(z)|^2e^{-\varphi}e^{-\psi_k} &\le 2\frac{\pi^l}{l!}\int_0^{e^{\frac{A_j}{2}}} e^{-\rho(t)}tdt\int_{H_j}|f|^2e^{-\varphi}dV_{H_j}\\ &\le 2\frac{\pi^l}{l!}\int_0^{e^{\frac{A}{2}}} e^{-\rho(t)}tdt\int_{H}|f|^2e^{-\varphi}dV_{H}.\end{aligned}$$

where $H_j = H\cap D_j$ and $A_j = \sup_{z\in D_j} l\log(\sum_{m=n-l+1}^{n}|z_m|^2)$. Note that for any $k \ge j$, $\varphi+\psi_k$ is bounded from above on $D_j$, by Proposition 2.2, we may find a subsequence of $\{F_{j,k}\}_k$, still denoted by $\{F_{j,k}\}_k$, which is convergent to a holomorphic function $F_j$ uniformly on any compact subset of $D_j$.

By Fatou's lemma, we have

$$\int_{D_j}|F_j|^2e^{-\varphi-\psi} \le \liminf_{k\to\infty}\int_{D_j}|F_{j,k}(z)|^2e^{-\varphi}e^{-\psi_k} \le 2\frac{\pi^l}{l!}\int_0^{e^{\frac{A}{2}}} e^{-\rho(t)}tdt\int_{H}|f|^2e^{-\varphi}dV_{H}.$$

Again, since $\varphi+\psi$ is bounded from above on any compact subset of $D$, by the same reason, we may find a subsequence of $\{F_j\}_j$, still denoted by $\{F_j\}_j$, which is convergent to a holomorphic function $F$ uniformly on any compact subset of $D$ and

$$\begin{aligned}\int_{D}|F|^2e^{-\varphi-\psi} &\le \liminf_{k\to\infty}\int_{D}\mathbb{I}_{\overline{D_j}}|F_j(z)|^2e^{-\varphi}e^{-\psi}\\ &\le 2\frac{\pi^l}{l!}\int_0^{e^{\frac{A}{2}}} e^{-\rho(t)}tdt\int_{H}|f|^2e^{-\varphi}dV_{H}.\end{aligned}$$

□

If we take $\psi(z) = \alpha|z_n|^2$ for $\alpha > 0$, then we have Theorem 1.7 in [10] and Theorem 4.1 in [17].

Corollary 2.4. *Let $\alpha > 0$ be a constant, $D$ be a pseudoconvex domain in $\mathbb{C}^n$, $\varphi$ be a plurisubharmonic function on $D$ and $H = D\cap\{z_n = 0\}$. Then for any holomorphic function*

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*f on H satisfying*

$$\int_H |f|^2 e^{-\varphi} dV_H < \infty,$$

*there exists a holomorphic function F on D satisfying F = f on H and*

$$\int_D |F|^2 e^{-\varphi-\alpha|z_n|^2} dV_D \leq \frac{\pi}{\alpha} \int_H |f|^2 e^{-\varphi} dV_H.$$

Let $\epsilon > 0$ be an arbitrary constant, if we take $\psi = (1+\epsilon)\log(1+|z_n|^2)$, we have Corollary 3.14 in [8] (see also Theorem 0.1 in [17]).

COROLLARY 2.5. *Let D be a pseudoconvex domain in* $\mathbb{C}^n$, $\varphi$ *be a plurisubharmonic function. For any plurisubharmonic function* $\varphi$ *on D, for any* $\varepsilon > 0$ *and for any holomorphic function f on* $H = D \cap \{z_n = 0\}$, *there exists a holomorphic function F on D such that F = f on H and*

$$\int_D \frac{|F|^2}{(1+|z_n|^2)^{1+\varepsilon}} e^{-\varphi} \leq \frac{\pi}{\varepsilon} \int_H |f|^2 e^{-\varphi}.$$

Now we prove Corollary 1.9.

*Proof.* For every $R > 0$, denote by $D(0,R)$ the disc centered at 0 of radius $R$. By Proposition 2.3, we can write $\psi$ on $D(0,R)$ as

$$\psi(z) = \frac{1}{2\pi} \int_{D(0,R)} \log|z-\zeta| i\partial\bar{\partial}\psi(\zeta) + h(z),$$

where $h$ is a harmonic function on $D(0,R)$.

Let $\psi_1(z) = \frac{1}{2\pi}\int_{D(0,R)} \log|z-\zeta| i\partial\bar{\partial}\psi(\zeta)$, then $\psi_1$ is a radial subharmonic function. By Theorem 1.7, we have a holomorphic function $f$ such that $f(0) = 1$ and

$$\int_{D(0,R)} |f|^2 e^{-\varphi-\psi_1-h} \leqslant \left(2\pi \int_0^R e^{-\psi_1(r)} r\mathrm{d}r\right) e^{-\varphi(0)-h(0)} = \left(\int_{D(0,R)} e^{-\psi_1}\right) e^{-\varphi(0)-h(0)}.$$

Because $h$ is a harmonic function and $e^x$ is an increasing convex function, $e^{-h}$ is a subharmonic function, thus

$$e^{-h(0)} \int_{D(0,R)} e^{-\psi_1} \leqslant \int_{D(0,R)} e^{-\psi_1-h}$$

and therefore we have

$$\int_{D(0,R)} |f|^2 e^{-\varphi-\psi} \leqslant e^{-\varphi(0)} \int_{D(0,R)} e^{-\psi} \leqslant e^{-\varphi(0)} \int_{\mathbb{C}} e^{-\psi}. \tag{2.1}$$

Now for every $R > 0$, we have a holomorphic function $F_R$ satisfying $F_R(0) = 1$ and (2.1). By Proposition 2.2 and a diagonal argument, we have a subsequence of $\{F_R\}$, still denoted by $\{F_R\}$, which is convergent to a holomorphic function $F$ uniformly on any compact subset of $\mathbb{C}$.

By Fatou's lemma, we have

$$\begin{aligned}\int_{\mathbb{C}} |F|^2 e^{-\varphi-\psi} &\leq \liminf_{R\to\infty} \int_{\mathbb{C}} \mathbb{I}_{\overline{D(0,R)}} |F_R| e^{-\varphi-\psi} \\ &\leq e^{-\varphi(0)} \int_{\mathbb{C}} e^{-\psi}.\end{aligned}$$

∎

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## §3. A counterexample of Ohsawa's question

The following example shows that the above-mentioned question posed by Ohsawa does not hold if $i\partial\bar\partial\psi$ is not rotationally invariant.

On $\mathbb{C}$, take $\varphi = 2(1-\alpha)\log|z+1|$, $\psi = 2\alpha\log|z+1| + h(|z+1|^2)$ where $0<\alpha<1$ is a constant and

$$h(x) = \begin{cases} R_1 + \frac{1}{4}, & x < R_1 \\ (x-R_1)^2 + R_1 + \frac{1}{4}, & R_1 \leqslant x \leqslant R_1 + \frac{1}{2} \\ x, & x > R_1 + \frac{1}{2} \end{cases}$$

where $R_1$ is a positive number that will be determined later.

Note that $h$ is an increasing convex function, so $h(|z+1|^2)$ is subharmonic.

We see that

$$\begin{aligned} \int_{\mathbb{C}} e^{-\psi} &= \int_{\{|z+1|^2<R_1+\frac{1}{2}\}} \frac{1}{|z+1|^{2\alpha}} e^{-h(|z+1|^2)} + \int_{\{|z+1|^2>R_1+\frac{1}{2}\}} \frac{1}{|z+1|^{2\alpha}} e^{-h(|z+1|^2)} \\ &= \int_{\{|z+1|^2<R_1+\frac{1}{2}\}} \frac{1}{|z+1|^{2\alpha}} e^{-h(|z+1|^2)} + \int_{\{|z+1|^2>R_1+\frac{1}{2}\}} \frac{1}{|z+1|^{2\alpha}} e^{-|z+1|^2} < \infty. \end{aligned}$$

The square integrable holomorphic functions with respect to $\varphi+\psi = 2\log|z+1| + h(|z+1|^2)$ have an orthogonal basis $\{(z+1)^k\}$, $k \geqslant 1$. So the minimal $L^2$-norm element such that $f(0)=1$ is just $f(z)=z+1$. We see

$$\begin{aligned} I := \int_{\mathbb{C}} |z+1|^2 \frac{1}{|z+1|^2} e^{-h(|z+1|^2)} &= \int_{\mathbb{C}} e^{-h(|z+1|^2)} \\ &= \int_{\{|z+1|^2<R_1\}} e^{-R_1-\frac{1}{4}} + \int_{\{|z+1|^2>R_1\}} e^{-h(|z+1|^2)} \\ &= e^{-R_1-\frac{1}{4}}\pi R_1 + \int_{\{|z+1|^2>R_1\}} e^{-h(|z+1|^2)}, \\ II := e^{-\varphi(0)} \int_{\mathbb{C}} e^{-\psi} &= \int_{\mathbb{C}} \frac{1}{|z+1|^{2\alpha}} e^{-h(|z+1|^2)} \\ &= \int_{\{|z+1|^2<R_1\}} \frac{1}{|z+1|^{2\alpha}} e^{-R_1-\frac{1}{4}} \\ &\quad + \int_{\{|z+1|^2>R_1\}} \frac{1}{|z+1|^{2\alpha}} e^{-h(|z+1|^2)} \\ &= e^{-R_1-\frac{1}{4}} \frac{\pi}{1-\alpha} R_1^{1-\alpha} + \int_{\{|z+1|^2>R_1\}} \frac{1}{|z+1|^{2\alpha}} e^{-h(|z+1|^2)}. \end{aligned}$$

We can choose $R_1$ large enough such that

$$I - II = e^{-R_1-\frac{1}{4}}\pi\left(R_1 - \frac{1}{1-\alpha}R_1^{1-\alpha}\right) + \int_{\{|z+1|^2>R_1\}} \left(1 - \frac{1}{|z+1|^{2\alpha}}\right) e^{-h(|z+1|^2)} > 0.$$

Thus the minimal $L^2$ extension with respect to $\psi+\varphi$ satisfying $f(0)=1$ exceeds $e^{-\varphi(0)} \cdot \int_{\mathbb{C}} e^{-\psi}$.

This example also shows that Theorem 1.7 fails in general without the assumptions posed in the theorem.

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