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THE DISTRIBUTION OF REFRACTED LÉVY PROCESSES WITH JUMPS HAVING RATIONAL LAPLACE TRANSFORMS

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Abstract

We consider a refracted jump diffusion process having two-sided jumps with rational Laplace transforms. For such a process, by applying a straightforward but interesting approach, we derive formulae for the Laplace transform of its distribution. Our formulae are presented in an attractive form and the approach is novel. In particular, the idea in the application of an approximating procedure is remarkable. In addition, the results are used to price variable annuities with state-dependent fees.

Keywords: Refracted Lévy process; rational Laplace transform; continuity theorem; Wiener–Hopf factorization; variable annuity; state-dependent fee

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1. Introduction

A refracted Lévy process $U = (U_t)_{t\geq 0}$ is derived from a Lévy process $X = (X_t)_{t\geq 0}$ and is described by the following equation (see [13]):

$$U_t = X_t - \delta \int_0^t \mathbf{1}_{\{U_s > b\}}\, \mathrm{d}s, \tag{1.1}$$

where $\delta, b \in \mathbb{R}$, and $\mathbf{1}_A$ is the indicator function of a set A. Refracted Lévy processes have been investigated in [13], [14], and [20] based on the assumption that X in (1.1) has negative jumps only; and in [23], the process X was assumed to be a double-exponential jump diffusion process. Many results, including formula for occupation times of U, have been obtained, and the interested reader is referred to the above papers for the details. In addition, in [22], [24], and [25], under several different assumptions on X, we have considered a similar process $U^s = (U^s_t)_{t\geq 0}$:

$$\mathrm{d}U^s_t = \mathrm{d}X_t - \delta \mathbf{1}_{\{U^s_t < b\}}\, \mathrm{d}t. \tag{1.2}$$

For the process U in (1.1) with X given by (2.1), we will show that $\mathbb{P}(U_t = b) = 0$ is Lebesgue for almost every $t > 0$ (see Remark 4.1), which means that $U_t = X_t - \delta t - (-\delta) \int_0^t \mathbf{1}_{\{U_s < b\}}\, \mathrm{d}s$. Thus, the two processes U^s and U are essentially equal.

In this paper we are interested in the distribution of U. When the process X_t in (1.1) is a Lévy process without positive jumps, the corresponding results can be found in [13, Theorem 6(iv)]. Thus, here we focus on the situation when X has both positive and negative jumps. Specifically,

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we assume that X in (1.1) is a jump diffusion process and its jumps have probability density functions whose Laplace transforms are rational functions. Such a Lévy process is popular and quite general, and two particular examples of it are a hyper-exponential jump diffusion process (see [5] and [6]) and a Lévy process with phase-type jumps (see [2] and [19]). Under the above assumption on X, we want to derive the expression of $\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}(U_t < y)\,\mathrm{d}t$ or in differential form

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}(U_t \in \mathrm{d}y)\,\mathrm{d}t, \qquad y \in \mathbb{R}, \tag{1.3}$$

where $q > 0$. One reason why we are interested in the above quantity is that it is closely related to occupation times of U since

$$q\int_0^\infty \mathrm{e}^{-qt}\mathbb{E}\bigg[\int_0^t \mathbf{1}_{\{U_s<y\}}\,\mathrm{d}s\bigg]\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}(U_t < y)\,\mathrm{d}t, \qquad y \in \mathbb{R},$$

which can be derived by applying integration by parts. This means that the occupation times of U, i.e. $\int_0^t \mathbf{1}_{\{U_s<y\}}\,\mathrm{d}s$, can be derived from (1.3).

In [22], under the same assumption on X as in this paper, we have derived formulae for $\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}(U_t^s < b)\,\mathrm{d}t$, where U^s and b are given by (1.2). In this paper, for a given b in (1.1), we combine the ideas of [22] with a novel and helpful approximating discussion to calculate $\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}(U_t < y)\,\mathrm{d}t$, where $y \in \mathbb{R}$. In particular, we obtain some attractive and uncommon formulae, which are written in terms of positive and negative Wiener–Hopf factors. These extraordinary expressions are important and are conjectured to hold for a general Lévy process X and the corresponding solution U to (1.1), providing that such a solution U exists.

The results in this paper have some applications. One application is to price equity-linked investment products or variable annuities with state-dependent fees as in [25]. Such a state-dependent fee charging method has been proposed recently and has several advantages (see [4] and [7]), e.g. it can reduce the incentive for a policyholder to surrender the policy. Equity-linked products are popular life insurance contracts and one reason for their popularity is that they typically provide a guaranteed minimum return. There are many papers devoted to studying such products; see, e.g. [10], [15], and [18]. Investigations on evaluating equity-linked products under a state-dependent fee structure are relatively new and we refer the reader to [17] for a recent work.

The remainder of this paper is organized as follows. In Section 2 some notation and some preliminary results are introduced. Next, we present an important proposition in Section 3 and state the main results in Section 4. Finally, the application of our main results is discussed in Section 5.

2. Notation and preliminary results

In this paper the process $X = (X_t)_{t\ge 0}$ in (1.1) is a jump diffusion process, where the jumps have rational Laplace transforms. Specifically,

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{k=1}^{N_t^+} Z_k^+ - \sum_{k=1}^{N_t^-} Z_k^-, \tag{2.1}$$

where X_0, μ, and $\sigma > 0$ are constants; $(W_t)_{t\ge 0}$ is a standard Brownian motion; $\sum_{k=1}^{N_t^+} Z_k^+$ and $\sum_{k=1}^{N_t^-} Z_k^-$ are compound Poisson processes with intensity λ^+ and λ^-, respectively; and the density functions of Z_1^+ and Z_1^- are given respectively by

$$p^+(z) = \sum_{k=1}^{m^+}\sum_{j=1}^{m_k} c_{kj}\frac{(\eta_k)^j z^{j-1}}{(j-1)!}\mathrm{e}^{-\eta_k z}, \qquad z > 0, \tag{2.2}$$

and

$$p^-(z) = \sum_{k=1}^{n^-}\sum_{j=1}^{n_k} d_{kj}\frac{(\vartheta_k)^j z^{j-1}}{(j-1)!}\mathrm{e}^{-\vartheta_k z}, \qquad z > 0, \tag{2.3}$$

with $\eta_i \neq \eta_j$ and $\vartheta_i \neq \vartheta_j$ for $i \neq j$; moreover, $(W_t)_{t\geq 0}$, $\sum_{k=1}^{N_t^+} Z_k^+$ and $\sum_{k=1}^{N_t^-} Z_k^-$ are mutually independent.

Remark 2.1. The parameters η_k and c_{kj} in (2.2) can take complex values as long as $p^+(z)$ satisfies $p^+(z) \geq 0$ and $\int_0^\infty p^+(z)\,\mathrm{d}z = 1$. In addition, if η_1 has the smallest real part among $\eta_1, \dots, \eta_{m^+}$ then $0 < \eta_1 < \mathrm{Re}(\eta_2) \leq \cdots \leq \mathrm{Re}(\eta_{m^+})$.

Remark 2.2. Equation (2.2) is quite general and, in particular, contains phase-type distributions. Thus, from Proposition 1 of [2], we know that for any given Lévy process X, there is a sequence of X^n with the form of (2.1) such that

$$\lim_{n\uparrow\infty}\sup_{s\in[0,t]}|X_s^n - X_s| = 0 \quad \text{almost surely.}$$

In what follows, the law of X starting from x is denoted by $\mathbb{P}_x$ with $\mathbb{E}_x$ denoting the corresponding expectation; when $x = 0$, we write $\mathbb{P}$ and $\mathbb{E}$ for convenience. And, as usual, for $T \geq 0$, define

$$\underline{X}_T := \inf_{0\leq t\leq T} X_t \quad \text{and} \quad \overline{X}_T := \sup_{0\leq t\leq T} X_t. \tag{2.4}$$

Throughout this paper, for a given $q > 0, e(q)$ is an exponential random variable whose expectation is equal to $1/q$. Furthermore, $e(q)$ is assumed to be independent of all stochastic processes appearing in this paper. In addition, for a complex value x, let $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ represent its real part and imaginary part, respectively.

For the Lévy process X given by (2.1), it has been shown that (1.1) has a unique strong solution $U = (U_t)_{t\geq 0}$ (see, e.g. Theorem 305 of [21]), which is a strong Markov process (see Remark 3 of [13]). For this unique solution U, our objective is deriving expression (1.3), i.e.

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(U_t \in \mathrm{d}y)\,\mathrm{d}t, \qquad y \in \mathbb{R},$$

and, more importantly, we try to derive some novel expression.

Similar to previous investigations on refracted Lévy processes (see, e.g. [13]), for a given $\delta \in \mathbb{R}$, we introduce a process Y, which is defined as $Y = \{Y_t = X_t - \delta t;\ t \geq 0\}$. For the process Y, the two quantities $\underline{Y}_T$ and $\overline{Y}_T$ are defined similarly as in (2.4). What is more, we denote by $\hat{\mathbb{P}}_y$ the law of Y such that $Y_0 = y$ and by $\hat{\mathbb{E}}_y$ the corresponding expectation, and abbreviate this to $\hat{\mathbb{P}}$ and $\hat{\mathbb{E}}$ when $y = 0$.

The following lemma gives the roots of $\psi(z) = q$ and $\hat{\psi}(z) = q$, where

$$\psi(z) := \mathrm{i}z\mu - \frac{\sigma^2}{2}z^2 + \lambda^+\Bigg(\sum_{k=1}^{m^+}\sum_{j=1}^{m_k}\frac{c_{kj}(\eta_k)^j}{(\eta_k - \mathrm{i}z)^j} - 1\Bigg) + \lambda^-\Bigg(\sum_{k=1}^{n^-}\sum_{j=1}^{n_k}\frac{d_{kj}(\vartheta_k)^j}{(\vartheta_k + \mathrm{i}z)^j} - 1\Bigg), \tag{2.5}$$

and $\hat{\psi}(z) := \psi(z) - \mathrm{i}\delta z$. Note that if $z \in \mathbb{R}$ then $\psi(z) := \ln(\mathbb{E}[\mathrm{e}^{\mathrm{i}zX_1}])$ and $\hat{\psi}(z) := \ln(\hat{\mathbb{E}}[\mathrm{e}^{\mathrm{i}zY_1}])$. Lemma 2.1 was developed in [16]; see Lemma 1.1 and Theorem 2.1 in that paper (note that $\sigma > 0$ here).

Lemma 2.1. (i) *For $q > 0$, $\psi(z) = q(\hat{\psi}(z) = q)$ has, in the set $\mathrm{Im}(z) < 0$, a total of $M^+(\hat{M}^+)$ distinct solutions $-\mathrm{i}\beta_1(-\mathrm{i}\hat{\beta}_1), -\mathrm{i}\beta_2(-\mathrm{i}\hat{\beta}_2), \dots, -\mathrm{i}\beta_{M^+}(-\mathrm{i}\hat{\beta}_{\hat{M}^+})$, with respective multiplicities $M_1 = 1(\hat{M}_1 = 1), M_2(\hat{M}_2), \dots, M_{M^+}(\hat{M}_{\hat{M}^+})$. Moreover, it holds that*

$$0 < \beta_1 < \mathrm{Re}(\beta_2) \le \cdots \le \mathrm{Re}(\beta_{M^+}), \qquad 0 < \hat{\beta}_1 < \mathrm{Re}(\hat{\beta}_2) \le \cdots \le \mathrm{Re}(\hat{\beta}_{\hat{M}^+}), \tag{2.6}$$

and

$$\sum_{k=1}^{M^+} M_k = \sum_{k=1}^{\hat{M}^+} \hat{M}_k = 1 + \sum_{k=1}^{m^+} m_k.$$

(ii) *For $q > 0$ and $s \ge 0$,*

$$\begin{aligned} \hat{\mathbb{E}}[\mathrm{e}^{-s\overline{Y}_{e(q)}}] &= \prod_{k=1}^{m^+}\Big(\frac{s+\eta_k}{\eta_k}\Big)^{m_k} \prod_{k=1}^{\hat{M}^+}\Big(\frac{\hat{\beta}_k}{s+\hat{\beta}_k}\Big)^{\hat{M}_k}, \\ \mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q)}}] &= \prod_{k=1}^{m^+}\Big(\frac{s+\eta_k}{\eta_k}\Big)^{m_k} \prod_{k=1}^{M^+}\Big(\frac{\beta_k}{s+\beta_k}\Big)^{M_k}. \end{aligned} \tag{2.7}$$

Next, consider a function $F_1(x)$ on $(0, \infty)$ with the Laplace transform

$$\int_0^\infty \mathrm{e}^{-sx} F_1(x)\,\mathrm{d}x = \frac{1}{s}\bigg(\frac{\hat{\mathbb{E}}[\mathrm{e}^{-s\overline{Y}_{e(q)}}]}{\mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q)}}]} - 1\bigg) := \hat{F}_1(s), \qquad s > 0. \tag{2.8}$$

From (2.7) and (2.8), applying rational expansion yields

$$\int_0^\infty \mathrm{e}^{-sx} F_1(x)\,\mathrm{d}x = \sum_{k=1}^{\hat{M}^+}\sum_{j=1}^{\hat{M}_k} \frac{1}{(s+\hat{\beta}_k)^j}\frac{1}{(\hat{M}_k - j)!}\frac{\partial^{\hat{M}_k - j}}{\partial s^{\hat{M}_k - j}}(\hat{F}_1(s)(s+\hat{\beta}_k)^{\hat{M}_k})_{s=-\hat{\beta}_k},$$

which leads to

$$F_1(x) = \sum_{k=1}^{\hat{M}^+}\sum_{j=1}^{\hat{M}_k} \frac{x^{j-1}}{(j-1)!}\mathrm{e}^{-\hat{\beta}_k x}\frac{1}{(\hat{M}_k - j)!}\frac{\partial^{\hat{M}_k - j}}{\partial s^{\hat{M}_k - j}}(\hat{F}_1(s)(s+\hat{\beta}_k)^{\hat{M}_k})_{s=-\hat{\beta}_k}, \qquad x > 0.$$

Since $\hat{M}_1 = 1$ and $0 < \hat{\beta}_1 < \mathrm{Re}(\hat{\beta}_2) \le \cdots \le \mathrm{Re}(\hat{\beta}_{\hat{M}^+})$ (see (2.6)), we have

$$\lim_{x\uparrow\infty}\frac{F_1(x)}{\mathrm{e}^{-\hat{\beta}_1 x}} = -\prod_{k=2}^{\hat{M}^+}\Big(\frac{\hat{\beta}_k}{\hat{\beta}_k - \hat{\beta}_1}\Big)^{\hat{M}_k}\prod_{k=1}^{M^+}\Big(\frac{\beta_k - \hat{\beta}_1}{\beta_k}\Big)^{M_k}. \tag{2.9}$$

In addition, it is obvious that

$$F_1(0) := \lim_{x\downarrow 0} F_1(x) = \lim_{s\uparrow\infty}\int_0^\infty s\mathrm{e}^{-sx}F_1(x)\,\mathrm{d}x = \frac{\prod_{k=1}^{\hat{M}^+}(\hat{\beta}_k)^{\hat{M}_k}}{\prod_{k=1}^{M^+}(\beta_k)^{M_k}} - 1. \tag{2.10}$$

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Remark 2.3. Owing to (2.9), $F_1(x)$ is absolutely integrable and the Laplace transform of $F_1(x)$ in (2.8) can be extended analytically to the half-plane $\mathrm{Re}(s) \geq 0$. When $s = 0$, the right-hand side of (2.8) is understood as

$$\lim_{s\downarrow 0} \frac{1}{s}\bigg(\frac{\hat{\mathbb{E}}[\mathrm{e}^{-s\overline{Y}_{e(q)}}]}{\mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q)}}]} - 1\bigg) = \frac{\partial}{\partial s}\bigg(\prod_{k=1}^{\hat{M}^+}\bigg(\frac{\hat{\beta}_k}{s+\hat{\beta}_k}\bigg)^{\hat{M}_k}\prod_{k=1}^{M^+}\bigg(\frac{s+\beta_k}{\beta_k}\bigg)^{M_k}\bigg)_{s=0}.$$

Besides, $F_1(x)$ is bounded on $[0, \infty]$ with $F_1(\infty) := \lim_{x\uparrow\infty} F_1(x) = 0$.

Lemma 2.2. *For the continuous function $F_1(x)$ given by (2.9), it holds that*

$$F_1(x) + 1 \geq 0 \quad \textit{for } x > 0. \tag{2.11}$$

Proof. For $q, s > 0$, we know that (see, e.g. [1, Equation (4)])

$$\mathbb{E}[\mathrm{e}^{-s(\overline{X}_{e(q)}-h)}\mathbf{1}_{\{\overline{X}_{e(q)}>h\}}] = \mathbb{E}[\mathrm{e}^{-q\tau_h^+ - s(X_{\tau_h^+}-h)}]\mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q)}}], \qquad h > 0, \tag{2.12}$$

where $\tau_h^+ := \inf\{t \geq 0 \colon X_t > h\}$. Exchanging the order of integration yields

$$\begin{aligned}\int_0^\infty \mathbb{E}[\mathrm{e}^{-s(\overline{X}_{e(q)}-h)}\mathbf{1}_{\{\overline{X}_{e(q)}>h\}}]\,\mathrm{d}h &= \int_0^\infty\int_0^x \mathrm{e}^{-s(x-h)}\,\mathrm{d}h\mathbb{P}(\overline{X}_{e(q)} \in \mathrm{d}x)\\ &= \frac{1}{s}(1 - \mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q)}}]).\end{aligned}$$

Then, on both sides of (2.12), integrating with respect to h from 0 to ∞ yields

$$\int_0^\infty \mathbb{E}[\mathrm{e}^{-q\tau_h^+ - s(X_{\tau_h^+}-h)}]\,\mathrm{d}h = \frac{1}{s}\bigg(\frac{1}{\mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q)}}]} - 1\bigg). \tag{2.13}$$

Equations (2.8) and (2.13) mean that $\int_0^\infty \mathrm{e}^{-sx}(F_1(x) + 1)\,\mathrm{d}x$ is a completely monotone function of s on $(0, \infty)$, then (2.11) follows from Theorem 1a of [8, p. 439]. □

The following lemma is taken from Proposition 1(v) of [11], which states that for almost all $q > 0$, $\psi(z) = q$ and $\hat{\psi}(z) = q$ only have simple solutions. Based on this lemma, we apply an approximating argument (which reduces the calculation in a large extent) to derive the final results.

Lemma 2.3. *There exists only finite numbers $q > 0$ such that $\psi(z) = q$ or $\hat{\psi}(z) = q$ has solutions of multiplicity greater than 1.*

In the following, let $\mathbb{S}$ be the set of $q > 0$ such that all the roots of $\psi(z) = q$ and $\hat{\psi}(z) = q$ are simple.

Remark 2.4. For $q \in \mathbb{S}$, Lemma 2.1 yields $M^+ = \hat{M}^+ = 1 + \sum_{k=1}^{m^+} m_k$.

Applying Lemma 2.1 to the dual processes $-X_t$ and $-Y_t$ gives the following result.

Lemma 2.4. (i) *For $q \in \mathbb{S}$, $\psi(z) = q(\hat{\psi}(z) = q)$ has, in the set $\mathrm{Im}(z) > 0$, a total of $N^-(\hat{N}^-) = \sum_{k=1}^{n^-} n_k + 1$ distinct simple roots $\mathrm{i}\gamma_1(\mathrm{i}\hat{\gamma}_1), \mathrm{i}\gamma_2(\mathrm{i}\hat{\gamma}_2), \ldots, \mathrm{i}\gamma_{N^-}(\mathrm{i}\hat{\gamma}_{\hat{N}^-})$, ordered such that*

$$0 < \gamma_1 < \mathrm{Re}(\gamma_2) \leq \cdots \leq \mathrm{Re}(\gamma_{N^-}), \qquad 0 < \hat{\gamma}_1 < \mathrm{Re}(\hat{\gamma}_2) \leq \cdots \leq \mathrm{Re}(\hat{\gamma}_{\hat{N}^-}).$$

(ii) *For* $q \in \mathbb{S}$ *and* $\mathrm{Re}(s) \geq 0$, *we have*

$$\begin{aligned} \mathbb{E}[\mathrm{e}^{s\underline{X}_{e(q)}}] &= \prod_{k=1}^{n^-}\Big(\frac{s+\vartheta_k}{\vartheta_k}\Big)^{n_k}\prod_{k=1}^{N^-}\Big(\frac{\gamma_k}{s+\gamma_k}\Big), \\ \hat{\mathbb{E}}[\mathrm{e}^{s\underline{Y}_{e(q)}}] &= \prod_{k=1}^{n^-}\Big(\frac{s+\vartheta_k}{\vartheta_k}\Big)^{n_k}\prod_{k=1}^{\hat{N}^-}\Big(\frac{\hat{\gamma}_k}{s+\hat{\gamma}_k}\Big). \end{aligned} \tag{2.14}$$

Remark 2.5. For $q > 0$ and $q \in \mathbb{S}^c$, a similar result to (2.14) holds, e.g.

$$\hat{\mathbb{E}}[\mathrm{e}^{s\underline{Y}_{e(q)}}] = \prod_{k=1}^{n^-}\Big(\frac{s+\vartheta_k}{\vartheta_k}\Big)^{n_k}\prod_{k=1}^{\hat{N}^-}\Big(\frac{\hat{\gamma}_k}{s+\hat{\gamma}_k}\Big)^{\hat{N}_k^-}, \tag{2.15}$$

where $\hat{N}_k^-$ is the multiplicity of $\hat{\gamma}_k$, and $\sum_{k=1}^{\hat{N}^-}\hat{N}_k^- = \sum_{k=1}^{n^-} n_k + 1$.

Remark 2.6. For any $q > 0$, from (2.7), (2.14), and (2.15), it can be concluded that both $\overline{X}_{e(q)}$ and $\underline{Y}_{e(q)}$ have probability density functions.

The following lemma is important. We remark that the result in Lemma 2.5 is not surprising as $\sigma > 0$ in (2.1), and its proof is omitted since it can be established by using almost the same discussion as in Theorem 2.1 of [25].

Lemma 2.5. *For* $q > 0$*, the function* $V_q(x)$*, defined as* $V_q(x) := \mathbb{P}_x(U_{e(q)} > y)$ *for given* $y > b$*, is continuously differentiable on* $\mathbb{R}$*. In particular, it holds that*

$$V_q(b-) = V_q(b+) \quad \textit{and} \quad V_q'(b-) = V_q'(b+). \tag{2.16}$$

For a given $y > b$ and $q \in \mathbb{S}$, the following proposition gives the expression of

$$\mathbb{P}_x(U_{e(q)} > y) = q\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(U_t > y)\,\mathrm{d}t.$$

The proof of Proposition 2.1 is long and is deferred to Appendix A.

Proposition 2.1. *For* $q \in \mathbb{S}$ *and* $y > b$*, we have*

$$\mathbb{P}_x(U_{e(q)} > y) = \begin{cases} \sum_{k=1}^{M^+} J_k \mathrm{e}^{\beta_k(x-b)}, & x \leq b, \\ \sum_{k=1}^{\hat{M}^+} \hat{H}_k \mathrm{e}^{\hat{\beta}_k(x-y)} + \sum_{k=1}^{\hat{N}^-} \hat{P}_k \mathrm{e}^{\hat{\gamma}_k(b-x)}, & b \leq x \leq y, \\ 1 + \sum_{k=1}^{\hat{N}^-} \hat{Q}_k \mathrm{e}^{\hat{\gamma}_k(y-x)} + \sum_{k=1}^{\hat{N}^-} \hat{P}_k \mathrm{e}^{\hat{\gamma}_k(b-x)}, & x \geq y, \end{cases} \tag{2.17}$$

where $\hat{H}_k$ *and* $\hat{Q}_k$ *are given by (A.13) and (A.14), respectively;* J_k *and* $\hat{P}_k$ *are given by rational* *expansion:*

$$\sum_{i=1}^{M^+}\frac{J_i}{x-\beta_i}-\sum_{i=1}^{\hat{N}^-}\frac{\hat{P}_i}{x+\hat{\gamma}_i}-\sum_{i=1}^{\hat{M}^+}\frac{\hat{H}_i}{x-\hat{\beta}_i}\mathrm{e}^{\hat{\beta}_i(b-y)}$$
$$=\frac{\prod_{k=1}^{m^+}(x-\eta_k)^{m_k}\prod_{k=1}^{n^-}(x+\vartheta_k)^{n_k}}{\prod_{i=1}^{M^+}(x-\beta_i)\prod_{i=1}^{\hat{N}^-}(x+\hat{\gamma}_i)}$$
$$\times\sum_{k=1}^{\hat{M}^+}\frac{\prod_{i=1}^{M^+}(\hat{\beta}_k-\beta_i)\prod_{i=1}^{\hat{N}^-}(\hat{\beta}_k+\hat{\gamma}_i)}{\prod_{i=1}^{m^+}(\hat{\beta}_k-\eta_i)^{m_i}\prod_{i=1}^{n^-}(\hat{\beta}_k+\vartheta_i)^{n_i}}\frac{-\hat{H}_k}{x-\hat{\beta}_k}\mathrm{e}^{\hat{\beta}_k(b-y)}. \tag{2.18}$$

Remark 2.7. Equation (2.17) contains the roots of $\psi(z)=q$ and $\hat{\psi}(z)=q$, i.e. β_k, $\hat{\beta}_k$, and $\hat{\gamma}_k$. This poses a limitation to extending the result in Proposition 2.1 to a refracted Lévy process U driven by other Lévy processes, because we cannot characterize the roots of $\psi(z)=q$ for a general Lévy process X (note that $\psi(z)=\ln(\mathbb{E}[\mathrm{e}^{\mathrm{i}zX_1}])$ if $z\in\mathbb{R}$). In Theorem 4.1 below, we will derive another expression for $\mathbb{P}_x(U_{e(q)}>y)$, which is free of β_k, $\hat{\beta}_k$, and $\hat{\gamma}_k$.

3. An important result

In this section the following proposition is derived, and we have to say that the ideas in the derivation are interesting.

Proposition 3.1. *For given* $y>b$, $q>0$, *and* $\mathrm{Re}(\phi)=0$, *we have*

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\phi(x-b)}(\mathbb{P}_x(U_{e(q)}>y)-\hat{\mathbb{P}}_x(Y_{e(q)}>y))\,\mathrm{d}x$$
$$=\hat{\mathbb{E}}[\mathrm{e}^{\phi\underline{Y}_{e(q)}}]\mathbb{E}[\mathrm{e}^{\phi\overline{X}_{e(q)}}]\int_0^{\infty}F_1(x+y-b)\mathrm{e}^{\phi x}\,\mathrm{d}x. \tag{3.1}$$

In the following, we first show that Proposition 3.1 holds for $q\in\mathbb{S}$ and then prove that it is also valid for $q\in\mathbb{S}^c$.

Proof of Proposition 3.1. (i) *With* $q\in\mathbb{S}$. From (A.20), we know that

$$\sum_{i=1}^{\hat{M}^+}\hat{H}_i\mathrm{e}^{\hat{\beta}_i(b-y)}+\sum_{i=1}^{\hat{N}^-}\hat{P}_i-\sum_{i=1}^{M^+}J_i=0.$$

From (2.17), for $\mathrm{Re}(\phi)=0$, some straightforward calculations yield

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\phi(x-b)}\frac{\partial}{\partial x}(\mathbb{P}_x(U_{e(q)}>y))\,\mathrm{d}x$$
$$=\int_{-\infty}^{\infty}\mathrm{e}^{-\phi(x-b)}\frac{\partial}{\partial x}(\mathbb{P}_x(U_{e(q)}>y))\,\mathrm{d}x+\sum_{i=1}^{\hat{M}^+}\hat{H}_i\mathrm{e}^{\hat{\beta}_i(b-y)}+\sum_{i=1}^{\hat{N}^-}\hat{P}_i-\sum_{i=1}^{M^+}J_i$$
$$=\sum_{i=1}^{M^+}\frac{J_i\phi}{\beta_i-\phi}+\sum_{i=1}^{\hat{N}^-}\frac{\hat{P}_i\phi}{\phi+\hat{\gamma}_i}-\sum_{i=1}^{\hat{M}^+}\frac{\hat{H}_i\phi}{\hat{\beta}_i-\phi}\mathrm{e}^{\hat{\beta}_i(b-y)}$$
$$+\mathrm{e}^{\phi(b-y)}\bigg(\sum_{i=1}^{\hat{M}^+}\frac{\hat{H}_i\hat{\beta}_i}{\hat{\beta}_i-\phi}-\sum_{i=1}^{\hat{N}^-}\frac{\hat{Q}_i\hat{\gamma}_i}{\hat{\gamma}_i+\phi}\bigg). \tag{3.2}$$

It can be proved that

$$\begin{aligned}\sum_{i=1}^{\hat{M}^+}\frac{\hat{H}_i\hat{\beta}_i}{\hat{\beta}_i-\phi}-\sum_{i=1}^{\hat{N}^-}\frac{\hat{Q}_i\hat{\gamma}_i}{\hat{\gamma}_i+\phi}&=1+\sum_{i=1}^{\hat{M}^+}\frac{\hat{H}_i\phi}{\hat{\beta}_i-\phi}+\sum_{i=1}^{\hat{N}^-}\frac{\hat{Q}_i\phi}{\hat{\gamma}_i+\phi}\\&=\hat{\psi}^+(-\phi)\hat{\psi}^-(\phi)\\&=\hat{\mathbb{E}}[\mathrm{e}^{\phi\overline{Y}_{e(q)}}]\hat{\mathbb{E}}[\mathrm{e}^{\phi\underline{Y}_{e(q)}}]\\&=\hat{\mathbb{E}}[\mathrm{e}^{\phi Y_{e(q)}}],\end{aligned}\tag{3.3}$$

where $\hat{\psi}^+(\cdot)$ and $\hat{\psi}^-(\cdot)$ are given by (A.3) and (A.4); the first equality is due to the fact that $\sum_{i=1}^{\hat{M}^+}\hat{H}_i-\sum_{i=1}^{\hat{N}^-}\hat{Q}_i-1=0$ (let $\theta\uparrow\infty$ in (A.24)); the second and the third equalities follow respectively from (A.24) and (A.5), and the final equality is a result of the well-known Wiener–Hopf factorization.

In addition, for $\mathrm{Re}(\phi)=0$, applying integration by parts yields

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{e}^{-\phi(x-b)}\frac{\partial}{\partial x}(\mathbb{P}_x(U_{e(q)}>y))\,\mathrm{d}x-\mathrm{e}^{\phi(b-y)}\hat{\mathbb{E}}[\mathrm{e}^{\phi Y_{e(q)}}]\\&\quad=\int_{-\infty}^{\infty}\mathrm{e}^{-\phi(x-b)}\frac{\partial}{\partial x}(\mathbb{P}_x(U_{e(q)}>y))\,\mathrm{d}x-\int_{-\infty}^{\infty}\mathrm{e}^{-\phi(x-b)}\frac{\partial}{\partial x}(\hat{\mathbb{P}}(Y_{e(q)}>y-x))\,\mathrm{d}x\\&\quad=\phi\int_{-\infty}^{\infty}\mathrm{e}^{-\phi(x-b)}(\mathbb{P}_x(U_{e(q)}>y)-\hat{\mathbb{P}}_x(Y_{e(q)}>y))\,\mathrm{d}x.\end{aligned}\tag{3.4}$$

Note that (see (3.10) in Lemma 3.1)

$$\int_{-\infty}^{\infty}|\mathbb{P}_x(U_{e(q)}>y)-\hat{\mathbb{P}}_x(Y_{e(q)}>y)|\,\mathrm{d}x\le\frac{|\delta|}{q}.$$

Besides, from (2.18), (A.1), (A.2), (A.4), (A.5), and (A.13), after some straightforward calculations, we derive (note that $M^+=\hat{M}^+$ if $q\in\mathbb{S}$)

$$\begin{aligned}&\sum_{i=1}^{M^+}\frac{J_i\phi}{\beta_i-\phi}+\sum_{i=1}^{\hat{N}^-}\frac{\hat{P}_i\phi}{\phi+\hat{\gamma}_i}-\sum_{i=1}^{\hat{M}^+}\frac{\hat{H}_i\phi}{\hat{\beta}_i-\phi}\mathrm{e}^{\hat{\beta}_i(b-y)}\\&\qquad=\phi\hat{\mathbb{E}}[\mathrm{e}^{\phi\underline{Y}_{e(q)}}]\mathbb{E}[\mathrm{e}^{\phi\overline{X}_{e(q)}}]\int_0^{\infty}F_0(x+y-b)\mathrm{e}^{\phi x}\,\mathrm{d}x,\end{aligned}\tag{3.5}$$

where

$$F_0(x)=\sum_{i=1}^{\hat{M}^+}\mathrm{e}^{-\hat{\beta}_i x}\prod_{k=1}^{M^+}\frac{\hat{\beta}_i-\beta_k}{\beta_k}\prod_{k=1,\,k\neq i}^{\hat{M}^+}\frac{\hat{\beta}_k}{\hat{\beta}_i-\hat{\beta}_k},\qquad x>0.\tag{3.6}$$

Finally, for $s>0$, it holds that

$$\begin{aligned}\int_0^{\infty}\mathrm{e}^{-sx}F_0(x)\,\mathrm{d}x&=\sum_{i=1}^{\hat{M}^+}\prod_{k=1}^{M^+}\frac{\hat{\beta}_i-\beta_k}{\beta_k}\prod_{k=1,k\neq i}^{\hat{M}^+}\frac{\hat{\beta}_k}{\hat{\beta}_i-\hat{\beta}_k}\frac{1}{\hat{\beta}_i+s}\\&=\frac{1}{s}\bigg(\prod_{k=1}^{M^+}\frac{s+\beta_k}{\beta_k}\prod_{k=1}^{\hat{M}^+}\frac{\hat{\beta}_k}{s+\hat{\beta}_k}-1\bigg)\end{aligned}$$

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$$= \frac{1}{s}\left(\frac{\hat{\mathbb{E}}[\mathrm{e}^{-s\overline{Y}_{e(q)}}]}{\mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q)}}]} - 1\right), \tag{3.7}$$

where the second equality follows from the rational expansion and the third equality is due to (A.2), (A.3), and (A.5).

Equations (2.8) and (3.7) confirm that $F_1(x) = F_0(x)$ for $x > 0$. Therefore, (3.1) for $q \in \mathbb{S}$ is derived from (3.2)–(3.5).

(ii) *For* $q > 0$ *and* $q \in \mathbb{S}^c$. First, for such a $q > 0$ and $q \in \mathbb{S}^c$, Lemma 2.3 implies that there exists a sequences of $q_n \in \mathbb{S}$ such that $\lim_{n\uparrow\infty} q_n \downarrow q$. In the proof of Proposition 3.1 (with $q \in \mathbb{S}$), we have shown that (3.1) holds for q_n, yielding

$$\begin{aligned}\int_{-\infty}^{\infty} &\mathrm{e}^{-\phi(x-b)}(\mathbb{P}_x(U_{e(q_n)} > y) - \hat{\mathbb{P}}_x(Y_{e(q_n)} > y))\,\mathrm{d}x \\ &= \hat{\mathbb{E}}[\mathrm{e}^{\phi \underline{Y}_{e(q_n)}}]\mathbb{E}[\mathrm{e}^{\phi\overline{X}_{e(q_n)}}]\int_0^{\infty} F_1^n(x+y-b)\mathrm{e}^{\phi x}\,\mathrm{d}x,\end{aligned} \tag{3.8}$$

where $\mathrm{Re}(\phi) = 0$ and

$$\int_0^{\infty} \mathrm{e}^{-sx} F_1^n(x)\,\mathrm{d}x = \frac{1}{s}\left(\frac{\hat{\mathbb{E}}[\mathrm{e}^{-s\overline{Y}_{e(q_n)}}]}{\mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q_n)}}]} - 1\right), \qquad s > 0. \tag{3.9}$$

To complete the proof, we want to show that (3.8) will reduce to (3.1) after letting $n \uparrow 1$. This will be carried out in the rest of this section. □

Lemma 3.1. *For a given* $\delta, y \in \mathbb{R}$*, we have*

$$\begin{aligned}\int_{-\infty}^{\infty} |\mathbb{P}_x(U_t > y) - \hat{\mathbb{P}}_x(Y_t > y)|\,\mathrm{d}x &\le |\delta| t, \\ \int_{-\infty}^{\infty} |\mathbb{P}_x(\overline{X}_t > y) - \hat{\mathbb{P}}_x(\overline{Y}_t > y)|\,\mathrm{d}x &\le |\delta| t.\end{aligned} \tag{3.10}$$

Proof. Recall (1.1) and $Y_t = X_t - \delta t$.

For $\delta > 0$, it holds that $\mathbb{P}_x(X_t > y) \ge \mathbb{P}_x(U_t > y) \ge \hat{\mathbb{P}}_x(Y_t > y)$, thus,

$$\begin{aligned}\int_{-\infty}^{\infty} |\mathbb{P}_x(U_t > y) - \hat{\mathbb{P}}_x(Y_t > y)|\,\mathrm{d}x &= \int_{-\infty}^{\infty} \mathbb{P}_x(U_t > y) - \hat{\mathbb{P}}_x(Y_t > y)\,\mathrm{d}x \\ &\le \int_{-\infty}^{\infty} \mathbb{P}_x(X_t > y) - \hat{\mathbb{P}}_x(Y_t > y)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \mathbb{E}[\mathbf{1}_{\{y-x<X_t\le y-x+\delta t\}}]\,\mathrm{d}x \\ &= \mathbb{E}\left[\int_{-\infty}^{\infty} \mathbf{1}_{\{y-x<X_t\le y-x+\delta t\}}\,\mathrm{d}x\right] \\ &= \delta t,\end{aligned}$$

where the penultimate equality is due to the Fubini theorem.

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For $\delta < 0$, it is obvious that $\mathbb{P}_x(X_t > y) \le \mathbb{P}_x(U_t > y) \le \hat{\mathbb{P}}_x(Y_t > y)$, which yields

$$\begin{aligned}\int_{-\infty}^{\infty} |\mathbb{P}_x(U_t > y) - \hat{\mathbb{P}}_x(Y_t > y)|\,\mathrm{d}x &\le \int_{-\infty}^{\infty} \hat{\mathbb{P}}_x(Y_t > y) - \mathbb{P}_x(X_t > y)\,\mathrm{d}x \\ &= \mathbb{E}\bigg[\int_{-\infty}^{\infty} \mathbf{1}_{\{y-x+\delta t < X_t \le y-x\}}\,\mathrm{d}x\bigg] \\ &= -\delta t,\end{aligned}$$

so the first inequality in (3.10) is derived.

Note that $\sup_{0\le s\le t}(X_s - \delta s) - |\delta|t \le \overline{X}_t \le \sup_{0\le s\le t}(X_s - \delta s) + |\delta|t$, which means that

$$\mathbb{P}_x(\overline{X}_t - |\delta|t > y) \le \hat{\mathbb{P}}_x(\overline{Y}_t > y) \le \mathbb{P}_x(\overline{X}_t + |\delta|t > y).$$

Hence, the second inequality in (3.10) can be proved similarly. □

Lemma 3.2. *For* $\mathrm{Re}(\phi) = 0$*,*

$$\begin{aligned}\lim_{n\uparrow\infty} \int_{-\infty}^{\infty} &\mathrm{e}^{-\phi(x-b)}(\mathbb{P}_x(U_{e(q_n)} > y) - \hat{\mathbb{P}}_x(Y_{e(q_n)} > y))\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{-\phi(x-b)}(\mathbb{P}_x(U_{e(q)} > y) - \hat{\mathbb{P}}_x(Y_{e(q)} > y))\,\mathrm{d}x, \end{aligned} \tag{3.11}$$

and, for $\mathrm{Re}(s) \ge 0$*,*

$$\lim_{n\uparrow\infty} \hat{\mathbb{E}}[\mathrm{e}^{s\underline{Y}_{e(q_n)}}] = \hat{\mathbb{E}}[\mathrm{e}^{s\underline{Y}_{e(q)}}] \quad \textit{and} \quad \lim_{n\uparrow\infty} \mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q_n)}}] = \mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q)}}]. \tag{3.12}$$

Proof. Since $q_n > q$ and $\mathbb{P}_x(U_{e(q_n)} > y) = \int_0^\infty q_n \mathrm{e}^{-q_n t}\mathbb{P}_x(U_t > y)\,\mathrm{d}t$, we derive, via the dominated convergence theorem,

$$\lim_{n\uparrow\infty} \mathbb{P}_x(U_{e(q_n)} > y) = \mathbb{P}_x(U_{e(q)} > y). \tag{3.13}$$

Similarly, we can prove (3.12) and the following result:

$$\lim_{n\uparrow\infty} \hat{\mathbb{P}}_x(Y_{e(q_n)} > y) = \hat{\mathbb{P}}_x(Y_{e(q)} > y). \tag{3.14}$$

In addition, it holds that

$$\begin{aligned}\int_{-\infty}^{\infty} &\mathrm{e}^{-\phi(x-b)}(\mathbb{P}_x(U_{e(q_n)} > y) - \hat{\mathbb{P}}_x(Y_{e(q_n)} > y))\,\mathrm{d}x \\ &= \int_0^{\infty} q_n \mathrm{e}^{-q_n t} \int_{-\infty}^{\infty} \mathrm{e}^{-\phi(x-b)}(\mathbb{P}_x(U_t > y) - \hat{\mathbb{P}}_x(Y_t > y))\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.15}$$

From (3.10) and (3.13)–(3.15), the dominated convergence theorem yields (3.11). □

Lemma 3.3. *For* $F_1^n(x)$ *in (3.9) and* $F_1(x)$ *in (2.8), it holds that*

$$\lim_{n\uparrow\infty} \int_0^{\infty} \mathrm{e}^{-sx} F_1^n(x)\,\mathrm{d}x = \int_0^{\infty} \mathrm{e}^{-sx} F_1(x)\,\mathrm{d}x, \qquad \mathrm{Re}(s) \ge 0. \tag{3.16}$$

Proof. Due to Remark 2.3, for each n, we have

$$\int_0^\infty \mathrm{e}^{-sx} F_1^n(x)\,\mathrm{d}x = \frac{1}{s}\bigg(\frac{\hat{\mathbb{E}}[\mathrm{e}^{-s\overline{Y}_{e(q_n)}}]}{\mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q_n)}}]} - 1\bigg), \qquad \mathrm{Re}(s) \geq 0.$$

Similar to the derivation of (3.12), it can be shown that

$$\lim_{n\uparrow\infty} \hat{\mathbb{E}}[\mathrm{e}^{-s\overline{Y}_{e(q_n)}}] = \hat{\mathbb{E}}[\mathrm{e}^{-s\overline{Y}_{e(q)}}], \qquad \mathrm{Re}(s) \geq 0. \tag{3.17}$$

Equations (3.12) and (3.17) lead to

$$\lim_{n\uparrow\infty} \int_0^\infty \mathrm{e}^{-sx} F_1^n(x)\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-sx} F_1(x)\,\mathrm{d}x = \frac{1}{s}\bigg(\frac{\hat{\mathbb{E}}[\mathrm{e}^{-s\overline{Y}_{e(q)}}]}{\mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q)}}]} - 1\bigg), \tag{3.18}$$

which holds for $\mathrm{Re}(s) \geq 0$ and $s \neq 0$.

Next, we consider the $s = 0$ case. It follows from (3.9) that

$$\begin{aligned}
\int_0^\infty F_1^n(x)\,\mathrm{d}x &= \lim_{s\downarrow 0} \frac{1}{s}\bigg(\frac{\hat{\mathbb{E}}[\mathrm{e}^{-s\overline{Y}_{e(q_n)}}]}{\mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q_n)}}]} - 1\bigg) \\
&= \lim_{s\downarrow 0} \frac{\hat{\mathbb{E}}[\mathrm{e}^{-s\overline{Y}_{e(q_n)}}] - \mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q_n)}}]}{s} \\
&= \lim_{s\downarrow 0} \int_0^\infty \mathrm{e}^{-sx}(\hat{\mathbb{P}}(\overline{Y}_{e(q_n)} \leq x) - \mathbb{P}(\overline{X}_{e(q_n)} \leq x))\,\mathrm{d}x \\
&= \int_0^\infty (\hat{\mathbb{P}}(\overline{Y}_{e(q_n)} \leq x) - \mathbb{P}(\overline{X}_{e(q_n)} \leq x))\,\mathrm{d}x,
\end{aligned} \tag{3.19}$$

where the third equality is due to integration by parts and the final equality follows from the dominated convergence theorem since (see (3.10))

$$\int_0^\infty |\hat{\mathbb{P}}(\overline{Y}_{e(q_n)} \leq x) - \mathbb{P}(\overline{X}_{e(q_n)} \leq x)|\,\mathrm{d}x \leq \int_0^\infty q\mathrm{e}^{-qt}|\delta|t\,\mathrm{d}t = \frac{|\delta|}{q}.$$

As $q_n > q$ and (3.10) holds, (3.19) yields

$$\begin{aligned}
\lim_{n\uparrow\infty} \int_0^\infty F_1^n(x)\,\mathrm{d}x &= \lim_{n\uparrow\infty} \int_0^\infty q_n \mathrm{e}^{-q_n t} \int_0^\infty (\hat{\mathbb{P}}(\overline{Y}_t \leq x) - \mathbb{P}(\overline{X}_t \leq x))\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{n\uparrow\infty} \int_0^\infty q \mathrm{e}^{-qt} \int_0^\infty (\hat{\mathbb{P}}(\overline{Y}_t \leq x) - \mathbb{P}(\overline{X}_t \leq x))\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\infty F_1(x)\,\mathrm{d}x,
\end{aligned}$$

which combined with (3.18), leads to (3.16). □

Proof of Proposition 3.1. (For $q \in \mathbb{S}^c$.) Since $F_1(x), F_1^n(x) \geq -1$ for $x > 0$ (see Lemma 2.2), we can define the following measures:

$$M_1^n(x) = \int_0^x (F_1^n(z) + 1)\,\mathrm{d}z \quad \text{and} \quad M_1(x) = \int_0^x (F_1(z) + 1)\,\mathrm{d}z.$$

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Equation (3.16) implies that

$$\lim_{n\uparrow\infty}\int_0^\infty e^{-sx}\,dM_1^n(x)=\int_0^\infty e^{-sx}\,dM_1(x),\qquad s>0,$$

which combined with the continuity theorem for the Laplace transform (see Theorem 2a of [8, p. 433]) yields

$$\lim_{n\uparrow\infty}\int_0^x (F_1^n(z)+1)\,dz=\int_0^x (F_1(z)+1)\,dz\quad\text{for all } x>0. \tag{3.20}$$

Next, for fixed $z>0$, introduce the following probability distribution functions:

$$P_1^n(x)=\frac{\int_0^x (F_1^n(t)+1)\,dt}{\int_0^z (F_1^n(t)+1)\,dt}\quad\text{and}\quad P_1(x)=\frac{\int_0^x (F_1(t)+1)\,dt}{\int_0^z (F_1(t)+1)\,dt},\qquad 0<x<z.$$

Then, (3.20) means that $P_1^n(x)$ converges to $P_1(x)$ in distribution. As a result,

$$\lim_{n\uparrow\infty}\int_0^z e^{\phi t}\,dP_1^n(t)=\int_0^z e^{\phi t}\,dP_1(t)\quad\text{for } \mathrm{Re}(\phi)=0,$$

so

$$\lim_{n\uparrow\infty}\frac{\int_0^z e^{\phi t}(F_1^n(t)+1)\,dt}{\int_0^z (F_1^n(t)+1)\,dt}=\frac{\int_0^z e^{\phi t}(F_1(t)+1)\,dt}{\int_0^z (F_1(t)+1)\,dt}. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$\lim_{n\uparrow\infty}\int_0^x e^{\phi t}F_1^n(t)\,dt=\int_0^x e^{\phi t}F_1(t)\,dt\quad\text{for any } x>0,$$

which combined with (3.16) yields

$$\lim_{n\uparrow\infty}\int_0^\infty e^{\phi x}F_1^n(x+y-b)\,dx=\int_0^\infty e^{\phi x}F_1(x+y-b)\,dx. \tag{3.22}$$

Therefore, the desired result that (3.1) holds also for $q\in\mathbb{S}^c$ follows from (3.8) by letting $n\uparrow\infty$ and using (3.11), (3.12), and (3.22). □

4. Main results

For the unique strong solution U to (1.1) with X given by (2.1), its probability distribution function is given by Theorems 4.1 and 4.2.

Theorem 4.1. *For $q>0$ and $y\geq b$,*

$$\mathbb{P}_x(U_{e(q)}>y)=1-K_q(y-x)-\int_{b-x}^{y-x}F_1(y-x-z)K_q(dz), \tag{4.1}$$

where $K_q(x)$ is the convolution of $\underline{Y}_{e(q)}$ under $\hat{\mathbb{P}}$ and $\overline{X}_{e(q)}$ under $\mathbb{P}$, i.e.

$$K_q(x)=\int_{-\infty}^{\min\{0,x\}}\mathbb{P}(\overline{X}_{e(q)}\leq x-z)\hat{\mathbb{P}}(\underline{Y}_{e(q)}\in dz),\qquad x\in\mathbb{R}, \tag{4.2}$$

and $F_1(x)$ is continuous and differentiable on $(0,\infty)$ with rational Laplace transform given by (2.8).

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Proof. First, the right-hand side of (3.1) can be rewritten as

$$\int_{-\infty}^{\infty} e^{\phi x} \int_{-\infty}^{x} F_1(x-z+y-b)\,dK_q(z)\,dx, \tag{4.3}$$

where $K_q(x)$ is given by (4.2). Since $\overline{X}_{e(q)}$ and $\underline{Y}_{e(q)}$ have density functions (see Remark 2.6) and $F_1(x)$ is continuous on $(0,\infty)$, we conclude that the integrand in (4.3), i.e. $\int_{-\infty}^{x} F_1(x-z+y-b)\,dK_q(z)$, is continuous with respect to x.

As $Y_{e(q)}$ is the convolution of $\overline{Y}_{e(q)}$ and $\underline{Y}_{e(q)}$ (which is due to the well-known Wiener–Hopf factorization), $\hat{\mathbb{P}}_x(Y_{e(q)} > y)$ is continuous with respect to x. This result and Lemma 2.5 lead to the fact that $\mathbb{P}_x(U_{e(q)} > y) - \hat{\mathbb{P}}_x(Y_{e(q)} > y)$ is also continuous. For $y > b$, it follows from (3.1) and (4.3) that

$$\mathbb{P}_x(U_{e(q)} > y) - \hat{\mathbb{P}}_x(Y_{e(q)} > y) = \int_{-\infty}^{b-x} F_1(y-x-z)\,dK_q(z), \qquad x \in \mathbb{R}. \tag{4.4}$$

In addition, for $\mathrm{Re}(\phi) = 0$, we have (recall (4.2) and Remark 2.3)

$$\begin{aligned}\int_{-\infty}^{\infty} e^{\phi x}\,dK_q(x) \int_{0}^{\infty} e^{\phi x} F_1(x)\,dx &= \frac{1}{\phi}\biggl(1 - \frac{\hat{\mathbb{E}}[e^{\phi \overline{Y}_{e(q)}}]}{\mathbb{E}[e^{\phi \overline{X}_{e(q)}}]}\biggr) \int_{-\infty}^{\infty} e^{\phi x}\,dK_q(x) \\ &= \frac{1}{\phi}\biggl(\int_{-\infty}^{\infty} e^{\phi x}\,dK_q(x) - \hat{\mathbb{E}}[e^{\phi Y_{e(q)}}]\biggr) \\ &= \int_{-\infty}^{\infty} e^{\phi x}(\hat{\mathbb{P}}(Y_{e(q)} \le x) - K_q(x))\,dx, \end{aligned} \tag{4.5}$$

where the second equality is due to the Wiener–Hopf factorization, and the third one follows from the application of integration by parts. Note that

$$\begin{aligned}&\int_{-\infty}^{\infty} |\hat{\mathbb{P}}(Y_{e(q)} \le x) - K_q(x)|\,dx \\ &\quad \le \int_{-\infty}^{\infty}\int_{-\infty}^{\min\{0,x\}} |\hat{\mathbb{P}}(\overline{Y}_{e(q)} \le x-z) - \mathbb{P}(\overline{X}_{e(q)} \le x-z)|\hat{\mathbb{P}}(\underline{Y}_{e(q)} \in dz)\,dx \\ &\quad \le \int_{-\infty}^{\infty}\int_{-\infty}^{0} |\hat{\mathbb{P}}(\overline{Y}_{e(q)} \le x-z) - \mathbb{P}(\overline{X}_{e(q)} \le x-z)|\hat{\mathbb{P}}(\underline{Y}_{e(q)} \in dz)\,dx \\ &\quad \le \frac{|\delta|}{q},\end{aligned}$$

where in the first inequality, we have used the Wiener–Hopf factorization and (4.2); the final inequality follows from (3.10).

For $x \in \mathbb{R}$, (4.5) yields

$$\int_{-\infty}^{y-x} F_1(y-x-z)\,dK_q(z) = \hat{\mathbb{P}}(Y_{e(q)} \le y-x) - K_q(y-x),$$

which combined with (4.4), leads to

$$\begin{aligned}&\mathbb{P}_x(U_{e(q)} > y) - \hat{\mathbb{P}}_x(Y_{e(q)} > y) \\ &\quad = \hat{\mathbb{P}}(Y_{e(q)} \le y-x) - K_q(y-x) - \int_{b-x}^{y-x} F_1(y-x-z)\,dK_q(z).\end{aligned} \tag{4.6}$$

This proves that (4.1) holds for $y > b$. Letting $y \downarrow b$ in (4.6) and using the fact that $\lim_{y\downarrow b}\int_{b-x}^{y-x} F_1(y-x-z)K_q(\mathrm{d}z) = 0$ (since $F_1(x)$ is bounded on $(0,\infty)$; see Remark 2.3), we deduce that (4.1) holds also for $y = b$. □

Similar derivations lead to the following Theorem 4.2, and for the sake of brevity, we omit the details.

Theorem 4.2. *For $q > 0$ and $y \le b$,*

$$\mathbb{P}_x(U_{e(q)} < y) = K_q(y-x) - \int_{y-x}^{b-x} F_2(y-x-z)K_q(\mathrm{d}z), \tag{4.7}$$

where $F_2(x)$ is continuous and differentiable on $(-\infty, 0)$ and its Laplace transform is given by

$$\int_{-\infty}^{0} \mathrm{e}^{sz}F_2(z)\,\mathrm{d}z = \frac{1}{s}\left(\frac{\mathbb{E}[\mathrm{e}^{s\underline{X}_{e(q)}}]}{\hat{\mathbb{E}}[\mathrm{e}^{s\underline{Y}_{e(q)}}]} - 1\right), \qquad s > 0. \tag{4.8}$$

Remark 4.1. It follows from (4.1) and (4.7) that $\mathbb{P}_x(U_{e(q)} > b) + \mathbb{P}_x(U_{e(q)} < b) = 1$, which implies that $\mathbb{P}_x(U_{e(q)} = b) = 0$ for all $q > 0$.

Remark 4.2. For fixed $b \in \mathbb{R}$, letting $y = b$ in (4.1), we arrive at $\mathbb{P}_x(U_{e(q)} > b) = 1 - K_q(b-x)$, which means that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\phi(x-b)}\,\mathrm{d}(\mathbb{P}_x(U_{e(q)} > b)) = \hat{\mathbb{E}}[\mathrm{e}^{\phi\overline{Y}_{e(q)}}]\mathbb{E}[\mathrm{e}^{\phi\underline{X}_{e(q)}}].$$

A similar result has already been derived in [22]; see Equation (4.1) in that paper.

Remark 4.3. Compared with (2.17), in (4.1), (4.2), (4.7), and (4.8), the roots of $\psi(z) = q$ and $\hat{\psi}(z) = q$ disappear. The forms of these results and Remark 2.2 lead to the following conjecture: (4.1) and (4.7) hold for a general Lévy process X and the corresponding solution U (if it exists) to (1.1). Proving this conjecture is a potential research direction.

Since both $F_1(x)$ and $F_2(x)$ are differentiable, from (4.1) and (4.7), the expression of $\mathbb{P}_x(U_{e(q)} \in \mathrm{d}y)$ can be derived.

Corollary 4.1. *We have*

$$\begin{aligned}
&\mathbb{P}_x(U_{e(q)} \in \mathrm{d}y)\\
&\quad = q\int_0^{\infty} \mathrm{e}^{-qt}\mathbb{P}_x(U_t \in \mathrm{d}y)\,\mathrm{d}t\\
&\quad = \begin{cases} (F_1(0)+1)K_q(\mathrm{d}y-x) + \displaystyle\int_{b-x}^{y-x} F_1'(y-x-z)K_q(\mathrm{d}z)\,\mathrm{d}y, & y > b,\\ (F_2(0)+1)K_q(\mathrm{d}y-x) - \displaystyle\int_{y-x}^{b-x} F_2'(y-x-z)K_q(\mathrm{d}z)\,\mathrm{d}y, & y < b,\end{cases}
\end{aligned} \tag{4.9}$$

where $F_1(0)$ is given by (2.10), $F_2(0) := \lim_{x\uparrow 0} F_2(x)$, and, moreover, $F_2(0) = F_1(0)$.

Proof. In (4.1) and (4.7), differentiating with respect to y yields (4.9). Noting that $\mathbb{P}(U_{e(q)} = b) = 0$ (see Remark 4.1), we can write $y > b$ or $y < b$ in (4.9) as $y \ge b$ or $y \le b$. An interesting result is $F_2(0) = F_1(0)$, which will be proved in the following.

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For simplicity, we only consider $q \in \mathbb{S}$ since the case of $q \in \mathbb{S}^c$ can be shown similarly. Similar to the derivation of (2.10), it follows from (2.14) and (4.8) that

$$F_2(0) = \frac{\prod_{k=1}^{N^-} \gamma_k}{\prod_{k=1}^{\hat{N}^-} \hat{\gamma}_k} - 1. \tag{4.10}$$

From (2.5), we can rewrite $\hat{\psi}(z) - q = \psi(z) - \mathrm{i}\delta z - q$ as

$$\hat{\psi}(z) - q = \frac{-(\sigma^2/2)(\mathrm{i})^{\sum_{k=1}^{n^-} n_k}(-\mathrm{i})^{\sum_{k=1}^{m^+} m_k}(z)^{2+\sum_{k=1}^{n^-} n_k + \sum_{k=1}^{m^+} m_k} + \mathcal{P}(z)}{\prod_{k=1}^{n^-}(\vartheta_k + \mathrm{i}z)^{n_k} \prod_{k=1}^{m^+}(\eta_k - \mathrm{i}z)^{m_k}},$$

where $\mathcal{P}(z)$ is a polynomial and its degree is $1 + \sum_{k=1}^{n^-} n_k + \sum_{k=1}^{m^+} m_k$.

For $q \in \mathbb{S}$, it follows from Lemma 2.1(i) and Lemma 2.4(i) that there is a constant $\mathcal{P}_0$ satisfying

$$-\frac{\sigma^2}{2}(\mathrm{i})^{\sum_{k=1}^{n^-} n_k}(-\mathrm{i})^{\sum_{k=1}^{m^+} m_k}\big(z\big)^{2+\sum_{k=1}^{n^-} n_k + \sum_{k=1}^{m^+} m_k} + \mathcal{P}(z) = \mathcal{P}_0 \prod_{k=1}^{\hat{M}^+}(\hat{\beta}_k - \mathrm{i}z)\prod_{k=1}^{\hat{N}^-}(\hat{\gamma}_k + \mathrm{i}z).$$

Lemma 2.1(i) and Lemma 2.4(i) also yield $\hat{M}^+ = \sum_{k=1}^{m^+} m_k + 1$ and $\hat{N}^- = \sum_{k=1}^{n^-} n_k + 1$. So we have $\mathcal{P}_0 = -\sigma^2/2$, thus, the last two displayed equations yield

$$\hat{\psi}(z) - q = -\frac{\sigma^2}{2}\frac{\prod_{k=1}^{\hat{M}^+}(\hat{\beta}_k - \mathrm{i}z)\prod_{k=1}^{\hat{N}^-}(\hat{\gamma}_k + \mathrm{i}z)}{\prod_{k=1}^{n^-}(\vartheta_k + \mathrm{i}z)^{n_k}\prod_{k=1}^{m^+}(\eta_k - \mathrm{i}z)^{m_k}},$$

which combined with the fact that $\hat{\psi}(0) = 0$, produces

$$q = \frac{\sigma^2}{2}\frac{\prod_{k=1}^{\hat{M}^+}\hat{\beta}_k \prod_{k=1}^{\hat{N}^-}\hat{\gamma}_k}{\prod_{k=1}^{n^-}(\vartheta_k)^{n_k}\prod_{k=1}^{m^+}(\eta_k)^{m_k}}.$$

Similarly, we have

$$q = \frac{\sigma^2}{2}\frac{\prod_{k=1}^{M^+}\beta_k \prod_{k=1}^{N^-}\gamma_k}{\prod_{k=1}^{n^-}(\vartheta_k)^{n_k}\prod_{k=1}^{m^+}(\eta_k)^{m_k}}.$$

Therefore,

$$\frac{\prod_{k=1}^{\hat{M}^+}\hat{\beta}_k}{\prod_{k=1}^{M^+}\beta_k} = \frac{\prod_{k=1}^{N^-}\gamma_k}{\prod_{k=1}^{\hat{N}^-}\hat{\gamma}_k}, \tag{4.11}$$

and the desired result follows from (2.10), (4.10), and (4.11). □

Remark 4.4. For a more general Lévy process X, the functions $F_1(x)$ and $F_2(x)$ given respectively by (2.8) and (4.8) may not be differentiable. Thus, it is better to understand (4.9) as

$$\mathbb{P}_x(U_{e(q)} \in \mathrm{d}y) = \begin{cases} (F_1(0)+1)K_q(\mathrm{d}y - x) + \displaystyle\int_{b-x}^{y-x} F_1(\mathrm{d}y - x - z)K_q(\mathrm{d}z), & y > b, \\ (F_2(0)+1)K_q(\mathrm{d}y - x) - \displaystyle\int_{y-x}^{b-x} F_2(\mathrm{d}y - x - z)K_q(\mathrm{d}z), & y < b. \end{cases}$$

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5. Applications in pricing variable annuities

As stated in the introduction, our results can be used to price variable annuities (VAs) with state-dependent fees. First of all, we provide some background.

VAs are life insurance products whose benefits are linked to the performance of a reference portfolio with guaranteed minimum returns. There are many kinds of guarantees such as guaranteed minimum death benefits (GMDBs) and guaranteed minimum maturity benefits (GMMBs), and we refer the reader to [3] for more details. Of course, the guaranteed benefits are not free. Traditionally, the corresponding fees are deducted at a fixed rate from the policyholder's account. This classical fee charging method has some disadvantages, which have been noted in [4]. Thus, in [4], the authors proposed a new fee deducting approach under which only when the policyholder's account value is lower than a prespecified level can the insurer charge fees. For more details and research on this new method, we refer the reader to [4], [7], [17], [24], and [25].

Let S_t and F_t represent, respectively, the value at time t of the reference portfolio and the policyholder's account. Under the state-dependent fee structure, we have (see [4, Equation (1)] or [25, Equation (2.3)])

$$\mathrm{d}F_t = F_{t-}\frac{\mathrm{d}S_t}{S_{t-}} - (-\delta)F_{t-}\mathbf{1}_{\{F_{t-}<B\}}\,\mathrm{d}t, \qquad t>0, \tag{5.1}$$

where $-\delta > 0$ is the fee rate and B is a prespecified level. Note that the case of $B=\infty$ corresponds to the classical fee charging method. Furthermore, assume that $S_t = S_0\mathrm{e}^{X_t-\delta t}$ with X_t given by (2.1).

For a VA with GMMBs, its payoff can be written as $G(F_T)$, where T is the maturity and $G(\cdot)$ is a payoff function. For a VA with GMDBs, its payment when the policyholder dies is given by $G(F_{T_x})$, where T_x is the time of the death of the insured. A simple example of $G(\cdot)$ is $G(x)=\max\{x, K\}$, where K is a constant. In order to price VAs with GMMBs or GMDBs, we need to compute the following expectations under an equivalent martingale measure:

$$\mathbb{E}[\mathrm{e}^{-rT}G(F_T)] \quad \text{or} \quad \mathbb{E}[\mathrm{e}^{-rT_x}G(F_{T_x})], \tag{5.2}$$

where $r>0$ denotes the continuously compounded constant risk-free rate.

As the market is incomplete, an equivalent martingale measure should be chosen to calculate (5.2). Similar to [25], we use the Cramér–Esscher transform (see [9]) to obtain the desired martingale measure. Specifically, first define

$$\frac{\mathrm{d}\mathbb{P}^c}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{e}^{cX_t}}{\mathbb{E}[\mathrm{e}^{cX_t}]},$$

where $c\in\mathbb{R}$ such that $\mathbb{E}[\mathrm{e}^{cX_t}]<\infty$. And for convenience, in (2.2) and (2.3), we assume that η_1 (ϑ_1) has the smallest real part among $\eta_1,\dots,\eta_{m^+}$ ($\vartheta_1,\dots,\vartheta_{n^-}$). As $S_t = S_0\mathrm{e}^{X_t-\delta t}$, it is reasonable to require that $\mathbb{E}[\mathrm{e}^{X_t}]<\infty$, this means that $\eta_1>1$ in (2.2). Note that $\lim_{c\uparrow\eta_1}\mathbb{E}[\mathrm{e}^{cX_t}]=\infty$ and $\lim_{c\downarrow-\vartheta_1}\mathbb{E}[\mathrm{e}^{cX_t}]=\infty$. We can choose c^* such that $\mathrm{e}^{-rt}S_t$ is a martingale under $\mathbb{P}^{c^*}$. It is obvious that X_t is still a Lévy process under $\mathbb{P}^{c^*}$, and, in particular, the process X has the same form as (2.1) under $\mathbb{P}^{c^*}$. So we drop the superscript c^* from $\mathbb{P}^{c^*}$ and assume that the expectations appearing in the following are calculated under the equivalent martingale measure $\mathbb{P}^{c^*}$.

For a VA with GMDBs, its price is $\mathbb{E}[\mathrm{e}^{-rT_x}G(F_{T_x})]$. Applying discussions similar to those presented in [25] (see the derivation of [25, Equation (2.9)]), we find that the computation of

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$\mathbb{E}[\mathrm{e}^{-rT_x}G(F_{T_x})]$ reduces to that of calculating $\mathbb{E}[\mathrm{e}^{-re(q)}G(F_{e(q)})] = (q/(r+q))\mathbb{E}[G(F_{e(r+q)})]$ for given $q > 0$. For a VA with GMDBs, we note that

$$\int_0^\infty \mathrm{e}^{-sT}\mathbb{E}[\mathrm{e}^{-rT}G(F_T)]\,\mathrm{d}T = \frac{1}{s+r}\mathbb{E}[G(F_{e(s+r)})],$$

from which $\mathbb{E}[\mathrm{e}^{-rT}G(F_T)]$ can be obtained by using a numerical Laplace inversion technique.

In summary, the key step to price a VA with GMDBs or GMMBs is deriving the expression of $\mathbb{E}[G(F_{e(q)})]$ for $q > 0$.

From (5.1), applying Itô's formula yields $F_t = F_0\mathrm{e}^{U_t}$ with

$$\mathrm{d}U_t = \mathrm{d}(X_t - \delta t) - (-\delta)\mathbf{1}_{\{U_t<b\}}\,\mathrm{d}t = \mathrm{d}X_t - \delta\mathbf{1}_{\{U_t>b\}}\,\mathrm{d}t, \tag{5.3}$$

where $b = \ln(B/F_0)$. So we arrive at

$$\mathbb{E}[G(F_{e(q)})] = \mathbb{E}[G(F_0\mathrm{e}^{U_{e(q)}})] = \int_{-\infty}^\infty G(F_0\mathrm{e}^y)\mathbb{P}(U_{e(q)} \in \mathrm{d}y).$$

It follows from (1.1), (4.9), and (5.3) that

$$\mathbb{P}(U_{e(q)} \in \mathrm{d}y) = \begin{cases} (F_1(0)+1)K_q(\mathrm{d}y) + \displaystyle\int_b^y F_1'(y-z)K_q(\mathrm{d}z)\,\mathrm{d}y, & y > b, \\ (F_2(0)+1)K_q(\mathrm{d}y) - \displaystyle\int_y^b F_2'(y-z)K_q(\mathrm{d}z)\,\mathrm{d}y, & y < b, \end{cases} \tag{5.4}$$

where $F_1(x)$, $F_2(x)$, and $K_q(x)$ are given by (2.8), (4.8), and (4.2), respectively.

By applying a rational expansion, we can obtain semiexplicit expressions for $F_1(x)$, $F_2(x)$, and $K_q(x)$ and, thus, for $\mathbb{P}(U_{e(q)} \in \mathrm{d}y)$. However, for $q \in \mathbb{S}^c$, formulae for $\mathbb{P}(U_{e(q)} \in \mathrm{d}y)$ are very long and complicated, and, more importantly, they are difficult to use in numerical computations since we need to handle multiple roots. Fortunately, due to Lemma 2.3, it is safe and convenient to consider only the case of $q \in \mathbb{S}$. The corresponding results will be given in the following corollary, from which we can obtain first the expression of $\mathbb{E}[G(F_{e(q)})]$ and then the price of a VA with GMDBs or GMMBs.

Corollary 5.1. *For $q \in \mathbb{S}$, defining $f_q(y) := \mathbb{P}(U_{e(q)} \in \mathrm{d}y)/\mathrm{d}y$, we have the following results.*

(i) *If $b \geq 0$ then*

$$f_q(y) = \begin{cases} \dfrac{\prod_{k=1}^{\hat{M}^+}\hat{\beta}_k}{\prod_{k=1}^{M^+}\beta_k}K_q(\mathrm{d}y) \\ \quad + \displaystyle\sum_{i=1}^{M^+}\sum_{j=1}^{\hat{N}^-}\sum_{m=1}^{\hat{M}^+}\frac{K_{i,j}F_{1,m}}{\hat{\beta}_m - \beta_i}(\mathrm{e}^{-\beta_i y} - \mathrm{e}^{-\beta_i b}\mathrm{e}^{\hat{\beta}_m(b-y)}), & y > b, \\ \dfrac{\prod_{k=1}^{N^-}\gamma_k}{\prod_{k=1}^{\hat{N}^-}\hat{\gamma}_k}K_q(\mathrm{d}y) \\ \quad - \displaystyle\sum_{i=1}^{M^+}\sum_{j=1}^{\hat{N}^-}\sum_{n=1}^{N^-}\frac{K_{i,j}F_{2,n}}{\gamma_n + \beta_i}(\mathrm{e}^{-\beta_i y} - \mathrm{e}^{-\beta_i b}\mathrm{e}^{\gamma_n(y-b)}), & 0 < y < b, \end{cases}$$

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and, for $y \le 0$,

$$f_q(y) = \frac{\prod_{k=1}^{N^-} \gamma_k}{\prod_{k=1}^{\hat{N}^-} \hat{\gamma}_k} K_q(\mathrm{d}y) - \sum_{i=1}^{M^+} \sum_{j=1}^{\hat{N}^-} \sum_{n=1}^{N^-} \frac{K_{i,j} F_{2,n}}{\hat{\gamma}_j - \gamma_n} \bigg(\mathrm{e}^{\gamma_n y} \bigg(\frac{\hat{\gamma}_j + \beta_i}{\gamma_n + \beta_i} - \frac{\hat{\gamma}_j - \gamma_n}{\gamma_n + \beta_i} \mathrm{e}^{-(\beta_i + \gamma_n) b} \bigg) - \mathrm{e}^{\hat{\gamma}_j y} \bigg).$$

(ii) *If $b < 0$ then*

$$f_q(y) = \begin{cases} \dfrac{\prod_{k=1}^{N^-} \gamma_k}{\prod_{k=1}^{\hat{N}^-} \hat{\gamma}_k} K_q(\mathrm{d}y) - \displaystyle\sum_{i=1}^{M^+} \sum_{j=1}^{\hat{N}^-} K_{i,j} \sum_{n=1}^{N^-} F_{2,n} \frac{\mathrm{e}^{\gamma_n (y-b) + \hat{\gamma}_j b} - \mathrm{e}^{\hat{\gamma}_j y}}{\hat{\gamma}_j - \gamma_n}, & y < b, \\ \dfrac{\prod_{k=1}^{\hat{M}^+} \hat{\beta}_k}{\prod_{k=1}^{M^+} \beta_k} K_q(\mathrm{d}y) + \displaystyle\sum_{i=1}^{M^+} \sum_{j=1}^{\hat{N}^-} K_{i,j} \sum_{m=1}^{\hat{M}^+} F_{1,m} \frac{\mathrm{e}^{\hat{\gamma}_j y} - \mathrm{e}^{\hat{\beta}_m (b-y) + \hat{\gamma}_j b}}{\hat{\gamma}_j + \hat{\beta}_m}, & b < y < 0, \end{cases}$$

and, for $y \ge 0$,

$$f_q(y) = \frac{\prod_{k=1}^{\hat{M}^+} \hat{\beta}_k}{\prod_{k=1}^{M^+} \beta_k} K_q(\mathrm{d}y) + \sum_{i=1}^{M^+} \sum_{j=1}^{\hat{N}^-} \sum_{m=1}^{\hat{M}^+} \frac{K_{i,j} F_{1,m}}{\beta_i - \hat{\beta}_m} \bigg\{ \mathrm{e}^{-\beta_m y} \bigg(\frac{\beta_i + \hat{\gamma}_j}{\hat{\gamma}_j + \hat{\beta}_m} + \frac{\hat{\beta}_m - \beta_i}{\hat{\beta}_m + \hat{\gamma}_j} \mathrm{e}^{(\hat{\beta}_m + \hat{\gamma}_j) b} \bigg) - \mathrm{e}^{-\beta_i y} \bigg\}.$$

In the above formulae,

$$F_{1,i} = -\hat{\beta}_i \prod_{k=1}^{M^+} \frac{\hat{\beta}_i - \beta_k}{\beta_k} \prod_{k=1,\, k \ne i}^{\hat{M}^+} \frac{\hat{\beta}_k}{\hat{\beta}_i - \hat{\beta}_k} \quad \textit{for } 1 \le i \le \hat{M}^+, \tag{5.5}$$

$$F_{2,i} = -\gamma_i \prod_{k=1}^{\hat{N}^-} \frac{\hat{\gamma}_k - \gamma_i}{\hat{\gamma}_k} \prod_{k=1,\, k \ne i}^{N^-} \frac{\gamma_k}{\gamma_k - \gamma_i} \quad \textit{for } 1 \le i \le N^-, \tag{5.6}$$

and

$$K_q(\mathrm{d}x) = \sum_{i=1}^{M^+} \sum_{j=1}^{\hat{N}^-} K_{i,j} \mathrm{e}^{-\beta_i x} \mathrm{e}^{(\beta_i + \hat{\gamma}_j)(x \wedge 0)}, \tag{5.7}$$

where

$$K_{i,j} = \frac{\beta_i \hat{\gamma}_j}{\beta_i + \hat{\gamma}_j} \prod_{k=1}^{m^+} \frac{(\eta_k - \beta_i)^{m_k}}{(\eta_k)^{m_k}} \prod_{k=1,\, k \ne i}^{M^+} \frac{\beta_k}{\beta_k - \beta_i} \prod_{k=1}^{n^-} \frac{(\vartheta_k - \hat{\gamma}_j)^{n_k}}{(\vartheta_k)^{n_k}} \prod_{k=1,\, k \ne j}^{\hat{N}^-} \bigg(\frac{\hat{\gamma}_k}{\hat{\gamma}_k - \hat{\gamma}_j} \bigg).$$

Proof. Since $q \in \mathbb{S}$, we have $F_1(x) = F_0(x)$ (see (3.6) and (3.7)). So, for $x > 0$, $F_1'(x) = \sum_{i=1}^{\hat{M}^+} F_{1,i} \mathrm{e}^{-\hat{\beta}_i x}$ with $F_{1,i}$ given by (5.5). In addition, (2.10) yields $F_1(0) = \prod_{k=1}^{\hat{M}^+} \hat{\beta}_k / \prod_{k=1}^{M^+} \beta_k - 1$.

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From (2.14) and (4.8), for $x < 0$, applying a partial fraction expansion leads to $F_2'(x) = \sum_{i=1}^{N^-} F_{2,i} \mathrm{e}^{\gamma_i x}$, where $F_{2,i}$ is given by (5.6). In addition, we know that (see (4.10)) $F_2(0) = \prod_{k=1}^{N^-} \gamma_k / \prod_{k=1}^{\hat{N}^-} \hat{\gamma}_k - 1$.

From Lemma A.1(i), Lemma A.1(iii), and (4.2), some straightforward calculations lead to (5.7).

Therefore, the desired results follow from (5.4) after some simple computations. □

Appendix A

The proof of Proposition 2.1 is given in this section, where some ideas used can also be found in [22]. For completeness and for the convenience of the reader, we present all the details rather than omit some of them even though we will repeat some preliminary results and calculation procedures that appeared in [22].

Recall that $\mathbb{S}$ is the set of $q > 0$ such that all roots of $\psi(z) = q$ and $\hat{\psi}(z) = q$ are simple. The following lemma follows directly from Lemmas 2.1 and 2.4.

Lemma A.1. *For $q \in \mathbb{S}$, the following results hold.*

(i) *For $y \geq 0$, $\mathbb{P}(\overline{X}_{e(q)} \in \mathrm{d}y) = \sum_{k=1}^{M^+} C_k \mathrm{e}^{-\beta_k y}\,\mathrm{d}y$, where*

$$\frac{C_i}{\beta_i} = \prod_{k=1}^{m^+} \bigg(\frac{\eta_k - \beta_i}{\eta_k}\bigg)^{m_k} \prod_{k=1,\,k\neq i}^{M^+} \frac{\beta_k}{\beta_k - \beta_i}, \qquad 1 \leq i \leq M^+.$$

(ii) *For $y \geq 0$, $\hat{\mathbb{P}}(\overline{Y}_{e(q)} \in \mathrm{d}y) = \sum_{k=1}^{\hat{M}^+} \hat{C}_k \mathrm{e}^{-\hat{\beta}_k y}\,\mathrm{d}y$, where*

$$\frac{\hat{C}_i}{\hat{\beta}_i} = \prod_{k=1}^{m^+} \bigg(\frac{\eta_k - \hat{\beta}_i}{\eta_k}\bigg)^{m_k} \prod_{k=1,\,k\neq i}^{\hat{M}^+} \frac{\hat{\beta}_k}{\hat{\beta}_k - \hat{\beta}_i}, \qquad 1 \leq i \leq \hat{M}^+. \tag{A.1}$$

(iii) *For $y \leq 0$, $\hat{\mathbb{P}}(\underline{Y}_{e(q)} \in \mathrm{d}y) = \sum_{k=1}^{\hat{N}^-} \hat{D}_k \mathrm{e}^{\hat{\gamma}_k y}\,\mathrm{d}y$, where*

$$\frac{\hat{D}_j}{\hat{\gamma}_j} = \prod_{k=1}^{n^-} \bigg(\frac{\vartheta_k - \hat{\gamma}_j}{\vartheta_k}\bigg)^{n_k} \prod_{k=1,\,k\neq j}^{\hat{N}^-} \bigg(\frac{\hat{\gamma}_k}{\hat{\gamma}_k - \hat{\gamma}_j}\bigg), \qquad 1 \leq j \leq \hat{N}^-.$$

Next, introduce the following three rational functions:

$$\psi^+(s) := \prod_{k=1}^{m^+} \bigg(\frac{s + \eta_k}{\eta_k}\bigg)^{m_k} \prod_{k=1}^{M^+} \bigg(\frac{\beta_k}{s + \beta_k}\bigg) = \sum_{k=1}^{M^+} \frac{C_k}{s + \beta_k}, \tag{A.2}$$

$$\hat{\psi}^+(s) := \prod_{k=1}^{m^+} \bigg(\frac{s + \eta_k}{\eta_k}\bigg)^{m_k} \prod_{k=1}^{\hat{M}^+} \bigg(\frac{\hat{\beta}_k}{s + \hat{\beta}_k}\bigg) = \sum_{k=1}^{\hat{M}^+} \frac{\hat{C}_k}{s + \hat{\beta}_k}, \tag{A.3}$$

$$\hat{\psi}^-(s) := \prod_{k=1}^{n^-} \bigg(\frac{s + \vartheta_k}{\vartheta_k}\bigg)^{n_k} \prod_{k=1}^{\hat{N}^-} \bigg(\frac{\hat{\gamma}_k}{s + \hat{\gamma}_k}\bigg) = \sum_{k=1}^{\hat{N}^-} \frac{\hat{D}_k}{s + \hat{\gamma}_k}. \tag{A.4}$$

For $q \in \mathbb{S}$ and $\mathrm{Re}(s) \geq 0$, note that (see (2.7) and (2.14))

$$\mathbb{E}[\mathrm{e}^{-s\overline{X}_{e(q)}}] = \psi^+(s), \qquad \hat{\mathbb{E}}[\mathrm{e}^{-s\overline{Y}_{e(q)}}] = \hat{\psi}^+(s), \qquad \hat{\mathbb{E}}[\mathrm{e}^{s\underline{Y}_{e(q)}}] = \hat{\psi}^-(s). \tag{A.5}$$

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In addition, for $a \in \mathbb{R}$, define

$$\tau_a^+ := \inf\{t \geq 0\colon X_t > a\} \quad \text{and} \quad \hat{\tau}_a^- := \inf\{t \geq 0\colon Y_t < a\}.$$

Results on the one-sided exit problems of X and Y are presented in the following lemma. Lemma A.2(i) can be established by applying Lemma A.1(i), (2.12), and (A.2); and Lemma A.2(ii) follows from Lemma A.1(iii), (A.4), and the following result (see Corollary 2 of [1]):

$$\hat{\mathbb{E}}[\mathrm{e}^{-q\hat{\tau}_x^- + s(Y_{\hat{\tau}_x^-} - x)}] = \frac{\hat{\mathbb{E}}[\mathrm{e}^{s(\underline{Y}_{e(q)} - x)}\mathbf{1}_{\{\underline{Y}_{e(q)} < x\}}]}{\hat{\mathbb{E}}[\mathrm{e}^{s\underline{Y}_{e(q)}}]}, \qquad x, s \geq 0.$$

Lemma A.2. (i) *For $q \in \mathbb{S}$ and $x, y \geq 0$,*

$$\mathbb{E}[\mathrm{e}^{-q\tau_x^+}\mathbf{1}_{\{X_{\tau_x^+} - x \in \mathrm{d}y\}}] = C_0(x)\delta_0(\mathrm{d}y) + \sum_{k=1}^{m^+}\sum_{j=1}^{m_k} C_{kj}(x)\frac{(\eta_k)^j y^{j-1}}{(j-1)!}\mathrm{e}^{-\eta_k y}\,\mathrm{d}y,$$

where $\delta_0(\mathrm{d}y)$ is the Dirac delta at $y = 0$, and $C_0(x)$ and $C_{kj}(x)$ are given by the rational expansion:

$$C_0(x) + \sum_{k=1}^{m^+}\sum_{j=1}^{m_k} C_{kj}(x)\bigg(\frac{\eta_k}{\eta_k + s}\bigg)^j = \frac{1}{\psi^+(s)}\sum_{k=1}^{M^+} C_k \frac{\mathrm{e}^{-\beta_k x}}{s + \beta_k}, \qquad x \geq 0. \tag{A.6}$$

(ii) *For $q \in \mathbb{S}$ and $x, y \leq 0$,*

$$\hat{\mathbb{E}}[\mathrm{e}^{-q\hat{\tau}_x^-}\mathbf{1}_{\{Y_{\hat{\tau}_x^-} - x \in \mathrm{d}y\}}] = \hat{D}_0(x)\delta_0(\mathrm{d}y) + \sum_{k=1}^{n^-}\sum_{j=1}^{n_k} \hat{D}_{kj}(x)\frac{(\vartheta_k)^j (-y)^{j-1}}{(j-1)!}\mathrm{e}^{\vartheta_k y}\,\mathrm{d}y,$$

where $\hat{D}_0(x)$ and $\hat{D}_{kj}(x)$ are given by the rational expansion:

$$\hat{D}_0(x) + \sum_{k=1}^{n^-}\sum_{j=1}^{n_k} \hat{D}_{kj}(x)\bigg(\frac{\vartheta_k}{\vartheta_k + s}\bigg)^j = \frac{1}{\hat{\psi}^-(s)}\sum_{k=1}^{\hat{N}^-} \hat{D}_k \frac{\mathrm{e}^{\hat{\gamma}_k x}}{s + \hat{\gamma}_k}, \qquad x \leq 0. \tag{A.7}$$

Remark A.1. A useful observation is that $C_0(x)$ and $C_{kj}(x)$ in (A.6) are linear combinations of $\mathrm{e}^{\beta_i x}$ for $1 \leq i \leq M^+$, and $\hat{D}_0(x)$ and $\hat{D}_{kj}(x)$ in (A.7) are linear combinations of $\mathrm{e}^{\hat{\gamma}_i x}$ for $1 \leq i \leq \hat{N}^-$.

Lemma A.3 is a straightforward result of (A.9) and (A.10), here the reader is reminded that $1/(\theta + \beta_k)(s + \beta_k)$ can be written as $(1/(s - \theta))(1/(\theta + \beta_k) - 1/(s + \beta_k))$.

Lemma A.3. *For any $\theta > 0$ and $s \neq -\eta_1, \dots, -\eta_{m^+}$ with $\theta \neq s$,*

$$\int_0^\infty \mathrm{e}^{-\theta x} C_0(x)\,\mathrm{d}x + \sum_{k=1}^{m^+}\sum_{j=1}^{m_k}\int_0^\infty \mathrm{e}^{-\theta x} C_{kj}(x)\,\mathrm{d}x\bigg(\frac{\eta_k}{\eta_k + s}\bigg)^j = \frac{1}{s - \theta}\bigg(\frac{\psi^+(\theta)}{\psi^+(s)} - 1\bigg), \tag{A.8}$$

and, for any $\theta > 0$ and $s \neq -\vartheta_1, \dots, -\vartheta_{n^-}$ with $\theta \neq s$,

$$\int_{-\infty}^0 \mathrm{e}^{\theta x}\hat{D}_0(x)\,\mathrm{d}x + \sum_{k=1}^{n^-}\sum_{j=1}^{n_k}\int_{-\infty}^0 \mathrm{e}^{\theta x}\hat{D}_{kj}(x)\,\mathrm{d}x\bigg(\frac{\vartheta_k}{\vartheta_k + s}\bigg)^j = \frac{1}{s - \theta}\bigg(\frac{\hat{\psi}^-(\theta)}{\hat{\psi}^-(s)} - 1\bigg). \tag{A.9}$$

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Proof of Proposition 2.1. For a given $y > b$, the function of x, $V_q(x)$, is defined as (see Lemma 2.5:

$$V_q(x) = \mathbb{P}_x(U_{e(q)} > y).$$

Recall (1.1). Note that $\{X_t,\ t < \tau_b^+\}$ and $\{U_t,\ t < \kappa_b^+\}$ with $\kappa_b^+ := \inf\{t \geq 0\colon U_t > b\}$ under $\mathbb{P}_x$ have the same law if $x < b$. Therefore, for $x < b$, the strong Markov property of U lead to

$$\begin{aligned}
V_q(x) &= \mathbb{E}_x[\mathbf{1}_{\{U_{e(q)}>y\}}\mathbf{1}_{\{e(q)>\kappa_b^+\}}] \\
&= \mathbb{E}_x[\mathrm{e}^{-q\kappa_b^+} V_q(U_{\kappa_b^+})] \\
&= \mathbb{E}_x[\mathrm{e}^{-q\tau_b^+} V_q(X_{\tau_b^+})] \\
&= \mathbb{E}[\mathrm{e}^{-q\tau_{b-x}^+} V_q(X_{\tau_{b-x}^+} + x)] \\
&= \sum_{k=1}^{m^+}\sum_{j=1}^{m_k} C_{kj}(b-x)\int_0^\infty \frac{(\eta_k)^j z^{j-1}}{(j-1)!}\mathrm{e}^{-\eta_k z} V_q(b+z)\,\mathrm{d}z + C_0(b-x)V_q(b) \\
&= \sum_{k=1}^{M^+} J_k \mathrm{e}^{\beta_k(x-b)}, \qquad x < b, \qquad\qquad \text{(A.10)}
\end{aligned}$$

where $J_1, \ldots, J_{M^+}$ are constants which are not dependent on x; the fifth and the sixth equalities follow from Lemma A.2(i) and Remark A.1, respectively.

For $x > b$, the strong Markov property of U and the fact that $\{Y_t,\ t < \hat{\tau}_b^-\}$ under $\hat{\mathbb{P}}_x$ and $\{U_t,\ t < \kappa_b^-\}$ with $\kappa_b^- := \inf\{t \geq 0\colon U_t < b\}$ under $\mathbb{P}_x$ have the same law (Strictly speaking, this statement should be written as follows: $\{Y_t,\ t < \tilde{\tau}_b^-\}$ with $\tilde{\tau}_b^- := \inf\{t \geq 0\colon Y_t \leq b\}$ under $\hat{\mathbb{P}}_x$ and $\{U_t,\ t < \tilde{\kappa}_b^-\}$ with $\tilde{\kappa}_b^- := \inf\{t \geq 0\colon U_t \leq b\}$ under $\mathbb{P}_x$ have the same law. But, since $\sigma > 0$, we have $\mathbb{P}_x(\hat{\tau}_b^- = \tilde{\tau}_b^-) = 1$ and $\mathbb{P}_x(\kappa_b^- = \tilde{\kappa}_b^-) = 1$.) yield

$$\begin{aligned}
V_q(x) &= \mathbb{E}_x\bigg[\int_0^{\kappa_b^-} q\mathrm{e}^{-qt}\mathbf{1}_{\{U_t>y\}}\,\mathrm{d}t + \int_{\kappa_b^-}^\infty q\mathrm{e}^{-qt}\mathbf{1}_{\{U_t>y\}}\,\mathrm{d}t\bigg] \\
&= \hat{\mathbb{E}}_x\bigg[\int_0^\infty q\mathrm{e}^{-qt}\mathbf{1}_{\{Y_t>y,t<\hat{\tau}_b^-\}}\,\mathrm{d}t\bigg] + \hat{\mathbb{E}}_x[\mathrm{e}^{-q\hat{\tau}_b^-}V_q(Y_{\hat{\tau}_b^-})] \\
&= \hat{\mathbb{P}}_x(Y_{e(q)} > y, \underline{Y}_{e(q)} \geq b) + \hat{\mathbb{E}}_x[\mathrm{e}^{-q\hat{\tau}_b^-}V_q(Y_{\hat{\tau}_b^-})]. \qquad\qquad \text{(A.11)}
\end{aligned}$$

Applying the Wiener–Hopf factorization (see Theorem 6.16 of [12]), we can rewrite the first item on the right-hand side of (A.11) as

$$\begin{aligned}
&\int_{b-x}^0 \hat{\mathbb{P}}(Y_{e(q)} - \underline{Y}_{e(q)} > y - x - z, \underline{Y}_{e(q)} \in \mathrm{d}z) \\
&\quad = \int_{b-x}^0 \hat{\mathbb{P}}(\overline{Y}_{e(q)} > y - x - z)\hat{\mathbb{P}}(\underline{Y}_{e(q)} \in \mathrm{d}z)
\end{aligned}$$

$$= \begin{cases} \sum_{k=1}^{\hat{M}^+} \hat{H}_k \mathrm{e}^{\hat{\beta}_k(x-y)} + \sum_{k=1}^{\hat{N}^-} \hat{P}_k^* \mathrm{e}^{\hat{\gamma}_k(b-x)}, & b < x \le y, \\ 1 + \sum_{k=1}^{\hat{N}^-} \hat{Q}_k \mathrm{e}^{\hat{\gamma}_k(y-x)} + \sum_{k=1}^{\hat{N}^-} \hat{P}_k^* \mathrm{e}^{\hat{\gamma}_k(b-x)}, & x \ge y, \end{cases} \tag{A.12}$$

where the second equality is due to Lemma A.1(ii) and A.1(iii) (note that $\hat{\mathbb{P}}(\overline{Y}_{e(q)} > z) = 1$ if $z \le 0$); for $k = 1, 2, \ldots, \hat{M}^+$,

$$\hat{H}_k = \frac{\hat{C}_k}{\hat{\beta}_k} \sum_{j=1}^{\hat{N}^-} \frac{\hat{D}_j}{\hat{\beta}_k + \hat{\gamma}_j}, \tag{A.13}$$

and, for $k = 1, 2, \ldots, \hat{N}^-$,

$$\hat{Q}_k = \hat{D}_k \sum_{i=1}^{\hat{M}^+} \frac{\hat{C}_i}{\hat{\beta}_i(\hat{\beta}_i + \hat{\gamma}_k)} - \frac{\hat{D}_k}{\hat{\gamma}_k} \quad \text{and} \quad \hat{P}_k^* = -\sum_{i=1}^{\hat{M}^+} \frac{\hat{C}_i}{\hat{\beta}_i} \frac{\hat{D}_k}{\hat{\beta}_i + \hat{\gamma}_k} \mathrm{e}^{\hat{\beta}_i(b-y)}. \tag{A.14}$$

Therefore, from (A.11), (A.12), Lemma A.2(ii), and Remark A.1, we conclude that there are some constants $\hat{P}_1, \ldots, \hat{P}_{\hat{N}^-}$ (independent of x) such that

$$\sum_{k=1}^{\hat{N}^-} \hat{P}_k \mathrm{e}^{\hat{\gamma}_k(b-x)} = \sum_{k=1}^{n^-} \sum_{j=1}^{n_k} \hat{D}_{kj}(b-x) \int_{-\infty}^{0} V_q(b+z) \frac{(\vartheta_k)^j (-z)^{j-1}}{(j-1)!} \mathrm{e}^{\vartheta_k z} \, \mathrm{d}z$$
$$+ \hat{D}_0(b-x) V_q(b) + \sum_{j=1}^{\hat{N}^-} \hat{P}_j^* \mathrm{e}^{\hat{\gamma}_j(b-x)} \quad \text{for all } x > b, \tag{A.15}$$

and

$$V_q(x) = \begin{cases} \sum_{k=1}^{\hat{M}^+} \hat{H}_k \mathrm{e}^{\hat{\beta}_k(x-y)} + \sum_{k=1}^{\hat{N}^-} \hat{P}_k \mathrm{e}^{\hat{\gamma}_k(b-x)}, & b < x \le y, \\ 1 + \sum_{k=1}^{\hat{N}^-} \hat{Q}_k \mathrm{e}^{\hat{\gamma}_k(y-x)} + \sum_{k=1}^{\hat{N}^-} \hat{P}_k \mathrm{e}^{\hat{\gamma}_k(b-x)}, & x \ge y, \end{cases} \tag{A.16}$$

For the constants J_k in (A.10) and $\hat{P}_k$ in (A.15), we will show in Lemma A.4 that (A.20)–(A.23) hold.

Next, consider a rational function of x as follows:

$$L(x) = \sum_{i=1}^{M^+} \frac{J_i}{x - \beta_i} - \sum_{i=1}^{\hat{N}^-} \frac{\hat{P}_i}{x + \hat{\gamma}_i} - \sum_{i=1}^{\hat{M}^+} \frac{\hat{H}_i}{x - \hat{\beta}_i} \mathrm{e}^{\hat{\beta}_i(b-y)}. \tag{A.17}$$

For fixed $1 \le k \le m^+$ and $0 \le j \le m_k - 1$, (A.22) yields $(\partial^j / \partial x^j)(L(x))_{x=\eta_k} = 0$. This implies that η_k is a root of $L(x) = 0$ and its multiplicity is m_k. Moreover, for $1 \le k \le n^-$, (A.23) means that $-\vartheta_k$ is a n_k-multiplicity root of $L(x) = 0$. From these results, $L(x)$ can be rewritten as

$$\frac{\prod_{k=1}^{m^+} (x - \eta_k)^{m_k} \prod_{k}^{n^-} (x + \vartheta_k)^{n_k} (l_0 + l_1 x + \cdots + l_{M^+-1} x^{M^+-1} + x^{M^+}(L_0 + L_1 x)}{\prod_{i=1}^{M^+} (x - \beta_i) \prod_{i=1}^{\hat{N}^-} (x + \hat{\gamma}_i) \prod_{i=1}^{\hat{M}^+} (x - \hat{\beta}_i)}, \tag{A.18}$$

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where $l_0, l_1, \dots, l_{M^+-1}$, L_0 and L_1 are constants, and we have used $M^+ = \sum_{k=1}^{m^+} m_k + 1 = \hat{M}^+$ (see Remark 2.4) and $\hat{N}^- = \sum_{k=1}^{n^-} n_k + 1$ (see Lemma 2.4(i)) in the above derivation.

Then, by applying (A.20) and (A.21), we derive $L_0 = 0$ and $L_1 = 0$ from (A.17) and (A.18). Finally, it can be seen from (A.17) that

$$\lim_{x \to \hat{\beta}_i} L(x)(x - \hat{\beta}_i) = -\hat{H}_i \mathrm{e}^{\hat{\beta}_i (b-y)}, \qquad 1 \le i \le \hat{M}^+.$$

Therefore, we arrive at the conclusion:

$$\begin{aligned} L(x) = {} & \frac{\prod_{k=1}^{m^+} (x - \eta_k)^{m_k} \prod_{k=1}^{n^-} (x + \vartheta_k)^{n_k}}{\prod_{i=1}^{M^+} (x - \beta_i) \prod_{i=1}^{\hat{N}^-} (x + \hat{\gamma}_i)} \\ & \times \sum_{k=1}^{\hat{M}^+} \frac{\prod_{i=1}^{M^+} (\hat{\beta}_k - \beta_i) \prod_{i=1}^{\hat{N}^-} (\hat{\beta}_k + \hat{\gamma}_i)}{\prod_{i=1}^{m^+} (\hat{\beta}_k - \eta_i)^{m_i} \prod_{i=1}^{n^-} (\hat{\beta}_k + \vartheta_i)^{n_i}} \frac{-\hat{H}_k}{x - \hat{\beta}_k} \mathrm{e}^{\hat{\beta}_k (b-y)}. \end{aligned} \tag{A.19}$$

Equations (2.17) and (2.18) are derived from (A.10), (A.16), (A.17), and (A.19). □

Lemma A.4. (i) *It holds that*

$$\sum_{i=1}^{M^+} J_i = V_q(b) = \sum_{i=1}^{\hat{M}^+} \hat{H}_i \mathrm{e}^{\hat{\beta}_i (b-y)} + \sum_{i=1}^{\hat{N}^-} \hat{P}_i \tag{A.20}$$

and

$$\sum_{i=1}^{M^+} J_i \beta_i = V_q'(b) = \sum_{i=1}^{\hat{M}^+} \hat{H}_i \hat{\beta}_i \mathrm{e}^{\hat{\beta}_i (b-y)} - \sum_{i=1}^{\hat{N}^-} \hat{P}_i \hat{\gamma}_i. \tag{A.21}$$

(ii) *For* $1 \le k \le m^+$ *and* $0 \le j \le m_k - 1$,

$$\sum_{i=1}^{M^+} \frac{J_i (-1)^j}{(\beta_i - \eta_k)^{j+1}} + \sum_{i=1}^{\hat{N}^-} \frac{\hat{P}_i}{(\eta_k + \hat{\gamma}_i)^{j+1}} - \sum_{i=1}^{\hat{M}^+} \frac{\hat{H}_i (-1)^j}{(\hat{\beta}_i - \eta_k)^{j+1}} \mathrm{e}^{\hat{\beta}_i (b-y)} = 0. \tag{A.22}$$

(iii) *For any given* $1 \le k \le n^-$ *and* $0 \le j \le n_k - 1$,

$$\sum_{i=1}^{M^+} \frac{J_i (-1)^j}{(\beta_i + \vartheta_k)^{j+1}} + \sum_{i=1}^{\hat{N}^-} \frac{\hat{P}_i}{(\hat{\gamma}_i - \vartheta_k)^{j+1}} - \sum_{i=1}^{\hat{M}^+} \frac{\hat{H}_i (-1)^j}{(\hat{\beta}_i + \vartheta_k)^{j+1}} \mathrm{e}^{\hat{\beta}_i (b-y)} = 0. \tag{A.23}$$

Proof. (i) These results follow from (2.16), (A.10), and (A.16).

(ii) First, as $\sum_{i=1}^{\hat{M}^+} \hat{C}_i / \hat{\beta}_i = 1 = \sum_{j=1}^{\hat{N}^-} \hat{D}_j / \hat{\gamma}_j$ (let $s = 0$ in (A.3) and (A.4)), for some proper θ, we have (see (A.14))

$$1 + \sum_{k=1}^{\hat{N}^-} \frac{\theta \hat{Q}_k}{\theta + \hat{\gamma}_k} = \sum_{i=1}^{\hat{M}^+} \sum_{j=1}^{\hat{N}^-} \frac{\hat{C}_i \hat{D}_j}{\hat{\beta}_i \hat{\gamma}_j} + \sum_{k=1}^{\hat{N}^-} \frac{\theta}{\theta + \hat{\gamma}_k} \left(\sum_{i=1}^{\hat{M}^+} \frac{\hat{D}_k \hat{C}_i}{\hat{\beta}_i (\hat{\beta}_i + \hat{\gamma}_k)} - \frac{\hat{D}_k}{\hat{\gamma}_k} \sum_{i=1}^{\hat{M}^+} \frac{\hat{C}_i}{\hat{\beta}_i} \right).$$

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Thus, for all $\theta \in \mathbb{C}$ except at $\hat{\beta}_1, \ldots, \hat{\beta}_{\hat{M}^+}$ and $-\hat{\gamma}_1, \ldots, -\hat{\gamma}_{\hat{N}^-}$, the last formula and (A.13) lead to

$$\begin{aligned}\sum_{k=1}^{\hat{M}^+} \frac{\theta \hat{H}_k}{\hat{\beta}_k - \theta} + \sum_{k=1}^{\hat{N}^-} \frac{\theta \hat{Q}_k}{\theta + \hat{\gamma}_k} + 1 &= \sum_{i=1}^{\hat{M}^+} \sum_{j=1}^{\hat{N}^-} \bigg\{ \frac{\theta \hat{C}_i \hat{D}_j}{\hat{\beta}_i(\hat{\beta}_i + \hat{\gamma}_j)(\hat{\beta}_i - \theta)} - \frac{\theta \hat{C}_i \hat{D}_j}{\hat{\gamma}_j(\hat{\beta}_i + \hat{\gamma}_j)(\hat{\gamma}_j + \theta)} + \frac{\hat{C}_i \hat{D}_j}{\hat{\beta}_i \hat{\gamma}_j} \bigg\} \\ &= \sum_{i=1}^{\hat{M}^+} \sum_{j=1}^{\hat{N}^-} \frac{\hat{C}_i \hat{D}_j}{(\hat{\beta}_i - \theta)(\theta + \hat{\gamma}_j)} \\ &= \hat{\psi}^+(-\theta)\hat{\psi}^-(\theta), \end{aligned} \tag{A.24}$$

where the last equality follows from (A.3) and (A.4). Note that

$$\frac{\partial^{j-1}}{\partial \eta^{j-1}} (\hat{\psi}^+(-\eta))_{\eta=\eta_k} = 0 \quad \text{for } 1 \le k \le m^+ \text{ and } 1 \le j \le m_k.$$

From the last two equations, we obtain

$$\frac{(\eta_k)^j(-1)^{j-1}}{(j-1)!} \frac{\partial^{j-1}}{\partial \eta^{j-1}} \bigg(\frac{1}{\eta} \mathrm{e}^{\eta(b-y)} \bigg(\sum_{i=1}^{\hat{M}^+} \frac{\hat{H}_i \eta}{\hat{\beta}_i - \eta} + \sum_{i=1}^{\hat{N}^-} \frac{\eta \hat{Q}_i}{\eta + \hat{\gamma}_i} + 1 \bigg) \bigg)_{\eta=\eta_k} = 0. \tag{A.25}$$

For $1 \le k \le m^+$ and $1 \le j \le m_k$, the integral $(-1)^{j-1} \int_{z_1}^{z_2} z^{j-1} \mathrm{e}^{-\eta_k z} \mathrm{e}^{\xi z} \,\mathrm{d}z$ can be understood as $(\partial^{j-1}/\partial \eta^{j-1})(\int_{z_1}^{z_2} \mathrm{e}^{-\eta z} \mathrm{e}^{\xi z} \,\mathrm{d}z)_{\eta=\eta_k}$ for some proper constants z_1, z_2, and ξ, then from (A.16) and (A.25), we have

$$\int_0^\infty \frac{(\eta_k)^j z^{j-1}}{(j-1)!} \mathrm{e}^{-\eta_k z} V_q(b+z) \,\mathrm{d}z = \sum_{i=1}^{\hat{N}^-} \frac{\hat{P}_i(\eta_k)^j}{(\eta_k + \hat{\gamma}_i)^j} + \sum_{i=1}^{\hat{M}^+} \frac{\hat{H}_i(\eta_k)^j}{(\eta_k - \hat{\beta}_i)^j} \mathrm{e}^{\hat{\beta}_i(b-y)},$$

which combined with (A.10) and the result of $V_q(b) = \sum_{i=1}^{\hat{M}^+} \hat{H}_i \mathrm{e}^{\hat{\beta}_i(b-y)} + \sum_{i=1}^{\hat{N}^-} \hat{P}_i$ (see (A.20)), yields

$$\begin{aligned}\sum_{k=1}^{M^+} J_k \mathrm{e}^{\beta_k(x-b)} &= \sum_{k=1}^{m^+} \sum_{j=1}^{m_k} C_{kj}(b-x) \bigg(\sum_{i=1}^{\hat{N}^-} \frac{\hat{P}_i(\eta_k)^j}{(\eta_k + \hat{\gamma}_i)^j} + \sum_{i=1}^{\hat{M}^+} \frac{\hat{H}_i(\eta_k)^j}{(\eta_k - \hat{\beta}_i)^j} \mathrm{e}^{\hat{\beta}_i(b-y)} \bigg) \\ &\quad + C_0(b-x) \bigg(\sum_{i=1}^{\hat{M}^+} \hat{H}_i \mathrm{e}^{\hat{\beta}_i(b-y)} + \sum_{i=1}^{\hat{N}^-} \hat{P}_i \bigg) \quad \text{for all } x < b. \end{aligned} \tag{A.26}$$

It follows from (A.8) and (A.26) that

$$\begin{aligned}\sum_{i=1}^{M^+} \frac{J_i}{\beta_i + \theta} &= \int_{-\infty}^b \mathrm{e}^{\theta(x-b)} \sum_{i=1}^{M^+} J_i \mathrm{e}^{\beta_i(x-b)} \,\mathrm{d}x \\ &= \sum_{i=1}^{\hat{M}^+} \frac{\hat{H}_i \mathrm{e}^{\hat{\beta}_i(b-y)}}{\theta + \hat{\beta}_i} \bigg(1 - \frac{\psi^+(\theta)}{\psi^+(-\hat{\beta}_i)} \bigg) + \sum_{i=1}^{\hat{N}^-} \frac{\hat{P}_i}{\hat{\gamma}_i - \theta} \bigg(\frac{\psi^+(\theta)}{\psi^+(\hat{\gamma}_i)} - 1 \bigg). \end{aligned} \tag{A.27}$$

We note that $\lim_{\theta\to-\hat{\beta}_i}(\psi^+(-\hat{\beta}_i)-\psi^+(\theta))/(\theta+\hat{\beta}_i)=-\psi^{+\prime}(-\hat{\beta}_i)$ and $\lim_{\theta\to\hat{\gamma}_i}(\psi^+(\theta)-\psi^+(\hat{\gamma}_i))/(\theta-\hat{\gamma}_i)=\psi^{+\prime}(\hat{\gamma}_i)$. In addition, noting that both sides of (A.27) are rational functions of θ, we can extend identity (A.27) to the whole plane except at $-\beta_1,\dots,-\beta_{M^+}$. Here, we omit the first equality in (A.27), i.e. the item $\int_{-\infty}^{b}\mathrm{e}^{\theta(x-b)}\sum_{i=1}^{M^+}J_i\mathrm{e}^{\beta_i(x-b)}\,\mathrm{d}x$ is omitted. Then, for given $1\le k\le m^+$ and $0\le j\le m_k-1$, (A.22) is derived by first taking a derivative on both sides of (A.27) with respect to θ up to the jth order and then letting θ be equal to $-\eta_k$, where we have used the fact that $(\partial^j/\partial\theta^j)(\psi^+(\theta))_{\theta=-\eta_k}=0$.

(iii) Similarly, for $1\le k\le n^-$ and $1\le j\le n_k$, it follows from (A.10) that

$$\int_{-\infty}^{0}V_q(b+z)\frac{(\vartheta_k)^j(-z)^{j-1}}{(j-1)!}\mathrm{e}^{\vartheta_k z}\,\mathrm{d}z=\sum_{i=1}^{M^+}\frac{J_i(\vartheta_k)^j}{(\vartheta_k+\beta_i)^j}. \tag{A.28}$$

From (A.9), (A.14), (A.15), (A.28), and the fact that $V_q(b)=\sum_{i=1}^{M^+}J_i$ (see (A.20)), it can be proved that

$$\begin{aligned}\sum_{i=1}^{\hat{N}^-}\frac{\hat{P}_i}{\theta+\hat{\gamma}_i}&=\sum_{i=1}^{\hat{N}^-}\hat{P}_i\int_b^\infty \mathrm{e}^{\theta(b-x)}\mathrm{e}^{\hat{\gamma}_i(b-x)}\,\mathrm{d}x\\&=-\sum_{i=1}^{\hat{M}^+}\frac{\hat{C}_i}{\hat{\beta}_i}\mathrm{e}^{\hat{\beta}_i(b-y)}\sum_{j=1}^{\hat{N}^-}\frac{\hat{D}_j}{\hat{\beta}_i+\hat{\gamma}_j}\frac{1}{\theta+\hat{\gamma}_j}+\sum_{i=1}^{M^+}\frac{J_i}{\beta_i-\theta}\bigg(\frac{\hat{\psi}^-(\theta)}{\hat{\psi}^-(\beta_i)}-1\bigg).\end{aligned}$$

In addition, we note that

$$\begin{aligned}&-\sum_{i=1}^{\hat{M}^+}\frac{\hat{C}_i}{\hat{\beta}_i}\mathrm{e}^{\hat{\beta}_i(b-y)}\sum_{j=1}^{\hat{N}^-}\frac{\hat{D}_j}{\hat{\beta}_i+\hat{\gamma}_j}\frac{1}{\theta+\hat{\gamma}_j}\\&\quad=-\sum_{i=1}^{\hat{M}^+}\frac{\hat{C}_i}{\hat{\beta}_i}\mathrm{e}^{\hat{\beta}_i(b-y)}\frac{1}{\hat{\beta}_i-\theta}\sum_{j=1}^{\hat{N}^-}\hat{D}_j\bigg(\frac{1}{\theta+\hat{\gamma}_j}-\frac{1}{\hat{\beta}_i+\hat{\gamma}_j}\bigg)\\&\quad=\sum_{i=1}^{\hat{M}^+}\frac{\hat{H}_i}{\hat{\beta}_i-\theta}\mathrm{e}^{\hat{\beta}_i(b-y)}-\sum_{i=1}^{\hat{M}^+}\frac{\hat{C}_i\hat{\psi}^-(\theta)}{\hat{\beta}_i(\hat{\beta}_i-\theta)}\mathrm{e}^{\hat{\beta}_i(b-y)},\end{aligned}$$

where the second equality follows from (A.4) and (A.13).

Hence, from the last two equations, we arrive at

$$\sum_{i=1}^{\hat{N}^-}\frac{\hat{P}_i}{\theta+\hat{\gamma}_i}=\sum_{i=1}^{M^+}\frac{J_i}{\beta_i-\theta}\bigg(\frac{\hat{\psi}^-(\theta)}{\hat{\psi}^-(\beta_i)}-1\bigg)+\sum_{i=1}^{\hat{M}^+}\frac{\hat{H}_i\mathrm{e}^{\hat{\beta}_i(b-y)}}{\hat{\beta}_i-\theta}-\sum_{i=1}^{\hat{M}^+}\frac{\hat{C}_i\hat{\psi}^-(\theta)}{\hat{\beta}_i(\hat{\beta}_i-\theta)}\mathrm{e}^{\hat{\beta}_i(b-y)}, \tag{A.29}$$

which holds for $\theta\in\mathbb{C}$ except at $-\hat{\gamma}_1,\dots,-\hat{\gamma}_{\hat{N}^-}$.

For given $1\le k\le n^-$, on both sides of (A.29), we take a derivative with respect to θ up to the jth order for $0\le j\le n_k-1$ and then let θ be equal to $-\vartheta_k$. This calculation leads to (A.23) since $(\partial^j/\partial\theta^j)(\hat{\psi}^-(\theta))_{\theta=-\vartheta_k}=0$, and the proof is completed. □

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