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Solvability of free boundary problems for steady groundwater flow

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In this paper the free boundary problem for groundwater phreatic surface is represented in the form of a variational principle. It is proved that the flow domain Ω that solves the problem is a minimizer of some functional $\Lambda(\Omega)$. Weak solutions are introduced as minimizers of the lower semi-continuous regularization of $\Lambda(\cdot)$. Within this approach the existence of weak solutions is proved for a wide class of input data.

Key words: Free boundary problems; Flows in porous media; Variational methods

1 Introduction

The classical free boundary problem for steady groundwater flow involves a linear elliptic equation with excessive number of boundary conditions on a part of the boundary which is called the free one. Namely, this part is endowed with both Neumann and Dirichlet boundary conditions simultaneously whereas the rest of the boundary is equipped with just one of them. Then the compatibility requirements for this overdetermined problem generate information source about the shape of the flow domain which is not given *a priori* and has to be found. Its worth mentioning that the same mathematical description is also available in some other physical applications. One of them is the stationary Hele-Shaw problem.

The first approach to the analysis of groundwater flows with free boundaries was suggested by J. Dupuit in the late 1840s. He considered water seepage through a dam between two reservoirs and got an explicit expression for the shape of the phreatic surface that separates wet and dry zones in the body of the dam. This expression, however, does not describe the exact solution to the problem. It is a result of an approximation which reduces the problem to a nonlinear differential equation in a fixed domain of a lesser dimension. Nevertheless, this approach initiated numerous studies of groundwater flows with free boundaries regarding the existence of solutions. The detailed review is beyond the aims of the presented paper, but a short classification of corresponding approaches seems to be reasonable.

First of all, a considerable number of particular solutions have been presented explicitly. Most of them are obtained by means of the conformal mapping method developed by G. Kirchhoff and N. Zhukovsky (see, for instance, [19] or [6], Chapter 7). This method is still in use in engineering hydrology [2,11]. Any solution of this kind can be regarded as a particular solvability result. Then, based on the techniques of small perturbations, the existence of closely spaced solutions can be studied. The majority of these studies, however, is related to unsteady versions of the free boundary problem, and their main subject is a stability analysis for moving wetting fronts (see, for example, [18] and [10]).

A large class of approaches relates to approximations of the original free boundary problem by mathematical models which are simpler with respect to solvability analysis. In particular, the above mentioned Dupuit approach is based on the thin layer approximation. Multilayered versions of this approach provide the basis for numerical algorithms for engineering studies of groundwater flows (see, for instance, [8]). Also groundwater flows are studied using various versions of Richards models which account for capillary forces ([21], see also [6], Chapter IX). The porous medium in these models includes a partially saturated zone which is called the capillary fringe. It separates the fully saturated flow domain and the dry zone of the medium. If the effective thickness of this fringe is small compared to the spatial scale of the flow, then Richards models can be regarded as an approximation for the classic free boundary problem. Unsaturated flows are described by degenerate elliptic or, in the non-stationary case, parabolic equations in a given domain. They are thoroughly studied with respect to existence and qualitative properties of solutions (see [3,22] and references therein).

A new step in the studies of free boundary problems comes from the research carried out by C. Baiocchi in the early 1970s. Considering the classic problem of seepage through a dam, he introduced its representation in the form of an elliptic problem with unilateral constraints in the fixed domain which includes the wet and dry zones of the dam. Then solvability of the problem is established by means of standard methods of analysis. It was the first proof of existence for a groundwater flow with phreatic boundary besides the problems which can be solved explicitly or semi-explicitly. The Baiocchi approach, however, is not flexible in view of generalizations because it is closely related to the particular problem under consideration. Some extensions of his method to other problems are available though. A number of them are given in [4]. Also several modifications of the Baiocchi method are developed in [5] and [12]. Nevertheless, all possible generalizations cover a small class of problems because the main generic trick within this method is very sensitive to the shape of fixed boundaries as well as to the structure of boundary conditions on them. Moreover, the extension of the method to problems in spatially heterogeneous porous media is attended by certain difficulties.

Further progress in the studies of the problem is based on the consideration of its weak solutions. They were introduced by Alt [1] and Brézis *et al.* [7] (see also [9], Chapter 4). Weak solutions of the problem are defined in terms of a variational inequality and involve partially saturated zones. A weak solution is consistent with the classic one if its partially saturated zone is a set of zero Lebesgue measure. Existence of weak solutions is proved in [1] and [7] by means of approximation of the groundwater flow problem by a sequence of Richards models. The existence is established under sufficiently general conditions. In particular, it is provided for porous domains with arbitrary Lipschitz boundary and arbitrary heterogeneous structure. Using solvability of the dam problem in the weak sense Chipot in [9] has established existence of its classic solutions under less restrictive conditions than it is done within Baiocchi method. These conditions, however, imply homogeneity of the medium because the proof makes use of some properties of subharmonic functions. Possible cases of non-existence of classic solutions for the dam problem in heterogeneous media are mentioned in [1] and [5].

In the present paper, the standard problem for stationary groundwater flow with free boundary is represented in the form of a variational principle. Namely, it is proved that a flow domain Ω is the solution to the problem if and only if it minimizes an explicitly introduced functional $\Lambda(\Omega)$ and, as an additional requirement, the minimum takes some prescribed value. The research is motivated by questions of rigorous justification of homogenized models or multilayered versions of the Dupuit approximation for flows in heterogeneous porous media. In this respect, the variational representation of the problem seems potentially favourable. The representation is available for problems with an arbitrary shape of the fixed boundary and structure of heterogeneity. These input data, as well as functions providing boundary conditions, are involved in the expression for the functional $\Lambda(\Omega)$.

The variational representation allows us to use straightforward methods of calculus of variations in studies of model problems with respect to existence and uniqueness of solutions. Some particular examples are presented and thoroughly analysed in Section 2 of this paper. It is important that the variational principle provides a criterion for non-existence of classical solutions and gives an opportunity to justify this non-existence for some problems.

The analysis of solvability becomes less sensitive to input data if the notion of solution is reasonably generalized. This generalization is developed in Section 3 of the paper using the lower semi-continuous regularization of the generating functional $\Lambda(\Omega)$. This idea is inspired by approaches to the problems of optimal design of composites in the homogenization theory [14, 17]. The point is the following. For some problems the minimal value of $\Lambda(\Omega)$ is not attainable, but its infimum exists and takes the above mentioned prescribed value. The corresponding minimizing sequences of flow domains demonstrate small-scale behaviour in the sense that they consist of subsets which are represented in the entire porous medium or locally by a large number of thin layers or fibres. This looks like the appearance of partially saturated subdomains in the porous medium. The limiting behaviour of these sequences in the appropriate topology can be described by water saturation fields ranging from zero to one. It is proved that the lower semi-continuous regularization of the generating functional $\Lambda(\cdot)$ results in its extension from the set of admissible flow domains to the set of all saturation fields. Then a saturation field may be interpreted as the generalized or relaxed solution for the free boundary problem if it minimizes the extended functional and the minimum is equal to the prescribed value. In this way, the class of solvable problems becomes much larger. The expression for the extended functional is introduced in an explicit form. The proof that this expression coincides with the lower semi-continuous regularization of the original functional is given in the Appendix because it is the most cumbersome part of the paper.

The generalized solutions for the free boundary problem coincide with the weak solutions introduced by Alt [1] and Brézis *et al.* [7]. Thus, the present approach represents the weak solutions as the minimizers of the regularized functional and provides a criterion for weak solvability of the problems under consideration.

In Section 4 of the paper, the generalized solutions are introduced through an approximation of the original model by a sequence of degenerate elliptic problems in a fixed domain. These problems are related to a version of Richards model for unsaturated groundwater flows which accounts for capillary effects. It is proved that no-capillary limits for solutions of the approximating problems correspond to the solutions defined by means of the relaxed variational principle. This provides an additional physical interpretation for the generalized form of the free boundary problem. Finally, the approximation of the problem by a sequence of Richards models allows us to develop a sufficient solvability condition which is expressed in explicit form and covers a considerable number of application-oriented problems.

2 Posing the problem in the form of a variational principle

Let $B \subset \mathbf{R}^d$ be a fixed bounded domain with Lipschitz boundary ∂B. The free boundary problem under consideration implies that a domain $\Omega \subset B$, a non-negative function $p \in W^{1,2}(B)$ and a vector-valued function $q \in (L_2(B))^d$ have to be determined from the following relations

$$q = K(g - \nabla p) \ \text{ in } \ \Omega, \tag{2.1}$$

$$q(\cdot) = 0, \quad p(\cdot) = 0 \ \text{ in } \ B \backslash \Omega, \tag{2.2}$$

$$\nabla \cdot q = 0, \quad p(\cdot) \geqslant 0 \ \text{ in } \ B, \tag{2.3}$$

$$q \cdot n = Q \cdot n \ \text{ on } \ \partial B_N \subset \partial B, \tag{2.4}$$

$$p = P \quad \text{on } \ \partial B_D = \partial B \backslash \partial B_N. \tag{2.5}$$

Here, g is a given vector, K denotes a positive definite matrix, the functions P and Q provide boundary conditions of Dirichlet and Neumann type on ∂B_D and ∂B_N respectively. Equality (2.1) is Darcy relation between water flux q, pressure gradient ∇p and external force g in the porous medium. Everywhere below it is assumed that $g(\cdot) \in (L_2(B))^d$, and the symmetric matrix of Darcy coefficients $K(\cdot)$ satisfies strong ellipticity conditions in the form of inequalities $k_1\xi^2 \leqslant K(x)\xi \cdot \xi \leqslant k_2\xi^2$ with some positive k_1 and k_2 for all $\xi \in \mathbf{R}^d$ and $x \in B$.

Boundary conditions (2.4) and (2.5) on the fixed boundary ∂B could be interpreted in the sense of traces if the distributions of subsets ∂B_N and ∂B_D on ∂B were sufficiently regular. To avoid any regularity assumptions with respect to these distributions let us introduce a subspace $W_0^{1,2}(B, \partial B_D) \subset W^{1,2}(B)$ as the closure of the set of smooth functions $C_0^\infty\left(\overline{B} \backslash \partial B_D\right)$ with compact support in $\overline{B} \backslash \partial B_D$. Then condition (2.5) implies that $p(\cdot) - P(\cdot) \in W_0^{1,2}(B, \partial B_D)$ for some given function $P \in W^{1,2}(B)$. Since the sought pressure $p(\cdot)$ is non-negative, the relation $P(\cdot) \geqslant 0$ for the boundary function everywhere on B can be presumed without loss of generality. In order to give a rigorous formulation for Neumann boundary condition (2.4), let us introduce spaces $E(B)$ and $E_0(B, \partial B_N)$ of vector-valued functions on B by the expressions

$$E(B) = \left\{ q \in (L_2(B))^d \, : \, \int_B q \cdot \nabla p \, dx = 0 \ \ \forall p \in W_0^{1,2}(B, \partial B) \right\},$$

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and

$$E_0(B, \partial B_N) = \left\{ q \in (L_2(B))^d : \int_B q \cdot \nabla p \, dx = 0 \;\; \forall p \in W_0^{1,2}(B, \partial B_D) \right\}.$$

Thus, $E(B)$ and $E_0(B, \partial B_N)$ are the orthogonal complements in $(L_2(B))^d$ to the gradients ∇p of all functions from $W_0^{1,2}(B, \partial B)$ and $W_0^{1,2}(B, \partial B_D)$ respectively. Then $E(B)$ is the subspace of all divergence-free vector fields in $(L_2(B))^d$, and vectors $q \in E_0(B, \partial B_N) \subset E(B)$ have, in a generalized sense, the normal to ∂B_N component $q \cdot n$ equal to zero. In terms of these subspaces Neumann boundary condition (2.4) implies that $q(\cdot) - Q(\cdot) \in E_0(B, \partial B_N)$ for a given divergence-free vector field $Q(\cdot) \in E(B)$.

Everywhere below the capacity of ∂B_D is assumed to be sufficient to provide Poincaré inequality

$$\int_B p^2 dx \leqslant \text{Const} \cdot \int_B (\nabla p)^2 \, dx, \tag{2.6}$$

for any $p \in W_0^{1,2}(B, \partial B_D)$. This is the only assumption on the geometry of boundary subsets ∂B_D and ∂B_N.

It is assumed that the porous medium in the dry zone $B \backslash \Omega$ is filled with atmospheric air, and air pressure is a given constant which is set to zero. These are the physical reasons for relations (2.2). The set $\Gamma = \partial\Omega \backslash \partial B$ separates the wet and dry zones of the porous medium. It is called "free boundary". If it is sufficiently regular then functions from $W^{1,2}(B)$ have traces on Γ, which can be defined as the continuous mappings from $W^{1,2}(\Omega)$ or $W^{1,2}(B \backslash \Omega)$ to $H^{1/2}(\Gamma)$, and the traces from both sides of Γ are equal to each other (see, for instance, [25], Chapter 1.) A divergence-free vector field $q \in E(B)$ has equal traces of its normal to Γ component which are defined as continuous mappings from $(L_2(\Omega))^d$ or $\left(L_2(B \backslash \Omega)\right)^d$ to $H^{-1/2}(\Gamma)$. Then condition (2.2) implies that $p = 0$ and $q \cdot n = 0$ on Γ in the sense of traces. Therefore, the fact that the extension of pressure field by zero from Ω onto the dry zone belongs to $W^{1,2}(B)$ is a natural generalization of Dirichlet condition $p = 0$ on the free boundary regardless its regularity. Analogously, the fact that a divergence-free vector field q on Ω extended by zero onto $B \backslash \Omega$ is still divergence-free generalizes Neumann boundary condition $q \cdot n = 0$ on Γ. This interpretation of the free boundary problem allows us to look for the solution among all measurable subsets $\Omega \subset B$. Of course, the subsets are presumed to be equivalent if they coincide almost everywhere on B. Within this agreement notation $\partial\Omega$ becomes meaningless, and the term "free boundary" has to be interpreted as "free measurable subset".

Let us introduce closed convex subsets $M_D \subset W^{1,2}(B)$ and $M_N \subset E(B)$ by the following expressions:

$$M_D = \left\{ p \in W^{1,2}(B) : \; p(\cdot) \geqslant 0, \; p(\cdot) - P(\cdot) \in W_0^{1,2}(B, \partial B_D) \right\},$$

$$M_N = \{ q \in E(B) : \; q(\cdot) - Q(\cdot) \in E_0(B, \partial B_N) \}.$$

Then conditions (2.3), (2.4) and (2.5) of the free boundary problem can be written briefly as $p \in M_D$, $q \in M_N$.

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In order to represent the problem in a variational form, let us define a couple of functionals. The first one, $I_D(\Omega,\cdot) : M_D \to \mathbf{R}\cup\{+\infty\}$, is defined by the formula

$$I_D(\Omega,p) = \frac{1}{2}\int_\Omega K(g-\nabla p)\cdot(g-\nabla p)\,dx + \int_B Q\cdot(\nabla p - \nabla P)\,dx,$$

if $p(x) = 0$ on $B\backslash\Omega$ almost everywhere, and $I_D(\Omega,p) = +\infty$ otherwise. Another one, $I_N(\Omega,\cdot) : M_N \to \mathbf{R}\cup\{-\infty\}$, is given by the expression

$$I_N(\Omega,q) = \int_\Omega \left[(g-\nabla P)\cdot q - \frac{1}{2}K^{-1}q\cdot q\right]dx,$$

if $q(x) = 0$ on $B\backslash\Omega$ almost everywhere, and $I_N(\Omega,q) = -\infty$ otherwise.

The subsets of functions $p \in M_D$ and $q \in M_N$ for which $I_D(\Omega,p) < +\infty$ and $I_N(\Omega,q) > -\infty$ are denoted by $\operatorname{dom} I_D(\Omega,\cdot)$ and $\operatorname{dom} I_N(\Omega,\cdot)$ respectively. These two sets are closed convex subsets of $W^{1,2}(B)$ and $(L_2(B))^d$ respectively. If $\operatorname{dom} I_D(\Omega,\cdot)$ is not empty then the functional $I_D(\Omega,\cdot)$ attains its minimal value, and the minimizer is unique. This follows from coerciveness and strong convexity of the functional. The coerciveness, in its turn, is a consequence of Poincaré inequality (2.6). Analogously, if $\operatorname{dom} I_N(\Omega,\cdot) \neq \emptyset$ then $I_N(\Omega,\cdot)$ has a unique maximizer.

Since relations $p \in \operatorname{dom} I_D(\Omega,\cdot)$ and $q \in \operatorname{dom} I_N(\Omega,\cdot)$ take into account conditions (2.2) in the dry zone, solutions of the free boundary problem can be defined as follows.

Definition 2.1 A measurable subset $\Omega \subset B$, a function $p(\cdot)$ and a vector field $q(\cdot)$ on B solve the free boundary problem under consideration if $p(\cdot) \in \operatorname{dom} I_D(\Omega,\cdot)$, $q \in \operatorname{dom} I_N(\Omega,\cdot)$ and Darcy equation (2.1) holds a.e. on Ω.

The goal of this section is the following.

Theorem 2.2 *Inequality*

$$I_N(\Omega,q) \leqslant I_D(\Omega,p),$$

holds for any $\Omega \subset B$, $p \in M_D$ *and* $q \in M_N$. *Furthermore,*

$$I_N(\Omega,q) = I_D(\Omega,p),$$

if and only if the triplet (Ω,p,q) *solves the free boundary problem.*

Proof It suffices to consider $p \in \operatorname{dom} I_D(\Omega,\cdot)$ and $q \in \operatorname{dom} I_N(\Omega,\cdot)$. Then the following equality is a straightforward consequence of definitions:

$$I_D(\Omega,p) - I_N(\Omega,q) = \int_B (Q-q)\cdot(\nabla p - \nabla P)\,dx$$
$$+\frac{1}{2}\int_\Omega \left[K^{-1}q\cdot q - 2q\cdot(g-\nabla p) + K(g-\nabla p)\cdot(g-\nabla p)\right]dx.$$

Here, the first integral equals 0 by the definitions of M_D and M_N. The second one is non-negative and vanishes if and only if Darcy relation (2.1) holds almost everywhere on Ω. This completes the proof. □

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Theorem 2.2 is followed by the fact that the free boundary problem (2.1)–(2.5) can be represented in the form of a variational principle. Indeed, according to the theorem a domain Ω solves the free boundary problem (2.1)–(2.5) if and only if it minimizes the non-negative functional defined by the formula

$$\Lambda(\Omega) = \inf_{M_D} I_D(\Omega, \cdot) - \sup_{M_N} I_N(\Omega, \cdot),$$

and the minimum is equal to 0. In this case, the pressure field $p(\cdot)$ and water flux $q(\cdot)$ are the minimizer and the maximizer of $I_D(\Omega, \cdot)$ and $I_N(\Omega, \cdot)$ respectively.

It is worth making a comment on possible extensions of this result to problems with other types of the boundary conditions on the boundary of the porous domain B in place of equations (2.4) and (2.5). In many applications to groundwater flows with a free boundary the following relations on a part of ∂B are imposed

$$p = 0, \quad q \cdot n \geqslant 0, \tag{2.7}$$

where n stands for the outer normal to the fixed boundary ∂B. This is called the outflow condition. The physical reasons for it are presented in [22] and [23]. This condition is imposed on the boundary of porous medium which is in contact with open space (occupied by air). In particular, it is used in [9] for the dam problem on the upper boundary of the dam. With some changes in definitions it is possible to modify the generating functional $\Lambda(\Omega)$ for problems with the outflow condition on a part of ∂B. Namely, conditions $p = 0$ and $q \cdot n \geqslant 0$ on the outflow part of ∂B should be involved into the definitions of sets M_D and M_N respectively. Then the same expressions can be used for the functionals I_D, I_N and Λ if the function $P(\cdot)$, which generates the Dirichlet boundary condition on ∂B_D, equals zero on the outflow part of ∂B.

Another example of boundary condition relates to the process of water withdrawal from the surface of the porous domain. This condition reads

$$p \geqslant 0, \quad 0 \leqslant q \cdot n \leqslant \varkappa, \quad (\varkappa - q \cdot n)\, p = 0,$$

where the parameter $\varkappa > 0$ stands for the maximal value of accessible discharge. This condition on a part of ∂B can be taken into account in the variational principle if the definitions of M_D and M_N are endowed with the additional restrictions $p \geqslant 0$ and $0 \leqslant q \cdot n \leqslant \varkappa$ respectively on this part of the boundary. Then the expressions for the functionals are maintained if the functions $P(\cdot)$ and $Q(\cdot)$, which generate boundary conditions on the rest of ∂B, satisfy equalities $P = 0$ and $Q \cdot n = \varkappa$ on this part of the boundary.

3 Some examples of existence and non-existence

In this section, some explicit examples are presented to show how the variational principle can be used in studies of the free boundary problem with respect to its solvability. In example problems the fixed domain B is assumed to be the unit square $[0, 1] \times [0, 1] \subset \mathbf{R}^2$, and coordinates in $\mathbf{R}^2$ are denoted by x and y. The vector of external force is taken in the

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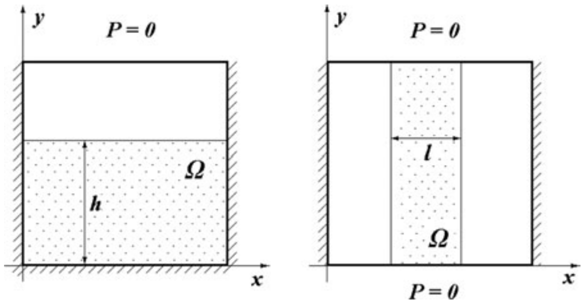

FIGURE 1. Examples of free boundary problems with multiple solutions.

form $g = (0, -1)$, and the matrix of Darcy coefficients $K = kI$ is supposed to be isotropic and uniform.

Conditions (2.1)–(2.5) in general do not provide uniqueness for the free boundary problem. This is noted in [7] and [9] in relation with the dam problem. Two examples with multiple solutions are shown in Figure 1. Equality $\Lambda(\Omega) = 0$ holds in both cases. It can be established by straightforward substitution of the solutions into expressions for functionals $I_D(\Omega, p)$ and $I_N(\Omega, q)$.

In the first problem, the top of the square stands for the Dirichlet boundary set $\partial B_D = [0, 1] \times \{1\}$. Both boundary functions, P and Q, are zero. Then for any h, $0 < h < 1$, the domain $\Omega = [0, 1] \times [0, h]$ solves the free boundary problem. The corresponding water flux and pressure field are given by expressions $q(x, y) = (0, 0)$ and $p(x, y) = \max\{0, h - y\}$.

Boundary functions P and Q in the second problem are still zero, but the Dirichlet boundary set includes the bottom of the square, as well as its top, namely, $\partial B_D = [0, 1] \times \{0, 1\}$. Then for any measurable subset $l \subset [0, 1]$ the domain $\Omega = l \times [0, 1]$ is a solution component for the problem. In this case the pressure field $p(x, y)$ is equal to zero in the square, water flux $q = (0, -k)$ in Ω and $q = (0, 0)$ in $B \backslash \Omega$.

A more valuable example is shown in Figure 2. In this problem, the Dirichlet boundary set $\partial B_D = [0.1] \times \{0\}$ is the bottom of the square, $P(x, y) = 0$ and $Q(x, y) = (0, -R)$, where R is a given constant. Darcy tensor $K = kI$ is still assumed isotropic and uniform. The problem describes, for instance, the process of water flow from land surface to water table. Then R specifies the water recharge. All possible solutions of this problem are described in the following proposition.

Proposition 3.1

(1) *If $R \geqslant k$ then the problem has a unique solution $\Omega = B$, $q(x, y) = (0, -R)$, $p(x, y) = y\left(R/k - 1\right)$;*

(2) *if $R = 0$ then the only solution is $\Omega = \emptyset$, $p \equiv 0$, $q \equiv (0, 0)$;*

(3) *if $R < 0$ then the problem has no solutions, and $\inf \Lambda(\Omega) > 0$;*

(4) *in the range $0 < R < k$ the problem has no solutions, $\Lambda(\Omega) > 0$ for any $\Omega \subset B$, but $\inf \Lambda(\Omega) = 0$.*

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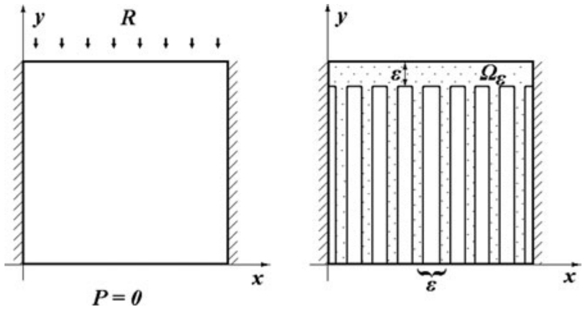

FIGURE 2. Example problem of water infiltration with given recharge (left) and a minimizing sequence of subsets (right).

Items 1–3 are not too informative. They are presented here for the sake of completeness. The most significant result is given in item 4. The complete proof of the proposition is given in the Appendix. Here, we establish the equality $\inf \Lambda(\Omega) = 0$ under the conditions of item 4. An example of a minimizing sequence $\Omega_\varepsilon \subset B$ for which $\lim \Lambda(\Omega_\varepsilon) = 0$ as $\varepsilon \to 0$ is shown in Figure 2 on the right. For any $\varepsilon, \varepsilon^{-1} \in \mathbf{N}$, the subset Ω_ε includes the horizontal stripe $[0, 1] \times [1 - \varepsilon, 1]$ and the set

$$\left\{(x, y) \in B : |x - \varepsilon(i - 1/2)| \leqslant \varepsilon R/2k,\ i = 1, \ldots, \varepsilon^{-1}\right\},$$

which consists of thin vertical fibres distributed with period ε along the horizontal coordinate axis.

To verify the fact that Ω_ε is the minimizing sequence, let us construct comparison functions $p_\varepsilon \in \operatorname{dom} I_D\,(\Omega_\varepsilon, \cdot)$ and $q_\varepsilon \in \operatorname{dom}\,(\Omega_\varepsilon, \cdot)$ in order to obtain an appropriate upper estimate for the value of $\Lambda\,(\Omega_\varepsilon)$. A suitable choice for the water pressure field is $p_\varepsilon(x, y) \equiv 0$. The straightforward substitution of this function into the expression for $I_D\,(\Omega_\varepsilon, \cdot)$ results in the equality $I_D\,(\Omega_\varepsilon, p_\varepsilon) = R/2 + \underline{O}(\varepsilon)$.

For the flux field let us set $q_\varepsilon(x, y) = 0$ for all $(x, y) \in B\backslash\Omega_\varepsilon$ and $q_\varepsilon(x, y) = (0, -k)$ if $y \leqslant 1 - \varepsilon$ and $(x, y) \in \Omega_\varepsilon$. Thus, the comparison function q_ε is defined in the square B everywhere besides the stripe $[0, 1] \times [1 - \varepsilon, 1]$. The natural extension of this piecewise constant vector field to the stripe does not satisfy the desired boundary condition on the top of the square. The stripe is included into Ω_ε in order to match small-scaled oscillations of the vector field q_ε in the interior zone of B with constant boundary flux through the top.

A convenient correcting function can be taken in the form $\nabla\varphi_\varepsilon$ where $\varphi_\varepsilon(x, y)$ is a harmonic function in the stripe with the relevant value of normal derivative at the boundary. Compatibility conditions for existence of this function are provided by the fact that the flux of q_ε through the lower border of the stripe $[0, 1] \times \{1 - \varepsilon\}$ is equal to the flux of boundary vector Q through the rest of the stripe's boundary. Then the extension of q_ε by $\nabla\varphi_\varepsilon$ is an admissible comparison function for functional $I_N\,(\Omega_\varepsilon, \cdot)$ because it is divergence-free in B and satisfies the desired boundary conditions.

It is easy to verify by means of rescaling that the L_2-norm of $\nabla\varphi_\varepsilon$ is sufficiently small, and the contribution of the corrector to the value of $I_N\,(\Omega_\varepsilon, \cdot)$ is of order ε.

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Then the substitution of the extended comparison field q_ε into the expression for this functional provides the equality $I_N(\Omega_\varepsilon, q_\varepsilon) = R/2 + \underline{O}(\varepsilon)$. Finally, this is followed by the relation

$$\Lambda(\Omega_\varepsilon) \leqslant I_D(\Omega_\varepsilon, p_\varepsilon) - I_N(\Omega_\varepsilon, q_\varepsilon) = \underline{O}(\varepsilon).$$

Therefore, $\lim_{\varepsilon\to 0} \Lambda(\Omega_\varepsilon) = 0$.

The presented example demonstrates that the free boundary problem can be unresolvable even if $\inf \Lambda(\cdot) = 0$. Minimizing sequences Ω_ε can have small-scale structure in the entire B or locally, so the limiting set Ω does not exist. This suggests the idea of involving such small-scaled sequences in the domain of functional $\Lambda(\cdot)$. The lowest value of the extended functional would be attainable in more cases, so the solvability of the problem could be established with more generality. An appropriate extension is developed in the next section.

4 Relaxed free boundary problem

Let $S(B)$ be the family of all measurable functions on B with values in $[0,1]$. Being endowed with weak topology of the space $L_2(B)$, the family $S(B)$ is a compact set. Any measurable domain $\Omega \subset B$ can be identified with its indicator $\mathbf{1}(\Omega,\cdot) : B \to \{0,1\}$, and $\mathbf{1}(\Omega,\cdot) \in S(B)$. In this sense, the set $S(B)$ is wider than the set of measurable subsets of B. It is easy to verify that the set of indicators on B is weakly dense in $S(B)$. This section is aimed at the generation of a lower semi-continuous regularization of $\Lambda(\cdot)$ which can be formally described by the expression

$$\tilde{\Lambda}(s) = \inf \left\{ \liminf_{\varepsilon\to 0} \Lambda(\Omega_\varepsilon) : \ \mathbf{1}(\Omega_\varepsilon,\cdot) \to s(\cdot) \in S(B) \text{ weakly} \right\}. \tag{4.1}$$

Note that, since B is a bounded domain and functions $s(\cdot) \in S(B)$ are uniformly bounded, the weak convergence in $L_2(B)$ for sequences from $S(B)$ is equivalent to their weak convergence in $L_1(B)$ or $*$-weak convergence in $L_\infty(B)$. Thus, the weak convergence in $L_2(B)$ for indicators in equation (4.1) is chosen for the sake of definiteness. The regularized functional $\tilde{\Lambda}(\cdot)$ in equation (4.1) is lower semi-continuous on the compact set $S(B)$ with respect to the weak topology. Therefore, in contrast to $\Lambda(\cdot)$, it attains its minimal value due to Weierstrass theorem, and this value is equal to $\inf \Lambda(\Omega)$. For instance, in the example problem which is presented by item 4 of Proposition 3.1, the minimum of $\tilde{\Lambda}(\cdot)$ is provided by the function $s(\cdot) \equiv R/k$ because it is the weak limit of $\mathbf{1}(\Omega_\varepsilon,\cdot)$ for the minimizing sequence of domains Ω_ε shown in Figure 2.

Equality (4.1) implies that if Ω is a minimizer of $\Lambda(\cdot)$, then $\tilde{\Lambda}(\mathbf{1}(\Omega,\cdot)) = \Lambda(\Omega)$. Below this equality is proved for any measurable $\Omega \subset B$. In this sense the regularized functional $\tilde{\Lambda}(\cdot)$ is an extension of $\Lambda(\cdot)$.

If the minimum value of $\tilde{\Lambda}(\cdot)$ equals zero, then its minimizers $s(\cdot) \in S(B)$ can be regarded as relaxed solutions to the free boundary problem. The minimizing function $s(\cdot)$ on B may be interpreted as water saturation of the porous medium. Thus, the relaxed solution splits the container B into wet, dry and partially saturated zones where $s(\cdot) = 1$, $s(\cdot) = 0$ and $0 < s(\cdot) < 1$ respectively.

In order to represent the relaxed problem in a more explicit form, let us introduce a functional $\tilde{I}_D(s,\cdot) : M_D \to \mathbf{R} \cup \{+\infty\}$ defined by the formula

$$\tilde{I}_D(s,p) = \int_B \left[Q \cdot (\nabla p - \nabla P) + \frac{s}{2} K \, (g - \nabla p) \cdot (g - \nabla p) \right] dx,$$

if functions $s \in S(B)$ and $p \in M_D$ satisfy the equality

$$(1 - s(x))\, p(x) = 0 \;\text{ a.e. on } B; \tag{4.2}$$

otherwise let us set $\tilde{I}_D(s,p) = +\infty$. Relation (4.2) can be written briefly as $s(\cdot) \in H(p(\cdot))$ where $H(\cdot)$ denotes the multi-valued Heaviside step function.

Analogously, let $\tilde{I}_N(s,\cdot) : M_N \to \mathbf{R} \cup \{-\infty\}$ be defined by the formula

$$\tilde{I}_N(s,q) = \int_{\{x:\, s(x)>0\}} \left[(g - \nabla P) \cdot q - \frac{1}{2s} K^{-1} q \cdot q \right] dx,$$

on the subset of vector fields $q \in M_N$ satisfying the relations

$$\int_{\{x:\, s(x)>0\}} \frac{q^2}{s} dx < \infty \text{ and } q(x) = 0 \text{ if } s(x) = 0; \tag{4.3}$$

and $\tilde{I}_N(s,q) = -\infty$ on the remaining part of M_N.

The set of pressure fields $p \in M_D$ with $\tilde{I}_D(s,p) < +\infty$ is denoted by $\operatorname{dom} \tilde{I}_D(s,\cdot)$, and notation $\operatorname{dom} \tilde{I}_N(s,\cdot)$ stands for the set of fluxes $q \in M_N$ satisfying conditions (4.3).

If $\operatorname{dom} \tilde{I}_D(s,\cdot)$ is not empty, then the existence and uniqueness of the minimizer of $\tilde{I}_D(s,\cdot)$ are provided by coerciveness and strong convexity of the functional. Analogously, if $\operatorname{dom} \tilde{I}_N(s,\cdot) \neq \emptyset$ then $\tilde{I}_N(s,\cdot)$ attains its maximal value, and the maximizer is unique.

Functionals $\tilde{I}_D(s,\cdot)$ and $\tilde{I}_N(s,\cdot)$ take the same values as $I_D(\Omega,\cdot)$ and $I_N(\Omega,\cdot)$ respectively if $s(x)$ is the indicator of Ω. In this sense they are extensions of the latter two. The main property of the extended functionals is given in the following theorem.

Theorem 4.1 *The relation*

$$\tilde{I}_N(s,q) \leqslant \tilde{I}_D(s,p),$$

holds for any $s \in S(B)$, $p \in M_D$ and $q \in M_N$. The equality

$$\tilde{I}_N(s,q) = \tilde{I}_D(s,p),$$

is satisfied if and only if the following equation holds

$$q(x) = s(x) K(x) \left(g(x) - \nabla p(x) \right), \tag{4.4}$$

almost everywhere on B, and $p \in \operatorname{dom} \tilde{I}_D(s,\cdot)$.

The proof is omitted because it is nearly the same as the proof of Theorem 2.2.

As a natural generalization of Definition 2.1, the relaxed solution to the free boundary problem can be defined as follows.

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Definition 4.2 Functions $s(\cdot)$, $p(\cdot)$ and $q(\cdot)$ on B solve the relaxed free boundary problem if $s \in S(B)$, $p \in \operatorname{dom} \tilde{I}_D(s, \cdot)$, $q \in M_N$ and the relaxed form of Darcy law (4.4) holds almost everywhere.

Note that the relation $q \in \operatorname{dom} \tilde{I}_N(s, \cdot)$ is not mentioned as a condition of Theorem 4.1 or Definition 4.2. It is satisfied automatically if the relaxed version of Darcy law equation (4.4) holds.

Let us define a functional $\tilde{\Lambda}(\cdot)$ on $S(B)$ by the formula

$$\tilde{\Lambda}(s) = \inf_{M_D} \tilde{I}_D(s, \cdot) - \sup_{M_N} \tilde{I}_N(s, \cdot).$$

Then Theorem 4.1 implies that a function $s \in S(B)$ solves the relaxed free boundary problem if and only if it provides zero minimal value of the extended functional $\tilde{\Lambda}(\cdot)$. The pressure field and water flux for this solution are the minimizer and the maximizer of the extended functionals $\tilde{I}_D(s, \cdot)$ and $\tilde{I}_N(s, \cdot)$ respectively.

The consistency of this formula for the extended functional with the former expression (4.1) for the regularized one is the straightforward consequence of the following two propositions.

Proposition 4.3 *Let $S(B) \subset L_2(B)$, $M_D \subset W^{1,2}(B)$ and $M_N \subset (L_2(B))^d$ be endowed with corresponding weak topologies. Then the functionals $\tilde{I}_D(s, p)$ and $\inf \tilde{I}_D(s, \cdot)$ are lower semi-continuous on $S(B) \times M_D$ and $S(B)$ respectively. Analogously, $\tilde{I}_N(s, q)$ and $\sup \tilde{I}_N(s, \cdot)$ are upper semi-continuous on $S(B) \times M_N$ and $S(B)$.*

Proposition 4.4 *For any $s \in S(B)$, $p \in M_D$ and $q \in M_N$ it is possible to construct sequences $\Omega_\varepsilon \subset B$, $p_\varepsilon \in M_D$ and $q_\varepsilon \in M_N$ with the following properties:*

(1) $\mathbf{1}(\Omega_\varepsilon, \cdot) \to s(\cdot)$, $p_\varepsilon(\cdot) \to p(\cdot)$ *and* $q_\varepsilon(\cdot) \to q(\cdot)$ *weakly as* $\varepsilon \to 0$;

(2) $\tilde{I}_D(s, p) = \lim_{\varepsilon \to 0} I_D(\Omega_\varepsilon, p_\varepsilon)$;

(3) $\tilde{I}_N(s, q) = \lim_{\varepsilon \to 0} I_N(\Omega_\varepsilon, q_\varepsilon)$.

Proofs of these two propositions are given in the Appendix. Proposition 4.3 guarantees that the extended functional $\tilde{\Lambda}(\cdot)$ does not exceed the expression on the right of equality (4.1). On the other hand, Proposition 4.4 is followed by the relations

$$\inf_{M_D} \tilde{I}_D(s, \cdot) \geqslant \lim_{\varepsilon \to 0} \inf_{M_D} I_D(\Omega_\varepsilon, \cdot) \quad \text{and} \quad \sup_{M_N} \tilde{I}_N(s, \cdot) \leqslant \lim_{\varepsilon \to 0} \sup_{M_N} I_N(\Omega_\varepsilon, \cdot),$$

for the sequence Ω_ε which is given by the proposition. Accounting for semi-continuity results, these relations imply equality (4.1) for the extended functional $\tilde{\Lambda}(\cdot)$. Thus, it is equal to the lower semi-continuous regularization of $\Lambda(\cdot)$.

The above construction provides existence of a saturation field $s \in S(B)$ that minimizes the extended functional $\tilde{\Lambda}(\cdot)$. However, this is still not a solution for the relaxed free boundary problem because the latter implies that the minimum should be equal to zero. The following proposition describes a sufficient solvability condition for the problem under consideration.

Proposition 4.5 *If the flux through Neumann part ∂B_N of the fixed boundary is non-positive in the sense that*

$$\int_B Q \cdot \nabla\varphi \, dx \leqslant 0, \tag{4.5}$$

for any $\varphi(\cdot) \geqslant 0$, $\varphi \in W_0^{1,2}(B, \partial B_D)$, then $\inf_{S(B)} \tilde{\Lambda}(s) = 0$.

This proposition is a straightforward consequence of Proposition 5.1 and Theorem 5.2 from the next section. Relation (4.5) suggests that $Q \cdot n \leqslant 0$ on ∂B_N if n is the outer normal to the boundary of B. This solvability condition is not necessary but it covers a reasonably wide class of free boundary problems. The example given in item 3 of Proposition 3.1 demonstrates that the presented sufficient condition is not just a technical one.

The relaxed solutions for the free boundary problem turn out to be the same as the weak solutions introduced by Alt [1] and Brézis *et al.* [7] (see also [9], Chapter 4), except for some details relating to the choice of boundary conditions on ∂B. Accounting for the settings and notation of the present paper, these weak solutions can be defined as follows. A pair of functions $p \in M_D$ and $s \in S(B)$ is called a weak solution to the problem under consideration if

$$s \in H(p) \ \text{ and } \ \int_B [K\,(sg - \nabla p) - Q] \cdot \nabla\varphi dx \equiv 0, \tag{4.6}$$

for any $\varphi \in W_0^{1,2}(B, \partial B_D)$. It is clear that the pair (s, p) is a weak solution in the sense of definition (4.6) if the triplet (s, p, q) solves the relaxed problem. Thus, Theorem 4.1 provides a representation of the weak solution components, s and p, as the minimizers of functionals $\tilde{\Lambda}(\cdot)$ and $\tilde{I}_D(s, \cdot)$.

The relaxed form of the free boundary problem is closely related to the groundwater flow model introduced by Green and Ampt [13] (see also [18] and [10]). In the stationary case it can be represented in the form of relations

$$p \geqslant 0, \ \ \nabla \cdot [\gamma(s) K\,(g - \nabla p)] = 0, \quad s \in H(p), \tag{4.7}$$

where $\gamma(\cdot)$ is a given monotone mapping from $[0, 1]$ onto $[0, 1]$ which corresponds physically to the relative permeability of the porous medium. The inclusion $s \in H(p)$ in this model is the extreme form of the capillary relation between pressure and saturation. In physically based models of groundwater flows this relation is a monotone continuous mapping from $(-\infty, +\infty)$ to $[0, 1]$ which accounts for capillary forces and involves the capillary rise as a parameter. If this parameter is small compared to the scale of length in the problem under consideration, then the Heaviside step seems to be a reasonable approximation of the realistic capillary relation.

If $\gamma(s) \equiv s$, then this model with appropriate boundary conditions on ∂B coincides with the relaxed form of the problem under consideration. In applications $\gamma(\cdot)$ is determined in experiments [16] or by means of numerical simulations of flows on the scale of pores [26]. The qualitative behaviour of this function depends on wettability of the liquid. Usually the function $\gamma(s)$ on the interval $\{s : 0 < \gamma(s) \leqslant 1\}$ is concave for nonwetting liquids and convex otherwise. As far as the starting problems (2.1)–(2.5) does not account for wettability effects, the relation $\gamma(s) \equiv s$ in its relaxed form is a naturally occurring result of the procedure of lower semi-continuous regularization. The stationary version of Green and Ampt model can be taken for the relaxed one even if $\gamma(s) \neq s$ because the inclusion $s \in H(p)$ in (4.7) implies the relation $\gamma(s) \in H(p)$. In this case, the physical interpretation of the variable s must be switched from water saturation to the relative permeability.

5 Approximation of the free boundary by a capillary fringe

Solvability of the relaxed problem under consideration is studied below by means of the approach developed by Alt [1] and Brézis *et al.* [7], where existence of weak solutions is established. This approach is based on an approximation of the weak version of the problem by a family of degenerate elliptic problems in the fixed domain B. The boundary conditions imposed on ∂B in these studies, however, were slightly different from equations (2.4) and (2.5). Namely, the Neumann part of the fixed boundary, ∂B_N, was assumed to be impermeable and, in addition, a part of the remaining boundary was equipped with outflow condition (2.7). These settings correspond to applications to the dam problem. Under condition (2.4) with $Q \cdot n \neq 0$ existence of weak solutions is not guaranteed as demonstrated by the counterexample from item 3 of Proposition 3.1. The choice of relation (2.4) as a boundary condition is partially motivated by a wish to present an example of non-existence.

For any $h > 0$ let us define functions $p \to \Theta_{(h)}(p)$ and $p \to U_{(h)}(p)$ of a non-negative variable $p \geqslant 0$ by the formulas

$$\Theta_{(h)}(p) = \min\{1, p/h\} \text{ and } U_{(h)}(p) = \int_0^p \Theta_{(h)}(p')dp'.$$

Then let us consider the following elliptic problem for a non-negative function $p(\cdot)$ on B:

$$\nabla \cdot q = 0, \quad q(\cdot) = \theta K\,(g - \nabla p), \quad \theta(\cdot) = \Theta_{(h)}(p), \tag{5.1}$$

with boundary conditions (2.4) and (2.5) for $q(\cdot)$ and $p(\cdot)$.

Being extended by zero for all $p < 0$, the function $\Theta_{(h)}(p)$ is a regular approximation from below of Heaviside step-function $H(p)$ as $h \to 0$. For a fixed value of $h > 0$ problem (5.1) conforms to a physical model for steady groundwater flows which accounts for capillary forces. According to this model the flow domain includes, besides wet and dry zones, a subdomain $\{x : 0 < \theta(x) < 1\}$ which is called the unsaturated zone or capillary fringe. The models of this type with various empirical relations between θ and p in the place of $\theta = \Theta_{(h)}(p)$ were introduced for unsaturated flows by Richards ([21], see also [6], Chapter IX). The Richards models of subsurface flows involve more physical parameters than the free boundary problem (2.1)–(2.5). This is a serious disadvantage with respect to some engineering applications. However, they are better suited for analysis. A lot of mathematical and numerical approaches have been developed in relation to Richards equations (see [3, 20, 22] and references therein). This is a good reason to use them as approximations for the problem (2.1)–(2.5).

The subject of the current section is an asymptotic analysis of problem (5.1) as $h \to 0$. It will be shown that solutions for equation (5.1) are close, in some sense, to solutions for the main problem under considerations (2.1)–(2.5). Physically, parameter h is responsible for the thickness of the capillary fringe. One could expect that in the limit the fringe degenerates into the free boundary between dry and wet zones. In reality the situation is more complicated. First of all, the solutions of boundary value problems for equation (5.1) are not unique in general. Also, the asymptotic behaviour of solutions for $h \to 0$ can depend on the particular form of the capillary approximation. At last, the unsaturated zone does not necessarily collapse as $h \to 0$ into a set of zero Lebesgue measure. In this respect, an explicit counterexample can be presented on the base of the particular problem which is considered in item 4 of Proposition 3.1. Hence the problem (2.1)–(2.5) itself can not be a candidate for the limiting one. The true limit behaviour of the solutions is described by its relaxed version given by Definition 4.2.

Boundary condition (2.5) in relation to equations (5.1) needs a rigorous interpretation. Since $\Theta_{(h)}(p)$ vanishes at $p = 0$, solutions $p(\cdot)$ to this problem do not necessarily belong to the space $W^{1,2}(B)$. In order to eliminate the degeneracy it is reasonable to change the sought variable in the following way: $p(\cdot) \to u(\cdot) \equiv U_{(h)}(p(\cdot))$. Then problem (5.1) can be interpreted precisely as follows. A measurable function $p(\cdot) \geqslant 0$ on B solves the problem if the function $u(\cdot) = U_{(h)}(p(\cdot))$ belongs to $W^{1,2}(B)$, meets Dirichlet boundary condition on ∂B_D in the sense that $u - U_{(h)}(P) \in W_0^{1,2}(B, \partial B_D)$ and satisfies the integral identity

$$\int_B K \left(\alpha_{(h)}(u)g - \nabla u\right) \cdot \nabla\varphi dx = \int_B Q \cdot \nabla\varphi dx, \tag{5.2}$$

for any $\varphi \in W_0^{1,2}(B, \partial B_D)$ where $\alpha_{(h)}(u)$ stands for the function $\Theta_{(h)}(p)$ expressed in terms of u, i.e. $\alpha_{(h)}\left(U_{(h)}(p)\right) \equiv \Theta_{(h)}(p)$.

For the family of elliptic problems (5.1) some auxiliary properties of its solutions are formulated as follows.

Proposition 5.1 *Any solution $p(x)$ to problem (5.1) satisfies inequalities*

$$\int_B q \cdot q\, dx \leqslant Const, \quad \int_B \nabla U_{(h)}(p) \cdot \nabla U_{(h)}(p)\, dx \leqslant Const, \tag{5.3}$$

where the constants are independent of h. Furthermore, if the boundary function Q satisfies relation $Q \cdot n \leqslant 0$ on ∂B_N in the sense of definition (4.5), then problem (5.1) has at least one solution.

Proof Estimates (5.3) are obtained in a standard way by substituting $\varphi = U_{(h)}(p) - U_{(h)}(P) \in W_0^{1,2}(B, \partial B_D)$ as a test function into identity (5.2).

In order to prove sufficiency of condition (4.5) for the solvability of problem (5.1), let us extend the function $\alpha_{(h)}(u)$ from identity (5.2) by zero for all $u < 0$. As soon as the restriction $u(\cdot) \geqslant 0$ is disregarded, the non-degenerate elliptic problem associated with identity (5.2) is solvable with respect to $u(x)$. This follows from coerciveness of the corresponding operator and boundedness of the continuous function $\alpha_{(h)}(\cdot)$ (see, for instance, [15], Chapter IV, Section 9). It suffices to prove existence of non-negative solutions $u(\cdot) \geqslant 0$ for the extended problem under condition (4.5). Then the restitution of the non-negative variable p by the formula $u = U_{(h)}(p)$ will provide solvability of the problem (5.1) with respect to $p(x)$.

For an arbitrary solution $u = u(\cdot)$, let us substitute into identity (5.2) the cut-off function $u_-(x) = \min\{u(x), 0\}$ in place of $\varphi(x)$. Since the boundary function $P(x)$ is assumed to be non-negative, then $u_-(\cdot) \in W_0^{1,2}(B, \partial B_D)$ is an admissible test function. This results in the following inequality:

$$-\int_B K\nabla u_- \cdot \nabla u_- dx = \int_B Q \cdot \nabla u_- dx.$$

The right-hand side here is non-negative due to condition (4.5) for the boundary function Q. Therefore, $\nabla u_- \equiv 0$ on B; this implies that $u_- \equiv 0$ thanks to Poincaré inequality (2.6), and $u(x) \geqslant 0$. This completes the proof of the proposition. □

The goal of this section is the following approximation result.

Theorem 5.2 *Let $p_h(\cdot)$ be a family of solutions of Richards problem (5.1) for some sequence $h \to 0$. Then, over a subsequence $h \to 0$,*

$p_h \to p$ in $L_2(B)$ strongly and $p \in W^{1,2}(B)$,

$q_h = K\left(\Theta_{(h)}(p_h)g - \nabla U_{(h)}(p_h)\right) \to q$ in $(L_2(B))^d$ weakly,

$\theta_h = \Theta_{(h)}(p_h) \to s$ in $L_2(B)$ weakly.

The functions s, p and q solve the relaxed free boundary problem in the sense of Definition 4.2.

Proof Uniform estimates (5.3) and uniform boundedness of θ_h on B are followed by the existence of a subsequence $h \to 0$ which provides weak convergence of θ_h in $L_2(B)$, weak convergence of q_h in $(L_2(B))^d$ and weak convergence of $U_{(h)}(p_h)$ in $W^{1,2}(B)$ over this subsequence. Let the limiting functions be s, q and p respectively. Compactness of embedding $W^{1,2}(B) \subset L_2(B)$ suggests that the convergence of $U_{(h)}(p_h)$ to p is strong in $L_2(B)$. The definition of $U_{(h)}(\cdot)$ results in the inequalities $U_{(h)}(p_h) \leqslant p_h \leqslant U_{(h)}(p_h) + h/2$. Hence p is the strong limit of p_h in $L_2(B)$.

It is evident that the limiting function s on B satisfies the inequalities $0 \leqslant s(x) \leqslant 1$. Therefore, $s \in S(B)$. Due to identity (5.2) functions $q_h(\cdot)$ for any h satisfy relation $q_h - Q \in E_0(B, \partial B_N)$. Since $E_0(B, \partial B_N)$ is a closed subspace of $(L_2(B))^d$, then the same inclusion holds for the limiting function q, i.e. $q \in M_N$. Furthermore, the inequality $p_h \geqslant 0$ and Dirichlet boundary condition $U_{(h)}(p_h) - U_{(h)}(P) \in W_0^{1,2}(B, \partial B_D)$ hold for any $h > 0$, hence $p - P \in W_0^{1,2}(B, \partial B_D)$ and $p \in M_D$.

Note that p is also the weak limit in $W^{1,2}(B)$ of cut-off functions $p'_h = \max\{p_h - h, 0\} \equiv \max\{U_{(h)}(p_h) - h/2, 0\}$ because $|\nabla p'_h| \leqslant |\nabla U_{(h)}(p_h)|$ a.e. on B. Due to the definition of the functions p'_h, the following equality holds for any $h > 0$ everywhere on B

$$(1 - \Theta(p_h(x)))\, p'_h(x) = 0.$$

The multipliers on the left of this equality converge in $L_2(B)$ weakly and strongly to $1 - s$ and p respectively. Consequently, the expression itself converges weakly in $L_1(B)$ to the product of the limiting functions. This results in equality (4.2) which means that $p \in \operatorname{dom} \tilde{I}_D(s, \cdot)$.

To complete the proof of the theorem it suffices to establish the relaxed Darcy equation (4.4) for the limiting functions. Since $K^{-1}q_h = \theta_h g - \nabla U_{(h)}(p_h)$ by definition, then one can

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take the weak limit in $(L_2(B))^d$ of both sides and obtain the equality $K^{-1}q = sg - \nabla p$. Compared with equation (4.4) the last term on the right of this equality has no multiplier s. However, the fact that $p(x) = 0$ a.e. on subdomain $\{x \in B : s(x) < 1\} \subset B$ suggests that $\nabla p(\cdot) = s(\cdot)\nabla p(\cdot)$ almost everywhere on B. Thus, the relaxed Darcy equation is established, and the proof of the theorem is completed. □

Concluding remarks

As can be seen from the above, the presented approach allows us to give a variational formulation for the free boundary problem that describes stationary groundwater flows with phreatic surfaces. It is proved that any solution to the problem, which includes a flow domain Ω, a water flux q and a pressure field p, is a minimizer of some functional. This functional is introduced in an explicit form, and its minimal value over q and p is denoted by $\Lambda(\Omega)$. Then the existence of solutions for the problem under consideration is equivalent to the fact that the minimal value of $\Lambda(\cdot)$ is attainable and equals zero. This representation of the free boundary problem provides opportunities to use standard methods of the calculus of variations in studies of particular problems with respect to existence or non-existence of solutions.

The variational approach makes it possible to introduce weak solutions to the free boundary problem as the minimizers of the lower semi-continuous regularization of the generating functional $\Lambda(\Omega)$. To this end the set of admissible domains is endowed with a natural weak topology, and the functional $\Lambda(\Omega)$ is extended onto the closure of this set. The latter is represented by water saturation fields ranging from zero to one. Then it is proved that the extended functional is the lower semi-continuous regularization of $\Lambda(\Omega)$.

Another way to introduce weak solutions for the free boundary problem is based on the approach developed by Alt [1] and Brézis *et al.* [7]. It consists of an approximation of the problem by a sequence of models for unsaturated groundwater flows accounting for capillary forces. These flows are described by degenerate elliptic equations in the entire porous medium. It is proved that the no-capillary limits of the solutions for the approximating problems correspond to the same sort of relaxed solutions as defined through the variational principle. This result gives an additional physical interpretation for the weak solutions. Another outcome is a sufficient solvability condition for the relaxed free boundary problem.

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Appendix A

Proof of Proposition 3.1

Let us denote the measure of Ω by $|\Omega|$. If $|\Omega| = 1$, i.e. $\Omega = B$ a.e., then the unique minimizer of $I_D(\Omega, \cdot)$ and the unique maximizer of $I_N(\Omega, \cdot)$ can be found explicitly. They are given by expressions $p(x, y) = \max\{y(R/k - 1), 0\}$ and $q(x, y) = (0, -R)$ respectively. Then straightforward calculations provide the equality $\Lambda(\Omega) = 0$ if $R \geqslant k$ or the inequality $\Lambda(\Omega) > 0$ if $R < k$.

If $|\Omega| = 0$, then $\Lambda(\Omega) = +\infty$ for all $R \neq 0$ because in this case $\operatorname{dom} I_N(\Omega, \cdot) = \emptyset$. If $|\Omega| = 0$ and $R = 0$ then $\operatorname{dom} I_D(\Omega, \cdot) = \{0\}$ and $\operatorname{dom} I_N(\Omega, \cdot) = \{0\}$. In this case, the unique minimizer of $I_D(\Omega, \cdot)$ is $p(x, y) \equiv 0$, and the unique maximizer of $I_N(\Omega, \cdot)$ is $q(x, y) \equiv (0, 0)$. Hence $\Lambda(\Omega) = 0$.

Thus, trivial sets $\Omega = B$ and $\Omega = \emptyset$ solve the free boundary problem if and only if the value of parameter R corresponds to items 1 and 2 of the proposition respectively. To continue the proof of the proposition it is necessary to check that the problem under consideration has no solutions if $0 < |\Omega| < 1$ regardless the value of R. To this end, let us establish the following estimates:

$$I_D(\Omega, p) \geqslant |\Omega| \max\left\{\frac{k}{2}, R - \frac{R^2}{2k}\right\}, \tag{A 1}$$

$$I_N(\Omega, q) \leqslant R - \frac{R^2}{2k|\Omega|}. \tag{A 2}$$

The value of $I_D(\Omega, p)$ can be estimated using convexity arguments. In this respect, the inequality

$$1/2\, K\, (g - \nabla p) \cdot (g - \nabla p) \geqslant \xi \cdot (g - \nabla p) - 1/2\, K^{-1} \xi \cdot \xi,$$

holds for any $\xi \in \mathbf{R}^2$. Therefore

$$I_D(\Omega, p) \geqslant \int_B Q \cdot \nabla p \, dx\, dy + \int_\Omega \left(\xi \cdot (g - \nabla p) - \frac{1}{2} K^{-1} \xi \cdot \xi\right) dx\, dy =$$

$$= \int_\Omega \left(\xi \cdot g - \frac{1}{2} K^{-1} \xi \cdot \xi\right) dx\, dy + \int_B (Q - \xi) \cdot \nabla p \, dx\, dy.$$

The last equality holds due to the fact that $\nabla p(\cdot) = 0$ a.e. on the set $B \backslash \Omega$ since $p(\cdot) = 0$ there.

We set $\xi = (0, -R')$ for an arbitrary $R' \geqslant R$. Then the last term on the right is non-negative for any $p \in M_D$ due to the specific structure of boundary conditions in the problem under consideration, and the first one equals $|\Omega| \left(R' - 1/2\, k^{-1} R'^2\right)$. Then maximization of this expression over all $R' \geqslant R$ proves estimate (A 1).

In order to establish estimate (A 2), we use the following equality for any divergence-free vector field $q = \left(q_x, q_y\right) \in M_N$

$$\int_\Omega q_y dx dy = -R. \tag{A 3}$$

It follows from the definition of M_N.

The value of $I_N(\Omega, q)$ satisfies the relation

$$I_N(\Omega, q) \leqslant -\int_\Omega \left[q_y + \frac{1}{2k}\left(q_x^2 + q_y^2\right)\right] dx dy, \tag{A 4}$$

where equality takes place if and only if $q \in \operatorname{dom} I_N(\Omega, \cdot)$. The maximum value of the

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expression on the right-hand side of relation (A 4) over all $q \in (L_2(\Omega))^2$ can be calculated explicitly if the differential equality $\nabla \cdot q = 0$ and boundary conditions for q are disregarded except for constraint (A 3) which is their consequence. Indeed, condition (A 3) can be taken into consideration by means of a Lagrange multiplier, then the maximization problem is reduced to an algebraic one for the expression under the integral sign. The desired maximum is equal to the right-hand side of inequality (A 2). The corresponding maximizer is $q' = \big(0, -R/|\Omega|\big)$. This provides estimate (A 2) because the maximum is taken over a wider set of vector fields than M_N.

Note that if $R \neq 0$, then the non-strict inequality in relation (A 2) can be replaced by the strict one. Indeed, the equality in relation (A 2) holds if and only if $q \in \operatorname{dom} I_N(\Omega, \cdot)$ is the maximizer of $I_N(\Omega, \cdot)$ and $q(x, y) = q'(x, y)$ in Ω because the expression for $I_N(\Omega, q)$ is strictly concave with respect to q. This suggests that the extension of q' by zero from Ω to B is divergence-free and satisfies Neumann boundary conditions (2.4) on ∂B_N. Since the horizontal component of q' is zero, the extension is divergence-free if and only if its vertical component does not depend on y. This is not compatible with the boundary condition on the top of the square if $R \neq 0$ and $|\Omega| \neq 1$.

Let (Ω, p, q) be a solution for the free boundary problem. If $R \geqslant k$, then inequalities (A 1) and (A 2) provide the following estimate:

$$\Lambda(\Omega) = I_D(\Omega, p) - I_N(\Omega, q) \geqslant |\Omega| \left(R - \frac{1}{2k} R^2 \right) - R + \frac{1}{2k|\Omega|} R^2.$$

Here, the expression on the right is positive for all values of $|\Omega|$, $0 < |\Omega| < 1$. This completes the proof of item 1 of the proposition concerning the uniqueness of the trivial solution.

If $R < k$, then the estimate for $\Lambda(\cdot)$ reads

$$\Lambda(\Omega) \geqslant |\Omega| \frac{k}{2} - R + \frac{1}{2k|\Omega|} R^2.$$

If $R \leqslant 0$ and $0 < |\Omega| \leqslant 1$, then the right-hand side here is positive, and this completes the proof of items 2 and 3 of the proposition.

Finally, in the range $0 < R < k$ the expression on the right is non-negative and attains zero at $|\Omega| = R/k < 1$. However, the above remark in the proof of inequality (A 2) suggests that $\Lambda(\Omega) > 0$ in this case as well. Thus, the solution does not exist. The example of minimizing sequence Ω_ε in Figure 2 completes the proof of the proposition. □

Proof of Proposition 4.3

Let sequences $s_j \in S(B)$ and $p_j \in M_D$ be weakly convergent as $j \to \infty$ to s and p respectively. In order to prove the lower semi-continuity of $\tilde{I}_D(s, p)$, it is necessary to establish the inequality

$$\liminf_{j \to \infty} \tilde{I}_D\left(s_j, p_j\right) \geqslant \tilde{I}_D(s, p).$$

In this respect, it suffices to consider sequences satisfying the relation $p_j \in \operatorname{dom} \tilde{I}_D(s_j, \cdot)$.

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In this case the value of $\tilde{I}_D\left(s_j, p_j\right)$ is determined by the expression

$$\tilde{I}_D\left(s_j, p_j\right) = \int_B \left[Q \cdot \left(\nabla p_j - \nabla P\right) + \frac{1}{2} K \nabla p_j \cdot \nabla p_j - K \nabla p_j \cdot g + \frac{s_j}{2} K g \cdot g\right] dx.$$

The quadratic term over ∇p_j in this expression is convex, and the sequence of ∇p_j converges weakly in $(L_2(B))^d$. Consequently, in the limit as $j \to \infty$ this equality turns into the relation

$$\liminf_{j\to\infty} \tilde{I}_D\left(s_j, p_j\right) \geqslant \int_B \left[Q \cdot (\nabla p - \nabla P) + \frac{1}{2} K \nabla p \cdot \nabla p - K \nabla p \cdot g + \frac{s}{2} K g \cdot g\right] dx.$$

It remains to verify that the right-hand side here is equal to $\tilde{I}_D(s,p)$. This is equivalent to the relation $p \in \operatorname{dom} \tilde{I}_D(s,\cdot)$. Since $p_j \in M_D$ for any j and M_D is a closed convex subset of $W^{1,2}(B)$, then the limiting function $p \in M_D$. In accordance with the assumptions made above, $p_j \in \operatorname{dom} \tilde{I}_D(s_j,\cdot)$. This implies that the equality $\left(1 - s_j(x)\right) p_j(x) = 0$ holds a.e. on B. The multipliers here converge in $L_2(B)$ weakly to $1 - s$ and strongly to p respectively. Therefore, $(1-s)p \equiv 0$, i.e. $p \in \operatorname{dom} \tilde{I}_D(s,\cdot)$. Thus, the lower semi-continuity of $\tilde{I}_D(s,p)$ is proved.

To prove that functional $s \to \inf_{M_D} \tilde{I}_D(s,\cdot)$ is lower semi-continuous, let us suppose, for a proof by contradiction, the existence of a sequence $s_j \in S(B)$ which converges weakly to some s and satisfies the inequalities

$$\inf_{M_D} \tilde{I}_D(s_j,\cdot) \leqslant \text{Const} < \tilde{I}_D(s,\cdot), \tag{A 5}$$

where $\text{Const} < +\infty$ is independent of j. Then $\operatorname{dom} \tilde{I}_D(s_j,\cdot) \neq \emptyset$ for any j. Let $p_j \in \operatorname{dom} \tilde{I}_D(s_j,\cdot)$ be the minimizer of $\tilde{I}_D(s_j,\cdot)$. The first inequality (A 5) provides boundedness of the sequence p_j in $W^{1,2}(B)$. Therefore, it is possible to find a weakly convergent subsequence $p_j \to p$. Since the lower semi-continuity of $\tilde{I}_D(\cdot,\cdot)$ in both variables is proved, this implies the relation

$$\liminf_{j\to\infty} \tilde{I}_D\left(s_j, p_j\right) \geqslant \tilde{I}_D(s,p).$$

This is in contradiction with assumption (A 5). Thus, $\inf_{M_D} \tilde{I}_D(s,\cdot)$ is a lower semi-continuous functional with respect to s.

To prove the upper semi-continuity of $\tilde{I}_N(s,q)$ we suppose, for a contradiction, the existence of weakly convergent sequences $s_j \in S(B)$ and $q_j \in M_N$ with the limiting functions s and q respectively, which satisfy the inequalities

$$\tilde{I}_N\left(s_j, q_j\right) \geqslant \text{Const} > \tilde{I}_N(s,q), \tag{A 6}$$

where $\text{Const} > -\infty$ is independent of j. Then $q_j \in \operatorname{dom} \tilde{I}_N(s_j,\cdot)$ for any j. Therefore, the value of $\tilde{I}_N(s_j,p_j)$ is given by the expression

$$\tilde{I}_N\left(s_j, p_j\right) = \int_{\{x:\, s_j(x)>0\}} \left[(g - \nabla P) \cdot q_j - \frac{1}{2 s_j} K^{-1} q_j \cdot q_j\right] dx.$$

For any ε, $0 < \varepsilon < 1$ let us introduce an auxiliary function $\varphi_\varepsilon(\cdot) : [0,1] \to [\varepsilon, 1]$ by the

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formula $\varphi_\varepsilon(s) = \varepsilon + (1-\varepsilon)s$. Since $\varphi_\varepsilon(s) \geqslant s$ for any $s \in [0,1]$, the following inequality holds by convexity arguments:

$$\int_{\{x:s_j(x)>0\}} \left[\frac{1}{2s_j} K^{-1} q_j \cdot q_j\right] dx \geqslant \int_B \left[q_j \cdot \xi - \frac{\varphi_\varepsilon(s_j)}{2} K\xi \cdot \xi\right] dx,$$

for any $\xi \in (L_\infty(B))^d$. The left-hand side here is bounded as $j \to \infty$ due to the first inequality (A 6). Then, as a result of sending j to ∞ and taking supremum over all $\xi(\cdot)$ after that, the last inequality is followed by the relation

$$\liminf_{j\to\infty} \int_{\{x:s_j(x)>0\}} \frac{1}{2s_j} K^{-1} q_j \cdot q_j dx \geqslant \left(\int_{\{x:s(x)=0\}} + \int_{\{x:s(x)>0\}}\right) \left[\frac{1}{2\varphi_\varepsilon(s)} K^{-1} q \cdot q\right] dx.$$

Therefore, both integrals on the right are bounded uniformly with respect to the small parameter ε. The value of the function $\varphi_\varepsilon(s(x))$ on the subset $\{x \in B : s(x) = 0\}$ equals ε identically. Hence, the first integral on the right is equal to zero, and $q(x) \equiv 0$ on this subset. This provides the second condition in definition (4.3) of $\operatorname{dom} \tilde{I}_N(s,\cdot)$. On the remaining part of B the functions $\varphi_\varepsilon(s(x))$ converge to $s(x) > 0$ pointwise as $\varepsilon \to 0$. Therefore, the sequence of expressions under the sign of the second integral satisfies conditions of Fatou theorem. The latter provides that the pointwise limit of these expressions is integrable and obeys inequality

$$\int_{\{x:s(x)>0\}} \left[\frac{1}{2s} K^{-1} q \cdot q\right] dx \leqslant \liminf_{j\to\infty} \int_{\{x:s_j(x)>0\}} \frac{1}{2s_j} K^{-1} q_j \cdot q_j dx.$$

This is equivalent to the first estimate in definition (4.3). Therefore, $q \in \operatorname{dom} \tilde{I}_N(s,\cdot)$. Accounting for the linear in q_j and q terms in the expressions for $\tilde{I}_N(s_j, q_j)$ and $\tilde{I}_N(s,q)$, this inequality can be transformed as follows:

$$\limsup_{j\to\infty} \tilde{I}_N\left(s_j, q_j\right) \leqslant \tilde{I}_N\left(s, q\right).$$

This is contrary to the second inequality of assumption (A 6). Thus, the upper semi-continuity of $\tilde{I}_N(\cdot,\cdot)$ is proved.

The upper semi-continuity of the functional $s \to \sup_{M_N} \tilde{I}_N(s,\cdot)$ can be proved in almost the same way as the lower semi-continuity for $\inf_{M_D} \tilde{I}_D(s,\cdot)$ is already established, and this completes the proof. □

Proof of Proposition 4.4

The proof is based on the triangulation procedure which is described at a greater length in [24]. Everywhere below symbol Δ stands for a simplex in $\mathbf{R}^d$, namely, for a triangle in $\mathbf{R}^2$ or tetrahedron in $\mathbf{R}^3$. Families of simplexes are always assumed to be non-degenerate in the sense that ratios of radii of inscribed and circumscribed balls for all simplexes are uniformly isolated from zero. Notations $E(\Delta)$ and $E_0(\Delta, \partial\Delta)$ are used for the sets of divergence-free vector fields from $(L_2(\Delta))^d$ and for divergence-free vector fields with the normal to $\partial\Delta$ component equal to zero respectively. Before the proof we want a couple of lemmas.

Lemma 1 *For any simplex $\Delta \subset B$, any constant c, $0 < c \leqslant 1$, and any vector $\xi \in \mathbf{R}^d$ it is possible to find sequences of measurable subsets $\Omega_\varepsilon \subset \Delta$ and vector fields $q_\varepsilon(\cdot) \in E(\Delta)$, such that the functions $\mathbf{1}(\Omega_\varepsilon, \cdot)$ and $q_\varepsilon(\cdot)$ are weakly convergent as $\varepsilon \to 0$ to c and ξ respectively, $q_\varepsilon(x) = 0$ on $\Delta \backslash \Omega_\varepsilon$ and $q_\varepsilon(\cdot) - \xi \in E_0(\Delta, \partial\Delta)$ for any ε, and*

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} K^{-1} q_\varepsilon \cdot q_\varepsilon dx = \frac{1}{c} \int_\Delta K^{-1} \xi \cdot \xi dx.$$

Proof Without loss of generality it can be assumed that the origin of coordinate system is strictly inside Δ, and vector ξ is collinear to the last coordinate axis, i.e. $\xi = (0, \dots, \xi_d)$. Let a set $V \subset \mathbf{R}^{d-1}$ be periodic with respect to all coordinates, and its $(d-1)$-dimensional volume fraction is equal to c. For a large parameter $m \in \mathbf{N}$ let us introduce a sequence of subsets $\Omega_{(m)} \subset \Delta$ by the formula

$$\Omega_{(m)} = \{x \in \Delta : \ (mx_1, \dots, mx_{(d-1)}) \in V\}.$$

For $m \to \infty$ these subsets have a small-scaled fibrous structure with fibres directed along the given vector ξ.

Let $q_{(m)}(\cdot)$ be a vector field in Δ such that $q_{(m)}(x) = 0$ if $x \in \Delta \backslash \Omega_{(m)}$ and $q_{(m)}(x) = \left(0, \dots, \xi_d / c\right)$ if $x \in \Omega_{(m)}$. Then sequences $\Omega_{(m)}$ and $q_{(m)}(\cdot)$ satisfy all the claims of the lemma except for the boundary condition $q_{(m)}(\cdot) \cdot n = \xi \cdot n$ on $\partial\Delta$. To match the oscillating vector fields $q_{(m)}$ with constant fluxes $\xi \cdot n$ on faces of the simplex Δ, let us consider a smaller simplex $\Delta_{(\varepsilon)}$ defined by the expression

$$\Delta_{(\varepsilon)} = \{x \in \Delta : \ (1+\varepsilon)x \in \Delta\}.$$

The boundary layer $\Delta \backslash \Delta_{(\varepsilon)}$ can be used for a desired correction of $q_{(m)}$. To construct an appropriate extension of $q_{(m)}(\cdot)$ from $\Delta_{(\varepsilon)}$ to $\Delta \backslash \Delta_{(\varepsilon)}$, let us consider the following auxiliary elliptic problem in $\Delta \backslash \Delta_{(\varepsilon)}$:

$$\nabla \cdot \nabla\varphi = 0, \ \ \nabla\varphi \cdot n = \xi \cdot n \text{ on } \partial\Delta, \ \ \nabla\varphi \cdot n = q_{(m)} \cdot n \text{ on } \partial\Delta_{(\varepsilon)}.$$

Since both vector fields $q_{(m)}(\cdot)$ and ξ are divergence-free, their total fluxes through $\partial\Delta_{(\varepsilon)}$ and $\partial\Delta$ respectively are equal to zero. Therefore, the compatibility conditions are satisfied, and the auxiliary problem has a solution $\varphi = \varphi_{(\varepsilon,m)}$ which is unique up to an additive constant.

Let us write $q_{(\varepsilon,m)}(x) = q_{(m)}(x)$ if $x \in \Delta_{(\varepsilon)}$ and $q_{(\varepsilon,m)}(x) = \nabla\varphi_{(\varepsilon,m)}(x)$ if $x \in \Delta \backslash \Delta_{(\varepsilon)}$. In accordance with the above definitions, the extended vector fields $q_{(\varepsilon,m)}(\cdot)$ are supported in subsets $\Omega_{(\varepsilon,m)} = \left(\Delta \backslash \Delta_{(\varepsilon)}\right) \cup \Omega_{(m)}$. They are divergence free in Δ, and $q_{(\varepsilon,m)} - \xi \in E_0(\Delta, \partial\Delta)$.

For a fixed ε and $m \to \infty$ the traces $q_{(m)}(\cdot) \cdot n$ on $\partial\Delta_{(\varepsilon)}$ are weakly convergent to $\xi \cdot n$ in $L_2(\partial\Delta_{(\varepsilon)})$. Since the embedding $L_2(\partial\Delta_{(\varepsilon)}) \subset H^{-1/2}(\partial\Delta_{(\varepsilon)})$ is compact, the sequence $m \to \nabla\varphi_{(\varepsilon,m)} \cdot n$ on $\partial(\Delta \backslash \Delta_{(\varepsilon)})$ for any fixed ε converges to $\xi \cdot n$ as $m \to \infty$ strongly in $H^{-1/2}(\partial(\Delta \backslash \Delta_{(\varepsilon)}))$. Then from properties of harmonic functions it follows that the sequence $\varphi_{(\varepsilon,m)}$ for a fixed ε is strongly compact in $W^{1,2}(\Delta \backslash \Delta_{(\varepsilon)})/\mathbf{R}$ (i.e. up to a constant), and the sequence of the gradients $\nabla\varphi_{(\varepsilon,m)}$ converges strongly to ξ in $\left(L_2(\Delta \backslash \Delta_{(\varepsilon)})\right)^d$. Thus, $q_{(\varepsilon,m)} \to \xi$ in $(L_2(\Delta_{(\varepsilon)}))^d$ weakly and $q_{(\varepsilon,m)} \to \xi$ in $(L_2(\Delta \backslash \Delta_{(\varepsilon)}))^d$ strongly. This implies that $q_{(\varepsilon,m)} \to \xi$

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weakly in $(L_2(\Delta))^d$ as $m \to \infty$. Then we obtain

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty}\int_\Delta K^{-1}q_{(\varepsilon,m)}\cdot q_{(\varepsilon,m)}dx =$$

$$= \lim_{\varepsilon\to 0}\left(\int_{\Delta\setminus\Delta_{(\varepsilon)}} K^{-1}\xi\cdot\xi dx + \frac{1}{c}\int_{\Delta_{(\varepsilon)}} K^{-1}\xi\cdot\xi dx\right) = \frac{1}{c}\int_\Delta K^{-1}\xi\cdot\xi dx,$$

where the limit $m \to \infty$ is taken first for a fixed ε, and ε tends to zero after that. In addition, the family of subsets $\Omega_{(\varepsilon,m)}$ satisfies the equality

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty}\mathbf{1}\left(\Omega_{(\varepsilon,m)},\cdot\right) = c,$$

where convergence with respect to m is weak, and the further limit with respect to ε is strong in $L_2(\Delta)$.

Thus, with an appropriate choice of a subsequence $m = m(\varepsilon) \to \infty$ as $\varepsilon \to 0$, the subsets $\Omega_\varepsilon = \Omega_{(\varepsilon,m(\varepsilon))}$ and vector fields $q_\varepsilon = q_{(\varepsilon,m(\varepsilon))}$ obey all the required properties. This completes the proof of Lemma 1. □

Lemma 2 *Let $q(\cdot)$ be a smooth divergence-free vector field in simplex $\Delta \subset B$, i.e. $q \in E(\Delta) \cap \left(C^\infty(\overline{\Delta})\right)^d$, and $s(\cdot)$ be a measurable function on Δ such that $\delta \leqslant s(x) \leqslant 1$ for some $\delta > 0$. Then it is possible to construct sequences of subsets $\Omega_\varepsilon \subset \Delta$ and divergence-free vector fields $q_\varepsilon \in E(\Delta)$ such that $q_\varepsilon(x) = 0$ on $\Delta\setminus\Omega_\varepsilon$, $q_\varepsilon - q \in E_0(\Delta, \partial\Delta)$, functions $\mathbf{1}(\Omega_\varepsilon,\cdot)$ and $q_\varepsilon(\cdot)$ are weakly convergent to $s(\cdot)$ and $q(\cdot)$ as $\varepsilon \to 0$, and*

$$\lim_{\varepsilon\to 0}\int_\Delta K^{-1}q_\varepsilon\cdot q_\varepsilon dx = \int_\Delta \frac{1}{s}K^{-1}q\cdot q dx. \tag{A 7}$$

Proof Below $\Delta_{(\varepsilon)} \subset \Delta$ stands again for the homothetic rescaling of Δ. Let us consider a partition of $\Delta_{(\varepsilon)}$ into a large number $m \in \mathbf{N}$ of smaller simplexes

$$\Delta_{(\varepsilon)} = \bigcup_{\alpha=1}^{m}\Delta^\alpha_{(\varepsilon,m)}.$$

The family $\{\Delta^\alpha_{(\varepsilon,m)}\}$ is assumed to be a regular triangulation of $\Delta_{(\varepsilon)}$. Regularity implies that the maximal diameter of the simplexes tends to zero as $m \to \infty$ and, for any m and $\alpha \neq \beta$, the intersection $\overline{\Delta}^\alpha_{(\varepsilon,m)} \cap \overline{\Delta}^\beta_{(\varepsilon,m)}$ is the empty set or a simplex of a dimension less than d.

For any $\alpha = 1, \ldots, m$ let us determine a constant vector $\xi^\alpha_{(\varepsilon,m)}$ which has the same integral flux through each face of $\Delta^\alpha_{(\varepsilon,m)}$ as the given vector field $q(\cdot)$. This implies $d+1$ linear equalities for d components of the sought vector $\xi^\alpha_{(\varepsilon,m)}$, but the equalities are compatible since $q(\cdot)$ and $\xi^\alpha_{(\varepsilon,m)}$ are divergence-free in $\Delta^\alpha_{(\varepsilon,m)}$ and, therefore, their total fluxes through $\partial\Delta^\alpha_{(\varepsilon,m)}$ are both equal to zero. Hence vectors $\xi^\alpha_{(\varepsilon,m)}$ are introduced correctly. Then the piecewise constant function $q_{(\varepsilon,m)}(\cdot)$ which is equal to $\xi^\alpha_{(\varepsilon,m)}$ in each simplex $\Delta^\alpha_{(\varepsilon,m)}$ is a divergence-free vector field on $\Delta_{(\varepsilon)}$ because the fluxes of $q_{(\varepsilon,m)}(\cdot)$ through common faces of adjoined simplexes are properly adjusted. It is clear that the sequence $q_{(\varepsilon,m)}(\cdot)$ converges to $q(\cdot)$ uniformly on $\overline{\Delta}_{(\varepsilon)}$ as $m \to \infty$.

Let $c^{\alpha}_{(\varepsilon,m)}$ be the mean value of $s(\cdot)$ over $\Delta^{\alpha}_{(\varepsilon,m)}$, and $s_{(\varepsilon,m)}(\cdot)$ stand for the aggregated piecewise constant function which equals $c^{\alpha}_{(\varepsilon,m)}$ in each simplex $\Delta^{\alpha}_{(\varepsilon,m)}$ of the triangulating family. Then functions $s_{(\varepsilon,m)}(\cdot)$ converge to $s(\cdot)$ strongly in $L_2(\Delta_{(\varepsilon)})$ as $m \to \infty$.

For a new small parameter h and for every simplex $\Delta^{\alpha}_{(\varepsilon,m)}$ by virtue of Lemma 1 it is possible to construct sequences of subsets $\Omega^{\alpha}_{(\varepsilon,m,h)} \subset \Delta^{\alpha}_{(\varepsilon,m)}$ and vector fields $q^{\alpha}_{(\varepsilon,m,h)} \in E(\Delta^{\alpha}_{(\varepsilon,m)})$ supported in $\Omega^{\alpha}_{(\varepsilon,m,h)}$ such that functions $\mathbf{1}(\Omega^{\alpha}_{(\varepsilon,m,h)},\cdot)$ and $q^{\alpha}_{(\varepsilon,m,h)}(\cdot)$ converge weakly to $c^{\alpha}_{(\varepsilon,m)}$ and $\xi^{\alpha}_{(\varepsilon,m)}$ respectively as $h \to 0$, the boundary conditions are satisfied in the form $q^{\alpha}_{(\varepsilon,m,h)} - \xi^{\alpha}_{(\varepsilon,m)} \in E_0(\Delta^{\alpha}_{(\varepsilon,m)}, \partial\Delta^{\alpha}_{(\varepsilon,m)})$, and

$$\lim_{h\to 0} \int_{\Delta^{\alpha}_{(\varepsilon,m)}} K^{-1} q^{\alpha}_{(\varepsilon,m,h)} \cdot q^{\alpha}_{(\varepsilon,m,h)} dx = \frac{1}{c^{\alpha}_{(\varepsilon,m)}} \int_{\Delta^{\alpha}_{(\varepsilon,m)}} K^{-1} \xi^{\alpha}_{(\varepsilon,m)} \cdot \xi^{\alpha}_{(\varepsilon,m)} dx.$$

Then, as a result of aggregation of subsets $\Omega^{\alpha}_{(\varepsilon,m,h)}$ and vector fields $q^{\alpha}_{(\varepsilon,m,h)}(\cdot)$ over all simplexes $\Delta^{\alpha}_{(\varepsilon,m)}$, $\alpha = 1, \ldots, m$, one can get sequences of subsets $\Omega_{(\varepsilon,m,h)} = \cup_{\alpha=1}^{m} \Omega^{\alpha}_{(\varepsilon,m,h)} \subset \Delta_{(\varepsilon)}$ and vector fields $q_{(\varepsilon,m,h)} \in E(\Delta_{(\varepsilon)})$ supported in them, which satisfy the equalities

$$\lim_{h\to 0} \int_{\Delta_{(\varepsilon)}} K^{-1} q_{(\varepsilon,m,h)} \cdot q_{(\varepsilon,m,h)} dx = \int_{\Delta_{(\varepsilon)}} \frac{1}{s_{(\varepsilon,m)}} K^{-1} q_{(\varepsilon,m)} \cdot q_{(\varepsilon,m)} dx. \tag{A 8}$$

Note that the right-hand side here converges to the right-hand side of equality (A 7) if $m \to \infty$ at first, and $\varepsilon \to 0$ after that. This conclusion holds due to the condition $s(\cdot) \geqslant \delta > 0$ of the lemma.

Let us construct a divergence-free extension of vector fields $q_{(\varepsilon,m,h)}(\cdot)$ from the scaled simplex $\Delta_{(\varepsilon)}$ to the boundary layer $\Delta \backslash \Delta_{(\varepsilon)}$. Since the boundary conditions for $q_{(\varepsilon,m,h)} \cdot n$ and $q_{(\varepsilon,m)} \cdot n$ on $\partial\Delta_{(\varepsilon)}$ are adjusted to each other, the extension may be independent of h. The desired corrector can be taken in the form $\nabla\varphi_{(\varepsilon,m)}$, where $\varphi_{(\varepsilon,m)}$ is a solution for the following auxiliary problem in the layer $\Delta \backslash \Delta_{(\varepsilon)}$:

$$\nabla \cdot \nabla\varphi = 0, \ \nabla\varphi \cdot n = q \cdot n \text{ on } \partial\Delta, \ \ \nabla\varphi \cdot n = q_{(\varepsilon,m)} \cdot n \text{ on } \partial\Delta_{(\varepsilon)}.$$

Compatibility conditions for this Neumann problem are satisfied because the piece-wise constant vector fields $q_{(\varepsilon,m)}$ have the same total flux through $\partial\Delta_{(\varepsilon)}$ as the given divergence-free vector field q. Hence the corrector $\varphi = \varphi_{(\varepsilon,m)}$ exists, and the extended vector fields are divergence-free in the simplex Δ. They are supported in $\Omega_{(\varepsilon,m,h)} \cup \left(\Delta \backslash \Delta_{(\varepsilon)}\right)$ and satisfy the desired boundary condition on $\partial\Delta$ in the form $q_{(\varepsilon,m,h)} - q \in E_0(\Delta, \partial\Delta)$, where the same notation $q_{(\varepsilon,m,h)}$ is kept for the extended functions.

Since the boundary functions $q_{(\varepsilon,m)}$ converge to q on $\partial\Delta_{(\varepsilon)}$ uniformly as $m \to \infty$, the sequence of correctors $\varphi_{(\varepsilon,m)}$ is compact in $W^{1,2}(\Delta \backslash \Delta_{(\varepsilon)})$, and the sequence $m \to \nabla\varphi_{(\varepsilon,m)}$ converges in $(L_2(\Delta \backslash \Delta_{(\varepsilon)}))^d$ strongly to the gradient $\nabla\varphi_{(\varepsilon)}$ of some harmonic function $\varphi_{(\varepsilon)}$ that satisfies Neumann boundary conditions $(\nabla\varphi_{(\varepsilon)} - q)\cdot n = 0$ on $\partial\Delta \cup \partial\Delta_{(\varepsilon)}$. Then standard integral estimates provide the fact that the L_2-norm of $\nabla\varphi_{(\varepsilon)}$ over the layer $\Delta \backslash \Delta_{(\varepsilon)}$ is not greater than the L_2-norm of q which is, in its turn, converges to zero as $\varepsilon \to 0$.

Thanks to the properties of extended vector fields $q_{(\varepsilon,m,h)}$, equation (A 8) imply the equalities

$$\lim_{\varepsilon\to 0} \lim_{m\to\infty} \lim_{h\to 0} q_{(\varepsilon,m,h)}(\cdot) = q(\cdot),$$

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$$\lim_{\varepsilon\to 0}\lim_{m\to\infty}\lim_{h\to 0}\mathbf{1}\left(\Omega_{(\varepsilon,m,h)}\cup(\Delta\backslash\Delta_{(\varepsilon)}),\cdot\right)=s(\cdot),$$

and

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty}\lim_{h\to 0}\int_\Delta K^{-1}q_{(\varepsilon,m,h)}\cdot q_{(\varepsilon,m,h)}dx=$$

$$=\lim_{\varepsilon\to 0}\left[\int_{\Delta\backslash\Delta_{(\varepsilon)}}K^{-1}\nabla\varphi_{(\varepsilon)}\cdot\nabla\varphi_{(\varepsilon)}dx+\int_{\Delta_{(\varepsilon)}}\frac{1}{s}K^{-1}q\cdot q dx\right]=\int_\Delta\frac{1}{s}K^{-1}q\cdot q dx.$$

To complete the construction of the desired subsets $\Omega_\varepsilon\subset\Delta$ and vector fields q_ε it remains to choose appropriate subsequences $m=m(\varepsilon)\to\infty$, $h=h(\varepsilon)\to 0$ and pose $\Omega_\varepsilon=\Omega_{(\varepsilon,m(\varepsilon),h(\varepsilon))}\cup\left(\Delta\backslash\Delta_{(\varepsilon)}\right)$, $q_\varepsilon=q_{(\varepsilon,m(\varepsilon),h(\varepsilon))}$. Thus, Lemma 2 is proved. □

Coming back to the proof of Proposition 4.4, let $B_\varepsilon\subset B$ be the set $B_\varepsilon=\{x\in B:\operatorname{dist}(x,\partial B)>\varepsilon\}$. Let us introduce a function $s_{(\varepsilon)}(\cdot)$ on B by the expressions $s_{(\varepsilon)}(x)=\max\{\varepsilon,s(x)\}$ for $x\in B_\varepsilon$ and $s_{(\varepsilon)}(x)=1$ for $x\in B\backslash B_\varepsilon$. Since $s_{(\varepsilon)}(\cdot)$ bounded away from zero, $\operatorname{dom}\tilde{I}_N(s_{(\varepsilon)},\cdot)=M_N$ and $\tilde{I}_N(s_{(\varepsilon)},q)>-\infty$. In addition, $\tilde{I}_N(s_{(\varepsilon)},q)\geqslant\tilde{I}_N(s,q)$ because the functional $\tilde{I}_N(s,q)$ is monotone with respect to s, and $s_{(\varepsilon)}(\cdot)\geqslant s(\cdot)$.

Any divergence-free vector field $q\in E(B)$ can be approximated by a smooth divergence-free vector function $q_{(\varepsilon)}\in E(B)\cap(C^\infty(\overline{B}\backslash\partial B))^d$ which satisfies relations $q_{(\varepsilon)}-q\in E_0(B,\partial B)$ and

$$\int_B\left(q_{(\varepsilon)}-q\right)^2dx\leqslant\varepsilon^4.$$

Then $\tilde{I}_N(s_{(\varepsilon)},q_{(\varepsilon)})=\tilde{I}_N(s_{(\varepsilon)},q)+\underline{O}(\varepsilon)$. Existence of the approximating function $q_{(\varepsilon)}$ is justified as follows. First, for the given q one can take the gradient $\nabla\varphi$ of a harmonic function φ such that $q-\nabla\varphi\in E_0(B,\partial B)$. Then it remains to approximate $q-\nabla\varphi$ by a smooth divergence-free vector function with the normal to ∂B component equal to zero. It is possible to do so because, in accordance with Theorem 1.4 from Chapter 1 of [25], for a bounded domain B with Lipschitz boundary the subspace $E_0(B,\partial B)\subset(L_2(B))^d$ is the closure of the set of all smooth compactly supported divergence-free vector fields on $\overline{B}\backslash\partial B$.

Let a family of simplexes $\{\Delta^\alpha_{(\varepsilon,m)}\subset B,\ \alpha=1,\dots,m\}$ be such a regular triangulation of B that satisfies the relation

$$B_\varepsilon\subset\bigcup_{\alpha=1}^m\Delta^\alpha_{(\varepsilon,m)}\subset B,$$

for any $\varepsilon>0$ and some $m=m(\varepsilon)\in\mathbf{N}$. Then, thanks to Lemma 2, for a new small parameter h it is possible to construct sequences of subsets $\Omega_{(\varepsilon,m,h)}\subset\cup_{\alpha=1}^m\Delta^\alpha_{(\varepsilon,m)}$ and vector fields $q_{(\varepsilon,m,h)}\in E(\cup_{\alpha=1}^m\Delta^\alpha_{(\varepsilon,m)})$ supported in $\Omega_{(\varepsilon,m,h)}$ such that for any simplex $\Delta^\alpha_{(\varepsilon,m)}$ functions $\mathbf{1}(\Omega_{(\varepsilon,m,h)},\cdot)$ and $q_{(\varepsilon,m,h)}(\cdot)$ converge weakly as $h\to 0$ to $s_{(\varepsilon)}(\cdot)$ in $L_2(\Delta^\alpha_{(\varepsilon,m)})$ and $q_{(\varepsilon)}(\cdot)$ in $(L_2(\Delta^\alpha_{(\varepsilon,m)}))^d$ respectively, $q_{(\varepsilon,m,h)}-q_{(\varepsilon)}\in E_0(\Delta^\alpha_{(\varepsilon,m)},\partial\Delta^\alpha_{(\varepsilon,m)})$, and

$$\lim_{h\to 0}\int_{\Delta^\alpha_{(\varepsilon,m)}}K^{-1}q_{(\varepsilon,m,h)}\cdot q_{(\varepsilon,m,h)}dx=\int_{\Delta^\alpha_{(\varepsilon,m)}}\frac{1}{s_{(\varepsilon)}}K^{-1}q_{(\varepsilon)}\cdot q_{(\varepsilon)}dx.$$

Let us attach the boundary layer $B\backslash\cup_{\alpha=1}^m\Delta^\alpha_{(\varepsilon,m)}$ to $\Omega_{(\varepsilon,m,h)}$ for any h and extend the vector fields $q_{(\varepsilon,m,h)}$ to the layer by $q_{(\varepsilon)}$. The enlarged sets and extended vector fields are denoted

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by the same symbols $\Omega_{(\varepsilon,m,h)}$ and $q_{(\varepsilon,m,h)}$. Since the boundary conditions for $q_{(\varepsilon,m,h)}$ and $q_{(\varepsilon)}$ on the inner boundary of the layer $\partial(\cup_{\alpha=1}^{m}\Delta_{(\varepsilon,m)}^{\alpha})$ are properly adjusted, the extended fields are still divergence-free. Also, the weak convergence of $\mathbf{1}(\Omega_{(\varepsilon,m,h)},\cdot)$ and $q_{(\varepsilon,m,h)}(\cdot)$ to $s_{(\varepsilon)}(\cdot)$ and $q_{(\varepsilon)}(\cdot)$ as $h \to 0$ still takes place. In addition, $q_{(\varepsilon,m,h)} - q \in E_0(B,\partial B)$ and

$$\lim_{h\to 0}\int_B K^{-1}q_{(\varepsilon,m,h)}\cdot q_{(\varepsilon,m,h)}dx = \int_B \frac{1}{s_{(\varepsilon)}}K^{-1}q_{(\varepsilon)}\cdot q_{(\varepsilon)}dx.$$

As a result, the following relations hold true for a subsequence $\varepsilon \to 0$, $m = m(\varepsilon)$:

$$\lim_{\varepsilon\to 0}\lim_{h\to 0} I_N\left(\Omega_{(\varepsilon,m,h)}, q_{(\varepsilon,m,h)}\right) = \lim_{\varepsilon\to 0}\tilde{I}_N\left(s_{(\varepsilon)}, q_{(\varepsilon)}\right) =$$

$$= \lim_{\varepsilon\to 0}\tilde{I}_N\left(s_{(\varepsilon)}, q\right) \geqslant \tilde{I}_N(s,q).$$

By their definitions, the functions $s_{(\varepsilon)}$ and $q_{(\varepsilon)}$ are strongly convergent to $s(\cdot)$ in $L_2(B)$ and $q(\cdot)$ in $(L_2(B))^d$ respectively as $\varepsilon \to 0$. Therefore, it is possible to choose subsequences $\varepsilon \to 0$, $m = m(\varepsilon)$, $h = h(\varepsilon)$, for which the functions $q_\varepsilon = q_{(\varepsilon,m(\varepsilon),h(\varepsilon))}$ converge weakly to q, the indicators of subsets $\Omega'_\varepsilon = \Omega_{(\varepsilon,m(\varepsilon),h(\varepsilon))}$ converge weakly to s and

$$\lim_{\varepsilon\to 0} I_N\left(\Omega'_\varepsilon, q_\varepsilon\right) \geqslant \tilde{I}_N\left(s,q\right).$$

Let us write $\Omega_0 = \{x \in B : s(x) = 1\}$ and $\Omega_\varepsilon = \Omega'_\varepsilon \cup \Omega_0$. From the identity

$$\mathbf{1}\left(\Omega_\varepsilon,\cdot\right) = \mathbf{1}\left(\Omega_0,\cdot\right) + \left(1 - \mathbf{1}\left(\Omega_0,\cdot\right)\right)\mathbf{1}\left(\Omega'_\varepsilon,\cdot\right),$$

it follows that sequence $\mathbf{1}(\Omega_\varepsilon,\cdot)$ has the same weak limit $s(\cdot)$ as the sequence $\mathbf{1}(\Omega'_\varepsilon,\cdot)$. Besides, since the functional $I_N(\Omega,q)$ is monotone with respect to Ω, the inequality

$$\lim_{\varepsilon\to 0} I_N\left(\Omega_\varepsilon, q_\varepsilon\right) \geqslant \tilde{I}_N\left(s,q\right),$$

is still valid. Accounting for the upper semi-continuity of $\tilde{I}_N(\cdot,\cdot)$, the sign "$\geqslant$" in this relation can be replaced by equality. Therefore, the subsets Ω_ε and vector fields q_ε satisfy the claims of the proposition.

It remains to construct a desired sequence of pressure fields $p_\varepsilon(\cdot)$. In this respect, an appropriate candidate is $p_\varepsilon(\cdot) \equiv p(\cdot)$. Indeed, if $p \notin \mathrm{dom}\tilde{I}_D(s,\cdot)$, then $\tilde{I}_D(s,p) = +\infty$, and the equality

$$\lim_{\varepsilon\to 0} I_D\left(\Omega_\varepsilon, p_\varepsilon\right) = \tilde{I}_D\left(s,p\right),$$

follows from the lower semi-continuity of $\tilde{I}_D(\cdot,\cdot)$. Furthermore, if $p \in \mathrm{dom}\tilde{I}_D(s,\cdot)$ then the supports of $p_\varepsilon(\cdot)$ are embedded into $\Omega_0 \subset \Omega_\varepsilon$, hence $p_\varepsilon \in \mathrm{dom}I_D(\Omega_\varepsilon,\cdot)$, and

$$\lim_{\varepsilon\to 0} I_D\left(\Omega_\varepsilon, p_\varepsilon\right) =$$

$$= \lim_{\varepsilon\to 0}\int_B \left(Q\cdot(\nabla p - P) + \frac{1}{2}K(g - \nabla p)\cdot(g - \nabla p)\mathbf{1}\left(\Omega_\varepsilon,\cdot\right)\right)dx = \tilde{I}_D(s,p).$$

Thus, the proof of Proposition 4.4 is completed. □