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Simplicial complexity of surface groups

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The simplicial complexity is an invariant for finitely presentable groups which was recently introduced by Babenko, Balacheff, and Bulteau to study systolic area. The simplicial complexity $\kappa(G)$ was proved to be a good approximation of the systolic area $\sigma(G)$ for large values of $\kappa(G)$. In this paper we compute the simplicial complexity of all surface groups (both in the orientable and in the non-orientable case). This partially settles a problem raised by Babenko, Balacheff, and Bulteau. We also prove that $\kappa(G * \mathbb{Z}) = \kappa(G)$ for any surface group G. This provides the first partial evidence in favor of the conjecture of the stability of the simplicial complexity under free product with free groups. The general stability problem, both for simplicial complexity and for systolic area, remains open.

Keywords: systolic area; simplicial complexity; surface groups

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1. Introduction

Let X be a connected, non-simply connected finite simplicial complex of dimension 2. Given a piecewise smooth Riemannian metric g on X, the *systole* $\mathrm{sys}(X, g)$ of (X, g) is the length of the shortest non-contractible loop in X. The *systolic area* $\sigma(X)$ of X is defined as

$$\sigma(X) := \inf_g \frac{\mathrm{area}(X, g)}{\mathrm{sys}(X, g)^2},$$

where the infimum is taken over all piecewise smooth Riemannian metrics on X. In [7], Gromov defined the systolic area of a finitely presentable group G as

$$\sigma(G) := \inf_X \sigma(X),$$

where the infimum is taken over the finite 2-dimensional simplicial complexes X with fundamental group isomorphic to G. Gromov proved that $\sigma(G)$ is strictly positive unless G is free (see [6, theorem 6.7.A]) and posed the problem of estimating the asymptotic behavior of the size of the set $\mathcal{G}_\sigma(C)$ of isomorphism classes of groups G for which $\sigma(G) \leqslant C$ [7]. By combining topological and geometric techniques, Rudyak and Sabourau proved in [10] an exponential upper bound to the size of $\mathcal{G}_\sigma(C)$, and considerably improved Gromov's lower bound for the systolic area of a non-free group. In [10] they raised the following question concerning the stability of the systolic area under free product with free groups: is $\sigma(G * \mathbb{Z}) = \sigma(G)$ for any non-free group G? (see [10, question 1.2]). The computation of $\sigma(G * T)$ for non-free groups G and T is a problem formulated by Gromov in [7, p. 337].

In [2] Babenko, Balacheff, and Bulteau introduced the *simplicial complexity* of a finitely presentable group, which is a combinatorial invariant that approximates the systolic area. The simplicial complexity $\kappa(G)$ of a finitely presentable group G is the minimum number of 2-simplices of a simplicial complex of dimension 2 with fundamental group isomorphic to G (see [2, definition 2.1]). Here, the number of 2-simplices of a simplicial complex of dimension 2 may be thought of as a combinatorial approximation for the area. They showed that the simplicial complexity of G constitutes a fairly good approximation to the systolic area of G for large values of $\kappa(G)$. More concretely, given $\varepsilon > 0$,

$$2\pi\sigma(G) \leqslant \kappa(G) \leqslant \sigma(G)^{(1+\varepsilon)}$$

for a group G of free index 0, whenever $\kappa(G)$ is big enough (see [2, theorem 1.2] for the precise statement). Using this comparison result, they provided a quite satisfactory answer to Gromov's question on the size of $\mathcal{G}_\sigma(C)$, improving the upper bound given in [10] (see [2, theorem 1.1]). In view of the close relationship between $\sigma(G)$ and $\kappa(G)$, they asked whether the equality $\kappa(G * \mathbb{Z}) = \kappa(G)$ holds for any group G (see [2, § 2]). Another problem posed in [2] is the exact computation of the value of κ for some groups ([2, examples 1,2]). To the best of our knowledge, the only groups for which the value of κ is known are the fundamental group of the projective plane $\mathbb{Z}_2$, of the torus $\mathbb{Z} \oplus \mathbb{Z}$, of the Klein bottle, and $\mathbb{Z}_3$, computed in [4] mainly by an exhaustive analysis.

The purpose of this article is to provide some partial answers to the problems described above. In the first place, we compute the simplicial complexity of the fundamental groups of all non-simply connected closed surfaces settling in the affirmative a conjecture from [4] (here by closed surface we mean a compact connected smooth 2-manifold, orientable or non-orientable, without boundary).

THEOREM 1.1. *Let S be a non-simply connected closed surface. Then, $\kappa(\pi_1(S)) = \delta(S)$, where $\delta(S)$ is the minimum number of 2-simplices in a triangulation of S.*

The numbers $\delta(S)$ were completely computed by Ringel [9], for non-orientable surfaces, and by Jungerman and Ringel [5] in the orientable case (see theorem 4.1 below).

We also show that $\kappa(G * \mathbb{Z}) = \kappa(G)$ for any surface group G. Here by a surface group we mean the fundamental group of any non-simply connected closed surface S. This provides some (admittedly partial) evidence in favor of the validity of $\kappa(G * \mathbb{Z}) = \kappa(G)$ for any group G. Actually, we prove the following stronger estimate.

THEOREM 1.2. *Let G be a surface group and T a finitely presentable group. Then, $\kappa(G * T) \geqslant \kappa(G)$. In particular, $\kappa(G * \mathbb{Z}) = \kappa(G)$.*

To prove these results, we derive a sharp lower bound of $\kappa(G * T)$ for any group G whose cohomology ring satisfies a certain regularity property. The proof of this estimate is based on rather elementary techniques that we developed previously in [**3**].

The outline of the article is as follows. In § 2, we recall the pertinent definitions and results from [**3**] and prove our main technical result. In § 3, we use the main lemma to estimate the mentioned lower bound on $\kappa(G * T)$. Finally, in § 4 we prove theorems 1.1 and 1.2. For almost all closed surfaces (*non-exceptional* surfaces in the vocabulary of [**3**]), these results will follow straightforwardly from the lower bound provided by theorem 3.2. In contrast, for handling the *exceptional* cases some ad-hoc arguments will be required (cf. [**3**, § 3]).

2. Main technical result

In this section we prove a central technical lemma that will allow us to give a lower bound of the simplicial complexity for groups of the form $G * T$, where G and T are finitely presentable groups and the cohomology ring of G satisfies a certain regularity property. By the cohomology ring of a group G we will understand its cohomology as a discrete group, i.e. the cohomology of an Eilenberg–MacLane space $K(G, 1)$. We will work with reduced (co)homology and the coefficient ring for (co)homology groups will be $\mathbb{F}_2$. Throughout the article, G and T will denote finitely presentable groups, and all the simplicial complexes that we deal with will be finite. We recall first the definition of property (A) from [**3**, definition 2.3].

Definition 2.1. Let X be a topological space. We say that the cohomology ring $H^*(X)$ (with coefficients in $\mathbb{F}_2$) satisfies *property (A)* if for every non-trivial α in $H^1(X)$, there exists $\beta \in H^1(X)$ such that $\alpha \cup \beta$ is non-trivial in $H^2(X)$.

We will say that the cohomology ring of a group satisfies property (A) whenever the cohomology ring of a $K(G, 1)$ space does.

Example 2.2. Any surface group (orientable or non-orientable) satisfies property (A) by Poincaré Duality.

More generally, any one-relator group G with $H_2(G) = \mathbb{F}_2$ and with a non-degenerate cup product form $H^1(G) \times H^1(G) \to H^2(G) = \mathbb{F}_2$, satisfies property (A). Using the computation of the cohomology ring of one-relator groups of [**8**], one may obtain many additional examples of such groups. As concrete examples, the Baumslag–Solitar groups $BS(m, n)$ satisfy property (A) whenever m and n are odd.

Let K be a connected simplicial complex of dimension 2 such that $\pi_1(K) = G * T$. Observe that K is the 2-skeleton of an aspherical CW-complex X (possibly infinite dimensional). Since the fundamental group of X is isomorphic to $G * T$, X is an Eilenberg–MacLane space $K(G * T, 1)$. By a theorem of Whitehead (see for example [**1**, theorem 7.3]), X is homotopy equivalent to a space of the form $K_G \vee K_T$, the wedge union of a $K(G, 1)$ space and a $K(T, 1)$ space. Informally speaking, our first objective is to obtain a subcomplex of K in which we have killed all the 2-dimensional homology classes of K that do not correspond to classes in $H_2(G)$. We start by giving a definition.

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Definition 2.3. Let X be a CW-complex of dimension at least 2, together with a homotopy equivalence $h : X \to K_G \vee K_T$, where K_G and K_T are defined as above. Assume further that its 2-skeleton $X^{(2)}$ is a finite simplicial complex. Let $M \leqslant X^{(2)}$ be a (simplicial) subcomplex satisfying the following properties:

(1) The inclusion $i : M \hookrightarrow X$ induces isomorphisms $i_* : H_n(M) \to H_n(X)$ for $n < 2$.

(2) The composition $H_2(M) \xrightarrow{i_*} H_2(X) \equiv H_2(K_G) \oplus H_2(K_T) \xrightarrow{p} H_2(K_G)$ is an epimorphism, where p is the projection and the isomorphism $H_2(X) \equiv H_2(K_G) \oplus H_2(K_T)$ is induced by h.

We will say that such a subcomplex M is *homologically G-full* with respect to h, or simply *homologically G-full* if the homotopy equivalence h is clear from the context.

The next result says, roughly, that we can kill the 'extra' homology classes in $H_2(X)$ one at a time. The result is inspired in [**3**, proposition 2.7].

Lemma 2.4. *Let X be a CW-complex of dimension at least 2 homotopy equivalent to a space of the form $K_G \vee K_T$ and such that its 2-skeleton $X^{(2)}$ is a finite simplicial complex. Let $M \leqslant X^{(2)}$ be a homologically G-full subcomplex. If $\dim H_2(M) > \dim H_2(G)$, there exists a 2-simplex $\sigma \in M$ such that $M \setminus \sigma$ is homologically G-full. Moreover, $\dim H_2(M \setminus \sigma) = \dim H_2(M) - 1$.*

Proof. Since by hypothesis $\dim H_2(M) > \dim H_2(K_G)$ there is a non-trivial class B in the kernel of the linear map $p \circ i_* : H_2(M) \to H_2(K_G)$. Let σ be a 2-simplex of M in the support of B. The topological boundary $\partial\sigma$ viewed as a chain in $C_1(M \setminus \sigma)$ is the boundary of the 2-chain $B - \sigma$. Hence the inclusion induces the zero morphism $H_1(\partial\sigma) \to H_1(M \setminus \sigma)$. It follows that the inclusion $M \setminus \sigma \hookrightarrow M$ induces isomorphisms $H_n(M \setminus \sigma) \to H_n(M)$ for $n < 2$. It remains to verify the surjectivity of $p \circ j_* : H_2(M \setminus \sigma) \to H_2(K_G)$, where j is the inclusion $j : M \setminus \sigma \hookrightarrow X$. Let $[Z]$ be a class in $H_2(K_G)$. By hypothesis, there is some class $C \in H_2(M)$ such that $p \circ i_*[C] = [Z]$. If σ does not belong to the support of C, when viewed as a class in $H_2(M \setminus \sigma)$ we have $p \circ j_*[C] = [Z]$. In the other case, consider the 2-chain $C + B$. Since the coefficients are taken in $\mathbb{F}_2$, this chain is a well defined 2-cycle in $M \setminus \sigma$ and $p \circ j_*[C + B] = p \circ i_*[C] + p \circ i_*[B] = p \circ i_*[C] = [Z]$. Hence, in any case $p \circ j_* : H_2(M \setminus \sigma) \to H_2(K_G)$ is an epimorphism. The fact that $\dim H_2(M \setminus \sigma) = \dim H_2(M) - 1$ follows immediately from the Euler characteristic, since $\chi(M \setminus \sigma) = \chi(M) - 1$. □

Notation. Given a finitely presentable group G, we will denote by $\overline{\chi}(G)$ the *2-truncated* Euler characteristic of G, that is $\overline{\chi}(G) := \dim H_2(G) - \dim H_1(G) + \dim H_0(G)$.

Recall that a simplex σ of a simplicial complex K is a *free face* of K if there is a unique simplex $\tau \in K$ containing σ properly. In that case, we say that there is an (elementary) collapse from K to the subcomplex L obtained from K by removing the free face σ together with τ. Note that the inclusion $L \subseteq K$ is, in particular, a strong deformation retract.

LEMMA 2.5. *Let K be a (finite) connected simplicial complex of dimension 2 with fundamental group isomorphic to $G * T$, and suppose that the cohomology ring of G satisfies property (A). Then, there is another simplicial complex L of dimension at most 2 with no more 2-simplices than K such that $\chi(L) \leqslant \overline{\chi}(G)$, $\dim H_2(L) = \dim H_2(G)$ and every edge of L is the face of at least two 2-simplices.*

Proof. Let X be an Eilenberg–MacLane space $K(G * T, 1)$ such that $X^{(2)} = K$. Then there is a map $i : K \to K_G \vee K_T$ inducing an isomorphism in H_n for $n = 0$ and 1 and an epimorphism in H_2, where K_G and K_T are respectively a $K(G, 1)$ and a $K(T, 1)$ space as before. Since the projection $H_2(X) \equiv H_2(K_G) \oplus H_2(K_T) \to H_2(K_G)$ is surjective, K is a homologically G-full subcomplex of X. By applying inductively lemma 2.4, we obtain a subcomplex M of K that is homologically G-full and such that $\dim H_2(M) = \dim H_2(G)$. After collapsing the free faces of M, we may assume that M has no edge that is the face of a unique 2-simplex. Suppose there is a maximal edge $e = \{a, b\}$ in M (otherwise we are done, since we may take the desired complex L as M). If there is no path between a and b in $M \setminus e$, the quotient M/e has a natural structure of a simplicial complex with one less maximal edge than M. If, on the contrary, a and b are joined by some path in $M \setminus e$, M is homotopy equivalent to a CW-complex of the form $Z \vee S^1$, where Z is the complex $M \setminus e$ and the S^1 results from attaching a 1-cell by a map that sends both vertices to $a \in Z$. After applying, if needed, finitely many of these moves, we get a CW-complex of the form $L \vee \bigvee_{i=1}^m S^1$ homotopy equivalent to M, where L is a simplicial complex formed by the 2-simplices of M (and hence, with no more 2-simplices than K) in which every edge is the face of at least two 2-simplices. It remains to verify the bound on the Euler characteristic of L. Since $L \vee \bigvee_{i=1}^m S^1$ is homotopy equivalent to M, clearly

$$\chi(M) = \chi\left(L \vee \bigvee_{i=1}^m S^1\right) = \chi(L) - m.$$

On the other hand, by construction $\chi(M) = \overline{\chi}(G) - \dim H_1(T)$, since $\dim H_2(M) = \dim H_2(G)$ and the first homology group of M is isomorphic to $H_1(K_G \vee K_T) = H_1(G) \oplus H_1(T)$. Now, by composing with the homotopy equivalence $L \vee \bigvee_{i=1}^m S^1 \simeq M$ we obtain a map $f : L \vee \bigvee_{i=1}^m S^1 \to K_G \vee K_T$ which induces an isomorphism in H_n for $n = 0$ and 1, and such that $p \circ f_* : H_2(L \vee \bigvee_{i=1}^m S^1) = H_2(L) \to H_2(K_G)$ is an epimorphism. In particular, dualizing we get an isomorphism

$$H^1(K_G \vee K_T) = H^1(G) \times H^1(T) \to H^1\left(L \vee \bigvee_{i=1}^m S^1\right) = H^1(L) \times H^1\left(\bigvee_{i=1}^m S^1\right).$$

Let $(0, a) \in H^1(L) \times H^1(\bigvee_{i=1}^m S^1)$ be a non-trivial class and suppose that $(\alpha, \delta) \in H^1(G) \times H^1(T)$ is the unique class such that $f^*(\alpha, \delta) = (0, a)$. We claim that $\alpha = 0$. Indeed, suppose that it was not the case. Then, since the cohomology ring of G satisfies property (A) there is a class $\beta \in H^1(G)$ with $\alpha \cup \beta \neq 0$. Consider the class $f^*((\alpha, \delta) \cup (\beta, 0)) = f^*(\alpha \cup \beta, 0) \in H^2(L) = H^2(L) \times H^2(\bigvee_{i=1}^m S^1)$. It is non-trivial: take a class $\lambda \in H_2(G)$ such that $(\alpha \cup \beta)\lambda \neq 0$ (here we use the identification

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$H^2(G) = \mathrm{Hom}(H_2(G), \mathbb{F}_2)$. Since M is homologically G-full, there is some class $\gamma \in H_2(L) \equiv H_2(M)$ such that $f_*(\gamma) = (\lambda, \eta)$, for some $\eta \in H_2(T)$. Then,

$$f^*(\alpha \cup \beta, 0)\gamma = (\alpha \cup \beta, 0) f_*(\gamma) = (\alpha \cup \beta, 0)(\lambda, \eta) \neq 0.$$

On the other hand, from the identity

$$f^*((\alpha, \delta) \cup (\beta, 0)) = (0, a) \cup f^*(\beta, 0) = 0$$

we obtain a contradiction, proving the claim. We conclude that the inverse of the map $f^* : H^1(G) \times H^1(T) \to H^1(L) \times H^1(\bigvee_{i=1}^m S^1)$ restricts to a monomorphism $H^1(\bigvee_{i=1}^m S^1) \to H^1(T)$. Hence, $m \leqslant \dim H_1(T)$ and, since $\chi(L) = \overline{\chi}(G) - (\dim H_1(T) - m)$, the result follows. □

3. The lower bound

This section is devoted to the proof of the announced lower bound on the simplicial complexity for groups of the form $G * T$, where G and T are finitely presentable groups and G satisfies property (A).

We begin by fixing some notations (see [**3**, § 2]).

Notation. Let $k \in \mathbb{Z}$, $k \leqslant 2$. We denote by $\rho(k)$ the integer number defined as

$$\rho(k) := \left\lceil \frac{7 + \sqrt{49 - 24k}}{2} \right\rceil.$$

By abuse of notation, if K is a simplicial complex of dimension 2 such that $\chi(K) \leqslant 2$ we will write $\rho(K)$ to mean $\rho(\chi(K))$. Also, we will denote by $\alpha_i(K)$ the number of i-simplices of K.

We prove now a simple result that links the special properties of the simplicial complex L from the statement of lemma 2.5 to a lower bound on its number of 2-simplices $\alpha_2(L)$. In what follows we will understand that a simplicial complex of dimension 2 is of strict dimension 2, i.e. it has at least one 2-simplex.

Lemma 3.1 cf. [**3**, lemma 2.2]. *Let L be a connected simplicial complex of dimension 2 in which every edge is the face of at least two 2-simplices. Then, if $\chi(L) \leqslant 2$, the complex L has at least $\rho(L)$ vertices and at least $2\rho(L) - 2\chi(L)$ 2-simplices.*

Proof. Consider the Euler characteristic formula,

$$\chi(L) = \alpha_0(L) - \alpha_1(L) + \alpha_2(L).$$

Since every edge of L is the face of at least two 2-simplices, we see that $3\alpha_2(L) \geqslant 2\alpha_1(L)$. On the other hand, since L is a simplicial complex it has at most $\binom{\alpha_0(L)}{2}$ edges. Then

$$6\chi(L) \geqslant 6\alpha_0(L) - \alpha_0(L)(\alpha_0(L) - 1).$$

If $\chi(L) \leqslant 0$, the minimum strictly positive integer that satisfies this inequality is precisely $\rho(L) = \rho(\chi(L))$ and therefore $\alpha_0(L) \geqslant \rho(L)$. An easy analysis shows that

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$\alpha_0(L) \geqslant \rho(L)$ also when $\chi(L) = 1, 2$. Finally, the claimed lower bound $\alpha_2(L) \geqslant 2\rho(L) - 2\chi(L)$ follows immediately from the Euler characteristic formula and the inequalities $3\alpha_2(L) \geqslant 2\alpha_1(L)$, $\alpha_0(L) \geqslant \rho(L)$. □

We are now ready to prove the main result of this section.

THEOREM 3.2. *Let G and T be finitely presented groups. If G satisfies property (A), $\overline{\chi}(G) \leqslant 2$, and $\dim H_2(G) > 0$, then $\kappa(G * T) \geqslant 2\rho(\overline{\chi}(G)) - 2\overline{\chi}(G)$.*

Proof. Let K be a simplicial complex of dimension 2 with fundamental group isomorphic to $G * T$. Since G satisfies property (A), from lemma 2.5 we obtain a simplicial complex L with $\alpha_2(L) \leqslant \alpha_2(K)$, $\chi(L) \leqslant \overline{\chi}(G)$ and such that every edge of L is in at least two 2-simplices. Furthermore, there is an epimorphism $H_2(L) \to H_2(G)$, so that $\dim H_2(L) > 0$ and hence L is of dimension 2. By lemma 3.1, L has at least $2\rho(L) - 2\chi(L)$ 2-simplices. Now, since $\chi(L) \leqslant \overline{\chi}(G)$ and ρ is a non-increasing function, we conclude that $\alpha_2(L) \geqslant 2\rho(\overline{\chi}(G)) - 2\overline{\chi}(G)$, as desired. □

We may apply theorem 3.2 to the one-relator groups from example 2.2. For instance, the theorem gives the lower bound $\kappa(BS(m, n)) \geqslant 14$ for Baumslag–Solitar groups with m, n odd since $\chi(BS(m, n)) = 0$. We know that this bound is not sharp except for the fundamental group of the torus $\mathbb{Z} \oplus \mathbb{Z} = BS(1, 1)$. But one would expect stronger lower bounds for one-relator groups with a large number of generators (and hence, small Euler characteristic). As we will see in the next section, the lower bound from theorem 3.2 is sharp for fundamental groups of surfaces.

4. The simplicial complexity of surface groups

In this section we derive the main results of the article. Recall that the number of 2-simplices in a minimal triangulation of a closed surface was computed by Ringel [**9**], in the non-orientable case, and by Jungerman and Ringel [**5**], in the orientable case.

THEOREM 4.1 Jungerman and Ringel. *Let S be a closed surface different from the orientable surface of genus 2 (M_2), the Klein bottle (N_2) and the non-orientable surface of genus 3 (N_3). Then, we have the following formula for the number $\delta(S)$ of 2-simplices in a minimal triangulation of S :*

$$\delta(S) = 2\rho(\chi(S)) - 2\chi(S).$$

For the exceptional cases M_2, N_2, and N_3, it is necessary to replace $\delta(S)$ by $\delta(S) - 2$ in the formula.

As it was observed in example 2.2, the fundamental group of a non-simply connected closed surface S satisfies property (A). Hence, we may apply theorem 3.2 to groups of the form $\pi_1(S) * T$ obtaining the following corollary.

PROPOSITION 4.2. *Let S be a non-simply connected closed surface. Then $\kappa(\pi_1(S) * T) \geqslant 2\rho(\chi(S)) - 2\chi(S)$. In particular, if S is non-exceptional, then $\kappa(\pi_1(S) * T) \geqslant \delta(S)$.*

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Proof. By theorem 3.2, we have that $\kappa(\pi_1(S) * T) \geqslant 2\rho(\overline{\chi}(\pi_1(S))) - \overline{\chi}(\pi_1(S))$. Hence, it is enough to see that $\overline{\chi}(\pi_1(S)) = \chi(S)$ for all non-simply connected closed surfaces S. This identity is clear for surfaces S different from the real projective plane $\mathbb{R}P^2$ since these surfaces are aspherical. For $\mathbb{R}P^2$, notice that the infinite real projective space $\mathbb{R}P^\infty$ is an Eilenberg–MacLane space for $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$, so we have $\overline{\chi}(\pi_1(S)) = \chi(S)$ also for $S = \mathbb{R}P^2$. □

It remains to handle the exceptional cases. Observe that for an exceptional surface S (i.e. $S = N_2, N_3$ or M_2), proposition 4.2 provides the lower bound $\kappa(\pi_1(S) * T) \geqslant 2\rho(\chi(S)) - 2\chi(S)$, which is slightly weaker than required because $\delta(S) = 2\rho(\chi(S)) - 2\chi(S) + 2$ in these cases. So, for the exceptional surfaces, we will need to refine the proof of the lower bound of theorem 3.2.

Lemma 4.3. *Let S be an aspherical closed surface (either exceptional or non-exceptional) and let K be a connected simplicial complex of dimension 2 with fundamental group isomorphic to $\pi_1(S) * T$. Let L be the simplicial complex obtained from K by applying lemma 2.5. If $\chi(L) = \chi(S)$, then there is a continuous map $L \to S$ that induces an isomorphism in (co)homology.*

Proof. From the proof of lemma 2.5 applied to K, we obtain a continuous map $f : L \vee \bigvee_{i=1}^m S^1 \to K_{\pi_1(S)} \vee K_T \simeq S \vee K_T$ (since S is aspherical) which induces an isomorphism in H_n for $n = 0$ and 1 and an epimorphism $p \circ f_* : H_2(L \vee \bigvee_{i=1}^m S^1) = H_2(L) \to H_2(S)$, and such that $\dim H_2(L) = \dim H_2(S)$. Consider the natural map $g : L \to S$ defined as the composition $L \hookrightarrow L \vee \bigvee_{i=1}^m S^1 \xrightarrow{f} S \vee K_T \to S$, where the first map is the inclusion and the last one is the projection to the quotient. Since the quotient map $S \vee K_T \to S$ induces the projection $H_*(S) \oplus H_*(K_T) \to H_*(S)$ in homology, g induces an isomorphism in H_2 and so it suffices to show $g_* : H_1(L) \to H_1(S)$ is an isomorphism. Note that from the proof of lemma 2.5 it follows that the inverse of $f^* : H^1(K_{\pi_1(S)}) \times H^1(K_T) \to H^1(L) \times H^1(\bigvee_{i=1}^m S^1)$ restricts to a monomorphism $h : H^1(\bigvee_{i=1}^m S^1) \to H^1(K_T)$. More concretely, the monomorphism h sends a class $a \in H^1(\bigvee_{i=1}^m S^1)$ to the unique class $\alpha \in H^1(K_T)$ such that $f^*(0, \alpha) = (0, a)$. On the other hand, $\chi(L) = \chi(S) - (\dim H_1(K_T) - m)$ and since $\chi(L) = \chi(S)$ by assumption, $m = \dim H_1(K_T)$ and so h is an isomorphism. It is clear then that also f^* restricts in an analogous way to an isomorphism $H^1(K_T) \to H^1(\bigvee_{i=1}^m S^1)$. Now, the morphism induced by g between the first cohomology groups coincides with the composition $H^1(S) \hookrightarrow H^1(S) \times H^1(K_T) \xrightarrow{f^*} H^1(L) \times H^1(\bigvee_{i=1}^m S^1) \to H^1(L)$, the first arrow being the inclusion and the last one the projection on the first coordinate. Using the fact that f^* restricts to an isomorphism between the second factors, it is not difficult to check that the described map between the first factors $H^1(S) \to H^1(L)$ is an isomorphism. Since $H^1(L)$ and $H^1(S)$ are vector spaces of finite dimension over $\mathbb{F}_2$, we conclude that g induces an isomorphism in (co)homology as desired. □

The next result, which was proved in [**3**], states roughly that if the complex L obtained from lemma 2.5 satisfies the hypothesis of lemma 4.3, it is close to being homeomorphic to a surface.

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PROPOSITION 4.4 see [**3**, proposition 3.1]. *Let K be a simplicial complex of dimension 2 such that each edge of K is the face of exactly two 2-simplices and let S be a closed surface. Suppose that there is a continuous map $K \to S$ inducing isomorphisms in all homology groups. Then K is homeomorphic to S.*

PROPOSITION 4.5. *Let $S = N_2, N_3$ or M_2. Then $\kappa(\pi_1(S) * T) \geqslant \delta(S)$.*

Proof. Let K be a simplicial complex with fundamental group isomorphic to $\pi_1(S) * T$. From lemma 2.5, and keeping the notations of the proof of theorem 3.2, we obtain a complex L with $\alpha_2(L) \leqslant \alpha_2(K)$, $\chi(L) \leqslant \overline{\chi}(\pi_1(S)) = \chi(S)$ and such that every edge of L is in at least two 2-simplices. By lemma 3.1, this implies that L has at least $\rho(L) \geqslant \rho(S)$ vertices and at least $2\rho(L) - 2\chi(L)$ 2-simplices. Note that if any of the strict inequalities $\alpha_0(L) > \rho(S)$, $\chi(L) < \chi(S)$ holds, we have

$$\alpha_2(L) \geqslant 2\rho(S) - 2\chi(S) + 2 = \delta(S)$$

and there is nothing to prove. In view of this, in what follows we will suppose that $\alpha_0(L) = \rho(S)$ and $\chi(L) = \chi(S)$. Observe that the homology of L is isomorphic to the homology of S via a continuous map $L \to S$ by lemma 4.3. Also, since $3\alpha_2(L) \geqslant 2\alpha_1(L)$, by the Euler characteristic formula for L we have

$$3(\alpha_0(L) - \chi(L)) \leqslant \alpha_1(L) \leqslant \binom{\alpha_0(L)}{2}.$$

We solve first the case $S = N_2$. By our assumption, we have that $\chi(L) = \chi(N_2) = 0$ and $\alpha_0(L) = \rho(N_2) = 7$. Hence, from the above inequality we learn that $\alpha_1(L) = 21$ and, since $\chi(L) = 0$, $\alpha_2(L) = 14$. Thus $3\alpha_2(L) = 2\alpha_1(L) = 42$, from where it follows that every edge of L is the face of exactly two 2-simplices. Since there is a map $L \to S$ inducing an isomorphism in homology, by proposition 4.4 L would be homeomorphic to N_2 contradicting theorem 4.1. Hence, $\alpha_0(L) > \rho(N_2)$ or $\chi(L) < \chi(S)$ and consequently $\alpha_2(L) \geqslant \delta(N_2)$.

For the surface $S = N_3$, we know that $\chi(L) = \chi(N_3) = -1$ and $\alpha_0(L) = \rho(N_3) = 8$. Hence,

$$3(\alpha_0(L) - \chi(L)) = 27 \leqslant \alpha_1(L) \leqslant 28 = \binom{\alpha_0(L)}{2}.$$

Suppose first that $\alpha_1(L) = 27$, so that $\alpha_2(L) = 18$. Hence every edge of L is the face of exactly two 2-simplices and from proposition 4.4, L is homeomorphic to N_3 in contradiction to theorem 4.1. Then, $\alpha_1(L) = 28$. In that case, $\alpha_2(L) = 19$ and since $57 = 3\alpha_2(L) = 2\alpha_1(L) + 1$, every edge of L is in two 2-simplices except for one that is the face of three 2-simplices of L. The link of a vertex of this edge is a graph in which every vertex has degree two except for one that has degree three. This is impossible because the sum of the degrees of an undirected graph is even. Therefore, $\alpha_0(L) > \rho(N_3)$ or $\chi(L) < \chi(N_3)$ and hence $\alpha_2(L) \geqslant \delta(N_3)$ as claimed.

Finally, when $S = M_2$ we have that $\chi(L) = \chi(M_2) = -2$ and $\alpha_0(L) = \rho(M_2) = 9$. In this case, we know that $\alpha_2(L) \geqslant 22 = \delta(M_2) - 2$ and we want to show that L has at least $\delta(M_2) = 24$ 2-simplices. We will see that the cases $\alpha_2(L) = 22$, $\alpha_2(L) = 23$ are not possible. Suppose first that $\alpha_2(L) = 22$. Then, by the Euler characteristic

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formula, $\alpha_1(L) = 33$. Therefore, every edge of L is the face of exactly two 2-simplices of L and so, by proposition 4.4 L should be homeomorphic to M_2, which contradicts theorem 4.1. If $\alpha_2(L) = 23$, it is $\alpha_1(L) = 34$, whence $69 = 3\alpha_2(L) = 2\alpha_1(L) + 1$. It follows that every edge of L is the face of exactly two 2-simplices except for one which is the face of three 2-simplices. The same argument as before shows that this is impossible. We conclude that $\alpha_2(L) \geqslant \delta(M_2)$. □

We obtain theorems 1.1 and 1.2 as corollaries of the previous propositions.

Proof of theorem 1.1. The upper bound $\kappa(\pi_1(S)) \leqslant \delta(S)$ is clear, while the lower bound follows from propositions 4.2 and 4.5. □

Note that, as a consequence of this result, the simplicial complexity of surface groups grows linearly on the genus. This was observed, in the orientable case, in [**2**, example 2].

Proof of theorem 1.2. Let T be a finitely presentable group. By propositions 4.2 and 4.5, $\kappa(\pi_1(S) * T) \geqslant \delta(S)$ and since $\kappa(\pi_1(S)) = \delta(S)$ by theorem 1.1, the first claim holds. For the second one, it is enough to observe that the upper bound $\kappa(G * \mathbb{Z}) \leqslant \kappa(G)$ holds trivially for every finitely presentable group G. □

Arguably, the conclusions of theorems 1.1 and 1.2 should hold at least for finitely presentable groups G satisfying property (A) in place of surface groups. Unfortunately, we have not been able to establish these results in the general case.

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