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Semi-focusing billiards: ergodicity

LEONID A. BUNIMOVICH† and GIANLUIGI DEL MAGNO‡

† *ABC Math Program and School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA*
(e-mail: bunimovh@math.gatech.edu)

‡ *Max Planck Institute for the Physics of Complex Systems, 01187 Dresden, Germany*
(e-mail: delmagno@mpipks-dresden.mpg.de)

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Abstract. In Bunimovich and Del Magno [Semi-focusing billiards: hyperbolicity. *Comm. Math. Phys.* **262** (2006), 17–32], we proved that billiards in certain three-dimensional convex domains are hyperbolic. In this paper, we continue the study of these systems, and prove that they enjoy the Bernoulli property. This result answers affirmatively a long-standing question on the existence of ergodic billiards in convex domains in dimensions greater than two. Besides, it shows that the chaotic components of the first rigorously investigated three-dimensional billiards with mixed phase space (mushroom billiards), introduced in Bunimovich and Del Magno, are in fact Bernoulli.

1. *Introduction*

Right after the discovery that the defocusing mechanism can produce chaotic behavior (hyperbolicity) in dimension two, doubts were raised as to whether this mechanism can generate chaotic behavior in higher dimensions as well. These doubts were based upon a fundamental optical phenomenon called astigmatism, which can be described as follows. Consider a family of rays lying in the same plane and being reflected by a curved mirror. Astigmatism occurs when the speed of focusing of the family of rays depends on the plane containing the rays; this speed could, in principle, be so slow as to disable the defocusing mechanism. Despite this complication, it was shown in [**BR1, BR2, BR3**] that the defocusing mechanism does work in any dimension. In those papers, hyperbolic billiards were constructed in every dimension by using spherical caps attached to the faces of a box. However, one does pay a price to the astigmatism: the spherical caps cannot be too big in dimension greater than two (whereas in dimension two, any arc arbitrarily close to an entire circle is allowed [**B1, B2**]).

The classes of high-dimensional ergodic nowhere dispersing billiards constructed in [**BR1, BR2, BR3**] do not contain convex billiard domains. Therefore the question of existence of convex ergodic billiards in dimensions greater than two remained open. Some specially shaped convex domains in dimension three were introduced and numerically investigated in [**P2**]. This study was motivated by a mechanical model of nuclei, consisting of point particles interacting via an attractive potential, which can be reduced to a billiard in a domain bounded by hyperplanes and focusing hypersurfaces in dimension greater than two [**P1**]. Such billiards could be considered as a natural counterpart of the semi-dispersing billiards generated by systems of particles interacting via a repulsing potential. This prompted us to introduce in [**BD**] a new class of billiards that we called semi-focusing. Billiards are semi-focusing if their boundary components consist of pieces of cylinders and possibly some flat components. It was shown in [**BD**] that semi-focusing billiards under certain conditions have non-vanishing Lyapunov exponents almost everywhere with respect to the Liouville measure, i.e. they are non-uniformly hyperbolic.

The purpose of this paper is to take a further step in the study of semi-focusing billiards by examining their main ergodic properties. Rather than considering the general case, which would necessarily force us to deal with many technical details, we restrict ourselves to the case (seemingly the simplest to treat) of a cylindrical semi-focusing domain which consists of a rectangular box and two orthogonal half-cylinders with their rectangular cross-sections attached to a pair of opposite faces of the box. This is the domain considered in [**P2**] and depicted in Figure 1. To avoid introducing further terminology, the expression 'cylindrical billiards' will be used throughout this paper to designate only billiards in these special domains. The main result of this paper is that if the distance between the cylinders is sufficiently large (which is more than required for hyperbolicity [**BD**]), then a cylindrical billiard is Bernoulli. In particular, it is mixing and ergodic. Our proof can be readily extended to a broader class of cylindrical billiards, and very likely to other high-dimensional billiards (see §5).

Even though we do not deal with the general case, the style of our exposition is nevertheless quite technical. This should not be surprising, because proving the Bernoulli property for non-uniformly hyperbolic systems, and particularly for billiards, is always rather involved. To obtain our results, we will re-prove in a more general setting many technical results established earlier for semi-dispersing hyperbolic billiards. This will often be done by concentrating only on those parts of the original proofs that need to be modified, rather than rewriting the entire proofs; in particular, we will constantly refer to the papers [**KSS, BCST1**]. Our decision not to show complete proofs makes this paper not self-contained, but reduces considerably its length.

The paper is organized as follows. Section 2 contains some background material on billiards. Definitions concerning cylindrical billiards and results from [**BD**] are recalled in §3, in which we also formulate the main result of the paper, Theorem 3.4. Its proof is given in §4, which consists of four subsections. Sections 4.1 and 4.2 contain further definitions and some preliminary results. In §4.3, we demonstrate that the Fundamental theorem of [**BCST1**], proved for semi-dispersing billiards, extends to cylindrical semi-focusing billiards. As a corollary of this theorem, we obtain a Local Ergodic Theorem for cylindrical semi-focusing billiards. Then, in §4.4, using the Local Ergodic Theorem, we prove that the cylindrical semi-focusing billiards are Bernoulli. One of the hypotheses of the Fundamental theorem is that the singular sets of the billiard map are Lipschitz decomposable. This property together with other properties concerning the regularity of the singular sets are proved in the Appendices, where we also establish the existence of local stable and unstable manifolds for three-dimensional cylindrical semi-focusing billiards by making use of the general theory of hyperbolic systems with singularities [**KS**].

2. *General definitions*

Let Q be a bounded domain, with piecewise C^3-differentiable boundary, of the Euclidean space $\mathbb{R}^k$ for $k \geq 2$. More precisely, we assume that $\partial Q = \bigcup_{i=1}^n \Gamma_i$ for some $n > 0$, where each Γ_i is a compact connected subset of $f_i^{-1}(0)$ for a proper C^3 function $f_i : \mathbb{R}^k \to \mathbb{R}$ that has 0 among its regular values. Furthermore, we assume that the boundary of $\partial\Gamma_i$ consists of finitely many smooth curves intersecting only at their boundaries. The sets Γ_i are called *boundary components* of Q.

The *billiard in* Q is the dynamical system associated to the mechanical system which consists of a point-particle moving inside Q along straight lines at unit speed. When the particle hits ∂Q, it gets reflected elastically so that the angle of incidence equals the angle of reflection.

In the rest of this section, we will make this definition formally precise.

2.1. *Phase space.*

Let $\mathcal{T}\mathbb{R}^k$ be the tangent bundle of $\mathbb{R}^k$, and denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the standard dot product and norm of $\mathbb{R}^k$, respectively.

For every $1 \leq i \leq n$, let us denote by $N_i(q)$ the unit normal of Γ_i at $q \in \Gamma_i$ directed toward the inner part of Q. The collection of all unit vectors of $\mathbb{R}^k$ attached to Γ_i and pointing inside Q, namely

$$\Sigma_i = \{(q, v) \in \mathcal{T}\mathbb{R}^k \mid q \in \Gamma_i, \langle v, v \rangle = 1 \text{ and } \langle N_i(q), v \rangle \geq 0\},$$

forms the billiard phase space over Γ_i. The full billiard phase space Σ is therefore given by

$$\Sigma = \bigcup_{i=1}^{n} \Sigma_i.$$

Clearly, each set Σ_i is a compact submanifold of $\mathcal{T}\mathbb{R}^k$, and inherits a Riemannian metric from the standard Riemannian metric on $\mathcal{T}\mathbb{R}^k$. Using such a metric, we can easily obtain by a standard procedure a distance ρ on the entire phase space Σ.

Let

$$\mathcal{R}_1 = \bigcup_{i=1}^{n} \{(q, v) \in \Gamma_i \mid \langle N_i(q), v \rangle = 0\},$$

$$\mathcal{R}_2 = \bigcup_{i=1}^{n} \{(q, v) \in \Gamma_i \mid q \in \partial\Gamma_i\}.$$

Owing to their geometric meaning, the elements of $\mathcal{R}_1$ and $\mathcal{R}_2$ are called *tangential collisions* and *multiple collisions*, respectively. The elements of $\mathcal{R}_1$ and $\mathcal{R}_2$ are also generically called *singular collisions*. These sets play an important role in the study of billiards, because they generate the singularities of the billiard dynamics. Finally, we define

$$\partial\Sigma = \mathcal{R}_1 \cup \mathcal{R}_2 \quad \text{and} \quad \text{int}\,\Sigma = \Sigma \setminus \partial\Sigma.$$

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It is easy to see that $\partial\Sigma$ is a finite union of compact submanifolds with boundary, and int Σ is a $(2k-1)$-dimensional manifold without boundary.

2.2. *Billiard map.* For every $(q, v) \in \text{int}\,\Sigma$, define

$$t(q, v) = \inf\{\tau > 0 \mid q + \tau v \in \partial Q\},$$
$$q_1(q, v) = q + t(q, v)v.$$

In general, these transformations are not C^3-differentiable everywhere on int Σ: the subset of int Σ where they fail to be of class C^3 is contained in the union of $\mathcal{R}_3$ and $\mathcal{R}_4$, where

$$\mathcal{R}_3 = \left\{(q, v) \in \text{int}\,\Sigma \mid q_1(q, v) \in \bigcup_{i=1}^{n} \partial\Gamma_i\right\}$$

is the set of elements of int Σ that q_1 sends into the intersection of any two boundary components of ∂Q, and

$$\mathcal{R}_4 = \{(q, v) \in \text{int}\,\Sigma \mid (q_1(q, v), v) \in \mathcal{R}_1\}$$

is the set of elements of int Σ such that the segment with endpoints q and $q_1(q, v)$ is tangent to ∂Q. Note that for a semi-focusing billiard table Q, the set $\mathcal{R}_4$ is empty. Denote by $N(q)$ the unit normal of ∂Q at $q \in \bigcup_i^n \text{int}\,\Gamma_i$ directed toward the inner part of Q. Then for every $(q, v) \in \text{int}\,\Sigma \setminus \mathcal{R}_3$, define

$$v_1(q, v) = v - 2\langle N(q_1(q, v)), v\rangle N(q_1(q, v)).$$

This gives exactly the velocity of the particle after a collision with ∂Q.

Let $\mathcal{R} = \bigcup_{i=1}^4 \mathcal{R}_i$ and $\Sigma' = \Sigma \setminus \mathcal{R}$. Since $\mathcal{R}$ is compact, the set Σ' is a manifold without boundary. The *billiard map T for the domain Q* is the transformation given by

$$T(q, v) = (q_1(q, v), v_1(q, v))$$

for every $(q, v) \in \Sigma'$. The set $\mathcal{R}$ is called the *singular set of T*. From previous considerations on the regularity of q_1, it follows that $T : \Sigma' \to T\Sigma'$ is a C^2 diffeomorphism. Another important property of T is that it preserves the probability measure of Σ whose restriction to each submanifold Σ_i is given by (see [**CFS, CM**])

$$d\mu = c|\langle v, N_i(q)\rangle|\, dq\, dv,$$

where dq and dv denote the Lebesgue measure of Γ_i and the Lebesgue measure of the unit sphere of $\mathbb{R}^k$, respectively. The constant c is a proper normalizing factor. Unless otherwise stated, we will always be using this measure in this paper.

2.3. *Singular sets.* The billiard dynamics is time-reversible. This means that $\mathcal{J}T = T^{-1}\mathcal{J}$, where $\mathcal{J}$ is the involution given by

$$\mathcal{J}(q, v) = (q, -v + 2\langle N(q), v\rangle N(q)) \quad \text{for } (q, v) \in \text{int}\,\Sigma.$$

Most of the time, we will write $-x$ in place of $\mathcal{J}(x)$. Accordingly, if $A \subset \text{int}\,\Sigma$, then $-A$ will denote the set $\{-x \mid x \in A\}$.

Let $\mathcal{S} = \mathcal{R}_3 \cup \mathcal{R}_4$. For every integer $m > 0$, let us define

$$\mathcal{S}_m^+ = \bigcup_{i=0}^{m-1} T^{-i}\mathcal{S} \quad \text{and} \quad \mathcal{S}_m^- = -\mathcal{S}_m^+.$$

The sets $\mathcal{S}_m^+$ and $\mathcal{S}_m^-$ consist of elements of int Σ which have a singular collision within the first m consecutive collisions with ∂Q in the future and in the past, respectively. Finally, set

$$\mathcal{S}_\infty^+ = \bigcup_{m\geq 1} \mathcal{S}_m^+ \quad \text{and} \quad \mathcal{S}_\infty^- = -\mathcal{S}_\infty^+.$$

Remark 2.1. Our definition of singular sets is slightly different from that of [**BCST1, KSS**]: the singular sets in those papers correspond to the closure of our singular sets.

2.4. *Cylindrical semi-focusing billiards.* A domain Q is called *semi-focusing* if its boundary components are subsets of convex hypersurfaces or hyperplanes. A billiard in such a domain is called semi-focusing as well. In this paper, we continue the study of semi-focusing billiards started in [**BD**]. These are billiards in domains obtained by properly gluing together three-dimensional domains that are direct products of planar semi-focusing domains and straight segments. More precisely, consider two planar semi-focusing domains B_1 and B_2 such that ∂B_i contains at least one straight segment I_i for each $i = 1, 2$. We require the curved components of ∂B_1 and ∂B_2 to be *absolutely focusing* [**B3, D, B4**]. These curves form a large family of focusing curves that can be used to construct hyperbolic billiards. Arcs of circles, their small perturbations (in the C^6 topology) and certain pieces of ellipses are examples of absolutely focusing curves. Let Q_1 be the direct product of B_1 and I_2, and similarly let Q_2 be the direct product of B_2 and I_1. We obtain three-dimensional convex domains that are unions of ‘half-cylinders’ and a rectangular box. If $F_1 \subset \partial Q_1$ and $F_2 \subset \partial Q_2$ are the rectangular faces containing I_1 and I_2, respectively, then we glue Q_1 and Q_2 together along F_1 and F_2 in such a way that the sides of F_1 and F_2 parallel to I_1 are identified. This can be done in two distinct ways; either way is fine for our purposes. The resulting domain Q is called *cylindrical semi-focusing*.

Basic assumption. In this paper, we are concerned with cylindrical semi-focusing billiards in $\mathbb{R}^3$ such that B_1 and B_2 are both C^1 half-stadia. In this case, Q is as in Figure 1. We further assume that the convex curve in the boundary of the half-stadia is a semi-circle. From now on, the expression *cylindrical semi-focusing billiards* (*cylindrical billiards* for short) will refer uniquely to these billiards.

3. *Main results*

In this section, we define some notation concerning cylindrical billiards, then we recall the results of [**BD**] and finally state the main results of this paper.

3.1. *Notation.* Denote by ∂Q_+ and ∂Q_0 the union of the cylindrical components and the union of the flat faces of ∂Q, respectively. Then define

$$\Sigma_+ = \{(q, v) \in \Sigma \mid q \in \partial Q_+\} \quad \text{and} \quad \text{int}\,\Sigma_+ = \text{int}\,\Sigma \cap \Sigma_+.$$

Similarly, define Σ_0 and int Σ_0.

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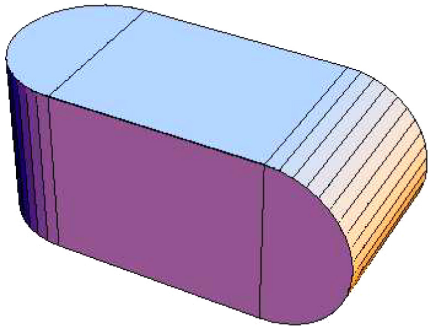

FIGURE 1. A three-dimensional cylindrical semi-focusing billiard.

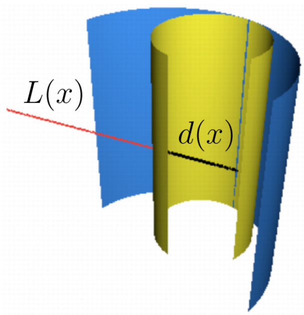

FIGURE 2. The dark curved sheet represents the boundary ∂Q_+ and the light curved sheet represents the surface of the solid cylinder $\Delta(x)$.

Given $x = (q, v) \in \operatorname{int} \Sigma_+$, let $r(x)$ be the radius of the section of the cylinder where x is attached. Also, denote by $L(x)$ the ray of $\mathbb{R}^3$ emerging from q in the direction of v, and by $\Delta(x)$ the solid cylinder tangent to ∂Q_+ at q such that the radius of its section equals $r(x)/2$. Then define $d(x)$ to be the length of the segment $\Delta(x) \cap L(x)$ (see Figure 2).

For each $x = (q, v) \in \operatorname{int} \Sigma$, let $\mathbb{V}_x$ be the plane of $\mathbb{R}^3$ passing through q and orthogonal to v, and let P_v be the orthogonal projection of $\mathcal{T}_x \partial Q$ onto $\mathbb{V}_x$. The isomorphism

$$\mathcal{T}_x \Sigma \ni \begin{pmatrix} \delta q \\ \delta v \end{pmatrix} \mapsto \begin{pmatrix} P_v \delta q \\ \delta v \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{V}_x \times \mathbb{V}_x \tag{1}$$

identifies $\mathcal{T}_x \operatorname{int} \Sigma$ with $\mathbb{V}_x \times \mathbb{V}_x$. Using this identification, we can write $u = (\xi, \eta) \in \mathcal{T}_x \operatorname{int} \Sigma$, and see that for $x \notin \mathcal{S}_1^+$, the matrix of $D_x T u$ takes the form

$$D_x T \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi + t(x)\eta \\ R(Tx)(\xi + t(x)\eta) + \eta \end{pmatrix}, \tag{2}$$

where $R(Tx) = 2\langle N(q_1(x)), v\rangle P^* K P$, P^* is the projection of $\mathcal{T}_{q_1(x)} \partial Q$ onto $\mathbb{V}_x$ along $N(q_1(x))$, and K is the second fundamental form of ∂Q evaluated at $q_1(x)$ [**BD**]. The operator $R(Tx)$ is self-adjoint, and an easy computation shows that its eigenvalues are 0 and either $-2/d(Tx)$ if $x \in \operatorname{int} \Sigma_+$, or 0 if $x \in \operatorname{int} \Sigma_0$ [**BD**].

Since $\operatorname{int} \Sigma$ is a submanifold of $\mathcal{T}\mathbb{R}^3$, it is naturally equipped with the Riemannian metric g induced by the standard Riemannian metric of $\mathcal{T}\mathbb{R}^3$. We will find it useful to equip int Σ with another Riemannian metric g' defined as follows. Let $x \in \text{int}\,\Sigma$, and take any two vectors $u_1, u_2 \in \mathcal{T}_x \text{ int}\,\Sigma$. In virtue of the identification $\mathcal{T}_x \text{ int}\,\Sigma \simeq \mathbb{V}_x \times \mathbb{V}_x$, we write $u_i = (\xi_i, \eta_i)$ with $\xi_i, \eta_i \in \mathbb{V}_x$ for each $i = 1, 2$. Then the new dot product $\langle \cdot, \cdot \rangle'$ on $\mathcal{T}_x$ int Σ is defined by

$$\langle u_1, u_2 \rangle' = \langle \xi_1, \xi_2 \rangle + \langle \eta_1, \eta_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard dot product in $\mathbb{R}^3$. Also, denote by $\| \cdot \|'$ the norm associated to $\langle \cdot, \cdot \rangle'$. Finally, let

$$\mathcal{T}_x^{(1)} \text{ int}\,\Sigma = \{u \in \mathcal{T}_x \text{ int}\,\Sigma \mid \|u\|' = 1\}$$

be the unit tangent space of int Σ at x with respect to the metric g'.

3.2. *Quadratic form and cone field.*

Definition 3.1. Let Σ^* be the collection of all $x \in \text{int}\,\Sigma$ such that $T^{-i}x$ is defined and belongs to int Σ_+ for some integer $i \geq 0$.

Clearly Σ^* is open, and $\text{int}\,\Sigma \setminus \Sigma^* \subset \mathcal{S}_\infty^-$. For each $x \in \Sigma^*$, let $m(x)$ be the smallest integer $i \geq 0$ for which $T^{-i}x \in \text{int}\,\Sigma_+$. Then let $d_-(x) = d(T^{-m(x)}x)$ and define $l_-(x)$ to be the length of the billiard orbit $\{T^{-m(x)}x, \ldots, x\}$; that is,

$$l_-(x) = \begin{cases} \displaystyle\sum_{i=0}^{m(x)-1} t(T^{-m(x)+i}x) & \text{if } m(x) > 0, \\ 0 & \text{if } m(x) = 0. \end{cases}$$

We now introduce two important tools used in this paper: the quadratic form $\mathcal{Q}$ and its associated cone field $\mathcal{C}$. The (indefinite) quadratic form $\mathcal{Q}$ at the point $x \in \Sigma^*$ is given by

$$\mathcal{Q}_x(u) = \langle \xi, \eta \rangle + (d_-(x) - l_-(x))\|\eta\|^2 \quad \text{for all } u \in \mathcal{T}_x \text{ int}\,\Sigma. \tag{3}$$

The cone field $\mathcal{C} = \{\mathcal{C}(x)\}_{x \in \Sigma^*}$ determined by $\mathcal{Q}$ is then defined by

$$\mathcal{C}(x) = \mathcal{Q}_x^{-1}([0, +\infty)) \quad \text{for all } x \in \Sigma^*.$$

The interior of $\mathcal{C}(x)$ is given by $\text{int}\,\mathcal{C}(x) = \mathcal{Q}_x^{-1}((0, +\infty))$. Similarly, we define the complementary cone field of $\mathcal{C}$ and its interior to be

$$\mathcal{C}'(x) = \mathcal{Q}_x^{-1}((-\infty, 0]) \quad \text{and} \quad \text{int}\,\mathcal{C}'(x) = \mathcal{Q}_x^{-1}((-\infty, 0)).$$

Let U be a chart of int Σ. A convenient topology on the space of cones $\{\mathcal{C}(x)\}_{x \in U \cap \Sigma^*}$ is obtained as follows. Since $\mathbb{V}_x \simeq \mathbb{R}^2$, we have $\mathcal{T}_x \text{ int}\,\Sigma \simeq \mathbb{R}^4$, therefore $\mathcal{C}(x)$ can be identified with a cone of $\mathbb{R}^4$. Intersecting each $\mathcal{C}(x)$ with the unit sphere of $\mathcal{T}_x$ int Σ, we then obtain a family of compact subsets of $\mathbb{R}^4$. To finish, we endow this family of sets with the Hausdorff topology.

Remark 3.2. It is easy to see that the quadratic form $\mathcal{Q}$ is continuous on its domain of definition Σ^*, and so is $\mathcal{C}$ in the topology introduced above.

The quadratic form $\mathcal{Q}$ is called *eventually strictly monotone* if it satisfies the following properties:

- for almost every $x \in \operatorname{int} \Sigma$, we have $\mathcal{Q}_{Tx}(D_x T u) \geq \mathcal{Q}_x(u)$ for all $u \in \mathcal{T}_x \operatorname{int} \Sigma$;
- for almost every $x \in \operatorname{int} \Sigma$, there exists an integer $i = i(x) > 0$ such that $\mathcal{Q}_{T^i x}(D_x T^i u) > \mathcal{Q}_x(u)$ and $\mathcal{Q}_{T^{-i} x}(D_x T^{-i} u) < \mathcal{Q}_x(u)$ for all $u \in \mathcal{T}_x \operatorname{int} \Sigma \setminus \{0\}$.

A quadratic form satisfying the first property is called *monotone*, and a monotone quadratic form $\mathcal{Q}$ such that $\mathcal{Q}_{T^i x}(D_x T^i u) > \mathcal{Q}_x(u)$ for all $u \in \mathcal{T}_x \operatorname{int} \Sigma \setminus \{0\}$ is called *strictly monotone along the orbit* $\{x, \ldots, T^i x\}$.

If $\mathcal{Q}$ is eventually strictly invariant, then we see that the associated cone field $\mathcal{C}$ has the following properties:

- $D_x T \mathcal{C}(x) \subset \mathcal{C}(Tx)$ for almost every $x \in \operatorname{int} \Sigma$;
- for almost every $x \in \operatorname{int} \Sigma$, there exists an integer $i = i(x) > 0$ such that $D_x T^i \mathcal{C}(x) \subset \operatorname{int} \mathcal{C}(T^i x) \cup \{0\}$ and $D_x T^{-i} \mathcal{C}'(x) \subset \operatorname{int} \mathcal{C}'(T^{-i} x) \cup \{0\}$.

Such a cone field is called *eventually strictly invariant*. If a cone field has only the first property, then it is called *invariant*; and if an invariant cone field $\mathcal{C}$ is such that $D_x T^i \mathcal{C}(x) \subset \operatorname{int} \mathcal{C}(T^i x) \cup \{0\}$, then it is called *strictly invariant along the orbit* $\{x, \ldots, T^i x\}$.

3.3. *Hyperbolicity.* Given a cylindrical semi-focusing domain Q, let r_1 and r_2 be the radii of the two half-cylinders of ∂Q, and let b be the length of the rectangular box along the direction perpendicular to the axes of both half-cylinders.

In [**BD**], we proved that the quadratic form $\mathcal{Q}$ defined in (3) is eventually strictly monotone if

$$b > r_1 + r_2. \tag{H}$$

This condition simply means that the full cylinders of ∂Q do not intersect. In fact, we proved that the strict monotonicity of $\mathcal{Q}$ (and therefore the strict invariance of $\mathcal{C}$) is attained along every orbit that crosses Q from one cylinder to the other at least twice. More precisely, if $\{x, \ldots, T^i x\}$ is defined for some $i > 0$, and $\pi(x)$, $\pi(T^i x)$ belong to the same cylinder while $T^j x$ belongs to the other cylinder for some $0 < j < i$, then $\mathcal{Q}$ is strictly invariant along $\{x, \ldots, T^i x\}$.

Remark 3.3. By general results [**W1, M1, KB**], the eventual strict monotonicity of $\mathcal{Q}$ implies that all the Lyapunov exponents of the billiard map T are non-zero almost everywhere. In §5 of this paper, we prove that the theory of hyperbolic systems with singularities [**KS**] applies to cylindrical semi-focusing billiards. From this, it follows that cylindrical semi-focusing billiards satisfying property (H) have, almost everywhere, two-dimensional local stable and unstable manifolds which are absolutely continuous.

3.4. *Statement of the main results.* We further restrict ourselves to cylindrical semi-focusing domains satisfying

$$b > 2 \max\{r_1, r_2\}. \tag{E}$$

This new condition (which clearly implies (H)) guarantees the monotonicity of the p-norm (see §4.1) along directions contained in $\mathcal{C}$. This property is precisely formulated in Proposition 4.8, and is required in proving the Fundamental theorem (see the proof of Proposition 4.10).

We can now formulate the main results of this paper.

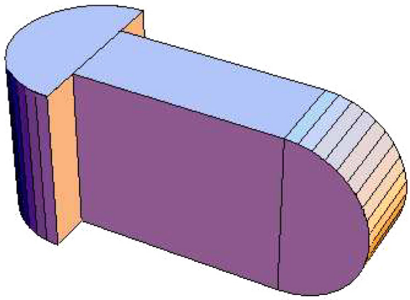

FIGURE 3. A three-dimensional mushroom billiard.

THEOREM 3.4. *For a cylindrical semi-focusing domain that satisfies Condition (E), the billiard map T is Bernoulli.*

Remark 3.5. The same is true for the billiard flow (see [**CFS, CM**] for definitions) of cylindrical semi-focusing billiards. Indeed, from Theorem 3.4 it follows immediately that the billiard flow is ergodic. To conclude that the billiard flow is also Bernoulli, one needs to observe that it is a contact flow and use [**KB**, Theorem 3.6].

In [**BD**], we also introduced a family of billiards with divided phase space, which are the three-dimensional analogs of the mushroom billiards [**B5**]; see Figure 3. Their phase space is the union of two invariant subsets of positive measure such that the restriction of the billiard map to one of them is integrable, whereas the restriction to the other one is hyperbolic. The following corollary is a direct consequence of Theorem 3.4.

COROLLARY 3.6. *The restriction of the billiard map to the hyperbolic component of a three-dimensional mushroom billiard considered in* [**BD**] *(Figure 3) is Bernoulli.*

4. *Local Ergodic Theorem*

Following an established approach in the theory of hyperbolic billiards, the first crucial step in the proof of Theorem 3.4 is to prove a local version of it. We explain what we mean by this after giving a few necessary definitions.

Definition 4.1. A point $x \in \text{int}\,\Sigma$ is called *sufficient* if there exist four integers $k_1 < k_2 < k_3 < k_4$ such that the orbit of x (not necessarily infinite) contains $T^{k_1}x, \ldots, T^{k_4}x$ and the points $\pi(T^{k_1}x)$, $\pi(T^{k_3}x)$ belong to one cylinder of ∂Q while $\pi(T^{k_2}x)$, $\pi(T^{k_4}x)$ belong to the other cylinder. Also, we say that the positive or the negative semi-orbit of x is sufficient if $k_1 \geq 0$ or $k_4 \leq 0$, respectively.

Let E be an ergodic component of T of positive measure. By general results [**KS**, Theorem 13.1 of Part II] and [**CH, OW**], it follows that E is the union of finitely many disjoint sets $B_1, \ldots, B_k = B_0$ such that $TB_i = B_{i+1}$ for each $i = 0, \ldots, k-1$ and the restriction of T^i to every set B_i is Bernoulli. The sets B_i are uniquely defined up to a set of zero measure, and are called *Bernoulli components* of T.

The following is the local version of Theorem 3.4 mentioned above.

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THEOREM 4.2. *If $x \in \text{int}\,\Sigma$ is a sufficient point, then it has a neighborhood contained* (mod 0) *in one Bernoulli component of T.*

Theorem 4.2 will be proved in this section by extending the Fundamental theorem of [**BCST1**] to cylindrical semi-focusing billiards. From now on, we will refer to Theorem 4.2 as the *Local Ergodic Theorem (LET)*. Because the formulation of the Fundamental theorem is extremely elaborate, we will not restate it here. The reader should refer to the results and proofs in [**BCST1, KSS**] if and where needed.

It is worth pointing out that there are other versions of the Fundamental theorem; see, e.g., [**LW, C, M3**]. All these versions go back to the groundbreaking paper by Sinai [**S**]. We have chosen the version of [**BCST1**] because it has the weakest hypotheses on the regularity (of the closure) of the singular sets $\mathcal{S}_n^{\pm}$; these sets are only required to be Lipschitz decomposable in [**BCST1**], whereas in the other versions they must be differentiable. Using the [**BCST1**] version is particularly convenient for us, because a proof of Lipschitz decomposability for the singular sets of cylindrical semi-focusing billiards can be obtained in a straightforward manner from the proof of the same property for semi-dispersing billiards with algebraic boundary components (see [**BCST1**]). We do not know whether the singular sets of cylindrical semi-focusing billiards are differentiable.

4.1. *Definitions: p-norm and semi-distance z_{tub}.* We now give some basic definitions needed in the proof of the Fundamental theorem. We have to warn the reader that some of these definitions may differ from those in [**BCST1**], which were specifically tailored to semi-dispersing billiards.

Recall that $\|\cdot\|$ denotes the standard norm in $\mathbb{R}^3$. In the proof of the Fundamental theorem, a special role is played by a semi-norm called the *p-norm*. Let $x \in \text{int}\,\Sigma$, and take $u \in \mathcal{T}_x\,\text{int}\,\Sigma$. If we write $u = (\xi, \eta)$ with $\xi, \eta \in \mathbb{V}_x \subset \mathbb{R}^3$, then the p-norm of u is defined by

$$\|u\|_p = \|\xi\|.$$

Definition 4.3. Fix an integer $n > 0$. To measure the least expansion rate of $D_{-T^n x}T^n$ over vectors contained in the cone $\mathcal{C}(-T^n x)$, we define

$$\kappa_{n,0}(x) = \inf_{\substack{u \in \mathcal{C}(-T^n x) \\ \|u\|_p > 0}} \frac{\|D_{-T^n x}T^n u\|_p}{\|u\|_p}.$$

This definition makes sense only if we assume that $x \in \text{int}\,\Sigma \setminus \mathcal{S}_n^+$ and $-T^n x \in \Sigma^*$.

Remark 4.4. It is easy to check that $\kappa_{n,0}$ is supermultiplicative; that is, if $n = n_1 + n_2$ for some integers $n_1, n_2 > 0$, then

$$\kappa_{n,0}(x) \geq \kappa_{n_1,0}(x)\kappa_{n_2,0}(T^{n_1}x).$$

It is also useful to measure the least expansion rate of $D_{-T^n x}T^n$ in the p-norm along proper hypersurfaces approximating local unstable manifolds that pass through $-T^n x$. This can be done as follows.

Definition 4.5. Let $x \in \text{int}\,\Sigma \setminus \mathcal{S}_n^+$ and $\delta > 0$. We say that a C^2 hypersurface $\Omega \subset \text{int}\,\Sigma$ is *δ-admissible through $-T^n x$* if:

(1) $-T^n x \in \Omega \subset B_\delta(-T^n x)$, where $B_\delta(y)$ is the ball with center y and radius δ in the p-norm;
(2) $\mathcal{T}_y \Omega \subset \mathcal{C}(y)$ for every $y \in \Omega$ (in particular, $\mathcal{C}(y)$ is defined for all $y \in \Omega$);
(3) $T^n|_\Omega$ is differentiable.

Definition 4.6. For every $x \in \operatorname{int} \Sigma \setminus \mathcal{S}_n^+$ and $\delta > 0$, let

$$\kappa_{n,\delta}(x) = \inf\left\{ \inf_{y \in -T^n\Omega} \kappa_{n,0}(y) \;\middle|\; \Omega \text{ is } \delta\text{-admissible through } -T^n x \right\}.$$

Another important tool used in the proof of the Fundamental theorem is a special semi-distance z_{tub} that measures the distance of elements of Σ from $\partial\Sigma$, the set of singular collisions. The definition of z_{tub} is as follows. For every $x = (q, v) \in \Sigma$ and real $r > 0$, let

$$U_{\text{tub}}(x, r) = \{(\tilde{q}, v) \in \Sigma \mid \|\tilde{q} - q\|^2 - (\langle \tilde{q} - q, v\rangle)^2 < r^2\},$$

which is the set of all elements of Σ, thought of as lines of $\mathbb{R}^3$, whose distance from the line passing through q and $q_1(x)$ is less than r.

Definition 4.7. For every $x \in \Sigma$, define

$$z_{\text{tub}}(x) = \inf\{r > 0 \mid U_{\text{tub}}(x, r) \cap \partial\Sigma = \emptyset\}.$$

Note that $\partial\Sigma = \mathcal{R}_2$ for convex semi-focusing domains.

4.2. *Expansion with respect to the p-norm.* The original proof of the Fundamental theorem is built on the assumption that $\kappa_{n,0}(x) \geq 1$ for every $n > 0$ and $x \notin \mathcal{S}_n^+$ (see [**KSS**, §5]), which is valid for semi-dispersing billiards but not (at least in that general form) for cylindrical semi-focusing billiards, as one can easily check. However, in this section, we will show that for cylindrical semi-focusing billiards the above assumption remains valid if $x \in \operatorname{int} \Sigma_+$. While this weaker property will not allow us to obtain an extension of the Fundamental theorem to cylindrical semi-focusing billiards for every sufficient point, only for those in $\operatorname{int} \Sigma_+$ with sufficient positive semi-orbit, it will be enough to prove Theorem 4.2, which is valid for every sufficient point. Even in the case of points of $\operatorname{int} \Sigma_+$ with sufficient positive semi-orbit, the Fundamental theorem does not extend immediately to cylindrical semi-focusing billiards; thus in §4.3 we will make several adjustments to the original proof.

In this subsection, together with the aforementioned property of $\kappa_{n,0}$ we will prove some other properties of $\kappa_{n,0}$ as well as some results concerning the growth of tangent vectors measured in the p-norm and the quadratic form $\mathcal{Q}$, which will be useful later.

PROPOSITION 4.8. *For every integer $n > 0$, we have:*
(1) *if $x \in \operatorname{int} \Sigma_+ \setminus \mathcal{S}_n^+$ and $-T^n x \in \Sigma^*$, then $\kappa_{n,0}(x) \geq 1$;*
(2) *if $x \in \operatorname{int} \Sigma_+ \setminus \mathcal{S}_n^+$ and $-T^n x \in \Sigma^*$, then $\kappa_{n,0}$ is continuous at x;*
(3) *if $T^{-n} x \in \operatorname{int} \Sigma_+$ and $-x \in \Sigma^*$, then*

$$\kappa_{n,0}(T^{-n}x) \geq \kappa_{n_1,0}(T^{-n_1}x) \quad \text{for every } 0 < n_1 < n.$$

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Proof. Part (1). Let $x \in \operatorname{int} \Sigma_+$ be as in the hypothesis. We claim that it suffices to prove property (1) for orbits of the type $\{y, \ldots, T^k y\}$, with $k > 0$, such that $T^k y \in \operatorname{int} \Sigma_+$ and $Ty, \ldots, T^{k-1} y \in \operatorname{int} \Sigma_0$ if $k > 1$. Indeed, the orbit $\{-T^n x, -T^{n-1} x, \ldots, -x\}$, for which we have to compute the coefficient $\kappa_{n,0}(x)$, is a finite union of orbits of the type just described, thus the claim follows from the supermultiplicativity of $\kappa_{n,0}$.

Consider one of the orbits $\{y, \ldots, T^k y\}$. Since $-T^n x \in \Sigma^*$, it follows immediately that $y \in \Sigma^*$. Let $u_0 = (\xi_0, \eta_0) \in \mathcal{C}(y)$ and $(\xi_1, \eta_1) = D_y T^k u_0$. Part (1) will be proved once we show that $\|\xi_1\| \geq \|\xi_0\|$.

Since $y \in \Sigma^*$, the definition of Σ^* implies that there exists $z \in \operatorname{int} \Sigma_+$ such that $y = T^j z$ for some $j \geq 0$ and $Tz, \ldots, T^{j-1} z \in \operatorname{int} \Sigma_0$ if $j > 0$. Furthermore, there exists $u_{-1} = (\xi_{-1}, \eta_{-1}) \in \mathcal{C}(z)$ such that $u_0 = D_z T^j u_{-1}$. Using equation (2), we easily obtain

$$\begin{aligned} \xi_1 &= \xi_0 + l_0 \eta_0, \\ \xi_0 &= \xi_{-1} + l_{-1} \eta_{-1}, \\ \eta_0 &= \eta_{-1}, \end{aligned}$$

where $l_{-1} \geq 0$ and $l_0 > 0$ are the length of the orbits $\{z, \ldots, T^j z\}$ and $\{y, \ldots, T^k y\}$, respectively. Note that $l_{-1} = 0$ if and only if $j = 0$. Recall that $\mathcal{Q}_z(u_{-1}) = \langle \xi_{-1}, \eta_{-1} \rangle + d(z) \|\eta_{-1}\|^2 \geq 0$. Hence

$$\|\xi_1\|^2 = \|\xi_0\|^2 + l_0(l_0 + 2l_{-1} - 2\, d(z)) \|\eta_{-1}\|^2 + 2 l_0 \mathcal{Q}_z(u_{-1}). \tag{4}$$

Consider now the orbit $\{z, \ldots, T^{j+k} z\}$, and recall that $z, T^{k+j} z \in \operatorname{int} \Sigma_+$ and $k + j > 0$. If all the collisions of this orbit are with the same cylinder, then we see that $j = 0$, $l_{-1} = 0$ and $l_0 = 2d(z)$, otherwise the orbit crosses Q at least once, and so $l_0 + 2l_{-1} \geq l_0 + l_{-1} \geq 2d(z)$ by condition (E). In both cases we obtain $l_0 + 2l_{-1} - 2d(z) \geq 0$, which together with (4) gives $\|\xi_1\| \geq \|\xi_0\|$.

Part (2). Let $x \in \operatorname{int} \Sigma_+$ be as in the hypothesis. It follows immediately that there exists a neighborhood U of x such that $U \cap \mathcal{S}_n^+ = \emptyset$. Let $U' = -T^n U$. Since Σ^* is open, we can further choose U so that $U' \subset \Sigma^*$. Every element of $\mathcal{T} U'$ can be identified with a pair $(-T^n y, u)$, where $y \in U$ and $u = (\xi, \eta) \in \mathbb{V}_{-T^n y} \times \mathbb{V}_{-T^n y}$. Next, recall the definition of $\mathcal{T}^{(1)}_{-T^n y} \operatorname{int} \Sigma$ from §3.1, and define

$$\tilde{\mathcal{C}}(y) = \mathcal{C}(-T^n y) \cap \mathcal{T}^{(1)}_{-T^n y} \operatorname{int} \Sigma.$$

Finally, for each $y \in U$ and $u \in \mathcal{T}^{(1)}_{-T^n y} \operatorname{int} \Sigma$, let

$$f(y, u) = \begin{cases} \dfrac{\|D_{-T^n y} T^n u\|_p}{\|u\|_p} & \text{if } \|u\|_p > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

From equation (4), one can easily deduce that $\|u\|_p = 0$ implies $\|D_{-T^n y} T^n u\|_p > 0$. It follows that f is continuous if we think of it as a function onto the extended real line with the order topology. Moreover, we see that $\kappa_{n,0}(y)$ equals the infimum of $f(y, u)$ over $u \in \tilde{\mathcal{C}}(y)$.

Fix a real $\epsilon > 0$. For every $u \in \tilde{\mathcal{C}}(x)$, we can find a neighborhood $U(u) \subset U$ of x and a neighborhood $V(u) \subset \mathcal{T}^{(1)}_{-T^n x}$ int Σ of u such that for every $y \in U(u)$ and $v \in V(u)$ we have

$$\begin{cases} |f(y, v) - f(x, u)| < \epsilon & \text{if } f(x, u) \text{ is finite,} \\ f(y, v) > \kappa_{n,0}(x) + 2\epsilon & \text{otherwise.} \end{cases}$$

Since $\tilde{\mathcal{C}}(x)$ is compact, there are $k > 0$ elements $u_1, \ldots, u_k \in \tilde{\mathcal{C}}(x)$ such that $\{V(u_1), \ldots, V(u_k)\}$ is a cover of $\tilde{\mathcal{C}}(x)$. Let

$$U_\epsilon = \bigcap_{i=1}^{k} U(u_i) \quad \text{and} \quad V_\epsilon = \bigcup_{i=1}^{k} V(u_i).$$

Clearly, $U_\epsilon \subset \operatorname{int} \Sigma_+$ is a neighborhood of x. We may assume without loss of generality that there exists $v \in V(u_1)$ such that $f(x, v) < \kappa_{n,0}(x) + \epsilon$, and so $f(x, u_1) < \kappa_{n,0}(x) + 2\epsilon$. By the continuity of $\mathcal{C}$ and taking a smaller U_ϵ if required, we may further assume that $\tilde{\mathcal{C}}(y) \subset V_\epsilon$ and $\tilde{\mathcal{C}}(y) \cap V(u_1) \neq \emptyset$ for every $y \in U_\epsilon$.

Let $y \in U_\epsilon$ and $v \in \tilde{\mathcal{C}}(y)$. Then either there exists an integer $1 \le i \le k$ such that $f(y, v) > f(x, u_i) - \epsilon \ge \kappa_{n,0}(x) - \epsilon$, or $f(y, v) > \kappa_{n,0}(x) + 2\epsilon$. In both cases, we have $f(y, v) > \kappa_{n,0}(x) - \epsilon$. On the other hand, if $v \in V(u_1) \cap \tilde{\mathcal{C}}(y)$, then $f(y, v) < f(x, u_1) + \epsilon < \kappa_{n,0}(x) + 3\epsilon$. We conclude that

$$|\kappa_{n,0}(y) - \kappa_{n,0}(x)| \le 3\epsilon,$$

which finishes the proof of part (2).

Part (3). Let $n_2 = n - n_1 > 0$. If $\kappa_{n_1,0}(T^{-n_1} x) = 0$, then there is nothing to prove. Hence we can assume that $\kappa_{n_1,0}(T^{-n_1} x) > 0$. Accordingly, if $u \in \mathcal{C}(-x)$ and $\|u\|_p > 0$, then $\|D_{-x} T^{n_1} u\|_p > 0$. Writing $D_{-x} T^n = D_{-T^{-n_1} x} T^{n_2} \circ D_{-x} T^{n_1}$, we obtain

$$\begin{aligned} \frac{\|D_{-x} T^n u\|_p}{\|u\|_p} &\ge \frac{\|D_{-T^{-n_1} x} T^{n_2} (D_{-x} T^{n_1} u)\|_p}{\|D_{-x} T^{n_1} u\|_p} \frac{\|D_{-x} T^{n_1} u\|_p}{\|u\|_p} \\ &\ge \inf_{\substack{v \in \mathcal{C}(-T^{-n_1} x) \\ \|v\|_p > 0}} \frac{\|D_{-T^{-n_1} x} T^{n_2} v\|_p}{\|v\|_p} \frac{\|D_{-x} T^{n_1} u\|_p}{\|u\|_p} \\ &\ge \kappa_{n_2,0}(T^{-n} x) \frac{\|D_{-T^n x} T^{n_2} u\|_p}{\|u\|_p}. \end{aligned}$$

By taking the infimum over $u \in \mathcal{C}(-T^n x)$ with $\|u\|_p > 0$, we obtain

$$\kappa_{n,0}(T^{-n} x) \ge \kappa_{n_2,0}(T^{-n} x) \kappa_{n_1,0}(T^{-n_1} x).$$

To conclude, we need only observe that $\kappa_{n_2,0}(T^{-n} x) \ge 1$ by part (1). □

Let us denote by F the two-dimensional linear subspace of $\mathbb{R}^3$ generated by the normal vectors of the flat faces of ∂Q. We then see that the set given by

$$\mathcal{N} = \{(q, v) \in \Sigma \mid q \in \partial Q_0 \text{ and } v \in F\}$$

consists of elements of int Σ whose orbits never hit ∂Q_+. Let

$$\Sigma^{**} = \Sigma^* \setminus (\mathcal{S}^+_\infty \cup \mathcal{N}).$$

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Remark 4.9. It is easy to see that Σ^{**} is the subset of int Σ where $\mathcal{Q}$ and $\mathcal{C}$ are defined, and the positive semi-orbit of each of its points hits both the cylinders of ∂Q infinitely many times. According to the results of [**BD**], the cone field $\mathcal{C}$ is then eventually strictly invariant along the positive semi-orbit of each element of Σ^{**}.

We say that a finite sequence of consecutive collisions of $x \in \operatorname{int} \Sigma$ with a cylinder $C \subset \partial Q$ is *complete* if the first element of the sequence enters C and the last element leaves C.

PROPOSITION 4.10. *For every $x \in \Sigma^{**}$, there exists a sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ such that $T^{n_k}x \in \operatorname{int}\Sigma_+$ for every $k > 0$, and*

$$\lim_{k\to+\infty} \inf_{\substack{u\in\mathcal{C}(x)\\ \|u\|_p>0}} \frac{\|D_x T^{n_k} u\|_p}{\|u\|_p} = +\infty.$$

Proof. Let $x \in \Sigma^{**}$. Since the positive semi-orbit of x visits both cylinders of ∂Q infinitely many times, we can choose two sequences of non-negative integers $\{m_k\}_{k\in\mathbb{N}}$ and $\{n_k\}_{k\in\mathbb{N}}$ as follows: for every $k > 0$, let $T^{m_k}x$ and $T^{n_k}x$ be, respectively, the first and the last collision of the kth complete sequence of collisions of x. Firstly, note that $T^{n_k}x \in \operatorname{int}\Sigma_+$ so that the first part of the proposition is proved. Secondly, since

$$\begin{aligned}\frac{\|D_x T^{n_k} u\|_p}{\|u\|_p} &= \frac{\|D_x T^{n_k} u\|_p}{\|u\|'} \frac{\|u\|'}{\|u\|_p}\\ &\geq \frac{\|D_x T^{n_k} u\|_p}{\|u\|'}\end{aligned}$$

for every $u \in \mathcal{T}_x \operatorname{int}\Sigma$ with $\|u\|_p > 0$, and $\|\cdot\|_p$ is a homogeneous function of degree 1 (see §3.1 for the definition of $\|\cdot\|'$), we have

$$\inf_{\substack{u\in\mathcal{C}(x)\\ \|u\|_p>0}} \frac{\|D_x T^{n_k} u\|_p}{\|u\|_p} \geq \inf_{\substack{u\in\mathcal{C}(x)\\ \|u\|'=1}} \|D_x T^{n_k} u\|_p.$$

Thus, to prove the second part of the proposition, it suffices to show that the right-hand side of the above inequality diverges as $k \to +\infty$.

Let $d_k = d(T^{n_k}x)$ (see §3.2 for the definition of the function d), and let l_k be the length of the finite orbit $\{T^{n_k}x, \ldots, T^{m_{k+1}}x\}$. Further, let $u = (\xi, \eta) \in \mathcal{T}_x \operatorname{int}\Sigma$, and for every $k > 0$ define

$$\begin{aligned}(\xi_k, \eta_k) &= D_x T^{n_k} u,\\ (\zeta_{k+1}, \gamma_{k+1}) &= D_x T^{m_{k+1}} u,\\ \mathcal{Q}_k &= \mathcal{Q}_{T^{n_k}x}(D_x T^{n_k} u).\end{aligned}$$

From equation (2), it follows easily that

$$\begin{aligned}\|\xi_k\|^2 \geq \|\zeta_k\|^2 &= \|\xi_{k-1}\|^2 + l_{k-1}\|\eta_{k-1}\|^2 + 2l_{k-1}\langle\xi_{k-1}, \eta_{k-1}\rangle\\ &\geq \|\xi_{k-1}\|^2 + l_{k-1}(l_{k-1} - 2d_{k-1})\|\eta_{k-1}\|^2 + 2l_{k-1}\mathcal{Q}_{k-1}\end{aligned}$$

for all $k > 1$. Condition (E) implies that $l_{k-1} - 2d_{k-1} > 0$ and $l_{k-1} \geq b$, so

$$\|\xi_k\|^2 \geq \|\xi_{k-1}\|^2 + 2b\mathcal{Q}_{k-1}.$$

Iterating this inequality, we obtain

$$\|\xi_k\|^2 \geq \|\xi_1\|^2 + 2b \sum_{i=1}^{k-1} \mathcal{Q}_i. \tag{5}$$

From the results of [**BD**], we know that there exists an integer $\bar{k} = \bar{k}(x) > 1$ such that

$$\mathcal{Q}_{\bar{k}}(D_x T^{n_{\bar{k}}} u) > \mathcal{Q}_1(D_x T^{n_1} u) \geq \mathcal{Q}_x(u) \quad \text{for all } u \in \mathcal{T}_x \text{ int } \Sigma \setminus \{0\}.$$

Since $\mathcal{Q}_{T^{n_{\bar{k}}} x}$ is continuous and homogeneous of degree two, it follows that

$$\mathcal{Q}_{\bar{k}}(D_x T^{n_{\bar{k}}} u) \geq \bar{c} \|u\|'^2 \quad \text{for all } u \in \mathcal{C}(x),$$

where

$$\bar{c} = \inf_{\substack{u \in \mathcal{C}(x) \\ \|u\|'=1}} \mathcal{Q}_{\bar{k}}(D_x T^{n_{\bar{k}}} u) > 0.$$

Finally, using the monotonicity of $\mathcal{Q}$, we obtain

$$\mathcal{Q}_k(D_x T^{n_k} u) \geq \bar{c} \|u\|'^2 \quad \text{for all } u \in \mathcal{C}(x) \text{ and } k \geq \bar{k}. \tag{6}$$

Combining (5) and (6), it follows that for $k > \bar{k}$,

$$\inf_{\substack{u \in \mathcal{C}(x) \\ \|u\|'=1}} \|D_x T^{n_k} u\|_p^2 \geq 2b\bar{c}(k - \bar{k} - 1) \xrightarrow[k \to +\infty]{} +\infty.$$

This finishes the proof. □

Remark 4.11. If we set $y = -x$, then another way of stating Proposition 4.10 is to say that

$$\lim_{k \to +\infty} \kappa_{n_k,0}(T^{-n_k} y) = +\infty.$$

PROPOSITION 4.12. *For every $x \in \Sigma^{**}$, we have*

$$\lim_{k \to +\infty} \mathcal{Q}_{T^k x}(D_x T^k u) = +\infty \quad \textit{for all } u \in \mathcal{C}(x) \setminus \{0\}.$$

Proof. Let $x \in \Sigma^{**}$ and let n_k, m_k, l_k, d_k be as in the proof of Proposition 4.10. Also, define $x_k = T^{n_k} x$.

For every $y \in \text{int } \Sigma_+$, consider the transformation on $\mathcal{T}_y$ int Σ given by

$$\begin{cases} \xi' = \xi + d(y)\eta, \\ \eta' = \eta. \end{cases} \tag{7}$$

The matrix form of $D_{x_k} T^{n_{k+1}-n_k}$ can be easily computed with respect to ξ', η' by using equations (2) and (7). Since this computation is straightforward, though cumbersome, we show only its final result, which is the following:

$$D_{x_k} T^{n_{k+1}-n_k} = \begin{pmatrix} A_k & 0 \\ 0 & A_k^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ B_k & I \end{pmatrix} \begin{pmatrix} I & C_k \\ 0 & I \end{pmatrix}, \tag{8}$$

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where

$$R_k = R(T^{m_{k+1}}x),$$
$$A_k = (I + d_{k+1}R_k)^{n_{k+1}-m_{k+1}+1},$$
$$B_k = (n_{k+1} - m_{k+1} + 1)R_k A_k,$$
$$C_k = (n_{k+1} - m_{k+1})d_{k+1}(I + A_k) + l_k - d_k + d_{k+1}A_k.$$

It is easy to check that the operators B_k, C_k are positive semi-definite and commute pairwise. Indeed, since the eigenvalues of R_k are 0 and $-2/d_{k+1}$, it follows immediately that those of A_k are -1 and 1, and those of C_k are $\sigma_k^{(1)} := l_k - d_k - d_{k+1}$ and $\sigma_k^{(2)} := l_k - d_k - d_{k+1} + 2(n_{k+1} - m_{k+1} + 1)\, d_{k+1}$. Hence

$$\|A_k\xi'\| \le \|\xi'\| \quad \text{for all } \xi' \in \mathbb{V}_{x_k}. \tag{9}$$

Moreover, since the length of the orbit starting at $T^{m_{k+1}}x$ and ending at x_{k+1} is equal to $2(n_{k+1} - m_{k+1})d_{k+1}$ (the length of the segment between two consecutive collisions of this orbit equals d_{k+1}) and is certainly less than the perimeter of the semi-circle forming the section of the cylinder where x_{k+1} is attached, it is not difficult to see that

$$\begin{aligned}\frac{\sigma_k^{(2)}}{\sigma_k^{(1)}} &= 1 + \frac{2(n_{k+1} - m_{k+1} + 1)d_{k+1}}{l_k - d_k - d_{k+1}} \\ &\le 1 + \frac{2\pi \max\{r_1, r_2\}}{b} =: \bar{\sigma}.\end{aligned} \tag{10}$$

Let $u = (\xi', \eta') \in \mathcal{T}_x \operatorname{int} \Sigma$. Then define $(\xi'_k, \eta'_k) = D_x T^{n_k} u$ and $\mathcal{Q}_k = \mathcal{Q}_{x_k}(D_x T^{n_k} u)$. It is easy to check that the quadratic form $\mathcal{Q}$ computed with respect to the pair (ξ', η') becomes the standard dot product between ξ' and η', and hence we have $\mathcal{Q}_k = \langle \xi'_k, \eta'_k \rangle$.

We now follow [**LW**, Proof of Proposition 6.9]. From (8) and (9), we obtain

$$\begin{aligned}\|\xi'_{k+1}\| &\le \|\xi'_k\| + \|C_k \eta'_k\| \\ &\le \|\xi'_k\| + \sigma_k^{(2)} \|\eta'_k\|,\end{aligned}$$

which once iterated gives

$$\|\xi'_{k+1}\| \le \|\xi'_1\| + \sum_{i=1}^{k} \sigma_i^{(2)} \|\eta'_i\|. \tag{11}$$

We claim that $\sum_{i=1}^{+\infty} \sigma_i^{(2)} \|\eta'_i\|$ is divergent. For contradiction, suppose that this is not true; from (11), we then see that $\|\xi'_i\|$ is bounded. Since $\|\eta'_i\| \ge \mathcal{Q}_i / \|\xi'_i\|$ and $\mathcal{Q}_i > 0$ for large i (recall that $\mathcal{Q}$ is eventually strictly monotone along the positive semi-orbit of x), it follows that $\|\eta'_i\|$ is bounded away from 0 if i is sufficiently large. Condition (E) clearly implies that $\sum_{k=1}^{+\infty} \sigma_k^{(2)}$ is divergent, thus we can deduce that the same is true for $\sum_{k=1}^{+\infty} \sigma_k^{(2)} \|\eta'_k\|$, obtaining a contradiction.

Next, from (8) and $B_k \ge 0$ it follows easily that

$$\mathcal{Q}_{k+1} \ge \mathcal{Q}_k + \sigma_k^{(1)} \|\eta'_k\|^2. \tag{12}$$

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Using (10)–(12) and $\|\eta'_k\| \geq \mathcal{Q}_k/\|\xi'_k\|$, we obtain

$$\begin{aligned}\mathcal{Q}_{k+1} &\geq \mathcal{Q}_k + \sigma_k^{(1)} \frac{\|\eta'_k\|}{\|\xi'_k\|} \mathcal{Q}_k \\ &\geq \mathcal{Q}_k \left(1 + \bar{\sigma} \frac{\sigma_k^{(2)} \|\eta'_k\|}{\|\xi'_1\| + \sum_{i=1}^{k-1} \sigma_i^{(2)} \|\eta'_i\|}\right).\end{aligned}$$

To finish the proof, it remains to demonstrate that

$$\sum_{k=1}^{+\infty} \frac{\sigma_k^{(2)} \|\eta'_k\|}{\|\xi'_1\| + \sum_{i=1}^{k-1} \sigma_i^{(2)} \|\eta'_i\|} = +\infty.$$

Indeed, for any j_1, j_2 such that $1 \leq j_1 \leq j_2$, we have

$$\sum_{k=j_1}^{j_2} \frac{\sigma_k^{(2)} \|\eta'_k\|}{\|\xi'_1\| + \sum_{i=1}^{k-1} \sigma_i^{(2)} \|\eta'_i\|} \geq \frac{\sum_{k=j_1}^{j_2} \sigma_k^{(2)} \|\eta'_k\|}{\|\xi'_1\| + \sum_{k=1}^{j_2} \sigma_k^{(2)} \|\eta'_k\|} \xrightarrow[j_2 \to +\infty]{} 1. \qquad \square$$

4.3. *Fundamental theorem.* We are now in a position to extend the Fundamental theorem of [**BCST1**] to cylindrical semi-focusing billiards. We stress that such an extension is valid only for points of int Σ_+ with sufficient positive semi-orbit. The two main hypotheses of this theorem are the Ansatz and the Lipschitz decomposability of the singular sets $\mathcal{S}_n^{\pm}$ [**BCST1**, Conditions 4.1 and 4.2]. The first hypothesis will be formulated and proved later in this subsection. For the formulation of the second hypothesis, we refer the reader to [**BCST1**]; its proof is given in Appendix A of this paper. The Fundamental theorem is proved in [**BCST1**, §4.2]; as its central argument (the part following Remark 4.11) and [**BCST1**, Lemma 4.5] remain true for cylindrical semi-focusing billiards, all we need to do to obtain the desired extension is re-prove for cylindrical semi-focusing billiards the following lemmas:

(1) [**BCST1**, Lemma 2.2] (reference neighborhood);
(2) [**BCST1**, Lemma 4.6] (parallelization lemma);
(3) [**BCST1**, Lemma 4.10] (tail bound lemma).

As regards [**KSS**, Lemmas 5.3 and 5.4], which also form part of the proof of the Fundamental theorem, we note that [**KSS**, Lemma 5.3] is superfluous (even for semi-dispersing billiards) in view of our improved proof of the tail bound lemma (see Lemma 4.18 of this paper), whereas [**KSS**, Lemma 5.4] remains true for cylindrical semi-focusing billiards as long as the reference neighborhood U is contained in int Σ_+ (which we require in this paper; recall that by Proposition 4.8 we have $\kappa_{n,0}(x) \geq 1$ if $x \in$ int Σ_+). Finally, we observe that the property formulated in [**BCST1**, Remark 4.7], namely $\lim_{n\to+\infty} \kappa_{n,0}(x) = +\infty$ for every $x \in \Sigma^{**}$, does not hold for cylindrical semi-focusing billiards, which have instead the weaker property described in Proposition 4.10; however, it turns out that this is enough to carry through the proof of the tail bound lemma.

The rest of this section will be devoted to proving the lemmas listed above and the Ansatz for cylindrical semi-focusing billiards.

4.3.1. *Reference neighborhood.* The following lemma is the analog of [**BCST1**, Lemma 2.2].

LEMMA 4.13. *For every point $x \in \operatorname{int} \Sigma_+$ with sufficient positive semi-orbit, there exist an integer $I > 0$, a real number $0 < \lambda < 1$ and a neighborhood $U \subset \operatorname{int} \Sigma_+$ of x such that:*
(i) *if $y \in U$ and $T^i y \in U$ with $i \geq I$, then $\kappa_{i,0}(y) > \lambda^{-1}$;*
(ii) *the local stable and unstable manifolds γ^s, γ^u of T are uniformly transversal in U.*

Proof. Part (i). Consider a point $x \in \operatorname{int} \Sigma_+$ whose positive semi-orbit is sufficient. First, suppose that there exist an integer $I > 0$ and a real number $0 < \lambda < 1$ such that $T^I x \in \operatorname{int} \Sigma_+$ and $\kappa_{I,0}(x) > \lambda^{-1}$. By part (2) of Proposition 4.8, $\kappa_{I,0}$ is then continuous at x, so we can find a neighborhood $U \subset \operatorname{int} \Sigma_+$ of x such that $\kappa_{I,0}(y) > \lambda^{-1}$ for every $y \in U$. Finally, arguing as in the proof of part (3) of Proposition 4.8, one can show that

$$\kappa_{i,0}(y) \geq \kappa_{I,0}(y) > \lambda^{-1} \quad \text{for all } i \geq I \text{ and } y \in U \cap T^{-i} U,$$

which is the desired conclusion. Thus, to finish the proof of part (i), it remains to demonstrate that there exist an integer I and a real number λ satisfying the property described above.

The proof of this fact is similar to the proof of Proposition 4.10. The positive semi-orbit of x must contain at least three complete sequences of collisions, because it is sufficient. Accordingly, we define n_1 to be the smallest non-negative integer such that $T^{n_1} x$ leaves the cylinder of ∂Q where $\pi(x)$ belongs. We also define m_j and n_j to be the positive integers such that $T^{m_j} x$ and $T^{n_j} x$ are, respectively, the first and the last collision of the jth complete sequence of collisions of x, for $j = 2, 3, 4$. Next, set $z_j = -T^{m_{5-j}} x$ for each $j = 1, 2, 3$, and $z_4 = -T^{n_1} x$. Given $u_1 = (\xi_1, \eta_1) \in \mathcal{C}(z_1)$, write $u_2 = (\xi_2, \eta_2) = D_{z_1} T^{m_4 - m_3} u_1$, $u_3 = (\xi_3, \eta_3) = D_{z_1} T^{m_4 - m_2} u_1$ and $u_4 = (\zeta_4, \gamma_4) = D_{z_1} T^{m_4 - n_1} u_1$. Finally, let l_j be the length of the piece of orbit $\{z_j, \ldots, -T^{n_{4-j}} x\}$ and write $\mathcal{Q}_j = \mathcal{Q}_{z_j}(u_j)$ for each $j = 1, 2, 3$.

Note that $\zeta_4 = \xi_3 + l_3 \eta_3$. Then, as in the proof of Proposition 4.10, Condition (E) allows us to show that

$$\|\zeta_4\|^2 \geq \|\xi_3\|^2 + 2b\mathcal{Q}_3,$$
$$\|\xi_3\|^2 \geq \|\xi_1\|^2 + 2b(\mathcal{Q}_1 + \mathcal{Q}_2),$$

which in turn give

$$\|\zeta_4\|^2 \geq \|\xi_1\|^2 + 2b \sum_{j=1}^{3} \mathcal{Q}_j.$$

Further, since $\mathcal{Q}_{z_3}$ is continuous, homogeneous of degree two and strictly monotone along $\{z_1, z_2, z_3\}$, we easily see that

$$\sum_{j=1}^{3} \mathcal{Q}_j \geq \mathcal{Q}_3 \geq \bar{c} \|u_1\|'^2,$$

where

$$\bar{c} = \inf_{\substack{u_1 \in \mathcal{C}(z_1) \\ \|u_1\|' = 1}} \mathcal{Q}_{z_3}(u_3) > 0.$$

Finally, it is straightforward to check that $\|D_{z_1}T^{m_4}u_1\|_p \geq \|\zeta_4\|$. Combining all the previous estimates, we obtain

$$\frac{\|D_{z_1}T^{m_4}u_1\|_p^2}{\|u_1\|_p^2} \geq \frac{\|\xi_1\|^2 + 2b\sum_{j=1}^{3}\mathcal{Q}_j}{\|\xi_1\|^2}$$
$$\geq 1 + 2b\bar{c}\frac{\|u_1\|'^2}{\|\xi_1\|^2}$$
$$\geq 1 + 2b\bar{c}.$$

Since $\bar{c}$ is independent of $u_1 \in \mathcal{C}(z_1)$, it follows that

$$\kappa_{m_4,0}(x) = \inf_{\substack{u_1\in\mathcal{C}(z_1)\\ \|u_1\|_p>0}} \frac{\|D_{z_1}T^{m_4}u_1\|_p}{\|u_1\|_p}$$
$$\geq (1+2b\bar{c})^{(1/2)} > 1.$$

The desired conclusion is now obtained by setting $I = m_4$ and choosing $1 < \lambda^{-1} < (1+2b\bar{c})^{1/2}$.

Part (ii). Let U be the same neighborhood of x as in part (i). If a local stable (unstable) manifold $\gamma^s(y)$ $(\gamma^u(y))$ through $y \in U$ exists, then let us denote by $E^s(y)$ $(E^u(y))$ the tangent space of $\gamma^s(y)$ $(\gamma^u(y))$ at y. Since $\mathcal{Q}$ is eventually strictly monotone, [**KB**, Theorem 2.1] (see also [**M2**, Theorem 1]) implies that for a.e. $y \in U$,

$$\begin{aligned} E^s(y) &= \bigcap_{k\geq 0} D_{T^k y}T^{-k}\mathcal{C}'(T^k y),\\ E^u(y) &= \bigcap_{k\geq 0} D_{T^{-k} y}T^{k}\mathcal{C}(T^{-k} y). \end{aligned} \tag{13}$$

By part (i), we know that $D_xT^I\mathcal{C}(x) \subset \operatorname{int}\mathcal{C}(T^Ix)\cup\{0\}$, or equivalently, $D_{T^Ix}T^{-I}\mathcal{C}'(T^Ix) \subset \operatorname{int}\mathcal{C}'(x)\cup\{0\}$. From this and the continuity of $\mathcal{C}'$, it is easy to deduce that there exist two cones† $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\mathbb{R}^4$ such that $\mathcal{C}(x) \subset \mathcal{C}_1 \subset \operatorname{int}\mathcal{C}_2\cup\{0\}$ and, by taking a smaller U if necessary, $\mathcal{C}(y) \subset \mathcal{C}_1$ and $D_{T^Iy}T^{-I}\mathcal{C}'(T^Iy) \subset \mathcal{C}_2'$ for every $y \in U$. Using (13), we then see that $E^u(y) \subset \mathcal{C}_1$ and $E^s(y) \subset \mathcal{C}_2'$ for a.e. $y \in U$. To finish the proof, we need only observe that any two non-zero vectors, one from $\mathcal{C}_1$ and one from $\mathcal{C}_2'$, form an angle that is uniformly bounded away from 0. □

4.3.2. *Parallelization.* We recall that the angle between two linear subspaces L_1, L_2 of $\mathbb{R}^k$ is defined by

$$\sphericalangle(L_1, L_2) = \sup_{0\neq v_1\in L_1}\ \inf_{0\neq v_2\in L_2} \sphericalangle(v_1, v_2).$$

The next proposition is the analog of the parallelization lemma in [**BCST1**].

† In this paper, a cone is the set defined as follows. Given a finite-dimensional linear space V with a symplectic form ω and two transversal Langrangian subspaces V_1, V_2 of V, the cone $\mathcal{C}(V_1, V_2)$ is the collection of all vectors $u \in V$ such that $\omega(u_1, u_2) \geq 0$, where $u_1 \in V_1$, $u_2 \in V_2$ and $u = u_1 + u_1$; for further details, see [**LW**]. A more general definition of a cone can be given in terms of quadratic forms or homogeneous functions; see [**M2, KB**].

PROPOSITION 4.14. *Consider* $x \in \operatorname{int} \Sigma_+ \setminus \mathcal{S}^+_\infty$. *For every* $\epsilon > 0$, *there exists a neighborhood* $U \subset \operatorname{int} \Sigma_+$ *of* x *such that:*

(i) *if* γ_1^s *and* γ_2^s *are local stable manifolds, then* $\sphericalangle(\mathcal{T}_{y_1}\gamma_1^s, \mathcal{T}_{y_2}\gamma_2^s) < \epsilon$ *for any* $y_1 \in \gamma_1^s \cap U$ *and* $y_2 \in \gamma_2^s \cap U$;

(ii) *let* H *be a Lipschitz component of* $T^{-n}\mathcal{S}_1^+$, $n \geq 0$, *and let* γ_1^s *be a local stable manifold. Then for any* $y \in H \cap U$ *such that* $\mathcal{T}_y H$ *exists, there is a two-dimensional subspace* $L_y \subset \mathcal{T}_y H$ *for which* $\sphericalangle(L_y, \mathcal{T}_{y_1}\gamma_1^s) < \epsilon$ *for every* $y_1 \in \gamma_1^s \cap U$.

Proof. For $k > 0$ and $y \in \operatorname{int} \Sigma_+ \setminus \mathcal{S}_k^+$, define

$$\mathcal{C}'_k(y) = D_{T^k y} T^{-k} \mathcal{C}'(T^k y).$$

Part (i). Consider $x \in \operatorname{int} \Sigma_+ \setminus \mathcal{S}^+_\infty$. Let $\operatorname{Lag} \mathcal{C}'_k(x)$ be the Grassmannian of all the Lagrangian† subspaces contained in $\operatorname{int} \mathcal{C}'_k(x) \cup \{0\}$. We endow this space with the complete distance used in [**LW**, Theorem 6.5], which will be denoted by $\bar{d}$ throughout this proof. The closure of $\operatorname{Lag} \mathcal{C}'_k(x)$ with respect to $\bar{d}$, denoted by $\overline{\operatorname{Lag} \mathcal{C}'_k(x)}$, turns out to be the collection of all the Lagrangian subspaces contained in $\mathcal{C}'_k(x)$ (see [**LW**, §5]). Since $\mathcal{C}$ is eventually strictly invariant along the positive semi-orbit of x, $\{\overline{\operatorname{Lag} \mathcal{C}'_k(x)}\}_k$ forms a nested sequence of compact subsets of $\operatorname{Lag} \mathcal{C}'(x)$, hence the intersection $\bigcap_{k\geq 0} \overline{\operatorname{Lag} \mathcal{C}'_k(x)}$ is non-empty. In fact, combining Proposition 4.12 of this paper and [**LW**, Theorem 6.5], we see that $\bigcap_{k\geq 0} \overline{\operatorname{Lag} \mathcal{C}'_k(x)}$ consists of a single (two-dimensional) Lagrangian subspace contained in $\operatorname{int} \mathcal{C}'(x) \cup \{0\}$. Fix $\epsilon > 0$. Since the metric $\bar{d}$ generates the standard topology of $\operatorname{Lag} \mathcal{C}'(x)$, there exists an integer $\bar{k} > 0$ such that if $V_1, V_2 \in \operatorname{Lag} \mathcal{C}'_{\bar{k}}$, then $\sphericalangle(V_1, V_2) < \epsilon/2$. Note that $T^{\bar{k}}$ is a local diffeomorphism at x, and $\mathcal{C}'$ is defined (and continuous) at $T^{\bar{k}} x \in \Sigma^*$. It follows that $\mathcal{C}'_{\bar{k}}$ is also continuous at x, thus we can find a neighborhood $U \subset \operatorname{int} \Sigma_+$ of x and a cone $\tilde{\mathcal{C}} \subset \mathbb{R}^4$ such that

$$\begin{gathered} T^{\bar{k}}|_U \text{ is a diffeomorphism,} \\ \mathcal{C}'_{\bar{k}}(y) \subset \operatorname{int} \tilde{\mathcal{C}} \cup \{0\} \quad \text{for all } y \in U, \\ \text{and} \quad \sphericalangle(V_1, V_2) < \epsilon \quad \text{for all } V_1, V_2 \in \operatorname{Lag} \tilde{\mathcal{C}}. \end{gathered} \tag{14}$$

Since $\bigcap_{k\geq 0} \mathcal{C}'_k(y)$ contains a single Lagrangian subspace for a.e. $y \in U$, by [**KB**, Theorem 2.1] we see that $\mathcal{T}_y \gamma^s(y)$ is Lagrangian and coincides with $\bigcap_{k\geq 0} \mathcal{C}'_k(y)$ for a.e. $y \in U$. In particular, $\mathcal{T}_y \gamma^s(y) \in \operatorname{Lag} \mathcal{C}'_{\bar{k}}(y)$ for a.e. $y \in U$. This together with (14) proves part (i).

Part (ii). Let U be the neighborhood of x as in part (i). Consider $y \in H \cap U$ such that $\mathcal{T}_y H$ exists. Note that T^n is a local diffeomorphism at y. Next, consider the subspace of $\mathcal{T}_{T^n y} H$,

$$\mathcal{L}_y = \operatorname{Null}(D_{T^n y} q_1 |_{\mathcal{T}_{T^n y} T^n H}).$$

It is easily seen that

$$\mathcal{L}_y = \{(-t(T^n y)\eta, \eta) \in \mathcal{T}_y \operatorname{int} \Sigma \mid \eta \in \mathbb{V}_{T^n y}\},$$

† In coordinates (ξ, η), the symplectic form of $\operatorname{int} \Sigma$ is given by the standard one, i.e. $\omega(u, w) = \langle \xi_1, \eta_2 \rangle - \langle \xi_2, \eta_1 \rangle$, where $u = (\xi_1, \eta_1)$ and $w = (\xi_2, \eta_2)$.

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which in turn implies that $\mathcal{L}_y$ is Lagrangian (in particular, it is a two-dimensional subspace). Since $T^n y \in \Sigma^*$, it follows that

$$\mathcal{Q}_{T^n y}(u) = (-t(T^n y) - l_-(T^n y) + d_-(T^n y))\|\eta\|^2 \quad \text{for all } u \in \mathcal{L}_y.$$

Note that $t(T^n y) + l_-(T^n y)$ is the length of the piece of orbit starting at $T^{n-m(T^n y)} y \in \operatorname{int} \Sigma_+$ and ending at a point belonging to the intersection of two or more boundary components of Q (for the definitions of the functions m, l_- and d_-, see §3.2). By the geometry of Q and condition (E) (actually (H) is enough here), we see that $t(T^n y) + l_-(T^n y) > d_-(T^n y)$ so that $\mathcal{Q}_{T^n u}(u) < 0$ for all $u \in \mathcal{L}_y \setminus \{0\}$. Hence $\mathcal{L}_y \in \operatorname{Lag} \mathcal{C}'(T^n y)$.

Let $L_y = D_{T^n y} T^{-n} \mathcal{L}_y$. This is clearly a two-dimensional subspace of $\mathcal{T}_y H$, and from the invariance of $\mathcal{C}$ it follows that $L_y \in \operatorname{Lag} \mathcal{C}'_n(y)$. But $n \geq \bar{k}$ because $T^{\bar{k}}|_U$ is a diffeomorphism, and so $L_y \in \operatorname{Lag} \mathcal{C}'_{\bar{k}}(y)$. Using part (i), we can now conclude that $\sphericalangle(L_y, \mathcal{T}_{y_1}\gamma_1^s) < \epsilon$ for every $y_1 \in \gamma_1^s \cap U$. □

4.3.3. *Ansatz.* In this subsection, we formulate and prove the Ansatz for cylindrical semi-focusing billiards.

We write

$$\bigcup_{i=1}^{n} \partial\Gamma_i = \bigcup_{j=1}^{m} \gamma_j,$$

where $\gamma_1, \ldots, \gamma_m$ are smooth curves intersecting only at their boundaries. For $1 \leq j \leq m$, let

$$A_j = \{(q, v) \in \mathcal{T}_1\mathbb{R}^3 \mid q \in \operatorname{int} \gamma_j \text{ and } \langle v, N_i(q)\rangle > 0 \;\forall\, i \text{ with } q \in \Gamma_i\}.$$

Each set A_j is a three-dimensional manifold, and any two sets A_j, A_k are disjoint if $j \neq k$. Denote by ν the volume measure on A_j (thought of as a submanifold of $\mathcal{T}\mathbb{R}^3$) generated by the Riemannian metric g (see §3.1). Next, let

$$\hat{\mathcal{R}}_2 = \bigcup_{j=1}^{m} A_j.$$

This is a subset of $\mathcal{R}_2$ (the set of multiple collisions) and clearly a three-dimensional manifold. The measure ν of A_j gives a measure on $\hat{\mathcal{R}}_2$, which is still denoted by ν.

Remark 4.15. We formulate the Ansatz in terms of $\hat{\mathcal{R}}_2$ instead of $\mathcal{R}_2$, and in this respect, our formulation differs slightly from that of [**BCST1**]. Nevertheless, this difference does not harm the proof of the tail bound lemma [**KSS**], which is the only part of the Fundamental theorem where the Ansatz is used.

Note that we have defined the involution $\mathcal{J}$ and the singular sets $\mathcal{S}_i^-$, $\mathcal{S}_i^+$ only on int Σ. In order to formulate the Ansatz, we need to define these objects on $\hat{\mathcal{R}}_2$ as well.

Let us start with $\mathcal{J}$. Consider $x = (q, v) \in \hat{\mathcal{R}}_2$ and suppose that $\Gamma_1, \ldots, \Gamma_k$ are all the boundary components of Q containing q. Then for each $i = 1, \ldots, k$, there exists a sequence $\{y_{i,j}\}_{j\in\mathbb{N}}$ such that

$$\operatorname{int} \Sigma_i \ni y_{i,j} \to x \quad \text{as } j \to +\infty.$$

Since $-y_{i,j}$ is well-defined, we consider its positive semi-orbit. Within a finite number of collisions, which is bounded above by a constant depending only on Q, this orbit will approach and successively leave the corner of ∂Q containing q. Denote by $x_{i,j}$ the first element of the positive semi-orbit of $-y_{i,j}$ 'leaving' the corner; in other words, let $x_{i,j} = (q_{i,j}, v_{i,j})$ be the first element of the positive semi-orbit of $-y_{i,j}$ such that $\langle v_{i,j}, N_l(q)\rangle > 0$ for every $l = 1, \ldots, k$. It is easy to see that the limit of $x_{i,j}$ as $j \to +\infty$ exists for all $i = 1, \ldots, k$; denote these limits by $x_1, \ldots, x_k$. We then define the involution $\mathcal{J}$ on $\hat{\mathcal{R}}_2$ as the multivalued transformation

$$\hat{\mathcal{R}}_2 \ni x \mapsto -x = \{x_1, \ldots, x_k\}.$$

We now define the analog of the sets $\mathcal{S}_i^+$ and $\mathcal{S}_i^-$ on $\hat{\mathcal{R}}_2$. First, let

$$\hat{\mathcal{S}}_1^+ = \bigcup_{j=1}^{k} \{(q, v) \in \hat{\mathcal{R}}_2 \mid q + \tau v \in \gamma_j \text{ for some } \tau > 0\}$$

and

$$\hat{\mathcal{S}}_1^- = -\hat{\mathcal{S}}_1^+.$$

Then define the transformation $\hat{T} : \hat{\mathcal{R}}_2 \setminus \hat{\mathcal{S}}_1^+ \to \operatorname{int} \Sigma$ as we defined T in §2, but replacing $\operatorname{int} \Sigma \setminus \mathcal{S}_1^+$ by $\hat{\mathcal{R}}_2 \setminus \hat{\mathcal{S}}_1^+$. Therefore, for each $i > 1$, let

$$\hat{\mathcal{S}}_{i+1}^+ = \hat{\mathcal{S}}_1^+ \cup \hat{T}^{-1} \mathcal{S}_i^+ \quad \text{and} \quad \hat{\mathcal{S}}_i^- = -\hat{\mathcal{S}}_i^+.$$

Finally, set

$$\hat{\mathcal{S}}_\infty^+ = \bigcup_{i>0} \hat{\mathcal{S}}_i^+ \quad \text{and} \quad \hat{\mathcal{S}}_\infty^- = -\hat{\mathcal{S}}_\infty^+.$$

We can now formulate the Ansatz.

Definition 4.16. We say that the *Ansatz* is satisfied if

$$\nu(\hat{\mathcal{S}}_\infty^- \cup \hat{\mathcal{S}}_\infty^+ \cup \mathcal{N}) = 0.$$

Another way of stating the Ansatz is to say that for ν-a.e. $x \in \hat{\mathcal{R}}_2$, the positive semi-orbits of x and $-x$ are infinite and hit each cylinder of ∂Q infinitely many times.

THEOREM 4.17. *The Ansatz is satisfied.*

Proof. It suffices to prove that

$$\nu(A_j \cap \mathcal{N}) = \nu(A_j \cap \hat{\mathcal{S}}_\infty^+) = \nu(A_j \cap \hat{\mathcal{S}}_\infty^-) = 0$$

for each $j = 1, \ldots, m$. The first equality from the left is a direct consequence of the fact that $A_j \cap \mathcal{N}$ is a codimension-one smooth submanifold of A_j.

We now prove the second equality. The first step is to show that $\nu(A_j \cap \hat{\mathcal{S}}_1^+) = 0$. To do this it suffices to demonstrate that $\nu(A_{j,k}) = 0$ for each $k = 1, \ldots, m$, where

$$A_{j,k} = \{(q, v) \in A_j \mid q + \tau v \in \gamma_k \text{ for some } \tau > 0\}.$$

Note that some of these sets are empty: for example, $A_{j,j} = \emptyset$ for all $j = 1, \ldots, m$. Given a non-empty $A_{j,k}$, it is easily seen by direct inspection that there is an open set $\hat{V} \subset f_k^{-1}(0)$

(the function f_k was introduced at the beginning of §2) such that $\gamma_k \subset \hat{V}$ and every ray $\{q + \tau v \mid \tau > 0\}$ with $(q, v) \in A_j$ is transversal to $\hat{V}$. Let

$$\hat{U} = \{(q, v) \in A_j \mid q + \tau v \in \hat{V} \text{ for some } \tau > 0\}.$$

Then for $(q, v) \in \hat{U}$, define

$$\hat{t}(q, v) = \inf\{\tau > 0 \mid q + \tau v \in V\},$$
$$\hat{q}_1(q, v) = q + \hat{t}(q, v)v.$$

It is easy to check that $\hat{U}$ is an open subset of A_j and, using the transversality of $\hat{V}$ and the rays emanating from A_j, that $\hat{q}_1$ is a smooth submersion from $\hat{U}$ to $\hat{V}$, i.e. $\hat{q}_1$ is a smooth transformation whose differential is surjective at every point of $\hat{U}$. From this and the fact that γ_k is a codimension-one submanifold of $\hat{V}$, it follows that $A_{j,k}$ is a codimension-one submanifold of A_j, which gives $\nu(A_{j,k}) = 0$. This proves that $\nu(A_j \cap \hat{\mathcal{S}}_1^+) = 0$.

We now proceed to show that $\nu(A_j \cap \hat{\mathcal{S}}_\infty^+) = 0$. Since $\hat{\mathcal{S}}_\infty^+ = \hat{\mathcal{S}}_1^+ \cup \hat{T}^{-1}\mathcal{S}_\infty^+$ and $\nu(A_j \cap \hat{\mathcal{S}}_1^+) = 0$, it suffices to demonstrate that $\nu(A_j \cap \hat{T}^{-1}\mathcal{S}_\infty^+) = 0$. We clearly have $A_j \cap \hat{T}^{-1}\mathcal{S}_\infty^+ = A_j \cap \hat{T}^{-1}(\mathcal{S}_\infty^+ \cap \mathcal{S}_1^-)$. From Propositions A.2 and A.4, $\mathcal{S}_\infty^+ \cap \mathcal{S}_1^-$ is a countable union of codimension-one submanifolds contained in $\mathcal{S}_1^-$. The restriction of $\hat{T}$ to $A_j \setminus \hat{\mathcal{S}}_1^+$ into $\hat{T}(A_j \setminus \hat{\mathcal{S}}_1^+)$ is easily checked to be a diffeomorphism; hence we have that $A_j \cap \hat{T}^{-1}(\mathcal{S}_\infty^+ \cap \mathcal{S}_1^-)$ is a countable union of codimension-one submanifolds contained in A_j. This gives the desired conclusion that $\nu(A_j \cap \hat{T}^{-1}\mathcal{S}_\infty^+) = 0$.

To finish, it remains to prove that $\nu(A_j \cap \hat{\mathcal{S}}_\infty^-) = 0$. We have

$$A_j \cap \hat{\mathcal{S}}_\infty^- = \bigcup_{i=1}^{k} \{x \in A_j \mid x_i \in \hat{\mathcal{S}}_\infty^+\},$$

where $-x = \{x_1, \ldots, x_k\}$ and $k > 0$ is the number of boundary components of Q containing $\pi(x)$. It is clear that k is bounded above by a number that depends only on Q; in fact, for cylindrical semi-focusing billiards, we have $k \leq 3$. Since each transformation $x \mapsto x_i$ preserves ν, and $\nu(A_j \cap \hat{\mathcal{S}}_\infty^+) = 0$, it follows that $\nu(A_j \cap \hat{\mathcal{S}}_\infty^-) \geq 3\nu(A_j \cap \hat{\mathcal{S}}_\infty^+) = 0$. This completes the proof. □

4.3.4. *Tail bound.* We now extend [**KSS**, Lemma 6.1], the so-called 'tail bound lemma', to cylindrical semi-focusing billiards. To do this, we need only revise [**KSS**, Lemma 6.7], because the rest of the proof of [**KSS**, Lemma 6.1] holds as well for cylindrical semi-focusing billiards. We assume the reader to be familiar with [**KSS**, §§5 and 6]; this allows us to provide only those definitions that are strictly required by the proof of [**KSS**, Lemma 6.7].

The hypotheses of the tail bound lemma are the following. Consider $x \in \operatorname{int} \Sigma_+ \setminus \mathcal{S}_\infty^+$, and let $0 < \lambda = \lambda(x) < 1$ and $U = U(x) \subset \operatorname{int} \Sigma_+$ be, respectively, the real number and the neighborhood of x as given in Lemma 4.13. Further, set $\Lambda = \lambda^{-1}$ and let

$$\hat{\mathcal{R}} = \hat{\mathcal{R}}_2 \setminus (\hat{\mathcal{S}}_\infty^+ \cup \hat{\mathcal{S}}_\infty^- \cup \mathcal{N})$$

be the full ν-measure set as in the Ansatz. Finally, for any pair of integers $m, n > 0$, let

$$U_{n,m}^b = \{y \in U \mid z_{\text{tub}}(T^n y) < 1/\kappa_{n,c_3\delta}(y) \text{ and } \kappa_{n,c_3\delta}(y) \in [\Lambda^m, \Lambda^{m+1})\}.$$

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The union of all the sets $U^b_{n,m}$ forms U^b, the set of points of U having 'short' local manifolds. For more details, see [**BCST1**, §4.2] and [**KSS**, §5].

The next lemma is the extension of [**KSS**, Lemma 6.7] that we need.

LEMMA 4.18. *For every integer $m > 0$ and every function $F(\delta) \nearrow +\infty$, we have*

$$\sum_{n \geq F(\delta)} \mu(U^b_{n,m}) = o(\delta).$$

Proof. Let $y \in \hat{\mathcal{R}}$. By the Ansatz, there exists a neighborhood $V_0 \subset \operatorname{int} \Sigma \cup \hat{\mathcal{R}}_2$ of y and an integer $l > 0$ such that T^l is continuous on V_0 and the closure of $T^l V_0$ (here we mean $T^l w = T^{l-1}(\hat{T} w)$ for every $w \in V_0 \cap \hat{\mathcal{R}}_2$) is contained in $\operatorname{int} \Sigma_+$. Let k be the cardinality of $-y$. If V_0 is sufficiently small, then $-V_0$ has exactly k connected components. Let Z be one of them, and let z be the element of $-y = \{y_1, \ldots, y_k\}$ contained in Z. The map T^{-l} is clearly a diffeomorphism on $Z \cap \operatorname{int} \Sigma$. Since $V_0 \cap \mathcal{R}_1 = \emptyset$, we see that T^{-l} extends to a diffeomorphism from $Z \cap \operatorname{int} \Sigma$ to Z. This and the fact that the closure of $T^{-l} Z$ is contained in $\operatorname{int} \Sigma_+$ allow us to extend continuously the quadratic form $\mathcal{Q}$, and therefore the cone field $\mathcal{C}$, from $Z \cap \operatorname{int} \Sigma$ to the entire Z. Such extensions will still be denoted by $\mathcal{Q}$ and $\mathcal{C}$.

It is easy to see that for every $n > 0$, we can further shrink V_0 in such a way that T^n also extends to a diffeomorphism from $Z \cap \operatorname{int} \Sigma$ to Z.

Since $\mathcal{C}(z)$ is defined and $z \notin \hat{\mathcal{S}}^+_\infty$, we can apply Proposition 4.10 to z. Fix an integer $m > 0$. It follows that there exists an integer $n_0 = n_0(z) > 0$ such that

$$T^{n_0} z \in \operatorname{int} \Sigma_+ \quad \text{and} \quad \inf_{\substack{u \in \mathcal{C}(z) \\ \|u\|_p > 0}} \frac{\|D_z T^{n_0} u\|_p}{\|u\|_p} > \Lambda^{m+2}.$$

Then, by the continuity of $\mathcal{C}$ on Z, part (2) of Proposition 4.8 and taking a smaller V_0 if necessary, we see that

$$\kappa_{n_0,0}(T^{-n_0} w) > \Lambda^{m+2} \tag{15}$$

for all $w \in V_0 \setminus \hat{\mathcal{R}}_2$ such that $-w \in Z$. Note that $\kappa_{n_0,0}$ is not defined on elements of $\hat{\mathcal{R}}_2$. Now suppose that $w \in V_0 \setminus \hat{\mathcal{R}}_2$ and that there is an integer $n \geq n_0$ such that $T^{-n} w \in \operatorname{int} \Sigma_+$. By part (3) of Proposition 4.8 and (15), we then obtain

$$\kappa_{n,0}(T^{-n} w) \geq \kappa_{n_0,0}(T^{-n_0} w) > \Lambda^{m+2}. \tag{16}$$

Inequalities (15) and (16) are valid for every connected component Z of $-V_0$. Thus if $n_1 = \max\{n_0(y_1), \ldots, n_0(y_k)\}$, then we see that there exists a neighborhood $V_1 \subset V_0$ of y such that

$$\kappa_{n,0}(T^{-n} w) > \Lambda^{m+2}$$

for all $w \in V_1 \setminus \hat{\mathcal{R}}_2$ such that $T^{-n} w \in \operatorname{int} \Sigma_+$ and $n \geq n_1$. By choosing carefully a neighborhood $V_2 \subset V_1$ of y and a small $\delta_1 > 0$, we further obtain

$$\kappa_{n,\delta}(T^{-n} w) \geq \Lambda^{m+2}$$

for all $w \in V_2 \setminus \hat{\mathcal{R}}_2$ such that $T^{-n} w \in \operatorname{int} \Sigma_+$, $n \geq n_1$ and for all $0 < \delta < \delta_1$.

Since $U \subset \operatorname{int} \Sigma_+$, it follows immediately that for every $n \geq n_1$,

$$T^n U \cap V_2 \subset \{w \in \operatorname{int} \Sigma \mid \kappa_{n,\delta}(T^{-n} w) \geq \Lambda^{m+2} \text{ for all } 0 < \delta < \delta_1\}.$$

On the other hand, by the definition of $U^b_{m,n}$, we get

$$T^n U^b_{n,m} \subset \{w \in \text{int}\,\Sigma \mid \Lambda^m \leq \kappa_{n,c_3\delta}(T^{-n}w) < \Lambda^{m+1}\}.$$

Therefore if we choose $c_3\delta < \delta_1$, we can conclude that

$$T^n U^b_{n,m} \cap V_2 = \emptyset \quad \text{for all } n \geq n_1.$$

The rest of the proof runs exactly as in [**KSS**]. □

4.4. *LET and the Bernoulli property.* We now proceed to prove the LET and the Bernoulli property for cylindrical semi-focusing billiards.

THEOREM 4.19. (LET) *If $x \in \text{int}\,\Sigma$ is a sufficient point, then it has a neighborhood $U \subset \text{int}\,\Sigma$ contained* (mod 0) *in one Bernoulli component of T.*

Proof. Let k_1 be the integer associated to x as in Definition 4.1. We see immediately that $T^{k_1}x$ belongs to int Σ_+ and its positive semi-orbit is sufficient. For $T^{k_1}x$, the theorem follows from the Fundamental theorem proved in §4.3, [**KSS**, Corollary 3.12] and general results concerning the Bernoulli property of hyperbolic systems [**CH, OW**]. To conclude that the theorem is true for x as well, we need only to observe that T^{k_1} is a local diffeomorphism at x. □

THEOREM 4.20. *The billiard map T is Bernoulli.*

Proof. First note that all the elements of int $\Sigma_+ \setminus (\mathcal{S}^-_\infty \cap \mathcal{S}^+_\infty)$ are sufficient points. From Propositions A.2 and A.4, it follows immediately that $\mathcal{S}^-_\infty \cap \mathcal{S}^+_\infty$ is a union of at most countably many codimension-two submanifolds of int Σ. Hence we see that the set of sufficient points has full measure, and is connected in each connected component of int Σ.

Consider a boundary component $\Gamma_i \subset \partial Q_+$ (one of the cylinders of ∂Q). From previous considerations, the subset of Σ_i consisting of sufficient points is connected, and its measure equals $\mu(\Sigma_i)$. Using the LET, we easily deduce that Σ_i is contained (mod 0) in one Bernoulli component B of T.

It follows from the definition of a Bernoulli component that either $TB = B$ or there exists an integer $k > 1$ such that $T^jB \cap B = \emptyset$ for every $1 \leq j \leq k-1$, and $T^kB = B$. Since $\mu(\Sigma_i \cap T\Sigma_i) > 0$, i.e. the subset of Σ_i consisting of points whose negative semi-orbit leaves Γ_i after at least two consecutive collisions with Γ_i has positive measure, we conclude that $\mu(B \cap TB) > 0$, and hence that $TB = B$. In particular, B is an ergodic component of T. From this, it follows immediately that $\bigcup_{j\in\mathbb{Z}} T^j\Sigma_i = B$ (mod 0), because $\bigcup_{j\in\mathbb{Z}} T^j\Sigma_i$ is T-invariant and contained (mod 0) in B. On the other hand, by the geometry of Q, we have $\bigcup_{k\in\mathbb{Z}} T^k\Sigma_i = \text{int}\,\Sigma$ (mod 0). Thus $B = \text{int}\,\Sigma$ (mod 0), which means that T is Bernoulli. □

5. *Concluding remarks*

We have given a detailed proof of the Bernoulli property for cylindrical semi-focusing billiards such that the domains B_1 and B_2 (see §3) are semi-stadia. We are confident

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that our proof can be extended in a straightforward manner to the case where B_1 and B_2 are general domains bounded by arcs of circles and by at least one segment for which the corresponding planar billiards are hyperbolic. The hyperbolicity of three-dimensional billiards obtained in this way and other high-dimensional billiards was proved by Wojtkowski in a recent preprint [**W4**]. Even dispersing components are allowed to be part of the boundary of B_1 and B_2. Of course, these must be algebraic in view of the Lipschitz decomposability of the singular sets. Since dispersing components produce singularities generated by tangential collisions, and we have only considered singular sets generated by multiple collisions, to deal with the case of mixed boundary components (focusing, dispersing and flat) one must combine the results of the present paper with those of [**BCST1**], where only singularities generated by tangential collisions were considered. We have not carried out the proof of the Bernoulli property in this general setting because of the many technicalities that would have obscured our arguments.

A natural question is whether our results remain valid when the sections of the cylinders are general absolutely focusing curves [**B3, D, B4**]. We recall that, under proper conditions, these billiards are hyperbolic [**BD**]. The problem faced in this case is related to the monotonicity of the p-norm. In particular, Proposition 4.8 does not hold in this general context, and therefore a new approach is required. We believe, however, that the answer to the question above is positive if the sections of the cylinders are algebraic convex scatterers [**W2**], planar convex algebraic curves for which $d^2r/ds^2 \leq 0$; here r is the radius of curvature and s is the arc-length of the curve. In this case, in fact, we should be able to recover Proposition 4.8 if we use the same trick as in [**Sz**]: replace the p-norm by the ratio of the p-norm with $d(x)$, where $d(x)$ is the length of the segment of $L(x)$ that is contained in the cylinder $\Delta(x)$ (see §3.2). In addition, z_{tub} also has to be modified properly. We refer the reader to [**Sz**] where these modification are described in detail.

Finally, we remark that the results of this paper can be extended to the billiards in domains made out of boxes and spherical caps studied in [**BR1, BR3**], thus obtaining an alternative proof of their Bernoulli property, which was first proved in [**BR2**]. The complication one faces in trying to adapt our arguments to these billiards is that the p-norm (along the directions in the invariant cone field) can decrease between two consecutive collisions with the same spherical cap. Note that for cylindrical semi-focusing billiards, the same happens for consecutive collisions with the flat boundary components. A solution to this problem is to prove the LET only for points entering the spherical caps, which is still enough to deduce the Bernoulli property. This can be done as for cylindrical semi-focusing billiards, the only difference being that now one has to make sure that the difference of the p-norm at two consecutive entrances of the orbit into a cap is non-negative, and this can be easily derived from the computations of [**W3, BR1, BR3**].

It would be interesting to check whether our results can be adapted to hyperbolic high-dimensional soft billiards such as those studied in [**BT**].

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Appendices

This part is independent of the previous content of the paper. The Fundamental theorem relies on the Lipschitz decomposability of the sets $\mathcal{S}_n^{\pm}$, the existence of local stable and unstable manifolds and the absolute continuity of these manifolds. In this section, we prove that cylindrical semi-focusing billiards have the above properties. To save space, we will not state the definition of Lipschitz decomposability, but instead refer the reader to [**BCST1**].

A. *Appendix. Regularity of the singular sets $\mathcal{S}_k^{\pm}$*

To show that $\mathcal{S}_n^{\pm}$ are Lipschitz decomposable, we will follow the argument used in [**BCST1**], where the same property is proved for semi-dispersing billiards with algebraic scatterers (the boundary components of cylindrical semi-focusing billiards are obviously algebraic).

PROPOSITION A.1. *Every set $\mathcal{S}_k^{\pm}$ is Lipschitz decomposable.*

Proof. In [**BCST1**, Theorem 5.7], it was shown that if the boundary components (scatterers) of a semi-dispersing billiard are measurable subsets of codimension-one algebraic varieties (SSAV), then every set $\mathcal{S}_k^{\pm}$ is a SSAV. In [**BCST1**, §§5.2–5.3], it was then proved that every SSAV is Lipschitz decomposable. While the last result is general and therefore also valid for cylindrical semi-focusing billiards, the fact that $\mathcal{S}_k^{\pm}$ is a SSAV was only explicitly proved for semi-dispersing billiards. The following remarks and modifications suffice to extend [**BCST1**, Theorem 5.7] to cylindrical semi-focusing billiards. We consider only the set $\mathcal{S}_k^{+}$, because $\mathcal{S}_k^{-}$ has the same properties as $\mathcal{S}_k^{+}$ in virtue of the time-reversing symmetry of billiards.

- The polynomials $B_1, \ldots, B_n$ in [**BCST1**] coincide with the functions $f_1, \ldots, f_n$, defining the boundary components of Q (see §2).
- When dealing with cylindrical semi-focusing billiards, we do not 'lift' the billiard dynamics to $\mathbb{R}^3$ as in [**BCST1**]. Therefore, the set $\mathcal{A}$, which keeps track of the position of the 'obstacles' in the lifted configuration space, has to be removed from the description of the collision sequence of the points of the phase space of cylindrical semi-focusing billiards.
- For cylindrical semi-focusing billiards, the singular sets $\mathcal{S}_k^{+}$ are generated by $\mathcal{R}_2$, the set of multiple collisions (whereas for semi-dispersing billiards they are generated by $\mathcal{R}_1$, the set of tangential collisions). This fact is used in the definition of the function Φ (see [**BCST1**, §5.1]), so we need to explain how to choose this function for cylindrical semi-focusing billiards. In the following, we will refer to the notation of [**BCST1**] (where $\mathcal{S}_k^{+}$ is denoted by $\mathcal{R}^{(k)}$). Given a point $(q_0, v_0) \notin \mathcal{S}_{k-1}^{+}$ of the billiard phase space, let $(q_i, v_i) = T^i(q_0, v_0)$ for $i = 1, \ldots, k$, and let $\sigma_1, \ldots, \sigma_k$ be its symbolic collision sequence up to the kth collision. If this collision is singular, i.e. $(q_0, v_0) \in \mathcal{S}_k^{+} \setminus \mathcal{S}_{k-1}^{+}$, then q_k must belong to the intersection of Γ_{σ_k} with another boundary component of Q, say Γ_j with $1 \leq j \neq \sigma_k \leq n$. We thus set $\Phi = f_j$. □

In the following propositions, we further investigate the regularity of $\mathcal{S}_k^{\pm}$. These results are used in the proof of the Ansatz (Theorem 4.17).

PROPOSITION A.2. *For every $k > 0$, the sets $\mathcal{S}_k^-$ and $\mathcal{S}_k^+$ are finite unions of codimension-one submanifolds-with-boundary of* int Σ, *which can intersect only along their boundaries.*

Proof. We only prove the proposition for $\mathcal{S}_k^+$. In fact, this implies immediately the result for $\mathcal{S}_k^-$, because $\mathcal{S}_k^- = -\mathcal{S}_k^+$ and the involution $\mathcal{J}$ is a smooth diffeomorphism from int Σ onto itself. Further, since $\mathcal{S}_k^+ = \mathcal{S}_1^+ \cup \cdots \cup T^{-k+1}\mathcal{S}_1^+$ and the sets $\mathcal{S}_1^+, \ldots, T^{-k+1}\mathcal{S}_1^+$ are pairwise disjoint (see their definition in §2), it suffices to prove the proposition for sets of the form $T^{-k}\mathcal{S}_1^+$ with $k \geq 0$.

We will use induction. Accordingly, we first show that the proposition is true for $k = 0$, and then proceed in much the same way as in the proof of Theorem 4.17. Write

$$\bigcup_{i=1}^{n} \partial\Gamma_i = \bigcup_{j=1}^{m} \gamma_j,$$

where $\gamma_1, \ldots, \gamma_m$ are smooth curves intersecting only at their boundary. Next, for $1 \leq i \leq n$ and $1 \leq j \leq m$, define

$$A_{i,j} = \{y \in \text{int}\, \Sigma_i \mid q_1(y) \in \gamma_j\}.$$

Note that $A_{i,j} \cap A_{i',j'} = \emptyset$ for every $i \neq i'$. Since

$$\mathcal{S}_1^+ = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} A_{i,j}$$

and any two sets $A_{i,j}$ and $A_{i,j'}$ with $j \neq j'$ can only intersect along their boundaries, the same being true for γ_j and $\gamma_{j'}$, we see at once that to show that the proposition holds for $\mathcal{S}_1^+$, it is enough to prove it for each set $A_{i,j}$.

Some of the sets $A_{i,j}$ are empty, namely, $A_{i,j} = \emptyset$ if and only if $\gamma_j \subset \partial\Gamma_i$. Given a non-empty $A_{i,j}$, it is easily seen by direct inspection that there is an open set $V \subset f_j^{-1}(0)$ such that $\gamma_j \subset V$ and every ray $\{q + \tau v \mid \tau > 0\}$ with $(q, v) \in A_i$ is transversal to V. For the purpose of this proof only, we redefine the transformation q_1 by replacing int Σ with int Σ_i and ∂Q with V in its original definition (see §2). In other words, $q_1(q, v)$ denotes, in this proof, the point of intersection of V and the ray emanating from $(q, v) \in \text{int}\, \Sigma_i$. Since, by construction, this intersection is transversal, it is not difficult to check that q_1 is a smooth submersion from int Σ_i to V. We then see that each $A_{i,j}$ is a codimension-one submanifold-with-boundary of A_i, because γ_j is a codimension-one submanifold of V. This completes the proof of the proposition for $\mathcal{S}_1^+$, and hence for $\mathcal{S}_1^-$.

From what we have just proved, it follows that there are finitely many open connected sets $B_1, \ldots, B_l$, for some integer $l > 0$, such that

- int $\Sigma \setminus \mathcal{S}_1^- = \bigcup_{i=1}^{l} B_i$;
- $T^{-1}|_{B_i}$ is a smooth diffeomorphism for each $i = 1, \ldots, l$.

Next, suppose that the proposition is true for $T^{-k+1}\mathcal{S}_1^+$ with $k > 0$. Thus

$$T^{-k+1}\mathcal{S}_1^+ = M_1 \cup \cdots \cup M_s,$$

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where $M_1, \ldots, M_s$ are codimension-one smooth submanifolds-with-boundary of int Σ, which can intersect only along their boundaries. Accordingly, we have

$$T^{-k}\mathcal{S}_1^+ = T^{-1}(T^{-k+1}\mathcal{S}_1^+) = T^{-1}\bigg(\bigcup_{i=1}^{l} B_i \cap T^{-k+1}\mathcal{S}_1^+\bigg)$$
$$= T^{-1}\bigg(\bigcup_{i=1}^{l}\bigcup_{j=1}^{s} B_i \cap M_j\bigg) = \bigcup_{i=1}^{l}\bigcup_{j=1}^{s} T^{-1}(B_i \cap M_j).$$

Since every set B_i is open and the restriction of T^{-1} to B_i is a diffeomorphism, it follows immediately that each set $T^{-1}(B_i \cap M_j)$ is a codimension-one smooth submanifold-with-boundary of int Σ. Further, any two sets $T^{-1}(B_i \cap M_j)$ and $T^{-1}(B_i \cap M_{j'})$ with $j \neq j'$ can only intersect along their boundaries, because the same is true for M_j and $M_{j'}$. We then conclude that the proposition is true for $T^{-k}\mathcal{S}_1^+$ as well. This finishes the induction argument and therefore the proof of the proposition. □

Remark A.3. We do not know whether the closure of $\mathcal{S}_k^{\pm}$ is a finite union of compact codimension-one smooth submanifolds. This is certainly not true for semi-dispersing billiards when $k > 1$; see [**BCST1, BCST2**].

Consider $\mathcal{S}_j^-$ and $\mathcal{S}_k^+$ with $j, k \geq 1$. By Proposition A.2, we have

$$\mathcal{S}_j^- = \bigcup_{i=1}^{m} Y_i \quad \text{and} \quad \mathcal{S}_k^+ = \bigcup_{l=1}^{n} Z_l,$$

where $Y_1, \ldots, Y_m$ and $Z_1, \ldots, Z_n$ are codimension-one smooth submanifolds-with-boundary intersecting only along their boundaries.

PROPOSITION A.4. *For every $1 \leq i \leq m$ and $1 \leq l \leq n$, the submanifolds Y_i and Z_l intersect transversally.*

Proof. Fix $i, l > 0$. We can assume that $Y_i \cap Z_l \neq \emptyset$, otherwise there is nothing to prove. Let $x \in Y_i \cap Z_l$. By definition of $\mathcal{S}_j^-$ and $\mathcal{S}_k^+$, there are two integers $0 \leq j' \leq j-1$ and $0 \leq k' \leq k-1$ such that $Y_i \subset T^{j'}\mathcal{S}_1^-$ and $Z_l \subset T^{-k'}\mathcal{S}_1^+$. Note that $T^{-j'}$ and $T^{k'}$ are local diffeomorphisms at x. Let

$$\mathcal{L}^+ = \mathrm{Null}(D_{T^{k'}x}q_1|_{\mathcal{T}_{T^{k'}x}T^{k'}Z_l}).$$

This is a two-dimensional subspace of the tangent space of $T^{k'}Z_l$ at the point $T^{k'}x$. In fact, as in the proof of Proposition 4.14, we see that

$$\mathcal{L}^+ = \{(-t(T^{k'}x)\eta, \eta) \mid \eta \in \mathbb{V}_{T^{k'}x}\}.$$

Further, let

$$\mathcal{L}^- = D_{-T^{-j'}x}\mathcal{J}\mathrm{Null}(D_{-T^{-j'}x}q_1|_{\mathcal{T}_{-T^{-j'}x}-T^{-j'}Y_i}).$$

Since

$$\mathrm{Null}(D_{-T^{-j'}x}q_1|_{\mathcal{T}_{-T^{-j'}x}-T^{-j'}Y_i}) = \{(-t(-T^{-j'}x)\eta, \eta) \mid \eta \in \mathbb{V}_{-T^{-j'}x}\}$$

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and it is easy to check that

$$D_{-T^{-j'}x}\mathcal{J}(\xi, \eta) = (\xi, R(T^{-j'}x)\xi - \eta)$$

for all $(\xi, \eta) \in \mathcal{T}_{-T^{-j'}x}$ int Σ, it follows that

$$\mathcal{L}^- = \{(t(-T^{-j'}x)\eta, t(-T^{-j'}x)R(T^{-j'}x)\eta + \eta) \mid \eta \in \mathbb{V}_{T^{-j'}x}\},$$

which is a two-dimensional subspace of the tangent space of $T^{-j'}Y_i$ at $T^{-j'}x$.

We will separately prove the proposition for the two complementary cases: $\{T^{-j'}x, \ldots, T^{k'}x\} \subset \text{int } \Sigma_0$ and $\{T^{-j'}x, \ldots, T^{k'}x\} \cap \text{int } \Sigma_+ \neq \emptyset$. Let us start with the first case. We set

$$L^+ = D_{T^{k'}x}T^{-k'}\mathcal{L}^+ \quad \text{and} \quad L^- = D_{T^{-j'}x}T^{j'}\mathcal{L}^-.$$

As these sets are two-dimensional subspaces of $\mathcal{T}_x Z_l$ and $\mathcal{T}_x Y_i$, respectively, we see that the proposition will be proved for the case under consideration if we show that $L^- \cap L^+ = \{0\}$. With this purpose in mind, define

$$\tilde{Q}(u) = \langle \xi, \eta \rangle \quad \text{for } u = (\xi, \eta) \in \mathcal{T}_x \text{ int } \Sigma.$$

Next, the assumption on $\{T^{-j'}x, \ldots, T^{k'}x\}$ implies at once that $R(T^{-j'}x)$ is the null operator, and the differentials $D_{T^{k'}x}T^{-k'}$ and $D_{T^{-j'}x}T^{j'}$ have the following matrix form in coordinates (ξ, η):

$$D_{T^{k'}x}T^{-k'} = \begin{pmatrix} I & -a_+ \\ 0 & I \end{pmatrix} \quad \text{and} \quad D_{T^{-j'}x}T^{j'} = \begin{pmatrix} I & a_- \\ 0 & I \end{pmatrix},$$

where $a_+, a_- \geq 0$ are the lengths of the pieces of the orbits $\{x, \ldots, T^{k'}x\}$ and $\{T^{-j'}x, \ldots, x\}$, respectively. An easy computation then gives

$$L^+ = \{(-(t(T^{k'}x) + a_+)\eta, \eta) \mid \eta \in \mathbb{V}_x\},$$
$$L^- = \{((t(-T^{-j'}x) + a_-)\eta, \eta) \mid \eta \in \mathbb{V}_x\}.$$

Since $t(T^{k'}x), t(-T^{-j'}x) > 0$, we conclude that

$$\tilde{Q}(u) = -(t(T^{k'}x) + a_+)\|\eta\|^2 < 0 \quad \text{for } u = (\xi, \eta) \in L^+ \setminus \{0\},$$
$$\tilde{Q}(u) = (t(T^{-j'}x) + a_-)\|\eta\|^2 > 0 \quad \text{for } u = (\xi, \eta) \in L^- \setminus \{0\},$$

which clearly means that $L^- \cap L^+ = \{0\}$.

It remains to consider the case $\{T^{-j'}x, \ldots, T^{k'}x\} \cap \text{int } \Sigma_+ \neq \emptyset$. Let i' be the smallest integer between $-j'$ and k' such that $T^{i'}x \in \text{int } \Sigma_+$. For this case, define

$$L^+ = D_{T^{k'}x}T^{i'-k'}\mathcal{L}^+ \quad \text{and} \quad L^- = D_{T^{-j'}x}T^{i'+j'}\mathcal{L}^-.$$

These are two-dimensional subspaces of $\mathcal{T}_{T^{i'}x}T^{i'}Z_l$ and $\mathcal{T}_{T^{i'}x}T^{i'}Y_i$, respectively. Since $T^{i'}$ is a local diffeomorphism at x, we see that it is enough show that $L^- \cap L^+ = \{0\}$. Note that the quadratic form $\mathcal{Q}$ is defined at each point of $\{T^{i'}x, \ldots, T^{k'}x\}$. From the definition of $\mathcal{Q}$ and $\mathcal{L}^+$, it follows that

$$\mathcal{Q}_{T^{k'}x}(D_x T^{k'}u) = -(t(T^{k'}x) + l_-(T^{k'}x))\|\eta\|^2 < 0$$

for all $u = (\xi, \eta) \in \mathcal{L}^+$. Using the monotonicity of $\mathcal{Q}$, we then obtain

$$\mathcal{Q}_{T^{i'}x}(u) < 0 \quad \text{for } u \in L^+ \setminus \{0\}. \tag{17}$$

We now observe that if $i' = -j'$, then $R(T^{-j'}x) \neq 0$, otherwise $T^{-j'}x \in \operatorname{int} \Sigma_0$ and hence $R(T^{-j'}x) = 0$. In the last case, in fact, we have $\{T^{-j'}x, \ldots, T^{i'-1}x\} \subset \operatorname{int} \Sigma_0$ and it is easy to see that, with respect to (ξ, η), the differential $D_{T^{-j'}x}T^{i'+j'}$ has the following matrix form:

$$D_{T^{-j'}x}T^{i'+j'} = \begin{pmatrix} I & 0 \\ R(T^{i'}x) & I \end{pmatrix} \begin{pmatrix} I & a_- \\ 0 & I \end{pmatrix}, \tag{18}$$

where $a_- > 0$ is the length of the orbit $\{T^{-j'}x, \ldots, T^{i'}x\}$. Let

$$\tilde{t}(x) = \begin{cases} t(-T^{-j'}x) & \text{if } i' = -j', \\ t(-T^{-j'}x) + a_- & \text{otherwise,} \end{cases}$$

and

$$\mathcal{A} = \tilde{t}(x)(\tilde{t}(x)R(T^{i'}x) + I) + d(T^{i'}x)(\tilde{t}(x)R(T^{i'}x) + I)^2.$$

A straightforward computation using (18) shows that

$$\mathcal{Q}_{T^{i'}x}(D_{T^{-j'}x}T^{i'+j'}u) = \langle \eta, \mathcal{A}\eta \rangle \quad \text{for all } u = (\xi, \eta) \in \mathcal{L}^-.$$

Since the eigenvalues of $R(T^{i'}x)$ are 0 and $-2/d(T^{i'}x)$, it follows that the eigenvalues of $\mathcal{A}$ are given by

$$\tilde{t}(x) + d(T^{i'}x) \quad \text{and} \quad \frac{1}{d(T^{i'}x)}(\tilde{t}(x) - d(T^{i'}x))(2\tilde{t}(x) - d(T^{i'}x)).$$

These are both positive numbers; while this is obvious for the first eigenvalue, the positivity of the second one is a consequence of the geometry of Q and condition (E) (actually condition (H) is enough here) which imply that $\tilde{t}(x) > d(T^{i'}x)$. Hence

$$\mathcal{Q}_{T^{i'}x}(u) > 0 \quad \text{for all } u \in L^- \setminus \{0\}.$$

From this and (17), it follows that $L^- \cap L^+ = \{0\}$, which concludes the proof. □

B. *Appendix. Stable and unstable manifolds*

The existence of local invariant manifolds is obtained by using the theory developed in [**KS**]. What we need to do is to demonstrate that conditions (1.1)–(1.4) of [**KS**, Part I] are satisfied by cylindrical semi-focusing billiards.

Recall that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{S}$ and Σ is endowed with the distance ρ (see §2). For every $x \in \operatorname{int} \Sigma$, let $\rho(x, \mathcal{R}) = \inf_{y \in \mathcal{R}} \rho(x, y)$. Furthermore, denote by $\|D_x T^{\pm 1}\|$ and $\|D_x^2 T^2\|$ the operator norm induced by the Riemannian metric g (see §3.1) of $D_x T^{\pm 1}$ and $D_x^2 T$, respectively. For billiards, conditions (1.1)–(1.4) of [**KS**, Part I] take the following form.

Condition (1.1). For each $\epsilon > 0$, let $\mathcal{R}^{[\epsilon]} = \{x \in \Sigma \mid \rho(x, \mathcal{R}) < \epsilon\}$. Then there exist two real numbers $a, c > 0$ such that

$$\mu(\mathcal{R}^{[\epsilon]}) \leq c\epsilon^a \quad \text{for } \epsilon > 0.$$

Condition (1.2).

$$\int_{\Sigma} \log^{+} \|D_x T\| \, d\mu < +\infty \quad \text{and} \quad \int_{\Sigma} \log^{+} \|D_x T^{-1}\| \, d\mu < +\infty.$$

Condition (1.3). There exist two real numbers $b_1, c_1 > 0$ such that

$$\|D_x T\| \leq c_1 \rho(x, \mathcal{R})^{-b_1} \quad \text{for all } x \in \Sigma'.$$

Condition (1.4). There exist two real numbers b_2, c_2 such that

$$\|D_x^2 T\| \leq c_2 \rho(x, \mathcal{R})^{-b_2} \quad \text{for all } x \in \Sigma'.$$

PROPOSITION B.1. *Condition (1.1) is satisfied by cylindrical semi-focusing billiards.*

Proof. Let m be the normalized Lebesgue measure on Σ generated by the Riemannian metric on each Σ_i induced by the standard Riemannian metric on $\mathcal{T}\mathbb{R}^3$. It suffices to prove the proposition with respect to m, because $\mu \leq m$. Given a subset A of Σ and a real number $\delta > 0$, let $A^{[\delta]}$ denote the δ-neighborhood of A with respect to the distance ρ. Since $\mathcal{R}_1 \cup \mathcal{R}_2$ is a finite union of codimension-one smooth compact submanifolds, we can find a real number $c_1 > 0$ such that the m-measure of $(\mathcal{R}_1 \cup \mathcal{R}_2)^{[\delta]}$ is less than $c_1\delta$ for a sufficiently small $\delta > 0$. Furthermore, we know that $\mathcal{S} = \mathcal{S}_1^{+}$ is Lipschitz decomposable by Proposition A.1. Using [**BCST1**, Lemma 3.8], we then see that there is a constant $c_2 > 0$ such that the m-measure of $\mathcal{S}^{[\delta]}$ is less than $c_2\delta$ for a sufficiently small $\delta > 0$. By taking $c = \max\{c_1, c_2\}$, we can then conclude that $m(\mathcal{R}) \leq c\delta$ for every sufficiently small $\delta > 0$. □

Although the map $T : \Sigma' \to T\Sigma'$ is a smooth diffeomorphism, $D_x T$ and $D_x T^2$ are not bounded in a neighborhood of $\mathcal{R}$. The following two propositions tell us how fast these operators can diverge as $\mathcal{R}$ is approached.

PROPOSITION B.2. *There exists a real number $a > 0$ such that for every $x = (q, v) \in \Sigma'$, we have*

$$\|D_x T\| \leq \frac{a}{|\langle N(q_1(x)), v\rangle|} \quad \textit{and} \quad \|D_x^2 T\| \leq \frac{a}{|\langle N(q_1(x)), v\rangle|^3}.$$

Proof. We will only give the proof of the first inequality, because the second one can be proved similarly.

The first inequality is proved if we can show that there exists $a > 0$ such that for every $x \in \Sigma'$, we can find a neighborhood $U \subset \Sigma'$ of x for which the inequality is satisfied on U. In fact, we will prove a stronger version of this property where U and T are replaced, respectively, by a neighborhood $\tilde{U} \subset \mathcal{T}\mathbb{R}^3$ of x and a smooth extension $\tilde{T}$ of T on $\tilde{U}$. This makes sense because U can be thought of as a subset of $\mathcal{T}\mathbb{R}^3$. The operator norm that we need to use with $D\tilde{T}$ and $D^2\tilde{T}$ is the one induced by the standard Riemannian metric on $\mathcal{T}\mathbb{R}^3$.

Consider $x = (q, v) \in \Sigma'$ such that $Tx \in \operatorname{int} \Sigma_m$ for some $1 \leq m \leq n$. In other words, $f_m(q + t(x)v) = 0$ (see §2). Let us apply the Implicit Function Theorem to f_m at x. Since $x \notin \mathcal{R}_4$, an easy computation gives

$$\frac{\partial f_m}{\partial t}(q_1(x)) = c(x)\langle N(q_1(x)), v\rangle \neq 0$$

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for some real $c(x) \neq 0$. It follows immediately that there exist a neighborhood $\tilde{U} \subset \{(\tilde{q}, \tilde{v}) \in \mathcal{T}\mathbb{R}^3 \mid 1/2 \leq \|\tilde{v}\| \leq 2\}$ of x and a smooth extension $\tilde{t}$ of t on $\tilde{U}$. Next, for every $\tilde{x} = (\tilde{q}, \tilde{v}) \in \tilde{U}$ define

$$\begin{aligned} \tilde{q}_1(\tilde{x}) &= \tilde{q} + \tilde{t}\tilde{v}, \\ \tilde{v}_1(\tilde{x}) &= \tilde{v} - 2\langle N(\tilde{q}_1(\tilde{x})), \tilde{v}\rangle N(\tilde{q}_1(\tilde{x})), \\ \tilde{T}(\tilde{x}) &= (\tilde{q}_1(\tilde{x}), \tilde{v}_1(\tilde{x})). \end{aligned}$$

These transformations are the desired smooth extensions of q_1, v_1, T on $\tilde{U}$.

Fix a cartesian coordinate system in $\mathcal{T}_{\tilde{x}}(\mathcal{T}\mathbb{R}^3) \simeq \mathbb{R}^3 \times \mathbb{R}^3$. The task is now to compute the partial derivatives of $\tilde{q}_1$, $\tilde{v}_1$, $\tilde{t}$ at $\tilde{x} \in \tilde{U}$ with respect to this system of coordinates. A straightforward computation shows that the partial derivatives of the components of $\tilde{q}_1$ and $\tilde{v}_1$ are given by

$$\begin{aligned} \frac{\partial \tilde{q}_{1,i}}{\partial \tilde{q}_j}(\tilde{x}) &= \delta_{ij} + \tilde{v}_i \frac{\partial \tilde{t}}{\partial \tilde{q}_j}(\tilde{x}), \\ \frac{\partial \tilde{q}_{1,i}}{\partial \tilde{v}_j}(\tilde{x}) &= \tilde{v}_i \frac{\partial \tilde{t}}{\partial \tilde{q}_j}(\tilde{x}) + \delta_{ij}\tilde{t}(\tilde{x}), \\ \frac{\partial \tilde{v}_{1,i}}{\partial \tilde{q}_j}(\tilde{x}) &= -2N_i(\tilde{q}_1) \sum_{k,l=1}^{3} \tilde{v}_k \frac{\partial N_k}{\partial \tilde{q}_{1,l}} \frac{\partial \tilde{q}_{1,l}}{\partial \tilde{q}_j} - 2\langle N(\tilde{q}_1), \tilde{v}\rangle \sum_{l=1}^{3} \frac{\partial N_i}{\partial \tilde{q}_{1,l}} \frac{\partial \tilde{q}_{1,l}}{\partial \tilde{q}_j} \\ \frac{\partial \tilde{v}_{1,i}}{\partial \tilde{v}_j}(\tilde{x}) &= \delta_{ij} - 2N_i(\tilde{q}_1) \sum_{k,l=1}^{3} \tilde{v}_k \frac{\partial N_k}{\partial \tilde{q}_{1,l}} \frac{\partial \tilde{q}_{1,l}}{\partial \tilde{v}_j} - 2N_i(\tilde{q}_1)N_j(\tilde{q}_1) \\ &\quad - 2\langle N(\tilde{q}_1), \tilde{v}\rangle \sum_{l=1}^{3} \frac{\partial N_i}{\partial \tilde{q}_{1,l}} \frac{\partial \tilde{q}_{1,l}}{\partial \tilde{v}_j}, \end{aligned}$$

where δ_{ij} is the Kronecker delta and N_k is the kth component of $N(\tilde{q}(\tilde{x}))$. From $f_m(\tilde{q} + \tilde{t}(\tilde{x})\tilde{v}) = 0$ for all $\tilde{x} \in \tilde{U}$, it follows immediately that

$$\begin{aligned} \frac{\partial \tilde{t}}{\partial \tilde{q}_j}(\tilde{x}) &= -\frac{1}{\langle N(\tilde{q}_1), \tilde{v}\rangle} N_j(\tilde{q}_1), \\ \frac{\partial \tilde{t}}{\partial \tilde{v}_j}(\tilde{x}) &= -\frac{\tilde{t}(\tilde{x})}{\langle N(\tilde{q}_1), \tilde{v}\rangle} N_j(\tilde{q}_1). \end{aligned}$$

To obtain the desired inequality, we only need to use the smoothness of $f_1, \ldots, f_n$, the compactness of $\Gamma_1, \ldots, \Gamma_n$ and the boundedness of $\|\tilde{v}\|$ to deduce that there exists a real number $a > 0$, depending only on Q, such that

$$\begin{aligned} \sup_{\tilde{x} \in \tilde{U}} \|D_{\tilde{x}}\tilde{T}\| &\leq \sup_{\tilde{x} \in \tilde{U}} \sum_{i,j=1}^{3} \left(\left| \frac{\partial \tilde{q}_{1,i}}{\partial \tilde{q}_j}(\tilde{x}) \right| + \left| \frac{\partial \tilde{q}_{1,i}}{\partial \tilde{v}_j}(\tilde{x}) \right| + \left| \frac{\partial \tilde{v}_{1,i}}{\partial \tilde{q}_j}(\tilde{x}) \right| + \left| \frac{\partial \tilde{v}_{1,i}}{\partial \tilde{v}_j}(\tilde{x}) \right| \right) \\ &\leq \frac{a}{|\langle N(\tilde{q}_1), \tilde{v}\rangle|}. \end{aligned}$$

This establishes the first inequality; as already mentioned, the second one can be proved similarly. □

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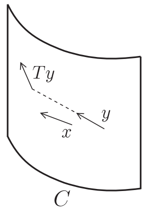

FIGURE B.1. The vector x is tangent to the cylinder C, and the neighborhood $W(x)$ consists of vectors y close to x such that the base points of y and Ty lie on C.

PROPOSITION B.3. *There exists a real number* $b > 0$ *such that*

$$|\langle N(q_1(y)), v\rangle| \geq b\rho(y, \mathcal{R})^2 \quad \textit{for all } y = (q, v) \in \Sigma'.$$

Proof. Since $|\langle N(q_1(y)), v_1(y)\rangle| = |\langle N(q_1(y)), v\rangle|$ and $\mathcal{R}$ is compact, it suffices to prove that for every $x \in \mathcal{R}$, there exist a neighborhood $U(x) \subset \Sigma$ of x and a real number $b(x) > 0$ such that

$$|\langle N(q_1(y)), v_1(y)\rangle| \geq b(x)\rho(y, \mathcal{R})^2 \quad \text{for } y = (q, v) \in U(x) \setminus \mathcal{R}. \tag{19}$$

This is certainly true for all $x \in \mathcal{R}$ with a neighborhood $W(x) \subset \Sigma$ such that $\inf_{y \in W(x)\setminus\mathcal{R}} |\langle N(q_1(y)), v_1(y)\rangle| > 0$. Thus to prove (19), we can restrict our analysis to those $x \in \mathcal{R}$ having a neighborhood $W(x)$ such that

$$\inf_{y \in W(x)\setminus\mathcal{R}} |\langle N(q_1(y)), v_1(y)\rangle| = 0.$$

Since the neighborhood $W(x)$ can be arbitrarily small, we assume that $W(x) \subset \{y \in \Sigma \mid \rho(y, \mathcal{R}) < 1\}$. It is not difficult to see that such an x must fall into one of the following classes:

(i) $x \in \Sigma_+ \cap \mathcal{R}_1$, and $W(x)$ is such that $\pi(y)$, $\pi(Ty)$ belong to the same cylinder C for every $y \in W(x) \setminus \mathcal{R}$ (see Figure B.1);

(ii) $x \in \Sigma_0 \cap \mathcal{R}_1$, and $W(x)$ is such that there are a cylinder C and a flat face F for which $\pi(y) \in F$ and $\pi(Ty) \in C$ for every $y \in W(x) \setminus \mathcal{R}$ (see Figure B.2);

(iii) $x \in \Sigma_0 \cap \mathcal{R}_1$, and $W(x)$ is such that there are a cylinder C and a flat face F for which $\pi(y) \in C$ and $\pi(Ty) \in F$ for every $y \in W(x) \setminus \mathcal{R}$ (see Figure B.3);

(iv) $x \in \Sigma_+ \cap \mathcal{R}_1$, and $W(x)$ is such that there are a cylinder C and a flat face F for which $\pi(y) \in F$ and $\pi(Ty) \in C$ for every $y \in W(x) \setminus \mathcal{R}$ (see Figure B.4);

(v) $x \in \Sigma_0 \cap \mathcal{R}_1$, and $W(x)$ is such that $\pi(y)$, $\pi(Ty)$ belong to distinct cylinders for every $y \in W(x) \setminus \mathcal{R}$ (see Figure B.5);

(vi) $x \in \Sigma_0 \cap \mathcal{R}_1$, and $W(x)$ is such that $\pi(y)$, $\pi(Ty)$ belong to distinct flat faces for every $y \in W(x) \setminus \mathcal{R}$ (see Figure B.6).

We will prove (19) for each of the cases above. With the exception of case (v), we will actually prove (19) with ρ^2 replaced by ρ; since $W(x) \subset \mathcal{R}^{[1]}$, this clearly implies (19) in the original form. Furthermore, we will assume without loss of generality that the radii of the circles forming the orthogonal sections of the cylinders of ∂Q are equal to 1. Let D be the diameter of Q.

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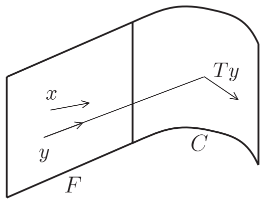

FIGURE B.2. The vector x is tangent to the flat face F, and the neighborhood $W(x)$ consists of vectors y close to x such that the base points of y and Ty lie on F and the cylinder C, respectively.

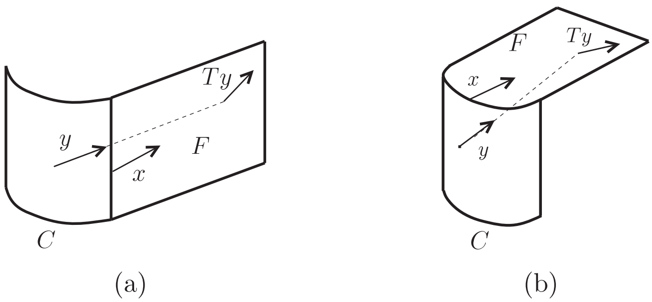

FIGURE B.3. The vector x is tangent to the flat face F in (a) and to the cylinder C in (b). The neighborhood $W(x)$ consists of vectors y close to x such that the base points of y and Ty lie on C and F, respectively.

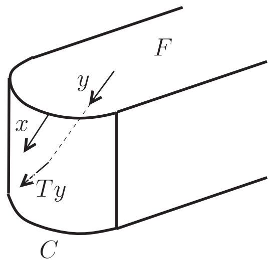

FIGURE B.4. The vector x is tangent to the cylinder C, and the neighborhood $W(x)$ consists of vectors y close to x such that the base points of y and Ty lie on the flat face F and C, respectively.

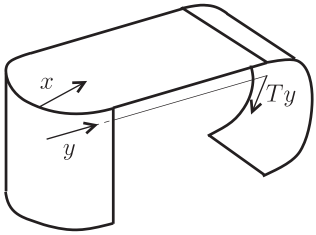

FIGURE B.5. This configuration is similar to the one depicted in Figure B.3(a), the only difference being that the base points of the vectors y and Ty lie on distinct cylinders.

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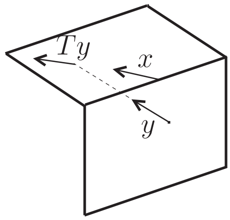

FIGURE B.6. The vector x is tangent to the flat face F, and the neighborhood $W(x)$ consists of vectors y close to x such that the base points of y and Ty lie on adjacent flat faces of ∂Q.

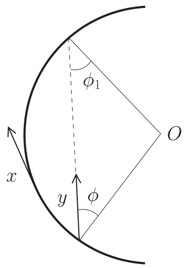

FIGURE B.7. Orthogonal projection of the configuration (i) onto the orthogonal section of the cylinder C. The vector x is tangent to the cylinder C, while ϕ and ϕ_1 are the angles formed by the normal of C with the projections of y and Ty, respectively.

Take $y = (q, v) \in \Sigma'$. Then fix a Cartesian coordinate system (X, Y, Z) of $\mathbb{R}^3$ such that $N(q)$ lies on the Y-axis and the Z-axis coincides with the axis of the cylinder C. Note that the origin of (X, Y, Z) lies somewhere on the axis of C. Denote by $\theta \in [0, \pi]$ the angle formed by v with the Z-axis, and denote by $\phi \in [-\pi/2, \pi/2]$ the angle formed by the orthogonal projection of v onto the XY-plane with the Y-axis. The pair θ, ϕ is a system of coordinates for v with parametrization

$$(\theta, \phi) \mapsto v = (\sin\theta \sin\phi, \sin\theta \cos\phi, \cos\theta).$$

We will denote by (θ_1, ϕ_1) the (θ, ϕ)-coordinates of $v_1(y)$.

Case (i). Let $y = (q, v) \in \Sigma_+$ as in (i). In Figure B.7, we have depicted the orthogonal projection of the configuration (i) onto the XY-plane; the point O is the center of the orthogonal section of C. Since $\theta_1 = \theta$ and $\phi = \phi_1$, C being a cylinder with circular section, it follows that

$$|\langle N(q_1(y)), v_1(y)\rangle| = \sin\theta_1 \cos\phi_1 = \sin\theta \cos\phi.$$

If ψ is the angle formed by v with the inner normal of C at q, then we easily obtain

$$\begin{aligned} \rho(y, \mathcal{R}) \le \rho(y, \mathcal{R}_1) &= \frac{\pi}{2} - \psi = \arcsin(\sin\theta \cos\phi) \\ &\le 2 \sin\theta \cos\phi = 2|\langle N(q_1(y)), v_1(y)\rangle|, \end{aligned}$$

which in turn implies (19) with $b(x) = 1/2$.

Case (ii). Figure B.8 shows the orthogonal projection of the configuration corresponding to (ii) onto the XY-plane, which by definition is the plane orthogonal to the axis of the

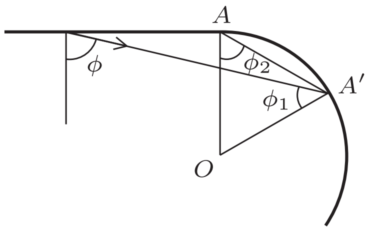

FIGURE B.8. Orthogonal projection of the configuration (ii) onto the orthogonal section of the cylinder C. The angle ϕ is formed by the projection of y and the normal of F. The point A belongs to the intersection of C with the flat face F, and the point A' is the projection of the base point of Ty.

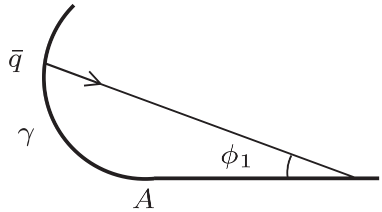

FIGURE B.9. Orthogonal projection of the configuration (iii) case (a) onto the orthogonal section of the cylinder C. The point A belongs to the intersection of C with the flat face F, the point $\bar{q}$ is the projection of the base point of y, and ϕ_1 is the angle formed by the projection of Ty and F.

cylinder C. The point O and the segments AO, $A'O$ are the center and two radii of the orthogonal section of C. From Figure B.8, we easily deduce that $\phi_1 \leq \phi_2 \leq \phi$. It is also clear that $\theta_1 = \theta$ and therefore

$$\sin\theta \cos\phi \leq \sin\theta_1 \cos\phi_1.$$

The desired inequality with $b(x) = 1/2$ is now obtained by arguing as in the final part of the analysis of case (i).

Case (iii). We further divide the analysis of this case into two subcases corresponding to the configurations depicted in Figures B.3(a) and B.3(b).

First we consider subcase (a). Referring to Figure B.9, which represents the orthogonal projection of the configuration corresponding to (a) onto the XY-plane, we easily deduce that

$$\begin{aligned}\rho(y, \mathcal{R}) &\leq \rho(y, \mathcal{R}_2) = \text{length}(\gamma)\\ &\leq 2t(y) \sin\theta_1 \sin\phi_1\\ &\leq 2D|\langle N(q_1(y)), v_1(y)\rangle|,\end{aligned}$$

where γ is the arc joining the point A, where the circle meets the segment, to the orthogonal projection $\bar{q}$ of q onto the XY-plane, and $t(y)$ is the length of the segment having endpoints q and $q_1(y)$ (see §2). This proves (19) with $b(x) = (2D)^{-1}$.

We now study subcase (b). It is easily seen that

$$|\langle N(q_1(y)), v_1(y)\rangle| = |\cos\theta|$$

and

$$\rho(y, \mathcal{R}) \le \rho(y, \mathcal{R}_2) = t(y)\cos\theta \\ \le D|\langle N(q_1(y)), v_1(y)\rangle|.$$

Thus, in this case, (19) is satisfied with $b(x) = 1/D$.

Case (iv). Denote by $\bar{q}_1$ the orthogonal projection of $q_1(y)$ onto the XY-plane, and denote by $\bar{v}$ the normalized orthogonal projection of v onto the XY-plane (Figure B.10). It follows that $v = \sin\theta_1\bar{v} + (0, 0, \cos\theta_1)$ and $\bar{q}_1 = q + \bar{t}\bar{v}$ for some real $\bar{t} > 0$. Let $\alpha = \langle q, \bar{v}\rangle$ and $\beta = 1 - \|q\|^2$. Since $\|\bar{q}_1\|^2 = 1$, an easy computation gives

$$\bar{t} = -\alpha + \sqrt{\alpha^2 + \beta^2}.$$

From $N(q_1(y)) = -\bar{q}_1$, we see that

$$|\langle N(q_1(y)), v_1(q)\rangle| = |\langle N(q_1(y)), v\rangle| = (\alpha + \bar{t})\sin\theta_1 \\ = \sqrt{\alpha^2 + \beta^2}\sin\theta_1 \ge |\alpha|\sin\theta_1,$$

which implies

$$2|\langle N(q_1(y)), v_1(q)\rangle| \ge (\alpha + |\alpha| + \bar{t})\sin\theta_1 \\ \ge \bar{t}\sin\theta_1.$$

Let h be the height of the cylinder C. It follows immediately that

$$|\tan\theta_1| \ge \frac{\bar{t}}{h}.$$

If $\theta_1 \in [0, \pi/3] \cup [2\pi/3, \pi]$, then $2\sin\theta_1 \ge |\tan\theta_1|$ and

$$|\langle N(q_1(y)), v_1(y)\rangle| \ge \frac{\bar{t}^2}{4h} \ge \frac{1}{4h}\rho^2(y, \mathcal{R}_2) \\ \ge \frac{1}{4h}\rho^2(y, \mathcal{R});$$

otherwise $\sin\theta_1 \ge \sqrt{3}/2$ and

$$|\langle N(q_1(y)), v_1(y)\rangle| \ge \frac{\sqrt{3}}{2}\bar{t} \ge \frac{\sqrt{3}}{2}\rho(y, \mathcal{R}_2) \\ \ge \frac{\sqrt{3}}{2}\rho(y, \mathcal{R}).$$

Therefore (19) is satisfied with $b(x) = \min\{(4h)^{-1}, \sqrt{3}/2\}$.

Case (v). In Figure B.11, we have drawn the orthogonal projection of the configuration onto the plane orthogonal to the cylinder C_2 containing $q_1(y)$. The point O denotes the center of the circle (section of C_2). A straightforward computation shows that

$$\rho(y, \mathcal{R}_2) = 1 - \cos(\theta - \phi_1) + t(y)\cos\theta.$$

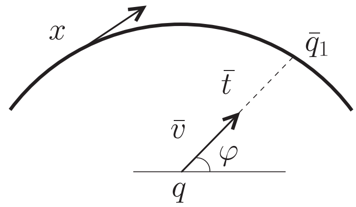

FIGURE B.10. Orthogonal projection of the configuration (iv) onto the orthogonal section of cylinder C. The vector x is tangent to C whose projection is a circle. The base point q of y belongs to the flat face F orthogonal to C, $\bar{q}_1$ is the projection of the base point of Ty, and $\bar{v}$ is the projection of the velocity of the particle.

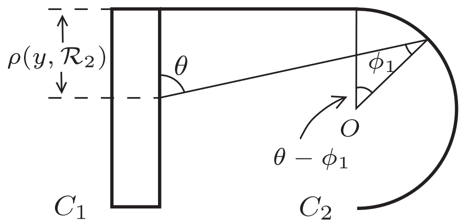

FIGURE B.11. Orthogonal projection of the configuration (v) onto the orthogonal section of the cylinder C_2. Here θ is the angle formed by the projection of y and the projection of the axis of the cylinder C_1, while ϕ_1 is the angle formed by the projection of Ty and the normal of C_2.

We also see that $\theta \geq \phi_1$. Thus

$$\begin{aligned} 1 - \cos(\theta - \phi_1) &= 1 - \sin\theta \sin\phi_1 + \cos\theta \cos\phi_1 \\ &\leq 1 - \sin^2\phi_1 + \cos^2\phi_1 = 2\cos^2\phi_1 \\ &\leq 2\cos\phi_1 \end{aligned}$$

and

$$t(y)\cos\theta \leq D\cos\phi_1.$$

For every y sufficiently close to x, it turns out that θ_1 is close to $\pi/2$, and so $\sin\theta_1 \geq 1/2$. Combining all the preceding statements, we can conclude that

$$\begin{aligned} \rho(y, \mathcal{R}) \leq \rho(y, \mathcal{R}_2) \leq (2 + D)\cos\phi_1 &= (2 + D)\frac{|\langle N(q_1(y)), v_1(y)\rangle|}{\sin\theta_1} \\ &\leq (4 + 2D)|\langle N(q_1(y)), v_1(y)\rangle|, \end{aligned}$$

and hence (19) is satisfied with $b(x) = (4 + 2D)^{-1}$.

Case (vi). The proof of (19) for this case is exactly the same as the proof for subcase (b) of (iii). □

COROLLARY B.4. *Conditions (1.2)–(1.4) are satisfied by cylindrical semi-focusing billiards.*

Proof. The proof of condition (1.2) resembles the proof of [**KS**, Part V, Theorem 5.1], after taking into account the conclusion of Lemma B.2. Conditions (1.3) and (1.4) follow immediately from Lemma B.2 and Proposition B.3. □

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