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INSTABILITY OF SUNSPOT EQUILIBRIA IN REAL BUSINESS CYCLE MODELS UNDER INFINITE HORIZON LEARNING

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This paper examines the stability of sunspot equilibria in one-sector RBC models under infinite horizon learning. We present general conditions under which the reduced-form model can possess E-stable sunspot equilibria and apply these conditions to three prominent one-sector RBC models. We find that the rational expectations sunspot equilibria are generally unstable under learning.

Keywords: Adaptive Learning, Real Business Cycle Models, Indeterminacy, Rational Expectations

1. INTRODUCTION

It is now well known that with an increasing-returns-to-scale production technology, tax distortions, or other types of nonconvexities, an otherwise standard real business cycle (RBC) model can exhibit multiple rational expectations equilibria (REE) that are locally nonunique or indeterminate. Near such equilibria, there exist self-fulfilling dynamics driven by extraneous stochastic processes known as "sunspots."[1]

One critique of sunspot equilibria in RBC models is that these equilibria are unstable under adaptive learning dynamics.[2] Suppose agents know the correct specification of a reduced-form model but do not know its parameter values and act like econometricians in estimating them using adaptive learning techniques,

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learning almost always leads agents away from sunspot equilibria. Evans and McGough (2005a, 2005b) find that stability results can differ by how the law of motion of the economy is represented. With a general form representation, sunspot solutions are not stable under learning. In the case of a common factor representation, sunspot equilibria can be stable under learning. However, when they restrict attention to versions of the reduced form model consistent with calibrations of three structural models, they find that sunspot equilibria are always unstable under adaptive learning. They identify this result as a "stability puzzle." Duffy and Xiao (2007) find that it is the structural parameter restrictions of this class of models that prevent the equilibria from being simultaneously indeterminate and stable under learning. Consequently, learnable sunspot equilibria only occur when the economy exhibits empirically implausible dynamics, regardless of which representation the agents use. In a recent paper, McGough et al. (2013) expand this class of models to include cases in which there is negative externality in capital inputs. They find that with this feature, it is possible to find learnable sunspot REEs that do not possess implausible system dynamics.

An important conclusion that emerges from recent research is that stability properties of REEs under learning can differ when agents' learning horizon changes. The standard approach is the Euler equation learning: It lets agents forecast future variables based on the Euler equation of the model, and the forecasts are typically one-step-ahead. All the aforementioned papers use this approach. An alternative learning algorithm is the infinite horizon learning approach put forth by Preston (2005). Agents are assumed to use their belief functions to make forecasts for the infinite horizon. Preston (2005) and Eusepi and Preston (2011) demonstrate that with infinite horizon learning, both the stability property and transitional dynamics of REEs can differ from those under the Euler equation learning. Infinite horizon learning has proved handy when the economic problems call for long-term forecasts. For example, Evans et al. (2013) and Evans and Mitra (2013) use infinite horizon learning to study fiscal policy and growth issues.

To our knowledge, no paper has studied the learnability of sunspot REEs in RBC models using the infinite horizon approach. This paper undertakes this task. We derive general conditions under which an REE is both indeterminate and stable under learning and apply them to three specific structural models: the Farmer and Guo (1994) model, the Schmitt-Grohé and Uribe (1997) model, and the Wen (1998) model. We find that the REEs are always unstable under learning if the agents do not know the steady state and include a constant term in their belief functions (perceived law of motion). If the belief functions do not have a constant term, then it is possible for the REE to be simultaneously E-stable and indeterminate. However, we find that the parameter values required for this result to hold are not within an empirically plausible range when applied to the three structural models.

This paper makes several contributions to the literature. First, this is the first paper that studies the learnability of sunspot equilibria in RBC models under infinite horizon learning. Evans et al. (2013) and Eusepi and Preston (2011) work on models that are structurally similar to ours, but their analyses focus on the unique, determinate REE and do not consider the possibility of sunspots. Second, our result complements those of Duffy and Xiao (2007) and Evans and McGough (2005a, 2005b). This series of papers deepen our understanding of how model dynamics surrounding the indeterminate equilibria of an RBC model may differ when rational agents are replaced with adaptive agents who have different learning horizons. Third, we contribute to the methodology for conducting this type of analyses. We derive a set of analytical conditions for the REE to be E-stable under infinite horizon learning. We prove analytically that when the perceived law of motion of agents is consistent with the REE solution in its functional form, the sunspot equilibria are always not stable under learning. We also demonstrate how the agents' optimal decision rule in infinite horizon is connected with the reduced form of the model. This set of tools will contribute to future research in this area.

The rest of the paper is organized as follows. Section 2 derives the conditions for learnability and indeterminacy using a generic model. Section 3 presents stability results for three structural models. Section 4 concludes.

2. STABILITY CONDITIONS UNDER INFINITE HORIZON LEARNING

In this section, we analytically derive the necessary conditions for both indeterminacy and E-stability under infinite horizon learning. The conditions for indeterminacy are standard. The main novelty here is to derive the conditions for E-stability. Under infinite horizon learning, agents attempt to solve long-term dynamic optimization problems given their beliefs about the law of motion of aggregate variables. Since the problem starts at the agent level, we cannot derive stability conditions with a general reduced form model of aggregate variables, as in Duffy and Xiao (2007) and Evans and McGough (2005a). Agent-level decisions require a specific economic environment. Our strategy is to use a well-known RBC model with nonconvexities, the Farmer and Guo (1994) model, to solve for the optimal behavioral equation and conduct stability analysis. We will show that the critical difference between the Euler equation approach and the infinite horizon approach lies in the functional form of the optimal consumption demand equation. Based on this equation, we derive conditions for stability under learning. Although the conditions are derived using a specific model, they can be easily generalized to cover similar one-sector RBC models, as we demonstrate in Section 3.

2.1. The Model

This is the business cycle model considered by Farmer and Guo (1994). There are a large number of identical agents. The representative agent solves

$$\max_{C_t, L_t} E_0 \sum_{t=0}^{\infty} \rho^t \left(\log C_t - A \frac{L_t^{1-\gamma}}{1-\gamma} \right),$$

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subject to the constraint

$$K_{t+1} = r_t K_t + w_t L_t + (1-\delta) K_t - C_t, \quad (1)$$

where K stands for the capital, Y the output, C the consumption, and L the labor. w and r stand for the real wage rate and real rental rate of capital prevalent in competitive factor markets, respectively. Parameters satisfy $\gamma \geq 0$, $0 < \rho < 1$, $0 < \delta < 1$. E stands for expectations that may or may not be rational.

Increasing returns are the result of externalities in the production technology

$$Y_t = K_t^a L_t^b \left(\bar{K}_t^{\alpha - a} \bar{L}_t^{\beta - b} \right), \quad (2)$$

where $\alpha + \beta > 1$, $\alpha = a/\lambda$, $\beta = b/\lambda$, $a + b = 1$, and $0 < \lambda < 1$. $\bar{K}_t$ and $\bar{L}_t$ are the average economy-wide levels of capital and labor. Without loss of generality, we assume there are no fundamental shocks.

Since the Farmer and Guo (1994) model is well known, and many model details have been presented in other publications, we will simplify our exposition here by directly presenting the log-linearized version of the economic system. A hatted variable $\hat{x}_t$ is defined as $\log(x_t) - \log(x)$, where x stands for the steady-state value of x_t.

The household's optimization problem yields the Euler equation

$$\hat{C}_t = E_t \hat{C}_{t+1} - E_t \hat{R}_{t+1}, \quad (3)$$

and the labor supply curve

$$\hat{w}_t = \hat{C}_t - \gamma \hat{L}_t, \quad (4)$$

where $\hat{R}_t$ stands for the gross real interest rate which, in linearized form, is related to the rental rate $\hat{r}_t$ by

$$R\hat{R}_t = r\hat{r}_t. \quad (5)$$

The representative firm's profit maximization problem yields the capital and labor demand curves:

$$\hat{r}_t = \hat{Y}_t - \hat{K}_t, \quad (6)$$

$$\hat{w}_t = \hat{Y}_t - \hat{L}_t. \quad (7)$$

Finally, the production function is linearized as

$$\hat{Y}_t = a\hat{K}_t + b\hat{L}_t + \Delta_t, \quad (8)$$

where $\Delta_t = (\alpha - a)\hat{\bar{K}}_t + (\beta - b)\hat{\bar{L}}_t$ represents the size of the externality effect.

In a symmetric equilibrium, all agents are identical. The economy-wide average level of capital and labor $\bar{K}_t$ and $\bar{L}_t$ will be equal to K_t and L_t, respectively. Using the three-factor market conditions (4)–(8), we can express output $\hat{Y}_t$, the real wage $\hat{w}_t$, and the gross real interest rate $\hat{R}_t$ as functions of $\hat{C}_t$ and $\hat{K}_t$:

$$\hat{Y}_t = \frac{\alpha(1-\gamma)}{1-\beta-\gamma}\hat{K}_t - \frac{\beta}{1-\beta-\gamma}\hat{C}_t, \quad (9)$$

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$$\hat{w}_t = \frac{1-\beta}{1-\beta-\gamma}\hat{C}_t - \frac{\alpha\gamma}{1-\beta-\gamma}\hat{K}_t, \tag{10}$$

$$\hat{R}_t = -\rho a \frac{Y}{K}\frac{\beta}{1-\beta-\gamma}\hat{C}_t - \rho a \frac{Y}{K}\left[1 - \frac{\alpha(1-\gamma)}{1-\beta-\gamma}\right]\hat{K}_t. \tag{11}$$

These conditions are identical to those presented in Duffy and Xiao (2007).

2.2. The REE and Conditions for Indeterminacy

The reduced form model is obtained by combining the above conditions and is given by

$$\hat{K}_{t+1} = d_1\hat{K}_t + d_2\hat{C}_t, \tag{12}$$

$$\hat{C}_t = d_3 E_t\hat{C}_{t+1} + d_4 E_t\hat{K}_{t+1}, \tag{13}$$

where $d_1 = [\frac{Y}{K}\frac{\alpha(1-\gamma)}{1-\beta-\gamma} + 1 - \delta]$, $d_2 = -(\frac{C}{K} + \frac{Y}{K}\frac{\beta}{1-\beta-\gamma})$, $d_3 = (1 + \rho a\frac{Y}{K}\frac{\beta}{1-\beta-\gamma})$ and $d_4 = \rho a\frac{Y}{K}[1 - \frac{\alpha(1-\gamma)}{1-\beta-\gamma}]$. The variables without time subscripts denote steady-state values.

To obtain conditions for indeterminacy, the general reduced-form system of equations (12) and (13) can be rewritten as

$$\begin{pmatrix} d_3 & d_4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_t\hat{C}_{t+1} \\ E_t\hat{K}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d_2 & d_1 \end{pmatrix}\begin{pmatrix} \hat{C}_t \\ \hat{K}_t \end{pmatrix},$$

or equivalently,

$$\begin{pmatrix} \hat{C}_{t+1} \\ \hat{K}_{t+1} \end{pmatrix} = \mathbf{J}\begin{pmatrix} \hat{C}_t \\ \hat{K}_t \end{pmatrix} + \mathbf{V}\varepsilon_t,$$

where $\mathbf{J}$ is a $2-$ by -2 matrix, and $\varepsilon_t = (\hat{C}_{t+1} - E_t\hat{C}_{t+1})$ represents forecast errors.

The necessary and sufficient conditions for indeterminacy are that both eigenvalues of $\mathbf{J}$ are less than one in modulus, which is the case if and only if the determinant and trace of the Jacobian matrix $\mathbf{J}$ satisfy

$$-1 < \det(\mathbf{J}) < 1, \tag{14}$$

$$-1 - \det(\mathbf{J}) < tr(\mathbf{J}) < 1 + \det(\mathbf{J}), \tag{15}$$

where the determinant and trace of the Jacobian matrix $\mathbf{J}$ are given by

$$\det(\mathbf{J}) = \frac{d_1}{d_3},$$

$$tr(\mathbf{J}) = \frac{1 - d_2 d_4 + d_1 d_3}{d_3}.$$

These conditions are identical to those derived in Duffy and Xiao (2007).

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2.3. Conditions for Stability Under Learning

Optimal decision rule. A key difference between Euler equation learning and infinite horizon learning is that in the latter case, agents explicitly incorporate their life-time budget constraint into the decision-making process. They not only make forecasts into the immediate future, but also make long-run forecasts that cover their life-time (infinite horizon). These long-run forecasts feed back into the actual law of motion (ALM) of the aggregate economy and can give rise to distinct stability properties of the equilibrium under learning. The following analysis closely follows the approach of other researchers who conduct stability analysis under infinite horizon learning—for instance, those of Evans and Mitra (2013) and Eusepi and Preston (2011).

The analysis starts with a derivation of the representative agent's life-time budget constraint. The agent's two-period budget constraint is (1), which can be rewritten as

$$\hat{K}_{t+1} = \frac{wL}{K}\hat{w}_t + R\hat{R}_t + \frac{wL}{K}\hat{L}_t + R\hat{K}_t - \frac{C}{K}\hat{C}_t. \tag{16}$$

We can iterate this equation forward, taking expectations in the process, and apply the transversality condition and the labor supply condition (4) to get

$$\begin{aligned}\hat{K}_t = {} & \frac{1-\gamma}{\gamma}\frac{wL}{KR}\left[E_t\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j\hat{w}_{t+j}\right] - \left[E_t\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right] \\ & + \left(\frac{C}{KR} - \frac{wL}{\gamma KR}\right)\left[E_t\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j}\right].\end{aligned} \tag{17}$$

Iterating the Euler equation (3), we can obtain

$$E_t\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j} = \frac{R}{R-1}\hat{C}_t + \frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}. \tag{18}$$

We use this expression to substitute out the infinite sum of expected future consumption from the agent's life-time budget constraint (17). The final expression is the agent's behavioral equation

$$\begin{aligned}\hat{C}_t = {} & \frac{\gamma K(R-1)}{\gamma C - wL}\hat{K}_t + \frac{\gamma K(R-1)}{\gamma C - wL}\hat{R}_t - \frac{wL(1-\gamma)(R-1)}{R(\gamma C - wL)}E_t\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j\hat{w}_{t+j} \\ & - \frac{\gamma C - wL - \gamma K(R-1)}{\gamma C - wL}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}.\end{aligned} \tag{19}$$

Hence, the agent's optimal consumption decision today is a function of current and expected future prices.

The behavioral equation indicates that the forecasts of future prices are crucial for the final solution of the model. Under rational expectations, these forecasts are simply the conditional expectations of the relevant variables based on the true law of motion of the economy. Under learning, they depend critically on the agents' subjective beliefs.

Beliefs. Following the literature, we assume that the functional forms of the agents' subjective beliefs coincide with those under rational expectations. They do not know the parameters of these functions and have to estimate them using observed data. In the RBC model, the indeterminate REE solution is distinct from the fundamental-based unique solution in two aspects. First, the solution takes the form of a VAR(1) in both consumption and capital, whereas in the case of a unique solution, consumption is uniquely pinned down by current capital. Second, there is a sunspot variable in the consumption decision rule. The following setup reflects these differences.

The agents' beliefs or perceived laws of motion (PLM) are

$$\hat{R}_t = n_1\hat{\mathbf{K}}_t + n_2\hat{\mathbf{C}}_t. \tag{20}$$

$$\hat{w}_t = m_1\hat{\mathbf{K}}_t + m_2\hat{\mathbf{C}}_t, \tag{21}$$

$$\hat{\mathbf{C}}_t = a_0 + a_k\hat{\mathbf{K}}_{t-1} + a_c\hat{\mathbf{C}}_{t-1} + a_f f_t + \varepsilon_t, \tag{22}$$

$$\hat{\mathbf{K}}_t = d_1\hat{\mathbf{K}}_{t-1} + d_2\hat{\mathbf{C}}_{t-1}, \tag{23}$$

where we use bold letters to denote *aggregate* variables. Equations (20) and (21) indicate that agents use aggregate consumption and capital data to forecast prices, which are important for their own decisions. Equations (22) and (23) are the VAR equations that agents use to forecast current consumption and capital. f_t is the sunspot variable, and ε_t is a white noise. This PLM system matches the functional form of the indeterminate REE solution. It is also the same belief function in Duffy and Xiao (2007) and Evans and McGough (2005a).

The above equations describe the behavior of one agent. To examine its economy-wide implications, we need to aggregate the behavior of all agents. Following the literature, we seek a symmetric solution in which all agents are identical. In that case, the optimal behavioral equation of an average agent looks exactly the same as (19), except that $\hat{C}_t$ and $\hat{K}_t$ now represent an *average* agent's consumption and capital stock. The PLMs (20)–(23) also keep their functional forms. But we can now replace aggregate consumption and capital with their economy-wide average values. For exposition purpose, we rewrite the PLM with new notations, in slightly different order, as follows:

$$\hat{C}_t = a_0 + a_k\hat{K}_{t-1} + a_c\hat{C}_{t-1} + a_f f_t + \varepsilon_t, \tag{24}$$

$$\hat{K}_t = d_1\hat{K}_{t-1} + d_2\hat{C}_{t-1}, \tag{25}$$

$$\hat{R}_t = n_1\hat{K}_t + n_2\hat{C}_t, \tag{26}$$

$$\hat{w}_t = m_1\hat{K}_t + m_2\hat{C}_t. \tag{27}$$

There is a fundamental difference between the belief (24) and the other three belief functions. Equation (24) is the agents' belief about the law of motion of

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consumption, which, when combined with the behavioral equation (19), is self-referential in nature: The forecast of future consumption can have an impact on current consumption decisions, which in turn affect how agents forecast the future. In essence, the data generating process is expectation dependent.[3] In this case, there is no guarantee that learning will converge to the equilibrium, or even converge at all. The PLMs (25), (26), and (27), on the other hand, all correspond to equilibrium conditions that are not self-referential. For example, (25) matches the true capital accumulation process in its functional form. The determination of K does not hinge on any expectations. In this case, its law of motion is usually learnable if the agents possess a sufficient amount of data. For this reason, researchers often assume that this type of PLMs is already learned by, or known to the agents when conducting stability analysis.[4] Evans and Mitra (2013), for example, make this assumption when they examine the learnability of the REE for the Ramsey model under infinite horizon learning. In this paper, we shall do the same: The true parameters of (25)–(27) are assumed known to the agents.

Note that in (24), the PLM includes a constant term. In the REE, the value of this constant term is always 0. The reason is that the variables in the linearized model are all expressed as percentage deviations from steady-state values. As long as the steady-state values are known, any constant terms would be eliminated when the model is linearized. But under learning, agents do not observe the steady-state values. It is natural to include a constant term in the PLM so that agents have an opportunity to learn the steady-state values. Duffy and Xiao (2007) test the robustness of their result by considering the case in which the steady-state values are known to learning agents, and there is no constant term in the PLM. For completeness of the inquiry, we consider this case too.

Suppose agents do observe the steady-state values accurately, and then the PLM for consumption can be written as

$$\hat{C}_t = a_k \hat{K}_{t-1} + a_c \hat{C}_{t-1} + a_f f_t + \varepsilon_t. \tag{28}$$

In the following sections, we show that this distinction can lead to different stability results.

E-stability conditions. Plugging the known laws of motion of capital, the wage rate, and the gross interest rate into the decision rule (19), we can simplify the average agent's behavioral equation to

$$\hat{C}_t = \frac{(1-R)\,d_1}{d_2 R}\hat{K}_t + \frac{(R-d_1)\,(R-1)+d_2 d_4 R}{d_2 R} E_t \sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{K}_{t+j} + (d_3-1)\,E_t \sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{C}_{t+j}. \tag{29}$$

Current consumption depends on agents' expected future consumption and capital in the infinite horizon. The appendix offers a step-by-step derivation of

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this equation. Note that except the steady-state gross interest rate R, all the other parameters are taken from the reduced form model (12) and (13). It turns out that this is not coincidental. In the next section, we show that the optimal decision rules for consumption in the Wen (1998) model and the Schmitt-Grohé and Uribe (1997) model can also be expressed in this form. It is the structure of the model that determines the functional form of this equation.

For stability analysis, we use the PLM (24) or (28) to compute the agent's forecasts of future consumption and capital, and plug them back into the behavioral equation (29) to obtain a map between the PLMs to the model's ALM.

To compute the agent's forecasts, define $\mathbf{Z}_t = \binom{\hat{C}_t}{\hat{K}_t}$. We do so because the endogenous variable C and the state variable K are both necessary in computing an equilibrium solution. The other two variables w and R are not required. The first two of the agent's PLMs take the following form:

$$\mathbf{Z}_t = \mathbf{A} + \mathbf{B}\mathbf{Z}_{t-1} + \mathbf{C} f_t + \eta_t, \quad \textbf{(30)}$$

where $\mathbf{A} = \binom{a_0}{0}$, $\mathbf{B} = \begin{pmatrix} a_c & a_k \\ d_2 & d_1 \end{pmatrix}$, $\mathbf{C} = \binom{a_f}{0}$, and $\eta_t = \binom{\varepsilon_t}{0}$.

The forecasts of future variables at any horizon T are determined as

$$E_t\mathbf{Z}_T = (\mathbf{I} - \mathbf{B})^{-1}\left(\mathbf{I} - \mathbf{B}^{T-t}\right)\mathbf{A} + \mathbf{B}^{T-t}E_t\mathbf{Z}_t, \quad \textbf{(31)}$$

where $\mathbf{I}$ is a 2×2 identity matrix. When $t \to \infty$, we can show that the infinite sum of expectations in the decision rule can be written as

$$E_t\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j \mathbf{Z}_{t+j} = (\mathbf{I} - \mathbf{B})^{-1}\left[\left(\frac{1}{R-1}\right)\mathbf{I} - (R\mathbf{I} - \mathbf{B})^{-1}\mathbf{B}\right]\mathbf{A}$$
$$+ (R\mathbf{I} - \mathbf{B})^{-1}\mathbf{B}E_t\mathbf{Z}_t. \quad \textbf{(32)}$$

Using equation (32) to evaluate expectations in equation (29) determines the ALM:

$$\hat{C}_t = T_0 + \mathbf{T}_z\mathbf{Z}_{t-1} + T_f f_t + \xi_t, \quad \textbf{(33)}$$

where T_0 and T_f are scalars, $\mathbf{T}_z$ is a 1×2 matrix, and ξ_t is a white noise. The appendix spells out the actual mathematical expressions of T_0, T_f, and $\mathbf{T}_z$. Note that they are functions of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in (30).

We can solve for the REE parameter values by applying the method of undetermined coefficients to the T-mapping. The T-mapping is

$$a_0 \to -\frac{Ra_0\left(d_1 - R + R^2d_3 - Rd_1d_3 + Rd_2d_4\right)}{d_3(R-1)\left[Rd_1 + a_kd_2 + a_c(R - d_1) - R^2\right]},$$

$$a_c \to -\frac{\begin{array}{c} a_cd_1 - Ra_c - a_kd_2 + R^2d_3a_c + d_4a_kd_2^2 - Rd_3a_cd_1 \\ + Rd_4a_cd_2 + Rd_3a_kd_2 + Rd_4d_1d_2 - d_4a_cd_1d_2 \end{array}}{d_3\left[Rd_1 + a_kd_2 + a_c(R - d_1) - R^2\right]},$$

$$a_k \to -\frac{R^2 d_3 a_k - R a_k + R d_4 d_1^2 - d_4 a_c d_1^2 + R d_4 a_k d_2 + d_4 a_k d_1 d_2}{d_3 \left[R d_1 + a_k d_2 + a_c \left(R - d_1\right) - R^2\right]},$$

$$a_f \to -\frac{a_f \left(d_1 - R + R^2 d_3 - R d_1 d_3 + R d_2 d_4\right)}{d_3 \left[R d_1 + a_k d_2 + a_c \left(R - d_1\right) - R^2\right]}.$$

And we can obtain

$$a_c = \frac{1 - d_2 d_4}{d_3}, \qquad (34)$$

$$a_k = -\frac{d_1 d_4}{d_3}, \qquad (35)$$

$$a_0 = 0, \qquad (36)$$

$$a_f \text{ is indeterminate.} \qquad (37)$$

These values are identical to the REE solution obtained by Duffy and Xiao (2007). The difference between their solution procedure and ours is that we take conditional expectations of the PLM in infinite horizons, whereas they do so for one future period. Under rational expectations, the two procedures should yield the same result, which we have confirmed with (34)–(37).

It remains to see whether or not forecasting horizons matter for the stability of the REE under learning. This is done by applying Evans and Honkapohja (2001)'s E-stability principle. Specifically, let $\theta = (a_0, a_c, a_k, a_f)'$ be a vector of parameters in the PLM and $\mathbf{T}(\theta)$ the vector of parameters in the ALM. A given sunspot solution θ is said to be E-stable if the differential equation

$$\frac{d\theta}{d\tau} = \mathbf{T}(\theta) - \theta,$$

evaluated at θ is locally asymptotically stable. Here, τ denotes "notional" or "virtual" time.

For this purpose, we compute

$$\frac{d\left[\mathbf{T}(\theta) - \theta\right]}{d\theta} = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & h_3 & 0 \\ 0 & h_4 & h_5 & 0 \\ 0 & h_6 & h_7 & 0 \end{pmatrix}, \qquad (38)$$

where the h's represent combinations of the parameters of the model, which are quite tedious and are provided in the appendix. The structure of this matrix guarantees that one eigenvalue is equal to h_1, which is the result of having a constant term in the PLM. It can be shown that

$$h_1 = \lambda_1 = \frac{1}{R - 1}.$$

The structure of the matrix also implies that there is always a 0 eigenvalue. Without loss of generality, let

$$\lambda_2 = 0,$$

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where λ_2 stands for an eigenvalue of this matrix. This 0 eigenvalue arises from the fact that the T-mapping of a_f yields an indeterminate solution, as in Duffy and Xiao (2007). The differential equation associated with this T-mapping is

$$\frac{da_f}{d\tau} = T\left(a_f\right) - a_f = -\frac{d_1 d_4}{d_3 a_k} a_f - a_f. \tag{39}$$

Integrating it yields

$$a_f(\tau) = a_f(0) \exp\left[\int_0^\tau \left(-\frac{d_1 d_4}{d_3 a_k} a_f - a_f\right) du\right].$$

As long as $a_k \to -\frac{d_1 d_4}{d_3}$ exponentially as $\tau \to \infty$, a_f will converge to a finite value. As Evans and Honkapohja (2001) demonstrate, one zero eigenvalue of $\frac{d[\mathbf{T}(\theta)-\theta]}{d\theta}$ does not alter the stability property of the REE.

E-stability of the REE requires that all other eigenvalues of the matrix (38) be negative. We state the conditions for stability in the next proposition.

PROPOSITION 1. *If the agent's PLM for consumption is (24), the necessary conditions for the Farmer and Guo (1994) model to be E-stable under infinite horizon learning are*

$$\lambda_1 = \frac{1}{R-1} < 0, \tag{40}$$

$$\lambda_3 \lambda_4 = \frac{d_1}{d_3 R^2 + (d_2 d_4 - d_1 d_3 - 1) R + d_1} > 0, \tag{41}$$

$$\lambda_3 + \lambda_4 = -\frac{d_1 - d_3 R^2}{d_3 R^2 + (d_2 d_4 - d_1 d_3 - 1) R + d_1} - 1 < 0. \tag{42}$$

Proof. See the appendix.

If the agent's PLM for consumption is (28), then the first condition in the above proposition will drop out. It is straightforward to state the following.

COROLLARY 1. *If the agent's PLM for consumption is (28), the necessary conditions for the Farmer and Guo (1994) model to be E-stable under infinite horizon learning are (41) and (42).*

3. STABILITY RESULTS FOR THREE STRUCTURAL MODELS

In this section, we apply the stability conditions to three structural models and examine whether or not sunspot equilibria can be stable under infinite horizon learning. The three models are the Farmer and Guo (1994) model, Wen (1998)'s RBC model with variable capacity utilization rate, and Schmitt-Grohé and Uribe (1997)'s RBC model with a balanced budget tax policy. It is well known that under the Euler equation learning, the indeterminate equilibria in these three structural models are not stable under learning.

3.1. The Farmer and Guo (1994) Model

Since we already derived the learnability conditions for this model in the previous section, all we need to do here is to apply them and check for stability under learning.

It turns out that we need not use all of the stability conditions in (40)–(42). E-stability requires that all three eigenvalues of the Jacobian in (38) be negative; if we can prove that one eigenvalue is positive, an instability conclusion follows. This is exactly the case here: The first eigenvalue of the matrix is $1/(R-1)$, where $R = 1 + r - \delta$ is the value of the *gross* interest rate in the steady state and is necessarily higher than 1 as long as the net interest rate $r - \delta$ is positive, which is required in an RBC model. Therefore, $1/(R-1)$ is positive. We can state the following proposition.

PROPOSITION 2. *The REE of the Farmer and Guo (1994) model cannot be both indeterminate and E-stable since condition (40) is always violated when the REE is indeterminate.*

A closer look at the Jacobian matrix in (38) reveals that the T-mapping that gives rise to the eigenvalue $\lambda_1 = 1/(R-1)$ is $T(a_0)$. In other words, the instability result arises because the agents' perceived law of motion contains a constant term. As we argue in the previous section, it is possible for the agents not to include a constant term in their forecasts, provided that they have knowledge of the steady-state values of economic aggregates. We consider this possibility next.

If the perceived law of motion is (28), then E-stability requires that conditions (41) and (42) hold. These two conditions are sufficiently complicated to rule out any analytical evaluation of their signs. We turn to numerical analyses. The model has three critical parameters: capital share a, labor share b, and the level of the externality $1/\lambda$ (note that $\alpha = a/\lambda$, $\beta = b/\lambda$). The combinations of these three parameters lead to various levels of increasing returns, and result in indeterminate REEs. Our strategy is to search over a range of plausible calibrations for these parameters to look for indeterminate REEs that are learnable. In order to do so, we also need to calibrate the rest of the parameters of the model. For those, we simply use the original values from Farmer and Guo (1994): $\rho = 0.99$, $\delta = 0.025$, $\gamma = 0$.

For the three parameters a, b, and λ, we do a comprehensive search over the range 0.01–0.99 at a step size of 0.01. For each combination of parameters, we check numerically if the REE is simultaneously indeterminate and E-stable by evaluating the conditions (14), (15) and (41)–(42). The results are presented in Table 1. Note that we only report the values for two parameters because the model's restriction is $a + b = 1$.

The results show that some REEs can qualitatively be simultaneously indeterminate and E-stable. For example, if labor share is 0.88 and capital share is 0.12, and the level of increasing returns is 1.19, then the indeterminate REE is stable under infinite horizon learning. However, the parameter values required for this

TABLE 1. Searched parameters and E-stability result

Parameter	Definition	Search range	Range for E-stability
a	Capital share	0.01–0.99	0.01–0.12
$1/\lambda$	Externality	1.01–5.00	1.19–3.57

result to hold are generally out of the range of those deemed empirically plausible by the literature. A labor share of 0.88 is much too high compared with 2/3—a standard value researchers use to calibrate the U.S. economy. The required size of the externality effect, 1.19, is also too high. Most research suggests that the level of increasing returns in the U.S. economy and other advanced economies is quite mild (see Basu and Fernald, 1997).

Hence, we conclude that for the Farmer and Guo (1994) model, indeterminate REEs are generally not stable under infinite horizon learning, even when agents' PLMs do not include a constant term.

3.2. A Variable Capacity Utilization Model

The model. The details of this model are presented in the appendix. Consider the reduced form of the model

$$\hat{K}_{t+1} = d_1\hat{K}_t + d_2\hat{C}_t,$$

$$\hat{C}_t = d_3E_t\hat{C}_{t+1} + d_4E_t\hat{K}_{t+1},$$

where $d_1 = [1 + (1 - \frac{\alpha}{\theta})\frac{Y}{K}\frac{\alpha^*(1+\gamma)}{1+\gamma-b^*}]$, $d_2 = -[\frac{C}{K} + (1 - \frac{\alpha}{\theta})\frac{Y}{K}\frac{b^*}{1+\gamma-b^*}]$, $d_3 = -\frac{\beta b^*-(1+\gamma)}{1+\gamma-b^*}$, and $d_4 = (1-\beta)\frac{1+\gamma-b^*-\alpha^*(1+\gamma)}{1+\gamma-b^*}$. All parameters are defined in the appendix.

Since this system is in exactly the same form as (12) and (13) in the Farmer and Guo (1994) model, the indeterminacy conditions remain (14) and (15).

For the purpose of deriving the optimal consumption decision rule in infinite horizons, we find it convenient to reformulate the agent's budget constraint as

$$C_t + K_{t+1} = R_tK_t + w_tL_t,$$

where $R_t = r_tu_t + 1 - \delta_t$ denotes the gross real rate of interest. Iterating the log-linearized version of this equation into infinite horizon, and apply the labor supply condition, we obtain

$$\hat{K}_t = -\frac{1+\gamma}{\gamma}\frac{wL}{KR}\left[\hat{w}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{w}_{t+j}\right] - \left[\hat{R}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right] + \left(\frac{C}{KR} + \frac{wL}{\gamma KR}\right)\left[\hat{C}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j}\right]. \quad \mathbf{(43)}$$

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The Euler equation is identical to the one in the Farmer and Guo (1994) model. Making use of it to substitute out the infinite sum of expected future consumption in (43), we have the agent's optimal decision rule

$$\hat{C}_t = \frac{\gamma K(R-1)}{\gamma C + wL}\hat{K}_t + \frac{\gamma K(R-1)}{\gamma C + wL}\hat{R}_t + \frac{wL(1+\gamma)(R-1)}{R(\gamma C + wL)}\hat{w}_t + \frac{wL(1+\gamma)(R-1)}{R(\gamma C + wL)}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{w}_{t+j} - \frac{\gamma C + wL - \gamma K(R-1)}{\gamma C + wL}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{R}_{t+j}. \quad \textbf{(44)}$$

As in the previous section, the agents' current consumption decisions are tied to expected future prices.

Agents' subjective beliefs coincide in their functional forms with the MSV solution of the model:

$$\hat{C}_t = a_0 + a_k\hat{K}_{t-1} + a_c\hat{C}_{t-1} + a_f f_t + \varepsilon_t, \quad \textbf{(45)}$$

$$\hat{K}_t = d_1\hat{K}_{t-1} + d_2\hat{C}_{t-1}, \quad \textbf{(46)}$$

$$\hat{R}_t = n_1\hat{K}_t + n_2\hat{C}_t, \quad \textbf{(47)}$$

$$\hat{w}_t = m_1\hat{K}_t + m_2\hat{C}_t. \quad \textbf{(48)}$$

As we explain in the previous section, beliefs (46)–(48) can be assumed known or learned by the agents. Plugging these conditions and the labor market and goods market equilibrium conditions into (44), we obtain a reduced form decision rule for the consumer

$$\hat{C}_t = \frac{(1-R)d_1}{d_2 R}\hat{K}_t + \frac{(R-d_1)(R-1) + d_2 d_4 R}{d_2 R}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{K}_{t+j} + (d_3 - 1)E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{C}_{t+j}. \quad \textbf{(49)}$$

This equation looks exactly the same as the optimal decision rule (29) that we derived for the Farmer and Guo (1994) model. A proof can be found in the appendix. Note that because the parameters $d_1 - d_4$ are defined differently, this decision rule is in fact not quantitatively the same.

Results. Because the agents' optimal decision rule and the PLMs or beliefs coincide with those of the Farmer and Guo (1994) model in functional forms, the derivation of the E-stability conditions follows the same process and the derived conditions are the same. We therefore state the following proposition.

PROPOSITION 3. *The necessary conditions for the REE in the Wen (1998) model to be E-stable are identical to those in Proposition 1 and Corollary 1.*

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Table 2. Searched parameters and E-stability result

Parameter	Definition	Search range	Range for E-stability
α	Capital share	0.01–0.99	0.01–0.22
η	Externality	0.01–5.00	0.03–1.70

If the agents' PLM includes a constant term, as in (45), the critical eigenvalue $\lambda = 1/(R-1)$ again governs the E-stability of the REE. In this model, the steady-state gross interest rate R is defined differently: $R = ru + 1 - \delta$. After substituting out r and u using the model's optimal conditions, we obtain $R = 1 + \alpha Y/K - \delta = 1 + \frac{\theta-1}{\theta} r^{\frac{\theta-1}{\theta}}$, which is restricted to be higher than 1 by the structure of the model. It immediately follows that the REE is not E-stable. We have the following proposition.

PROPOSITION 4. *If the perceived law of motion is (45), the REE of the Wen (1998) model cannot be both indeterminate and E-stable since condition (40) is always violated when the REE is indeterminate.*

It remains to check if the REE can be E-stable if we allow the agents to use a PLM of the form

$$\hat{C}_t = a_k \hat{K}_{t-1} + a_c \hat{C}_{t-1} + a_f f_t + \varepsilon_t.$$

As in the Farmer and Guo (1994) model, a numeric search is necessary to analyze the stability properties of the REE under learning. For this model, two parameters are critical: capital share α (labor share is $1-\alpha$) and the size of the externality η. We calibrate the rest of the parameters as in Wen (1998): $\beta = 0.99$, $\delta = 0.025$, $\gamma = 0$. We then search through the other two parameters. The results are presented in Table 2.

We find that the REE in the Wen (1998) model can qualitatively be simultaneously indeterminate and E-stable. For example, when capital share is 0.22 and the size of the externality effect is 0.09, the sunspot equilibria are stable under infinite horizon learning. Our concern is still about empirical plausibility. Although 0.09 is a reasonable value for the size of increasing returns, the maximum value for the capital share required for the REE to be E-stable is 0.22. This is close to but still not quite consistent with the value that the literature suggests.

3.3. A Model of Tax Distortions

The model. As shown in Schmitt-Grohé and Uribe (1997), indeterminacy can arise from an endogenous distortionary labor-income tax under a balanced-budget fiscal policy rule even with a constant-returns-to-scale production technology.

It can be shown that the reduced form of the model is again in the form

$$\hat{K}_{t+1} = d_1 \hat{K}_t + d_2 \hat{C}_t,$$

$$\hat{C}_t = d_3 E_t \hat{C}_{t+1} + d_4 E_t \hat{K}_{t+1},$$

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TABLE 3. Searched parameters and E-stability result

Parameter	Definition	Search range	Range for E-stability
a	Capital share	0.01–0.99	0.01–0.07
τ	Tax rate	0.01–0.50	0.06–0.32

where $d_1 = [\frac{Y}{K}\frac{(1-\tau)a}{1-b-\tau} + 1 - \delta]$, $d_2 = [\frac{Y}{K}\frac{b(1-\tau)}{b-1+\tau} - \frac{C}{K}]$, $d_3 = [1 - \beta a\frac{Y}{K}\frac{(1-\tau)b}{b-1+\tau}]$, and $d_4 = a\beta\frac{Y}{K}[1 + \frac{(1-\tau)a}{b-1+\tau}]$. All parameters and model details are defined in the appendix.

To obtain conditions for stability under infinite horizon learning, we need to derive the agent's optimal decision rule for consumption. Since the process involved is very similar to that of the Farmer and Guo (1994) model and the Wen (1998) model, we skip the details here. As in those two models, the optimal decision rule can again be written as

$$\hat{C}_t = \frac{(1-R)\,d_1}{d_2 R}\hat{K}_t + \frac{(R-d_1)\,(R-1) + d_2 d_4 R}{d_2 R}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{K}_{t+j}$$

$$+ (d_3 - 1)\,E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j}. \qquad \textbf{(50)}$$

The appendix provides a proof.

Under learning, the perceived laws of motion will again take the form of (45)–(48). It immediately follows that the following proposition is true.

PROPOSITION 5. *The necessary conditions for the REE in the Schmitt-Grohé and Uribe (1997) model to be E-stable are identical to those in Proposition 1 and Corollary 1.*

Results. If the perceived law of motion includes a constant term, the critical eigenvalue will be $\lambda = 1/(R-1)$, where R is the steady-state gross interest rate and is necessarily greater than 1. We therefore have the following proposition.

PROPOSITION 6. *If the perceived law of motion has a constant term, the REE of the Schmitt-Grohé and Uribe (1997) model cannot be both indeterminate and E-stable since condition (40) is always violated when the REE is indeterminate.*

It remains to check if the REE can be E-stable if we allow the agents to use a PLM of the form

$$\hat{C}_t = a_k\hat{K}_{t-1} + a_c\hat{C}_{t-1} + a_f f_t + \varepsilon_t.$$

We turn to numerical analysis. For this model two parameters are critical: capital share a (labor share is $b = 1 - a$) and the tax rate τ. We calibrate the rest of the parameters as in Schmitt-Grohé and Uribe (1997): $\beta = 0.99$, $\delta = 0.025$. We then search through the other two parameters. The results are presented in Table 3.

As in the previous two models, the REEs can qualitatively be simultaneously indeterminate and E-stable, but the required parameters for this to happen are generally out of the range of empirical plausible values. A capital share of 0.07 is much too low, compared with empirical findings.

4. CONCLUDING REMARKS

Previous research examines the stability of sunspot equilibria in one-sector RBC models under Euler equation learning. Inspired by Preston (2005), who finds that E-stability conditions under infinite horizon learning can be different, we conduct stability analysis of sunspot equilibria in three prominent RBC models using infinite horizon learning techniques. We find that at least for these three models, sunspot equilibria are generally not stable under learning. Our result holds when the law of motion of the economy is written in a general form representation. If a common factor representation is adopted, as in Evans and McGough (2005a), then stability results may be different. This is a possible direction for future research.

NOTES

1. An incomplete list of important works in this area includes Benhabib and Farmer (1994), Farmer and Guo (1994), Schmitt-Grohé and Uribe (1997), and Wen (1998) among others. The closest theoretical and empirical precursors to this line of work are Shell (1977), Azariadis (1981), Cass and Shell (1983), and Guesnerie (1986).

2. This has been examined extensively in the literature, e.g., by Evans and Honkapohja (2001) for the Farmer and Guo (1994) model, Rudanko (2002) for the Schmitt-Grohé and Uribe (1997) model, Duffy and Xiao (2007) for the Farmer and Guo (1994), Schmitt-Grohé and Uribe (1997), and Wen (1998) models.

3. As our narration above has made clear, the agents are not aware of this mechanism—they do not think that their own forecast is affecting the true consumption process. From their point of view, they are merely using aggregate data to forecast prices.

4. If the purpose of the analysis is not just equilibrium stability, then the distinction of a known PLM and an unknown PLM still matters. For example, if the goal is to understand the short-run dynamics of capital under adaptive learning, then the simulation of the learning process itself is useful.

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APPENDIX A

A.1. DERIVATION OF THE DECISION RULE (29) FOR THE FARMER AND GUO (1994) MODEL

We start by iterating forward equations (11) and (10):

$$\hat{w}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R}\right)^j \hat{w}_{t+j} = -\frac{\alpha\gamma}{1-\gamma-\beta}\left(\hat{K}_t + E_t \sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{K}_{t+j}\right) + \frac{1-\beta}{1-\gamma-\beta}\left(\hat{C}_t + E_t \sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{C}_{t+j}\right), \tag{A.1}$$

$$\hat{R}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R}\right)^j \hat{R}_{t+j} = -\rho a \frac{Y}{K}\left[1 - \frac{\alpha(1-\gamma)}{1-\gamma-\beta}\right]\left(\hat{K}_t + E_t \sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{K}_{t+j}\right) - \rho a \frac{Y}{K}\frac{\beta}{1-\gamma-\beta}\left(\hat{C}_t + E_t \sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{C}_{t+j}\right). \tag{A.2}$$

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To derive the decision rule (29), we find it convenient to keep (19) in a form similar to (17). More specifically, it looks like

$$\hat{K}_t = -\frac{\gamma-1}{\gamma}\frac{wL}{KR}\left(\hat{w}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{w}_{t+j}\right) - \left(\hat{R}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{R}_{t+j}\right) + \left(\frac{C}{KR} - \frac{wL}{\gamma KR}\right)\left[\frac{R}{R-1}\hat{C}_t + \frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{R}_{t+j}\right]. \tag{A.3}$$

Substituting equations (A.1) and (A.2) into equation (A.3), we can reduce it to the following form:

$$\hat{K}_t = \left\{\frac{\gamma-1}{\gamma}\frac{wL}{KR}\times\frac{\alpha\gamma}{1-\gamma-\beta} + \rho a\frac{Y}{K}\left[1-\frac{\alpha(1-\gamma)}{1-\gamma-\beta}\right]\right\}\left(\hat{K}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{K}_{t+j}\right)$$
$$+\left[-\frac{\gamma-1}{\gamma}\frac{wL}{KR}\times\frac{1-\beta}{1-\gamma-\beta} + \rho a\frac{Y}{K}\frac{\beta}{1-\gamma-\beta}\right]\left(\hat{C}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{C}_{t+j}\right)$$
$$+\left(\frac{C}{KR} - \frac{wL}{\gamma KR}\right)\left[\frac{R}{R-1}\hat{C}_t + \frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{R}_{t+j}\right]$$

$$= \left\{\frac{1}{R}\frac{wL}{K}\frac{\alpha(\gamma-1)}{1-\gamma-\beta} + \frac{1}{R}a\frac{Y}{K}\left[1-\frac{\alpha(1-\gamma)}{1-\gamma-\beta}\right]\right\}\left(\hat{K}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{K}_{t+j}\right)$$
$$+\left[\frac{wL}{\gamma KR}\left(1+\frac{\gamma\beta}{1-\gamma-\beta}\right) + \rho a\frac{Y}{K}\frac{\beta}{1-\gamma-\beta}\right]\left(\hat{C}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{C}_{t+j}\right)$$
$$+\left(\frac{C}{KR} - \frac{wL}{\gamma KR}\right)\left[\frac{R}{R-1}\hat{C}_t + \frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{R}_{t+j}\right]$$

$$= \left\{\frac{1}{R}\left[a\frac{Y}{K}\left[1-\frac{\alpha(1-\gamma)}{1-\gamma-\beta}\right] - \frac{wL}{K}\frac{\alpha(1-\gamma)}{1-\gamma-\beta}\right]\right\}\left(\hat{K}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{K}_{t+j}\right)$$
$$+\left[\frac{wL}{\gamma KR} + \frac{1}{R}b\frac{Y}{K}\frac{\beta}{1-\gamma-\beta} + \rho a\frac{Y}{K}\frac{\beta}{1-\gamma-\beta}\right]\left(\hat{C}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{C}_{t+j}\right)$$
$$+\left(\frac{C}{KR} - \frac{wL}{\gamma KR}\right)\left[\frac{R}{R-1}\hat{C}_t + \frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{R}_{t+j}\right]$$

$$= \left\{ \frac{1}{R} \left[r - r \frac{\alpha(1-\gamma)}{1-\gamma-\beta} - \left(\frac{Y}{K} - r \right) \frac{\alpha(1-\gamma)}{1-\gamma-\beta} \right] \right\} \left(\hat{K}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{K}_{t+j} \right)$$

$$+ \left[\frac{wL}{\gamma K R} + \frac{1}{R} \frac{Y}{K} \frac{\beta}{1-\gamma-\beta} + \frac{1}{R} \frac{C}{K} - \frac{1}{R} \frac{C}{K} \right] \left(\hat{C}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{C}_{t+j} \right)$$

$$+ \left(\frac{C}{KR} - \frac{wL}{\gamma K R} \right) \left[\frac{R}{R-1} \hat{C}_t + \frac{R}{R-1} E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{R}_{t+j} \right]$$

$$= \left\{ \frac{1}{R} \left[R - \left[\frac{Y}{K} \frac{\alpha(1-\gamma)}{1-\gamma-\beta} + 1 - \delta \right] \right] \right\} \left(\hat{K}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{K}_{t+j} \right)$$

$$+ \left[\frac{1}{R} \left(\frac{C}{K} + \frac{Y}{K} \frac{\beta}{1-\gamma-\beta} \right) + \frac{wL}{\gamma K R} - \frac{C}{KR} \right] \left(\hat{C}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{C}_{t+j} \right)$$

$$+ \left(\frac{C}{KR} - \frac{wL}{\gamma K R} \right) \left[\frac{R}{R-1} \hat{C}_t + \frac{R}{R-1} E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{R}_{t+j} \right]$$

$$= \left\{ \frac{1}{R} \left[R - \left[\frac{Y}{K} \frac{\alpha(1-\gamma)}{1-\gamma-\beta} + 1 - \delta \right] \right] \right\} \left(\hat{K}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{K}_{t+j} \right)$$

$$+ \left[\frac{1}{R} \left(\frac{C}{K} + \frac{Y}{K} \frac{\beta}{1-\gamma-\beta} \right) + \frac{wL}{\gamma K R} - \frac{C}{KR} \right]$$

$$\times \left[\frac{R}{R-1} \hat{C}_t + \frac{R}{R-1} E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{R}_{t+j} \right]$$

$$+ \left(\frac{C}{KR} - \frac{wL}{\gamma K R} \right) \left[\frac{R}{R-1} \hat{C}_t + \frac{R}{R-1} E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{R}_{t+j} \right]$$

$$= \left\{ \frac{1}{R} \left[R - \left[\frac{Y}{K} \frac{\alpha(1-\gamma)}{1-\gamma-\beta} + 1 - \delta \right] \right] \right\} \left(\hat{K}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{K}_{t+j} \right)$$

$$+ \frac{1}{R} \left(\frac{C}{K} + \frac{Y}{K} \frac{\beta}{1-\gamma-\beta} \right) \left[\frac{R}{R-1} \hat{C}_t + \frac{R}{R-1} E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{R}_{t+j} \right]$$

$$= \frac{R-d_1}{R} E_t \left(\hat{K}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R}\right)^j \hat{K}_{t+j} \right) - \frac{d_2}{R} \left[\frac{R}{R-1} \hat{C}_t + \frac{R}{R-1} E_t \sum_{j=1}^{\infty} \left(\frac{1}{R}\right)^j \hat{R}_{t+j} \right]. \tag{A.4}$$

The derivation takes many steps. Throughout the steps, we repeatedly use the steady-state version of the optimal and equilibrium conditions to simplify the parameters.

(A.2) can be rewritten as

$$\hat{R}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R}\right)^j \hat{R}_{t+j} = -d_4 \left(\hat{K}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R}\right)^j \hat{K}_{t+j} \right) + (1-d_3) \left(\hat{C}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R}\right)^j \hat{C}_{t+j} \right). \tag{A.5}$$

Substituting (A.5) for $E_t \sum_{j=1}^{\infty} (\frac{1}{R})^j \hat{R}_{t+j}$ into equation (A.4), and moving $\hat{C}_t$ to the left-hand side, we obtain the decision rule (29).

A.2. PROOF OF PROPOSITION 1.

By combining (A.2) and (18), we obtain

$$\hat{C}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R}\right)^j \hat{C}_{t+j} = \frac{Rd_3}{Rd_3 - 1} \hat{C}_t - \frac{Rd_3}{Rd_3 - 1} \frac{d_4}{d_3} E_t \sum_{j=1}^{\infty} \left(\frac{1}{R}\right)^j \hat{R}_{t+j}. \tag{A.6}$$

We then use this expression to substitute out the infinite sum of consumption in (29). The behavioral equation now becomes

$$\hat{C}_t = \frac{(1-d_3 R)\, d_1}{d_2 d_3 R} \hat{K}_t + \frac{d_2 d_4 R + (d_1 - R)(1 - d_3 R)}{d_2 d_3 R} E_t \sum_{j=1}^{\infty} \left(\frac{1}{R}\right)^j \hat{K}_{t+j}. \tag{A.7}$$

We then plug in the PLMs, and obtain the following T-mappings from the PLMs to the ALMs:

$$T_0 = -\frac{Ra_0 \left(d_1 - R + R^2 d_3 - Rd_1 d_3 + Rd_2 d_4\right)}{d_3 (R-1) \left[Rd_1 + a_k d_2 + a_c (R - d_1) - R^2\right]}, \tag{A.8}$$

$$\mathbf{T}'_z = \begin{pmatrix} -\frac{a_c d_1 - Ra_c - a_k d_2 + R^2 d_3 a_c + d_4 a_k d_2^2 - Rd_3 a_c d_1 + Rd_4 a_c d_2 + Rd_3 a_k d_2 + Rd_4 d_1 d_2 - d_4 a_c d_1 d_2}{d_3 \left[Rd_1 + a_k d_2 + a_c (R - d_1) - R^2\right]} \\ -\frac{R^2 d_3 a_k - Ra_k + Rd_4 d_1^2 - d_4 a_c d_1^2 + Rd_4 a_k d_2 + d_4 a_k d_1 d_2}{d_3 \left[Rd_1 + a_k d_2 + a_c (R - d_1) - R^2\right]} \end{pmatrix}, \tag{A.9}$$

$$T_f = -\frac{a_f \left(d_1 - R + R^2 d_3 - Rd_1 d_3 + Rd_2 d_4\right)}{d_3 \left[Rd_1 + a_k d_2 + a_c (R - d_1) - R^2\right]}. \tag{A.10}$$

The sunspot solution (34)–(37) is obtained by solving the fixed points for these T-mappings.

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By applying Evans and Honkapohja (2001)'s E-stability principle, we obtain the differential matrix (38). h's are given by $h_1 = \frac{1}{R-1}$, $h_2 = \frac{R(1-d_2d_4)-d_1}{d_3R^2+(d_2d_4-d_1d_3-1)R+d_1}$, $h_3 = \frac{Rd_2d_3}{d_3R^2+(d_2d_4-d_1d_3-1)R+d_1}$, $h_4 = -\frac{Rd_1d_4}{d_3R^2+(d_2d_4-d_1d_3-1)R+d_1}$, $h_5 = \frac{d_1(Rd_3-1)}{d_3R^2+(d_2d_4-d_1d_3-1)R+d_1}$, $h_6 = \frac{d_3(R-d_1)}{d_3R^2+(d_2d_4-d_1d_3-1)R+d_1}\bar{a}_f$, and $h_7 = \frac{d_2d_3}{d_3R^2+(d_2d_4-d_1d_3-1)R+d_1}\bar{a}_f$, where $\bar{a}_f$ denotes the REE value of a_f. The eigenvalues of this matrix are $\lambda_1 = 0$, $\lambda_2 = \frac{1}{R-1}$ and

$$\lambda_3 = \frac{R-2d_1+Rd_1d_3-Rd_2d_4+R\sqrt{d_1^2d_3^2-2d_1d_2d_3d_4-2d_1d_3+d_2^2d_4^2-2d_2d_4+1}}{2\left[d_3R^2+(d_2d_4-d_1d_3-1)\,R+d_1\right]}, \tag{A.11}$$

$$\lambda_4 = \frac{R-2d_1+Rd_1d_3-Rd_2d_4-R\sqrt{d_1^2d_3^2-2d_1d_2d_3d_4-2d_1d_3+d_2^2d_4^2-2d_2d_4+1}}{2\left[d_3R^2+(d_2d_4-d_1d_3-1)\,R+d_1\right]}. \tag{A.12}$$

It is immediate that (41) is obtained by multiplying (A.11) by (A.12) and (42) is obtained by summing (A.11) and (A.12). ■

A.3. DERIVATIONS FOR THE WEN (1998) MODEL

A.3.1. The Model

In the model economy, the representative agent chooses sequences of consumption $\{C\}$, hours to work $\{L\}$, rate of capacity utilization $\{u\}$, and capital stock $\{K\}$ to maximize her life-time utility. Wen's original model setup is a social planner's problem. In order to consider infinite horizon learning, we slightly modify the model and consider a decentralized maximization problem as follows. The agent's problem is

$$\max_{\{C_t,L_t,K_{t+1},u_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t - \frac{L_t^{1+\gamma}}{1+\gamma} \right),$$

subject to the constraint

$$C_t + K_{t+1} - (1-\delta_t)\,K_t = r_t\,(u_tK_t) + w_tL_t,$$

$$\delta_t = \frac{1}{\theta}u_t^{\theta},$$

where $0 < \alpha < 1$, $0 < \beta < 1$, $\gamma \geq 0$, $\eta > 0$, and $\theta > 1$. The depreciation rate of capital stock, $\delta_t \in (0, 1)$ is increasing in capacity utilization rate, u_t. $\theta > 1$ ensures that the optimal capacity utilization rate u_t lies in (0, 1). w_t represents the wage rate. r_t is the rental rate paid to utilized capital stocks, in line with the firm's profit maximization problem where the cost of each firm is expressed in terms of the utilized capital. The firm's problem is

$$\max_{\{L_t,K_t\}} Y_t - w_tL_t - r_t\,(u_tK_t)\,,$$

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subject to the production technology $Y_t = \bar{e}_t(u_t K_t)^\alpha L_t^{1-\alpha}$, $\bar{e}_t = (\bar{u}_t \bar{K}_t)^{\alpha\eta} \bar{L}_t^{(1-\alpha)\eta}$ where $0 < \alpha < 1$ and $\eta > 0$.

Of particular note is the production externality, $\bar{e}_t$, a function of mean productive capacity, $\bar{u}_t \bar{K}_t$, and mean labor hours, $\bar{L}_t$. The optimal capacity utilization rate is $u_t = (\alpha \frac{Y_t}{K_t})^{1/\theta}$, which implies $u_t = (\alpha K_t^{\alpha(1+\eta)-1} L_t^{(1-\alpha)(1+\eta)})^{1/[\theta-\alpha(1+\eta)]}$. So the reduced-form aggregate production function evaluated at this optimal rate is of the form $Y_t = K_t^{a^*} L_t^{b^*}$, where $a^* = \alpha(1+\eta)\tau_K$, $b^* = (1-\alpha)(1+\eta)\tau_N$, $\tau_K = \dfrac{\theta-1}{\theta-\alpha(1+\eta)}$, and $\tau_N = \dfrac{\theta}{\theta-\alpha(1+\eta)}$.

A.3.2. Derivation of the Behavioral Equation

We offer a step-by-step derivation for the decision rule. In this model, the two equilibrium conditions for the prices are

$$\hat{w}_t = \frac{a^*\gamma}{1+\gamma-b^*}\hat{K}_t + \frac{1-b^*}{1+\gamma-b^*}\hat{C}_t, \tag{A.13}$$

$$\hat{R}_t = -(1-\beta)\left[1 - \frac{a^*(1+\gamma)}{1+\gamma-b^*}\right]\hat{K}_t - (1-\beta)\frac{b^*}{1+\gamma-b^*}\hat{C}_t. \tag{A.14}$$

We iterate forward these two equations to obtain

$$\begin{aligned}\hat{w}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{w}_{t+j} = {}& \frac{a^*\gamma}{1+\gamma-b^*}\left(\hat{K}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{K}_{t+j}\right)\\ &+\frac{1-b^*}{1+\gamma-b^*}\left(\hat{C}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{C}_{t+j}\right),\end{aligned} \tag{A.15}$$

$$\begin{aligned}\hat{R}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{R}_{t+j} = {}& -(1-\beta)\left[1-\frac{a^*(1+\gamma)}{1+\gamma-b^*}\right]\left(\hat{K}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{K}_{t+j}\right)\\ &-(1-\beta)\frac{b^*}{1+\gamma-b^*}\left(\hat{C}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{C}_{t+j}\right).\end{aligned} \tag{A.16}$$

We rewrite the agent's decision rule (44) as

$$\begin{aligned}\hat{K}_t = {}& -\frac{\gamma+1}{\gamma}\frac{wL}{KR}\left(\hat{w}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{w}_{t+j}\right) - \left(\hat{R}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{R}_{t+j}\right)\\ &+\left(\frac{C}{KR}+\frac{wL}{\gamma KR}\right)\left[\frac{R}{R-1}\hat{C}_t + \frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{R}_{t+j}\right].\end{aligned} \tag{A.17}$$

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We then plug (A.15) and (A.16) into (A.17) to get

$$\hat{K}_t = \left\{-\frac{\gamma+1}{\gamma}\frac{wL}{KR}\times\frac{a^*\gamma}{1+\gamma-b^*}-\left[-(1-\beta)\left[1-\frac{a^*(1+\gamma)}{1+\gamma-b^*}\right]\right]\right\}$$
$$\times\left(\hat{K}_t+E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{K}_{t+j}\right)$$
$$+\left[-\frac{\gamma+1}{\gamma}\frac{wL}{KR}\times\frac{1-b^*}{1+\gamma-b^*}+(1-\beta)\frac{b^*}{1+\gamma-b^*}\right]\left(\hat{C}_t+E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j}\right)$$
$$+\left(\frac{C}{KR}+\frac{wL}{\gamma KR}\right)\left[\frac{R}{R-1}\hat{C}_t+\frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right]$$

$$=\left\{-\frac{1}{R}\frac{wL}{K}\frac{a^*(\gamma+1)}{1+\gamma-b^*}+\frac{1}{R}\alpha\frac{Y}{K}\frac{\theta-1}{\theta}\left[1-\frac{a^*(1+\gamma)}{1+\gamma-b^*}\right]\right\}\left(\hat{K}_t+E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{K}_{t+j}\right)$$
$$+\left[-\frac{wL}{\gamma KR}\left(1-\frac{\gamma b^*}{1+\gamma-b^*}\right)+\frac{1}{R}\alpha\frac{Y}{K}\frac{\theta-1}{\theta}\frac{b^*}{1+\gamma-b^*}\right]\left(\hat{C}_t+E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j}\right)$$
$$+\left(\frac{C}{KR}+\frac{wL}{\gamma KR}\right)\left[\frac{R}{R-1}\hat{C}_t+\frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right]$$

$$=\left\{\frac{1}{R}\left[\alpha\frac{Y}{K}\frac{\theta-1}{\theta}\left[1-\frac{a^*(\gamma+1)}{1+\gamma-b^*}\right]-\frac{wL}{K}\frac{a^*(\gamma+1)}{1+\gamma-b^*}\right]\right\}\left(\hat{K}_t+E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{K}_{t+j}\right)$$
$$+\left[-\frac{wL}{\gamma KR}+\frac{1}{R}(1-\alpha)\frac{Y}{K}\frac{b^*}{1+\gamma-b^*}+\frac{1}{R}\alpha\frac{Y}{K}\frac{\theta-1}{\theta}\frac{b^*}{1+\gamma-b^*}\right]$$
$$\times\left(\hat{C}_t+E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j}\right)$$
$$+\left(\frac{C}{KR}+\frac{wL}{\gamma KR}\right)\left[\frac{R}{R-1}\hat{C}_t+\frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right]$$

$$=\left\{\frac{1}{R}\left[\alpha\frac{Y}{K}\frac{\theta-1}{\theta}-\alpha\frac{Y}{K}\frac{\theta-1}{\theta}\frac{a^*(\gamma+1)}{1+\gamma-b^*}-\frac{Y}{K}\frac{a^*(\gamma+1)}{1+\gamma-b^*}+\alpha\frac{Y}{K}\frac{a^*(\gamma+1)}{1+\gamma-b^*}\right]\right\}$$
$$\times\left(\hat{K}_t+E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{K}_{t+j}\right)$$

$$+\left[-\frac{wL}{\gamma KR}+\frac{1}{R}\left(1-\frac{\alpha}{\theta}\right)\frac{Y}{K}\frac{b^*}{1+\gamma-b^*}+\frac{1}{R}\frac{C}{K}-\frac{1}{R}\frac{C}{K}\right]\left(\hat{C}_t+E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j}\right)$$

$$+\left(\frac{C}{KR}+\frac{wL}{\gamma KR}\right)\left[\frac{R}{R-1}\hat{C}_t+\frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right]$$

$$=\left\{\frac{1}{R}\left[\frac{1}{\beta}-1-\left(1-\frac{\alpha}{\theta}\right)\frac{Y}{K}\frac{a^*(\gamma+1)}{1+\gamma-b^*}\right]\right\}\left(\hat{K}_t+E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{K}_{t+j}\right)$$

$$+\left[-\frac{wL}{\gamma KR}+\frac{1}{R}\left(1-\frac{\alpha}{\theta}\right)\frac{Y}{K}\frac{b^*}{1+\gamma-b^*}+\frac{1}{R}\frac{C}{K}-\frac{1}{R}\frac{C}{K}\right]\left(\hat{C}_t+E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j}\right)$$

$$+\left(\frac{C}{KR}+\frac{wL}{\gamma KR}\right)\left[\frac{R}{R-1}\hat{C}_t+\frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right]$$

$$=\left\{\frac{1}{R}\left[R-\left[1+\left(1-\frac{\alpha}{\theta}\right)\frac{Y}{K}\frac{a^*(\gamma+1)}{1+\gamma-b^*}\right]\right]\right\}\left(\hat{K}_t+E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{K}_{t+j}\right)$$

$$+\left[\frac{1}{R}\left[\frac{C}{K}+\left(1-\frac{\alpha}{\theta}\right)\frac{Y}{K}\frac{b^*}{1+\gamma-b^*}\right]-\frac{wL}{\gamma KR}-\frac{C}{KR}\right]\left(\hat{C}_t+E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j}\right)$$

$$+\left(\frac{C}{KR}+\frac{wL}{\gamma KR}\right)\left[\frac{R}{R-1}\hat{C}_t+\frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right]$$

$$=\left\{\frac{1}{R}\left[R-\left[1+\left(1-\frac{\alpha}{\theta}\right)\frac{Y}{K}\frac{a^*(\gamma+1)}{1+\gamma-b^*}\right]\right]\right\}\left(\hat{K}_t+E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{K}_{t+j}\right)$$

$$+\left[\frac{1}{R}\left[\frac{C}{K}+\left(1-\frac{\alpha}{\theta}\right)\frac{Y}{K}\frac{b^*}{1+\gamma-b^*}\right]-\frac{wL}{\gamma KR}-\frac{C}{KR}\right]$$

$$\times\left[\frac{R}{R-1}\hat{C}_t+\frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right]$$

$$+\left(\frac{C}{KR}-\frac{wL}{\gamma KR}\right)\left[\frac{R}{R-1}\hat{C}_t+\frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right]$$

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$$= \left\{ \frac{1}{R} \left[R - \left[1 + \left(1 - \frac{\alpha}{\theta} \right) \frac{Y}{K} \frac{a^* (\gamma + 1)}{1 + \gamma - b^*} \right] \right] \right\} \left(\hat{K}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{K}_{t+j} \right)$$

$$+ \frac{1}{R} \left[\frac{C}{K} + \left(1 - \frac{\alpha}{\theta} \right) \frac{Y}{K} \frac{b^*}{1 + \gamma - b^*} \right] \left[\frac{R}{R-1} \hat{C}_t + \frac{R}{R-1} E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{R}_{t+j} \right]$$

$$\begin{aligned} &= \frac{R - d_1}{R} \left(\hat{K}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{K}_{t+j} \right) \\ &\quad - \frac{d_2}{R} \left[\frac{R}{R-1} \hat{C}_t + \frac{R}{R-1} E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{R}_{t+j} \right]. \end{aligned} \tag{A.18}$$

Equation (A.16) can be rewritten as

$$\begin{aligned} \hat{R}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{R}_{t+j} &= -d_4 \left(\hat{K}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{K}_{t+j} \right) \\ &\quad + (1 - d_3) \left(\hat{C}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{C}_{t+j} \right). \end{aligned} \tag{A.19}$$

We then plug this equation into (A.18) to obtain the behavioral equation.

A.4. DERIVATIONS FOR THE SCHMITT-GROHÉ AND URIBE (1997) MODEL

A.4.1. The Model

The representative agent chooses paths for consumption C_t and hours L_t to solve her infinite-horizon utility maximization problem. The agent's problem is reformulated as

$$\max_{C_t, L_t} E_0 \sum_{t=0}^{\infty} \beta^t (\log C_t - A L_t),$$

subject to the budget constraint

$$K_{t+1} = w_t L_t + r_t K_t + (1 - \delta) K_t - C_t - G,$$

$$G = \tau_t w_t L_t.$$

where G denotes government revenue that is financed with taxes on labor income at rate $\tau_t \in (0, 1)$. Parameters satisfy $0 < \beta < 1$, $0 < \delta < 1$, and $A > 0$. The firm's problem is standard.

A.4.2. Derivations of the Behavioral Equation

The equilibrium conditions for the prices are

$$\hat{w}_t = -\frac{a\tau}{1 + \tau - b} \hat{K}_t + \frac{(1 - b)(1 - \tau)}{1 + \tau - b} \hat{C}_t, \tag{A.20}$$

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$$\hat{R}_t = -\beta a \frac{Y}{K}\left[1 - \frac{a(1-\tau)}{1-\tau-b}\right]\hat{K}_t - \beta a \frac{Y}{K}\frac{b(1-\tau)}{1-\tau-b}\hat{C}_t. \tag{A.21}$$

We iterate these two equations to get

$$\begin{aligned}\hat{w}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{w}_{t+j} = &-\frac{a\tau}{1-\tau-b}\left(\hat{K}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{K}_{t+j}\right)\\ &+\frac{(1-b)(1-\tau)}{1-\tau-b}\left(\hat{C}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{C}_{t+j}\right),\end{aligned} \tag{A.22}$$

$$\begin{aligned}\hat{R}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{R}_{t+j} = &-\beta a\frac{Y}{K}\left[1 - \frac{a(1-\tau)}{1-\tau-b}\right]\left(\hat{K}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{K}_{t+j}\right)\\ &-\beta a\frac{Y}{K}\frac{b(1-\tau)}{1-\tau-b}\left(\hat{C}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{C}_{t+j}\right).\end{aligned} \tag{A.23}$$

The optimal decision of the agent is

$$\begin{aligned}\hat{K}_t = &-\frac{\tau-1}{\tau}\frac{wL}{KR}\left(\hat{w}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{w}_{t+j}\right) - \left(\hat{R}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{R}_{t+j}\right)\\ &+\left(\frac{C}{KR} - \frac{1-\tau}{\tau}\frac{wL}{KR}\right)\left[\frac{R}{R-1}\hat{C}_t + \frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{R}_{t+j}\right].\end{aligned} \tag{A.24}$$

We then plug (A.22) and (A.23) into (A.24) to get

$$\begin{aligned}\hat{K}_t = &\left\{\frac{\tau-1}{\tau}\frac{wL}{KR}\times\frac{a\tau}{1-\tau-b} - \left[-\beta a\frac{Y}{K}\left[1-\frac{a(1-\tau)}{1-\tau-b}\right]\right]\right\}\left(\hat{K}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{K}_{t+j}\right)\\ &+\left[-\frac{\tau-1}{\tau}\frac{wL}{KR}\times\frac{(1-b)(1-\tau)}{1-\tau-b} + \beta a\frac{Y}{K}\frac{b(1-\tau)}{1-\tau-b}\right]\left(\hat{C}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{C}_{t+j}\right)\\ &+\left(\frac{C}{KR} - \frac{1-\tau}{\tau}\frac{wL}{KR}\right)\left[\frac{R}{R-1}\hat{C}_t + \frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j \hat{R}_{t+j}\right]\end{aligned}$$

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$$= \left\{ \frac{1}{R}\frac{wL}{K}\frac{a(\tau-1)}{1-\tau-b} + \frac{1}{R}a\frac{Y}{K}\left[1-\frac{a(1-\tau)}{1-\tau-b}\right]\right\}\left(\hat{K}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{K}_{t+j}\right)$$

$$+\left\{\frac{wL}{\tau KR}\left[(1-\tau)+\frac{b\tau(1-\tau)}{1-\tau-b}\right]+\frac{1}{R}a\frac{Y}{K}\frac{b(1-\tau)}{1-\tau-b}\right\}\left(\hat{C}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j}\right)$$

$$+\left(\frac{C}{KR}-\frac{1-\tau}{\tau}\frac{wL}{KR}\right)\left[\frac{R}{R-1}\hat{C}_t+\frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right]$$

$$= \left\{ \frac{1}{R}\left[a\frac{Y}{K}\left[1-\frac{a(1-\tau)}{1-\tau-b}\right]-\frac{wL}{K}\frac{a(1-\tau)}{1-\tau-b}\right]\right\}\left(\hat{K}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{K}_{t+j}\right)$$

$$+\left[\frac{1-\tau}{\tau}\frac{wL}{KR}+\frac{1}{R}b\frac{Y}{K}\frac{b(1-\tau)}{1-\tau-b}+\frac{1}{R}a\frac{Y}{K}\frac{b(1-\tau)}{1-\tau-b}\right]\left(\hat{C}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j}\right)$$

$$+\left(\frac{C}{KR}-\frac{1-\tau}{\tau}\frac{wL}{KR}\right)\left[\frac{R}{R-1}\hat{C}_t+\frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right]$$

$$= \left\{ \frac{1}{R}\left[r-r\frac{a(1-\tau)}{1-\tau-b}-\frac{Y}{K}\frac{a(1-\tau)}{1-\tau-b}+r\frac{a(1-\tau)}{1-\tau-b}\right]\right\}\left(\hat{K}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{K}_{t+j}\right)$$

$$+\left[\frac{1-\tau}{\tau}\frac{wL}{KR}+\frac{1}{R}\frac{Y}{K}\frac{b(1-\tau)}{1-\tau-b}+\frac{1}{R}\frac{C}{K}-\frac{1}{R}\frac{C}{K}\right]\left(\hat{C}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j}\right)$$

$$+\left(\frac{C}{KR}-\frac{1-\tau}{\tau}\frac{wL}{KR}\right)\left[\frac{R}{R-1}\hat{C}_t+\frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right]$$

$$= \left\{ \frac{1}{R}\left[R-\left[\frac{Y}{K}\frac{a(1-\tau)}{1-\tau-b}+1-\delta\right]\right]\right\}\left(\hat{K}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{K}_{t+j}\right)$$

$$+\left[\frac{1}{R}\left(\frac{C}{K}+\frac{Y}{K}\frac{b(1-\tau)}{1-\tau-b}\right)+\frac{1-\tau}{\tau}\frac{wL}{KR}-\frac{C}{KR}\right]\left(\hat{C}_t + E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{C}_{t+j}\right)$$

$$+\left(\frac{C}{KR}-\frac{1-\tau}{\tau}\frac{wL}{KR}\right)\left[\frac{R}{R-1}\hat{C}_t+\frac{R}{R-1}E_t\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j\hat{R}_{t+j}\right]$$

$$= \left\{ \frac{1}{R} \left[R - \left[\frac{Y}{K} \frac{a(1-\tau)}{1-\tau-b} + 1 - \delta \right] \right] \right\} \left(\hat{K}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{K}_{t+j} \right)$$

$$+ \frac{1}{R} \left(\frac{C}{K} + \frac{Y}{K} \frac{b(1-\tau)}{1-\tau-b} \right) \left[\frac{R}{R-1} \hat{C}_t + \frac{R}{R-1} E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{R}_{t+j} \right]$$

$$= \frac{R-d_1}{R} \left(\hat{K}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{K}_{t+j} \right) - \frac{d_2}{R} \left[\frac{R}{R-1} \hat{C}_t + \frac{R}{R-1} E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{R}_{t+j} \right]. \tag{A.25}$$

Equation (A.23) can be rewritten as

$$\hat{R}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{R}_{t+j} = -d_4 \left(\hat{K}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{K}_{t+j} \right) + (1-d_3) \left(\hat{C}_t + E_t \sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j \hat{C}_{t+j} \right). \tag{A.26}$$

Substituting (A.26) for $E_t \sum_{j=1}^{\infty} (\frac{1}{R})^j \hat{R}_{t+j}$ into equation (A.25) and collecting terms with $\hat{C}_t$ on the left-hand side yields the decision rule.