FIBRED ALGEBRAIC SEMANTICS FOR A VARIETY OF NON-CLASSICAL FIRST-ORDER LOGICS AND TOPOLOGICAL LOGICAL TRANSLATION

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Abstract. Lawvere hyperdoctrines give categorical algebraic semantics for intuitionistic predicate logic. Here we extend the hyperdoctrinal semantics to a broad variety of substructural predicate logics over the Typed Full Lambek Calculus, verifying their completeness with respect to the extended hyperdoctrinal semantics. This yields uniform hyperdoctrinal completeness results for numerous logics such as different types of relevant predicate logics and beyond, which are new results on their own; i.e., we give uniform categorical semantics for a broad variety of non-classical predicate logics. And we introduce an analogue of Lawvere–Tierney topology and cotopology in the hyperdoctrinal setting, which gives a unifying perspective on different logical translations, in particular allowing for a uniform treatment of Girard's exponential translation between linear and intuitionistic logics and of Kolmogorov's double negation translation between intuitionistic and classical logics. In the hyerdoctrinal conception, type theories are categories, logics over type theories are functors, and logical translations between them, then, are natural transformations, in particular Lawvere–Tierney topologies and cotopologies on hyperdoctrines. The view of logical translations as hyperdoctrinal Lawvere–Tierney topologies and cotopologies has not been elucidated before, and may be seen as a novel contribution of the present work. From a broader perspective, this work may be regarded as taking first steps towards interplay between algebraic and categorical logics; it is, technically, a combination of substructural (or Lambekian) algebraic logic and hyperdoctrinal (or Lawverian) categorical logic, as the hyperdoctrinal completeness theorem is shown via the integration of the Lindenbaum–Tarski algebra construction with the syntactic category construction. As such this work lays a foundation for further interactions between algebraic and categorical logics.

§1. Introduction. The present paper aims to connect algebraic substructural logic with categorical predicate logic via Lawvere hyperdoctrines; in particular we combine the algebraic approach to substructural logics as in Galatos–Jipsen–Kowalski–Ono [10] with the categorical approach to (typed) predicate logic as in Pitts [35], which originates in Lawvere [23]. While the correspondence between propositional logics and their algebras is transparent and almost obvious in many cases, there is no widely agreed conception of the correspondence between predicate logics and their algebras; given a predicate logic, there is no broadly applicable, automatic method to define its algebras.

We consider Lawvere hyperdoctrines to be a suitable framework to do this; given a predicate logic, the corresponding hyperdoctrines can be defined in a canonical manner, serving as the fibred algebras of it, as we shall explain below. Although

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Henkin–Monk–Tarski [12] give the classic concept of cylindric algebras as the algebras of predicate logic, there is, in general, no canonical way to define cylindric algebras for a given predicate logic, and cylindric algebras can actually be subsumed under the framework of hyperdoctrines (see Jacobs [16]; note that fibrations and hyperdoctrines are linked with each other via the Grothendieck construction).

The general framework of substructural logics over the Full Lambek calculus (FL for short) encompasses a wide variety of logical systems (classical, intuitionistic, linear, fuzzy, relevant, etc.), allowing for a uniform treatment of different non-classical logics as well as classical logics. Logics over FL have been studied extensively in the past few decades, especially by algebraic logicians interested in residuated lattices, a monoidal extension of Heyting algebras (see, e.g., Galatos–Jipsen–Kowalski–Ono [10] and Galatos–Ono [11]). Among other things, Ciabattoni–Galatos–Terui [4, 5] have successfully applied the methods of algebraic logic and universal algebra to the proof theory of logics over FL, proving that a propositional logic admits cut elimination if and only if the variety of its algebras is closed under algebraic completion (for more detailed, precise formulations, see Ciabattoni–Galatos–Terui [4, 5]); this has led to developments of algebraic proof theory.

Yet there is no algebraic proof theory systematically developed for predicate logics because there are no adequate algebraic models of them in the first place. Although some efforts have been made towards the algebraic treatment of logics over quantified FL (see, e.g., Ono [33, 34] and references therein), however, it seems that there has been no adequate concept of algebraic models of them so far. Note that complete residuated lattices can only give complete semantics for those classes of substructural predicate logics for which completions (such as Dedekind–MacNeille or Crawley) of Lindenbaum–Tarski algebras work in an adequate manner (see Ono [33, 34]); for this reason, complete residuated lattices (or quantales) cannot serve the purpose.

In the context as clarified above, we propose to employ Lawvere hyperdoctrines and their extensions as fibred algebraic models of predicate logic, especially substructural logics over quantified FL. Hyperdoctrines may be seen as fibred algebras indexed by categories, which are supposed to represent type theories (or sort structures). According to Pitts' formulation [35] (see also [2, 3, 28, 29, 31, 32, 37]), a hyperdoctrine is a functor (or algebra-valued presheaf)

$$P: \mathbf{C}^{\mathrm{op}} \to \mathbf{HA},$$

where **HA** is the category of Heyting algebras and their homomorphisms; there are additional conditions on P (and **C**) to express quantifiers and other logical structures, which shall be clarified below. The intuitive meaning of the base category **C** is the category of types (aka. sorts) or domains of discourse, and then $P(C)$ is the algebra of predicates on a type C. We can regard a hyperdoctrine as a fibred Heyting algebra

$$(P(C))_{C \in \mathbf{C}},$$

a bunch of Heyting algebras indexed by a category **C**. Note that toposes amount to higher-order hyperdoctrines; they are interlaced by two functors, one taking subobject hyperdoctrines and the other given by the tripos-to-topos construction (see, e.g., Frey [9], Hyland–Johnstone–Pitts [14], and Pitts [36]).

In general, a fibred algebra is a universal algebra indexed by a category; fibred universal algebra studies a functor of form

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}$$

apart from logical conditions to express quantifiers and others, where **Alg** is an algebraic category (for basic concepts in category theory, see, e.g., Adámek–Herrlich–Strecker [1]; but we do not use category theory in any heavy way, in order to make the paper accessible to the algebraic logician as well as the categorical logician). There is a general principle of completeness lifting: if a propositional logic L is sound and complete with respect to a variety (or algebras of a monad, which is a possibly infinitary variety) **Alg**, then the corresponding fibred algebras or hyperdoctrines

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}$$

give sound and complete semantics for the predicate logic that extends L with quantifiers; that is to say, the soundness and completeness of propositional logic with respect to algebras in **Alg** lifts to the soundness and completeness of predicate logic with respect to fibred algebras $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}$.

In the present paper we demonstrate this in the fairly general context of arbitrary logics over the base system FL; this constitutes the first result of the paper. The second result of the paper is concerned with Lawvere–Tierney topologies and cotopologies on hyperdoctrines regarded as logical translations. In topos theory, the double negation topology has been applied broadly for different purposes. The generalised notion of Lawvere–Tierney topologies and cotopologies on hyperdoctrines allows for a unified treatment of different logical translations, in particular Girard's exponential translation in linear logic and Kolmogorov's double negation translation in intuitionistic logic, as we shall see below (we call the latter the Kolmogorov translation for historical reasons; see Ferreira–Oliva [8]). The former allows us to embed intuitionistic logic into linear logic via the exponential operator !; the latter allows us to embed classical logic into intuitionistic logic via the double negation operator ¬¬. Since logics (or theories) are dual to algebraic semantics (or models), we construct intuitionistic (resp. classical) hyperdoctrines from linear (resp. intuitionistic) hyperdoctrines on the basis of suitable Lawvere–Tierney (co)topologies on them. Lawvere–Tierney topologies on toposes work for higher-order logic; Lawvere–Tierney topologies on hyperdoctrines, by contrast, work for first-order logic as well as higher-order logic. In this sense, the latter is more general than the former, making the notion of topology available in a more general setting. And Lawvere–Tierney topologies and cotopologies on hyperdoctrines allow us to obtain hyperdoctrinal analogues of logical translation theorems in a uniform and modular manner. This constitutes the second result of the paper.

From a broader perspective, the hope of the present paper is to bridge between algebraic logic, in which substructural logics over FL have been studied, and categorical logic, in which Lawvere hyperdoctrines have been pursued. The two disciplines are currently separated to some extent, but were tightly intertwined with each other in their early days. For example, Lawvere's original ideas on categorical logic were seemingly of algebraic nature; in particular, Lawvere's functorial semantics [22] was directly concerned with universal algebra. Note also

that Lambek's categorical duality theory [20], too, was based on certain ideas of algebraic logic and universal algebra, such as primal (or functionally complete) algebras and their induced Stone dualities, combining them with categorical methods to give what is called the general Stone–Gelfand duality. It would be fruitful to restore vital interactions between the two fields yet again (note that interactions between categorical and traditional universal algebras have been revitalised since Hyland–Power [15]). This paper aims at taking modest first steps towards them.

A final remark before proceeding to the following sections is that we do not aim at giving so-called semantics of proofs; we give semantics of provability just as the topos semantics does, a principal reason for which is as follows. Not all logics over FL have well-behaved proof theories, and yet we would like to develop a uniform general theory for them. A well-known (notorious) case of this is classical logic, which basically makes the identity of proofs trivialise as the Joyal Lemma shows (i.e., Cartesian closed categories with dualising objects, the existence of which means the validity of the double negation elimination, trivialise; see, e.g., Lambek–Scott [21]); some fuzzy logics suffer from similar collapsing phenomena. And there is no general method to remedy them (remedies for them are highly dependent upon the nature of each logic; or there may be no such remedies). For this reason we do not aim at semantics of proofs; rather we develop uniform semantics of provability for all logics over the base system FL. And by doing so, we get special benefits: i.e., we can relate and analyse different sorts of categorical logics in a single setting; the uniform framework for various logics allows us to compare different categorical logics on the one setting. The results about Lawvere–Tierney topologies and cotopologies indeed embody such a comparison in terms of translations between different types of hyperdoctrines.

The rest of the paper is organised as follows. In the following, we first give an introduction to typed FL (Section 2). We then define FL hyperdoctrines, and prove the soundness and completeness of the hyperdoctrinal semantics for logics over typed FL (Section 3). And we introduce Lawvere–Tierney topologies and cotopologies on hyperdoctrines, and give hyperdoctrinal translation theorems based on them (Section 4); this may be regarded as the most original part of the present work. We finally conclude by illustrating several directions of further research (Section 5). Prior knowledge of category theory is not much required, and we keep the formulation as simple as possible while explaining the categorical details that the purely algebraic logician may not be familiar with.

§2. Typed Full Lambek calculus.
In this section we introduce the Typed Full Lambek calculus with quantifiers, denoted TFL, which is basically a typed or many-sorted version of the first-order Full Lambek calculus FL as in Ono [34]; we follow the typing style of Pitts [35], and so this is basically the combination of Pitts [35] and Ono [34].

Standard categorical logic discusses a typed version of intuitionistic (or coherent or regular) logic, as seen, for example, in Pitts [35], Lambek–Scott [21], Jacobs [16], and Johnstone [18]. Typed logic is more natural than single-sorted one from a categorical point of view, and is more expressive in general, since it can be equipped with various type constructors, which make the logic more powerful (if they do not

make it inconsistent). If one prefers single-sorted logic to typed logic, the latter can be reduced to the former by allowing for one type (or sort) only.

To put it differently, typed logic is the combination of logic and type theory, and has not only a logic structure but also a type structure, and the latter itself has a rich structure as well as the former. For this reason, syntactic hyperdoctrines constructed from typed systems of logic (which are discussed in relation to completeness in the next section) are amalgamations of syntactic categories obtained from their type theories on the one hand, and Lindenbaum–Tarski algebras obtained from their logic parts on the other; in a nutshell, syntactic hyperdoctrines are type-fibred Lindenbaum–Tarski algebras.

Another merit of typed logic is that the problem of empty domains is resolved because it allows us to have explicit control on type contexts. This was discovered by Joyal, according to Marquis and Reyes [24], and shall be touched upon later in more detail.

TFL has the following logical connectives:

$$\otimes, \wedge, \vee, \backslash, /, 1, 0, \top, \bot, \forall, \exists.$$

Note that there are two kinds of implication connectives $\backslash$ and $/$, owing to the non-commutative nature of TFL.

In TFL, every variable x comes with its type σ. That is, TFL has basic types, which are denoted by letters like σ, τ, and

$$x : \sigma$$

is a formal expression meaning that a variable x is of type σ. Then, a (type) context is a finite list of type declarations on variables:

$$x_1 : \sigma_1, \dots, x_n : \sigma_n.$$

A context is often denoted Γ.

Accordingly, TFL has typed predicate symbols (aka. predicates in context) and typed function symbols (aka. function symbols in context):

$$R(x_1, \dots, x_n) \, [x_1 : \sigma_1, \dots, x_n : \sigma_n]$$

is a formal expression meaning that R is a predicate with n variables $x_1, \dots, x_n$ of types $\sigma_1, \dots, \sigma_n$ respectively; likewise,

$$f : \tau \, [x_1 : \sigma_1, \dots, x_n : \sigma_n]$$

is a formal expression meaning that f is a function symbol with n variables $x_1, \dots, x_n$ of types $\sigma_1, \dots, \sigma_n$ and with its values in τ. Then, formulae-in-context $\varphi \, [\Gamma]$ and terms-in-context $t : \tau \, [\Gamma]$ are defined in the usual, inductive way.

In the present paper, we do not consider any specific type constructor, but even if type constructors such as products, sums, and function spaces are added, they just make the syntactic category construction below more complex (since they do not affect the propositional structure), and all proofs below can be directly adapted to such cases in a modular manner. Note that if type constructors affect the propositional structure, then proofs get essentially more difficult; this is the case in higher-order logic with the proposition type Prop (aka. truth value type Ω), which interlace the term structure with the propositional structure. Although we

focus upon plainly typed predicate logic with no complicated type structure, still, products (not as types but as categorical structures) shall be used in categorical semantics in the next section, to the end of interpreting predicate and function symbols (of arity greater than one).

TFL thus has both a type structure and a logic structure, dealing with sequents-in-contexts:

$$\Phi \vdash \varphi \ [\Gamma],$$

where Γ is a type context, and Φ is a finite list of formulae: $\varphi_1, \dots, \varphi_n$. Although it is common to write $\Gamma \mid \Phi \vdash \varphi$ rather than $\Phi \vdash \varphi \ [\Gamma]$, we employ the latter notation in this paper, following Pitts [35], since TFL is an adaptation of Pitts' typed system for intuitionistic logic to the system of the Full Lambek calculus.

The syntax of type contexts Γ in TFL is the same as that of typed intuitionistic logic in Pitts [35]. Note that it is allowed to add a fresh $x : \sigma$ to a context Γ: e.g.,

$$\Phi \vdash \varphi \ [\Gamma, x : \sigma]$$

whenever $\Phi \vdash \varphi \ [\Gamma]$. On the other hand, it is not permitted to delete redundant variables; the reason becomes clear in latter discussion on empty domains. It is allowed to change the order of contexts (e.g., $[\Gamma, \Gamma']$ into $[\Gamma', \Gamma]$). Now, in the following, we explain logical rules of inference.

TFL has no structural rule other than the following cut rule:

$$\frac{\Phi_1 \vdash \varphi \ [\Gamma] \quad \Phi_2, \varphi, \Phi_3 \vdash \psi \ [\Gamma]}{\Phi_2, \Phi_1, \Phi_3 \vdash \psi \ [\Gamma]} \ (cut),$$

where ψ may be empty; this is allowed in the following L (left) rules as well. As usual, we have the rule of identity

$$\frac{}{\varphi \vdash \varphi \ [\Gamma]} \ (id).$$

In the following, we list the rules of inference for the logical connectives of TFL. The following formulation is intuitionistic in the sense that only one formula is allowed to appear on the right-hand side of sequents; nevertheless, we can treat classical logic as an axiomatic extension of TFL, by adding to TFL exchange, weakening, contraction, and the excluded middle (note that structural rules can be expressed as axioms). There are two kinds of conjunction in TFL: multiplicative or monoidal $\otimes$ and additive or Cartesian $\wedge$:

$$\frac{\Phi, \varphi, \psi, \Psi \vdash \chi \ [\Gamma]}{\Phi, \varphi \otimes \psi, \Psi \vdash \chi \ [\Gamma]} \ (\otimes L) \qquad \frac{\Phi \vdash \varphi \ [\Gamma] \quad \Psi \vdash \psi \ [\Gamma]}{\Phi, \Psi \vdash \varphi \otimes \psi \ [\Gamma]} \ (\otimes R)$$

$$\frac{\Phi, \varphi, \Psi \vdash \chi \ [\Gamma]}{\Phi, \varphi \wedge \psi, \Psi \vdash \chi \ [\Gamma]} \ (\wedge L_1) \qquad \frac{\Phi, \varphi, \Psi \vdash \chi \ [\Gamma]}{\Phi, \psi \wedge \varphi, \Psi \vdash \chi \ [\Gamma]} \ (\wedge L_2)$$

$$\frac{\Phi \vdash \varphi \ [\Gamma] \quad \Phi \vdash \psi \ [\Gamma]}{\Phi \vdash \varphi \wedge \psi \ [\Gamma]} \ (\wedge R).$$

There is only one disjunction in TFL, which is additive, since TFL is intuitionistic in the aforementioned sense.

$$\frac{\Phi, \varphi, \Psi \vdash \chi \, [\Gamma] \quad \Phi, \psi, \Psi \vdash \chi \, [\Gamma]}{\Phi, \varphi \vee \psi, \Psi \vdash \chi \, [\Gamma]} \ (\vee L)$$

$$\frac{\Phi \vdash \varphi \, [\Gamma]}{\Phi \vdash \varphi \vee \psi \, [\Gamma]} \ (\vee R_1) \qquad \frac{\Phi \vdash \varphi \, [\Gamma]}{\Phi \vdash \psi \vee \varphi \, [\Gamma]} \ (\vee R_2).$$

Due to non-commutativity, there are two kinds of implication in TFL, $\backslash$ and $/$, which are a right adjoint of $\varphi \otimes (\text{-})$ and a right adjoint of $(\text{-}) \otimes \psi$ respectively.

$$\frac{\Phi \vdash \varphi \, [\Gamma] \quad \Psi_1, \psi, \Psi_2 \vdash \chi \, [\Gamma]}{\Psi_1, \Phi, \varphi \backslash \psi, \Psi_2 \vdash \chi \, [\Gamma]} \ (\backslash L) \qquad \frac{\varphi, \Phi \vdash \psi \, [\Gamma]}{\Phi \vdash \varphi \backslash \psi \, [\Gamma]} \ (\backslash R)$$

$$\frac{\Phi \vdash \varphi \, [\Gamma] \quad \Psi_1, \psi, \Psi_2 \vdash \chi \, [\Gamma]}{\Psi_1, \psi/\varphi, \Phi, \Psi_2 \vdash \chi \, [\Gamma]} \ (/L) \qquad \frac{\Phi, \varphi \vdash \psi \, [\Gamma]}{\Phi \vdash \psi/\varphi \, [\Gamma]} \ (/R).$$

There are two kinds of truth and falsity constants, monoidal and Cartesian ones.

$$\frac{\Psi_1, \Psi_2 \vdash \varphi \, [\Gamma]}{\Psi_1, 1, \Psi_2 \vdash \varphi \, [\Gamma]} \ (1L) \qquad \frac{}{\vdash 1 \, [\Gamma]} \ (1R)$$

$$\frac{}{0 \vdash \, [\Gamma]} \ (0L) \qquad \frac{\Phi \vdash \, [\Gamma]}{\Phi \vdash 0 \, [\Gamma]} \ (0R)$$

$$\frac{}{\Phi \vdash \top \, [\Gamma]} \ (\top R) \qquad \frac{}{\Phi_1, \bot, \Phi_2 \vdash \varphi \, [\Gamma]} \ (\bot L).$$

Finally, we have the following rules for quantifiers $\forall$ and $\exists$, in which type contexts explicitly change; notice that type contexts do not change in the rest of the rules presented above.

$$\frac{\Phi_1, \varphi, \Phi_2 \vdash \psi \, [x:\sigma, \Gamma]}{\Phi_1, \forall x \varphi, \Phi_2 \vdash \psi \, [x:\sigma, \Gamma]} \ (\forall L) \qquad \frac{\Phi \vdash \varphi \, [x:\sigma, \Gamma]}{\Phi \vdash \forall x \varphi \, [\Gamma]} \ (\forall R)$$

$$\frac{\Phi_1, \varphi, \Phi_2 \vdash \psi \, [x:\sigma, \Gamma]}{\Phi_1, \exists x \varphi, \Phi_2 \vdash \psi \, [\Gamma]} \ (\exists L) \qquad \frac{\Phi \vdash \varphi \, [x:\sigma, \Gamma]}{\Phi \vdash \exists x \varphi \, [x:\sigma, \Gamma]} \ (\exists R).$$

As usual, there are eigenvariable conditions on the rules above: x does not appear as a free variable in the bottom sequent of Rule $\forall R$; likewise, x does not appear as a free variable in the bottom sequent of Rule $\exists L$. The other two rules do not have eigenvariable conditions, and this is why contexts do not change in them.

The deducibility of sequents-in-context in TFL is defined in the usual way. In this paper, we denote by FL the propositional (and hence no contextual) part of TFL. Note that what is called FL in the literature often lacks $\bot$ and $\top$.

As is well known, the following propositional (resp. predicate) logics can be represented as axiomatic (to be precise, axiom-schematic) extensions of FL (resp. TFL): classical logic, intuitionistic logic, linear logic (without exponentials), relevance logics, and fuzzy logics such as Gödel–Dummett logic (see, e.g., Galatos–Jipsen–Kowalski–Ono [10]). Given a set of axioms (to be precise, axiom schemata), say X, we denote by FL_X (resp. TFL_X) the corresponding extension of FL (resp. TFL) via X.

LEMMA 1. *The following sequents-in-context are deducible in* TFL :

- $\varphi \otimes (\exists x\psi) \vdash \exists x(\varphi \otimes \psi) \, [\Gamma]$ *and* $\exists x(\varphi \otimes \psi) \vdash \varphi \otimes (\exists x\psi) \, [\Gamma]$,
- $(\exists x\psi) \otimes \varphi \vdash \exists x(\psi \otimes \varphi) \, [\Gamma]$ *and* $\exists x(\psi \otimes \varphi) \vdash (\exists x\psi) \otimes \varphi \, [\Gamma]$,

where it is supposed that φ *does not contain* x *as a free variable, and* Γ *contains type declarations on those free variables that appear in* φ *and* $\exists x\psi$.

A striking feature of typed predicate logic is that domains of discourse in semantics can be empty; they are assumed to be non-empty in the usual Tarski semantics of predicate logic. We therefore need no ad hoc condition on domains of discourse if we work with typed predicate logic. This resolution of the problem of empty domains is due to Joyal as noted in Marquis and Reyes [24].

A proof-theoretic manifestation of this feature is that the following sequent-in-context is not necessarily deducible in TFL:

$$\forall x\varphi \vdash \exists x\varphi \, [\,],$$

where the context is empty. Nonetheless, the following is deducible in TFL:

$$\forall x\varphi \vdash \exists x\varphi \, [x : \sigma, \Gamma],$$

where Γ is an appropriate context including type declarations on free variables in φ. This means that we can prove the sequent above when a type σ is inhabited. Here, it is crucial that it is not allowed to delete redundant free variables (e.g., $[x : \sigma, \Gamma]$ cannot be reduced into $[\Gamma]$ even if x does not appear as a free variable in formulae involved); however, it is allowed to add fresh free variables to a context.

§3. Full Lambek hyperdoctrine. It is well known that FL algebras (defined below) provide sound and complete semantics for propositional logic FL (see, e.g., Galatos–Jipsen–Kowalski–Ono [10]). In this section we show that fibred FL algebras, or FL hyperdoctrines (defined below), yield sound and complete semantics for typed (or many-sorted) predicate logic TFL.

We emphasise the simple, algebro-logical idea that single algebras (symbolically, A with no indexing) correspond to propositional logic, and fibred algebras (symbolically, $(A_C)_{C \in \mathbf{C}}$ indexed by a category $\mathbf{C}$) correspond to predicate logic. As universal algebra gives foundations for algebraic propositional logic, so fibred universal algebra lays a foundation for algebraic predicate logic.

DEFINITION 2 [10].

$$(A, \otimes, \wedge, \vee, \backslash, /, 1, 0, \top, \bot)$$

is called an FL algebra iff

- $(A, \otimes, 1)$ is a monoid ; 0 is a (distinguished) element of A;
- $(A, \wedge, \vee, \top, \bot)$ is a bounded lattice, which induces a partial order $\leq$ on A;
- for any $a \in A$, $a\backslash(\text{-}) : A \to A$ is a right adjoint of $a \otimes (\text{-}) : A \to A$: i.e., $a \otimes b \leq c$ iff $b \leq a\backslash c$ for any $a, b, c \in A$;
- for any $b \in A$, $(\text{-})/b : A \to A$ is a right adjoint of $(\text{-}) \otimes b : A \to A$: i.e., $a \otimes b \leq c$ iff $a \leq c/b$ for any $a, b, c \in A$.

A homomorphism of FL algebras is defined as a map preserving all the operations $(\otimes, \wedge, \vee, \backslash, /, 1, 0, \top, \bot)$. **FL** denotes the category of FL algebras and their homomorphisms.

Although 0 is just a neutral element of A with no axiom, the rules for 0 are automatically valid by the definition of interpretations defined below.

FL is an algebraic category (i.e., a category monadic over **Set**), or a variety in terms of universal algebra, since the two adjointness conditions can be rephrased by equations (see, e.g., Galatos–Jipsen–Kowalski–Ono [10]). An axiomatic extension FL_X of FL corresponds to an algebraic full subcategory (or sub-variety) of **FL**, denoted $\mathbf{FL}_X$ (algebraicity follows from definability by axioms); this is the well-known, logic-variety correspondence for logics over FL (see Galatos–Jipsen–Kowalski–Ono [10]).

DEFINITION 3. An FL (Full Lambek) hyperdoctrine is an **FL**-valued presheaf

$$P: \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$$

such that **C** is a category with finite products, and the following conditions on quantifiers hold:

- For any projection $\pi: X \times Y \to Y$ in **C**, $P(\pi): P(Y) \to P(X \times Y)$ has a right adjoint, denoted

$$\forall_\pi : P(X \times Y) \to P(Y).$$

And the corresponding Beck–Chevalley condition holds, i.e., the following diagram commutes for any arrow $f: Z \to Y$ in $\mathbf{C}(\pi': X \times Z \to Z$ below denotes a projection):

$$
\begin{array}{ccc}
P(X \times Y) & \xrightarrow{\forall_\pi} & P(Y) \\
{\scriptstyle P(X\times f)}\downarrow & & \downarrow{\scriptstyle P(f)} \\
P(X \times Z) & \xrightarrow[\forall_{\pi'}]{} & P(Z)
\end{array}
$$

- For any projection $\pi: X \times Y \to Y$ in **C**, $P(\pi): P(Y) \to P(X \times Y)$ has a left adjoint, denoted

$$\exists_\pi : P(X \times Y) \to P(Y).$$

The corresponding Beck–Chevalley condition holds:

$$
\begin{array}{ccc}
P(X \times Y) & \xrightarrow{\exists_\pi} & P(Y) \\
{\scriptstyle P(X\times f)}\downarrow & & \downarrow{\scriptstyle P(f)} \\
P(X \times Z) & \xrightarrow[\exists_{\pi'}]{} & P(Z)
\end{array}
$$

Furthermore, the Frobenius Reciprocity conditions hold: for any projection $\pi : X \times Y \to Y$ in $\mathbf{C}$, any $a \in P(Y)$, and any $b \in P(X \times Y)$,

$$a \otimes (\exists_\pi b) = \exists_\pi (P(\pi)(a) \otimes b),$$
$$(\exists_\pi b) \otimes a = \exists_\pi (b \otimes P(\pi)(a)).$$

For an axiomatic extension FL_X of FL, an FL_X hyperdoctrine is defined by restricting the value category $\mathbf{FL}$ into $\mathbf{FL}_X$. An FL(resp. FL_X) hyperdoctrine is also called a fibred FL(resp. FL_X) algebra.

The category $\mathbf{C}$ of an FL hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$ is called its base category or type category, and P is also called its predicate functor; intuitively, $P(C)$ is the algebra of predicates on a type, or domain of discourse, C.

Note that, in the definition above, we need two Frobenius Reciprocity conditions due to the non-commutativity of FL algebras.

An FL hyperdoctrine may be seen as an indexed category, and thus as a fibration via the Grothendieck construction. Although we formulate everything in terms of indexed categories in this paper, we can do the same in terms of fibrations as well (working with indexed categories would probably be simpler to those who are not familiar with category theory). From a fibrational point of view, each $P(C)$ is called a fibre of an FL hyperdoctrine P.

The FL (resp. FL_X) hyperdoctrine semantics for TFL (resp. TFL_X) is defined as follows.

DEFINITION 4. Fix an FL hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$. An interpretation [[-]] of TFL in the FL hyperdoctrine P consists of the following:

- assignment of an object

$$[[\sigma]]$$

  in $\mathbf{C}$ to each basic type σ in TFL;
- assignment of an arrow

$$[[f : \tau \,[\Gamma]]] : [[\sigma_1]] \times \cdots \times [[\sigma_n]] \to [[\sigma]]$$

  in $\mathbf{C}$ to each typed function symbol $f : \tau \,[\Gamma]$ in TFL where Γ is supposed to be $x_1 : \sigma_1, \ldots, x_n : \sigma_n$ (note that $[[\sigma_1]] \times \cdots \times [[\sigma_n]]$ makes sense because $\mathbf{C}$ has finite products);
- assignment of an element

$$[[R \,[\Gamma]]]$$

  in $P([[\Gamma]])$, which is an FL algebra, to each typed predicate symbol $R \,[\Gamma]$ in TFL; if the context Γ is $x_1 : \sigma_1, \ldots, x_n : \sigma_n$, then $[[\Gamma]]$ denotes $[[\sigma_1]] \times \cdots \times [[\sigma_n]]$.

Then, terms are inductively interpreted in the following way:

- $[[x : \sigma \,[\Gamma_1, x : \sigma, \Gamma_2]]]$ is defined as the following projection in $\mathbf{C}$:

$$\pi : [[\Gamma_1]] \times [[\sigma]] \times [[\Gamma_2]] \to [[\sigma]].$$

- $[[f (t_1, \ldots, t_n) : \tau \,[\Gamma]]]$ is defined as:

$$[[f]] \circ \langle [[t_1 : \sigma_1 \,[\Gamma]]], \ldots, [[t_n : \sigma_n \,[\Gamma]]] \rangle,$$

where it is supposed that $f : \tau$ $[x_1 : \sigma_1, \dots, x_n : \sigma_n]$, and $t_1 : \sigma_1$ $[\Gamma], \dots, t_n : \sigma_n$ $[\Gamma]$. Note that $\langle [[t_1 : \sigma_1 \, [\Gamma]]], \dots, [[t_n : \sigma_n \, [\Gamma]]] \rangle$ above is the product (or pairing) of arrows in $\mathbf{C}$.

Formulae are then interpreted inductively in the following manner:

- $[[R(t_1, \dots, t_n) \, [\Gamma]]]$ is defined as

$$P(\langle [[t_1 : \sigma_1[\Gamma]]], \dots, [[t_n : \sigma_n[\Gamma]]] \rangle)([[R \, [x : \sigma_1, \dots, x_n : \sigma_n]]]),$$

  where R is a predicate symbol in context $x_1 : \sigma_1, \dots, x_n : \sigma_n$.

- $[[\varphi \otimes \psi \, [\Gamma]]]$ is defined as

$$[[\varphi \, [\Gamma]]] \otimes [[\psi \, [\Gamma]]].$$

  The other binary connectives $\wedge, \vee, \backslash, /$ are interpreted in the same way. $[[1 \, [\Gamma]]]$ is defined as the monoidal unit of $P([[\Gamma]])$. The other constants $0, \top, \bot$ are interpreted in the same way.

- $[[\forall x \varphi \, [\Gamma]]]$ is defined as

$$\forall_\pi([[\varphi \, [x : \sigma, \Gamma]]]),$$

  where $\pi : [[\sigma]] \times [[\Gamma]] \to [[\Gamma]]$ is a projection in $\mathbf{C}$, and φ is a formula in context $[x : \sigma, \Gamma]$. Similarly, $[[\exists x \varphi \, [\Gamma]]]$ is defined as

$$\exists_\pi([[\varphi \, [x : \sigma, \Gamma]]]).$$

Finally, satisfaction of sequents is defined as follows:

- $\varphi_1, \dots, \varphi_n \vdash \psi \, [\Gamma]$ is satisfied in an interpretation $[[\text{-}]]$ in an FL hyperdoctrine P iff the following holds in $P([[\Gamma]])$:

$$[[\varphi_1 \, [\Gamma]]] \otimes \cdots \otimes [[\varphi_n \, [\Gamma]]] \leq [[\psi \, [\Gamma]]].$$

  In case the right-hand side of a sequent is empty, $\varphi_1, \dots, \varphi_n \vdash [\Gamma]$ is satisfied in $[[\text{-}]]$ iff

$$[[\varphi_1 \, [\Gamma]]] \otimes \cdots \otimes [[\varphi_n \, [\Gamma]]] \leq 0$$

  in $P([[\Gamma]])$. In case the left-hand side of a sequent is empty, $\vdash \varphi \, [\Gamma]$ is satisfied in $[[\text{-}]]$ iff $1 \leq [[\varphi[\Gamma]]]$ in $P([[\Gamma]])$.

An interpretation of TFL_X in an FL_X hyperdoctrine is defined by replacing $\mathbf{FL}$ and TFL above with $\mathbf{FL}_X$ and TFL_X respectively.

In the following, we show that the FL (resp. FL_X) hyperdoctrine semantics is sound and complete for TFL (resp. TFL_X). Let $[[\Phi \, [\Gamma]]]$ denote

$$[[\varphi_1 \, [\Gamma]]] \otimes \cdots \otimes [[\varphi_n \, [\Gamma]]]$$

when Φ is $\varphi_1, \dots, \varphi_n$.

Intuitively, an arrow f in $\mathbf{C}$ is a term, and $P(f)$ is a substitution operation (this is exactly true in syntactic hyperdoctrines defined later); then, the Beck–Chevalley conditions and the functoriality of P tell us that substitution commutes with all the logical operations (namely, both propositional connectives and quantifiers). From such a logical point of view, the meaning of the Beck–Chevalley conditions is transparent; they just say that substitution after quantification is the same as quantification after substitution.

PROPOSITION 5. *If $\Phi \vdash \psi$ [Γ] is deducible in* TFL(*resp.* TFL$_X$), *then it is satisfied in any interpretation in any* FL(*resp.* FL$_X$) *hyperdoctrine.*

PROOF. Fix an FL or FL$_X$ hyperdoctrine P and an interpretation $[[-]]$ in P. Initial sequents in context are satisfied because $a \leq a$ in any fibre $P(C)$. The cut rule preserves satisfaction, since tensoring preserves $\leq$ and $\leq$ has transitivity. It is easy to verify that all the rules for the logical connectives preserve satisfaction.

Let us consider universal quantifier $\forall$. To show the case of Rule $\forall R$, assume that

$$[[\Phi \, [x : \sigma, \Gamma]]] \leq [[\varphi \, [x : \sigma, \Gamma]]]$$

in $P([[\sigma]] \times [[\Gamma]])$. It then follows that

$$[[\Phi \, [x : \sigma, \Gamma]]] = P(\pi : [[\sigma]] \times [[\Gamma]] \to [[\Gamma]])([[\Phi \, [\Gamma]]]),$$

where π is a projection in **C**, and note that Φ does not include x among its free variables by the eigenvariable condition. We thus have

$$P(\pi)([[\Phi \, [\Gamma]]]) \leq [[\varphi \, [x : \sigma, \Gamma]]].$$

Since $\forall_\pi : P([[\sigma]] \times [[\Gamma]]) \to P([[\Gamma]])$ is a right adjoint of $P(\pi)$, it follows that

$$[[\Phi \, [\Gamma]]] \leq \forall_\pi([[\varphi \, [x : \sigma, \Gamma]]]) = [[\forall x \varphi \, [\Gamma]]].$$

We next show the case of $\forall L$. Assume that

$$[[\Phi_1 \, [x : \sigma, \Gamma]]] \otimes [[\varphi \, [x : \sigma, \Gamma]]] \otimes [[\Phi_2 \, [x : \sigma, \Gamma]]] \leq [[\psi \, [x : \sigma, \Gamma]]].$$

The adjunction condition for universal quantifier gives us

$$P(\pi)(\forall_\pi([[\varphi \, [x : \sigma, \Gamma]]])) \leq [[\varphi \, [x : \sigma, \Gamma]]],$$

where $\pi : [[\sigma]] \times [[\Gamma]] \to [[\Gamma]]$ is a projection. Yet we also have

$$P(\pi)(\forall_\pi([[\varphi \, [x : \sigma, \Gamma]]])) = P(\pi)([[\forall x \varphi \, [\Gamma]]]) = [[\forall x \varphi \, [x : \sigma, \Gamma]]].$$

Since tensoring respects $\leq$, these together imply that

$$[[\Phi_1 \, [x : \sigma, \Gamma]]] \otimes [[\forall x \varphi \, [x : \sigma, \Gamma]]] \otimes [[\Phi_2 \, [x : \sigma, \Gamma]]] \leq [[\psi \, [x : \sigma, \Gamma]]].$$

It remains to show the case of existential quantifier $\exists$. In order to prove that Rule $\exists L$ preserves satisfaction, assume that

$$[[\Phi_1 \, [x : \sigma, \Gamma]]] \otimes [[\varphi \, [x : \sigma, \Gamma]]] \otimes [[\Phi_2 \, [x : \sigma, \Gamma]]] \leq [[\psi \, [x : \sigma, \Gamma]]].$$

This is equivalent to the following:

$$[[\Phi_1 \, [x : \sigma, \Gamma]]] \otimes [[\varphi \, [x : \sigma, \Gamma]]] \otimes [[\Phi_2 \, [x : \sigma, \Gamma]]] \leq P(\pi)([[\psi \, [\Gamma]]]),$$

where $\pi : [[\sigma]] \times [[\Gamma]] \to [[\Gamma]]$ is a projection. Since

$$\exists_\pi : P([[\sigma]] \times [[\Gamma]]) \to P([[\Gamma]])$$

is left adjoint to $P(\pi)$, it follows that

$$\exists_\pi([[\Phi_1 \, [x : \sigma, \Gamma]]] \otimes [[\varphi \, [x : \sigma, \Gamma]]] \otimes [[\Phi_2 \, [x : \sigma, \Gamma]]]) \leq [[\psi \, [\Gamma]]].$$

This is equivalent to the following:

$$\exists_\pi(\, P(\pi)([[\Phi_1 \, [\Gamma]]]) \otimes [[\varphi \, [x : \sigma, \Gamma]]] \otimes P(\pi)([[\Phi_2 \, [\Gamma]]]) \,) \leq [[\psi \, [\Gamma]]].$$

Repeated applications of the two Frobenius Reciprocity conditions give us

$$[[\Phi_1 \ [\Gamma]]] \otimes \exists_\pi([[\varphi \ [x : \sigma, \Gamma]]]) \otimes [[\Phi_2 \ [\Gamma]]] \leq [[\psi \ [\Gamma]]].$$

Then we finally have the following:

$$[[\Phi_1 \ [\Gamma]]] \otimes [[\exists x \varphi \ [\Gamma]]] \otimes [[\Phi_2 \ [\Gamma]]] \leq [[\psi \ [\Gamma]]].$$

To show the case of $\exists R$, assume that

$$[[\Phi \ [x : \sigma, \Gamma]]] \leq [[\varphi \ [x : \sigma, \Gamma]]].$$

The adjunction condition for existential quantifier tells us that

$$[[\varphi \ [x : \sigma, \Gamma]]] \leq P(\pi)(\exists_\pi([[\varphi \ [x : \sigma, \Gamma]]])),$$

where $\pi : [[\sigma]] \times [[\Gamma]] \to [[\Gamma]]$ is a projection. We thus have the following:

$$[[\Phi \ [x : \sigma, \Gamma]]] \leq P(\pi)(\exists_\pi([[\varphi \ [x : \sigma, \Gamma]]])) = [[\exists x \varphi \ [x : \sigma, \Gamma]]].$$

This completes the proof. ⊣

Syntactic hyperdoctrines are then defined as follows towards the goal of proving completeness. They are the hyperdoctrinal categorificaton of Lindenbaum–Tarski algebras.

DEFINITION 6. The syntactic hyperdoctrine of TFL is defined as follows ; that of TFL_X is defined by replacing **FL** and TFL below with $\textbf{FL}_X$ and TFL_X.

We first define the base category **C**. An object in **C** is a context Γ up to α-equivalence (i.e., the naming of variables does not matter). An arrow in **C** from an object Γ to another Γ' is a list of terms $[t_1, \ldots, t_n]$ (up to equivalence) such that $t_1 : \sigma_1 \ [\Gamma], \ldots, t_n : \sigma_n \ [\Gamma]$ where Γ' is supposed to be $x_1 : \sigma_1, \ldots, x_n : \sigma_n$.

The syntactic hyperdoctrine $P : \mathbf{C}^{\text{op}} \to \textbf{FL}$ is then defined in the following way. For an object Γ in **C**, let

$$\text{Form}_\Gamma = \{\varphi \mid \varphi \text{ is a formula in context } \Gamma\}.$$

Define an equivalence relation $\sim$ on Form_Γ as follows: for $\varphi, \psi \in \text{Form}_\Gamma$, $\varphi \sim \psi$ iff both $\varphi \vdash \psi \ [\Gamma]$ and $\psi \vdash \varphi \ [\Gamma]$ are deducible in TFL. We then define

$$P(\Gamma) = \text{Form}_\Gamma/\sim$$

with an FL algebra structure induced by the logical connectives.

The arrow part of P is defined as follows. Let $[t_1, \ldots, t_n] : \Gamma \to \Gamma'$ be an arrow in **C** where Γ' is $x_1 : \sigma_1, \ldots, x_n : \sigma_n$. Then we define $P([t_1, \ldots, t_n]) : P(\Gamma') \to P(\Gamma)$ by

$$P([t_1, \ldots, t_n])(\varphi) = \varphi[t_1/x_1, \ldots, t_n/x_n],$$

where it is supposed that $t_1 : \sigma_1 \ [\Gamma], \ldots, t_n : \sigma_n \ [\Gamma]$, and that φ is a formula in context $x_1 : \sigma_1, \ldots, x_n : \sigma_n$.

Intuitively, $P(\Gamma)$ above is a Lindenbaum–Tarski algebra sliced with respect to each Γ. It is straightforward to verify that the operations of $P(\Gamma)$ above are well defined, and $P(\Gamma)$ forms an FL algebra. We still have to check that P defined above is a hyperdoctrine; this is done in the following lemma.

LEMMA 7. *The syntactic hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$(resp. $\mathbf{FL}_X$) is an FL(resp. FL_X) hyperdoctrine. In particular, it has quantifier structures satisfying the Beck–Chevalley and Frobenius Reciprocity conditions.*

PROOF. Since substitution commutes with all the logical connectives, $P([t_1, \ldots, t_n])$ defined above is always a homomorphism of FL algebras. Thus, P is a contravariant functor.

We have to verify that the base category $\mathbf{C}$ has finite products, or equivalently, binary products. For objects Γ, Γ' in $\mathbf{C}$, we define their product $\Gamma \times \Gamma'$ as follows. Suppose that Γ is $x_1 : \sigma_1, \ldots, x_n : \sigma_n$, and Γ' is $y_1 : \tau_1, \ldots, y_m : \tau_m$. Then, $\Gamma \times \Gamma'$ is defined as

$$x_1 : \sigma_1, \ldots, x_n : \sigma_n, y_1 : \tau_1, \ldots, y_m : \tau_m.$$

An associated projection $\pi : \Gamma \times \Gamma' \to \Gamma'$ is defined as

$$[y_1, \ldots, y_m] : \Gamma \times \Gamma' \to \Gamma',$$

where the context of each y_i is taken to be $x_1 : \sigma_1, \ldots, x_n : \sigma_n, y_1 : \tau_1, \ldots, y_m : \tau_m$ (rather than $y_1 : \tau_1, \ldots, y_m : \tau_m$). The other projection is defined in a similar way. It is easily verified that these indeed form a categorical product in $\mathbf{C}$.

In order to show that P has quantifier structures, let $\pi : \Gamma \times \Gamma' \to \Gamma'$ denote the projection in $\mathbf{C}$ defined above, and then consider $P(\pi)$, which we have to show has right and left adjoints. The right and left adjoints of $P(\pi)$ can be constructed as follows. Recall Γ is $x : \sigma_1, \ldots, x_n : \sigma_n$. Let $\varphi \in P(\Gamma \times \Gamma')$; here we are identifying φ with the equivalence class to which φ belongs, since every argument below respects the equivalence. Then define $\forall_\pi : P(\Gamma \times \Gamma') \to P(\Gamma')$ by

$$\forall_\pi(\varphi) = \forall x_1 \ldots \forall x_n \varphi,$$

where the formula on the right-hand side actually denotes the corresponding equivalence class. Similarly, we define $\exists_\pi : P(\Gamma \times \Gamma') \to P(\Gamma')$ by

$$\exists_\pi(\varphi) = \exists x_1 \ldots \exists x_n \varphi.$$

Let us show that $\forall_\pi$ is the right adjoint of $P(\pi)$. We first assume $P(\pi)(\psi) \leq \varphi$ in $P(\Gamma \times \Gamma')$ for $\psi \in P(\Gamma')$ and $\varphi \in P(\Gamma \times \Gamma')$. Then it follows from the definition of P and π that

$$P(\pi)(\psi \, [\Gamma]) = \psi \, [\Gamma, \Gamma'],$$

where we are making explicit the two different contexts of ψ; the role of $P(\pi)$ just lies in changing contexts. Since the $\leq$ of $P(\Gamma \times \Gamma')$ is induced by its lattice structure, we have $\varphi \wedge \psi = \psi$. It follows from the definition of $P(\Gamma \times \Gamma')$ that $\varphi \wedge \psi \vdash \psi \, [\Gamma, \Gamma']$ and $\psi \vdash \varphi \wedge \psi \, [\Gamma, \Gamma']$ are deducible in TFL (resp. TFL_X), whence $\psi \vdash \varphi \, [\Gamma, \Gamma']$ is deducible as well. By repeated applications of rule $\forall R$, it follows that

$$\psi \vdash \forall x_1 \ldots \forall x_n \varphi \, [\Gamma']$$

is deducible. This implies that both $\psi \vdash \psi \wedge \forall x_1 \ldots \forall x_n \varphi \, [\Gamma']$ and $\psi \wedge \forall x_1 \ldots \forall x_n \varphi \vdash \psi \, [\Gamma']$ are deducible, whence $\psi \leq \forall x_1 \ldots \forall x_n \varphi$ in $P(\Gamma')$.

We show the converse. Assume that $\psi \leq \forall x_1...\forall x_n \varphi$ in $P(\Gamma')$. By arguing as in the above,

$$\psi \vdash \forall x_1...\forall x_n \varphi \ [\Gamma']$$

is deducible. By enriching the context,

$$\psi \vdash \forall x_1...\forall x_n \varphi \ [\Gamma, \Gamma']$$

is deducible. Since

$$\forall x_1...\forall x_n \varphi \vdash \varphi \ [\Gamma, \Gamma']$$

is deducible by rule $\forall L$, the cut rule tells us that $\psi \vdash \varphi \ [\Gamma, \Gamma']$ is deducible; note that the contexts of two sequents-in-context must be the same when applying the cut rule to them. It finally follows that $P(\pi)(\psi) \leq \varphi$ in $P(\Gamma \times \Gamma')$. Thus, $\forall_\pi$ is the right adjoint of $P(\pi)$. Similarly, $\exists_\pi$ can be shown to be the left adjoint of $P(\pi)$.

The Beck–Chevalley condition for $\forall$ can be verified as follows. Let $\varphi \in P(\Gamma \times \Gamma')$, $\pi : \Gamma \times \Gamma' \to \Gamma'$ a projection in $\mathbf{C}$, and $\pi' : \Gamma \times \Gamma'' \to \Gamma''$ another projection in $\mathbf{C}$ for objects $\Gamma, \Gamma', \Gamma''$ in $\mathbf{C}$. Then, we have

$$P([t_1, ..., t_n]) \circ \forall_\pi(\varphi) = (\forall x_1...\forall x_n \varphi)[t_1/y_1, ..., t_n/y_m],$$

where it is supposed that Γ is $x_1 : \sigma_1, ..., x_n : \sigma_n$, Γ' is $y_1 : \tau_1, ..., y_m : \tau_m$, and $t_1 : \tau_1 \ [\Gamma''], ..., t_m : \tau_m \ [\Gamma'']$. We also have the following:

$$\forall_{\pi'} \circ P([t_1, ..., t_n])(\varphi) = \forall x_1...\forall x_n(\varphi[t_1/y_1, ..., t_n/y_m]).$$

The Beck–Chevalley condition for $\forall$ thus follows. The Beck–Chevalley condition for $\exists$ can be verified in a similar way. The two Frobenius Reciprocity conditions for $\exists$ follow immediately from Lemma 1. ⊣

The syntactic hyperdoctrine is a free or classifying hyperdoctrine; it is the combination of the classifying category $\mathbf{C}$ and the free algebras $P(\Gamma)$'s.

Now, there is the obvious, canonical interpretation of TFL (resp. TFL_X) in the syntactic hyperdoctrine of TFL (resp. TFL_X); it is straightforward to see:

LEMMA 8. *If $\Phi \vdash \psi \ [\Gamma]$ is satisfied in the canonical interpretation in the syntactic hyperdoctrine of* TFL*(resp.* TFL_X*), it is deducible in* TFL*(resp.* TFL_X*).*

The lemmata above give us the completeness result: If $\Phi \vdash \psi \ [\Gamma]$ is satisfied in any interpretation in any FL (resp. FL_X) hyperdoctrine, then it is deducible in TFL (resp. TFL_X). Combining soundness and completeness, we obtain:

THEOREM 9. *The following are equivalent*:

- $\Phi \vdash \psi \ [\Gamma]$ *is deducible in* TFL*(resp.* TFL_X*),*
- $\Phi \vdash \psi \ [\Gamma]$ *is satisfied in any interpretation in any* FL*(resp.* FL_X*) hyperdoctrine.*

Let us finally give set-theoretical examples of hyperdoctrines and remark on what set-theoretical semantics look like in light of the hyperdoctrinal approach to semantics.

Any complete FL algebra yields an FL hyperdoctrine as a contravariant Hom functor into the algebra in the following way:

PROPOSITION 10.   *Let $\Omega \in$ **FL** with Ω being complete. Then*,

$$\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{FL}(\textit{resp. } \mathbf{FL}_X)$$

is an $\mathrm{FL}(\textit{resp. } \mathrm{FL}_X)$ *hyperdoctrine.*

PROOF.   Let $\pi : X \times Y \to Y$ be a projection in **Set**. We define $\forall_\pi$ and $\exists_\pi$ as follows: given $v \in \mathrm{Hom}(X \times Y, \Omega)$ and $y \in Y$, let

$$\forall_\pi(v)(y) := \bigwedge \{v(x, y) \mid x \in X\}$$

and

$$\exists_\pi(v)(y) := \bigvee \{v(x, y) \mid x \in X\}.$$

These morphisms give the required quantifier structures satisfying the Beck–Chevalley and Frobenius Reciprocity properties.   ⊣

If Ω is the two element Boolean algebra, the functor above comes down to the contravariant powerset functor, the hyperdoctrinal interpretation with which amounts to the standard Tarski semantics of classical logic. So the hyperdoctrinal semantics may be seen as a vast generalisation of the classic Tarski semantics.

The above hyperdoctrine and corresponding semantics may be regarded as the Ω-valued powerset functor and Ω-valued semantics, respectively; this is related to algebra-valued models of set theory as we shall remark in the concluding section below. The Ω-valued semantics of substructural logic have been developed essentially in the context of algebraic substructural logic, independently of any categorical semantics (see, e.g., Ono [33, 34]); yet the hyperdoctrinal semantics even encompass those semantics.

The algebraic completeness of algebras Ω actually prevents us from proving the logical completeness of the semantics with respect to those complete algebras. To be precise, assuming completeness prevents us from obtaining completeness for any axiomatic extension TFL_X of TFL; such incompleteness phenomena have been observed in various settings (see, e.g., Ono [33]). Yet at the same time, if Ω is not complete, in general, $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ cannot interpret quantifiers; this is because $\forall$ and $\exists$ in $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ are actually infinitary meets and joins in Ω. In algebraic predicate logic, thus, the notion of safe valuations has been applied in order to address this issue; yet they require ad hoc constraints to interpret quantifiers in non-complete algebras (see, e.g., Ono [34]). If we restrict the hyperdoctrine semantics to the Ω-valued models $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$, we can encounter the same problem, and we do need the same, safe valuation technique to show the completeness theorem with respect to those models. Yet from a categorical point of view, there is no a priori reason to stick to set-theoretical models, and in the general hyperdoctrinal setting as above, completeness comes for free (i.e., without any cost of ad hoc manipulations) via the syntactic hyperdoctrine construction, just as completeness holds for free in algebraic propositional logic via the Lindenbaum–Tarski algebra construction. This would suggest that hyperdoctrines or fibred algebras give the right notion of algebras of predicate logic that directly extend the salient features of algebraic propositional logic; there is, to the best of the author's knowledge, no other such concept of algebras of predicate logic that works for a broad variety of logical systems in a uniform and modular manner.

The above functor is actually part of a dual adjunction: given $\Omega \in \mathbf{FL}$, the following dual adjunction holds between $\mathbf{Set}$ and $\mathbf{FL}$, induced by Ω as a dualising object:

$$\mathrm{Hom}_{\mathbf{FL}}(\text{-}, \Omega)^{\mathrm{op}} \dashv \mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{FL}.$$

As this example suggests, hyperdoctrines tend to arise from the predicate functor parts of dual adjunctions induced by dualising objects. The dual adjunction between frames and topological spaces (see Johnstone [17]) gives a hyperdoctrine for geometric predicate logic; likewise, the dual adjunction between convex algebras and convex spaces as in [25] gives a hyperdoctrine for convex geometric logic. Coalgebraic dualities as in [19, 26] give hyperdoctrines for modal predicate logics. There could be a systematic study of these phenomena at the interface of duality theory and categorical logic. In general, categorical dualities for propositional logics (see, e.g., [27, 30]) often give hyperdoctrinal models of the predicate logics that extend them with typed quantifiers (as studied in [29]).

§4. Hyperdoctrinal Lawvere–Tierney topology as Girard and Kolmogorov translation.
In the hyerdoctrinal setting, type theories are categories, logics over them are functors, and logical translations between them are natural transformations; we elaborate on this idea in the present section, introducing the generalised notion of Lawvere–Tierney topologies and cotopologies on hyperdoctrines (rather than toposes), which allow for a uniform treatment of logical translations, such as the Kolmogorov's double negation translation and the Girard's exponential translation as we shall see below.

In the following, $\mathbf{HA}$ and $\mathbf{BA}$ denote the category of Heyting algebras and the category of Boolean algebras, respectively. IL and CL hyperdoctrines are defined as FL hyperdoctrines with values in $\mathbf{HA}$ and $\mathbf{BA}$, respectively. Note that both IL and CL hyperdoctrines are FL_X hyperdoctrines for suitable sets of axioms X.

Given a Heyting algebra, the fixpoints of double negation, namely those elements that validate the double negation elimination, form a Boolean algebra; this is an algebraic version of the Kolmogorov translation between intuitionistic and classical propositional logics. In the following, we explore how this extends to predicate logics and hyperdoctrines through the concept of Lawvere–Tierney topologies and cotopologies, which give a unifying perspective on different logical translations. In particular, exponential ! is understood as a Lawvere–Tierney cotopology on substructural hyperdoctrines.

Lawvere–Tierney topologies originally come from topos theory, and they induce topologies on hyperdoctrines (this is just for motivating the following discussion, which itself does not require any substantial knowledge of topos theory): given a topos $\mathbf{C}$ with a subobject classifier Ω, a Lawvere–Tierney topology on Ω corresponds to a natural transformation on the subobject hyperdoctrine $\mathrm{Sub} : \mathbf{C} \to \mathbf{HA}$

$$j : \mathrm{Sub}_{\mathbf{C}}(\text{-}) \to \mathrm{Sub}_{\mathbf{C}}(\text{-})$$

such that j_C is a closure operator on $\mathrm{Sub}_{\mathbf{C}}(C)$ for every $C \in \mathbf{C}$ (see, e.g., Jacobs [16]). Having this correspondence in mind, let us define the concept of Lawvere–Tierney topologies and cotopologies on hyperdoctrines in the following manner,

which allow us to capture different logical translations uniformly in the general hyperdoctrinal setting.

DEFINITION 11. For an FL hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$, a Lawvere–Tierney topology on P is a natural transformation

$$j : P \to P$$

such that

$$j_C : P(C) \to P(C)$$

is a closure operator on $P(C)$ for each $C \in \mathbf{C}$. We also call a Lawvere–Tierney topology a Lawvere–Tierney operator.

Dually, a Lawvere–Tierney cotopology (or co-operator) for P is a natural transformation

$$j : P \to P$$

such that

$$j_C : P(C) \to P(C)$$

is an interior operator on $P(C)$ for each $C \in \mathbf{C}$.

Examples we are going to discuss below would clarify what the definition above means in logical contexts. The above transformation j on the subobject hyperdoctrine Sub : $\mathbf{C} \to \mathbf{HA}$ of a topos, of course, forms a Lawvere–Tierney topology on Sub.

Let us introduce intuitionistic linear logic with exponential !, which allows us to have control over structural rules and simulate intuitionistic logic within linear logic. An exponential ! on an FL algebra A is defined in algebraic terms as follows (see Coumans–Gehrke–Rooijen [7]):

DEFINITION 12. An exponential ! on an FL algebra A is a unary operation satisfying the following:

- $a \leq b$ implies $!a \leq !b$;
- $!!a = !a$;
- $!a \leq a$;
- $!\top = 1$;
- $!a \otimes !b = !(a \wedge b)$.

We denote by $\mathbf{FL}_e^!$ (where e means exchange) the category of commutative FL algebras with exponential ! and maps preserving both exponential ! and FL algebra operations ; they give the algebraic counterpart of intuitionistic linear logic with !, which we denote simply by ILL.

ILL hyperdoctrines are defined as FL hyperdoctrines with values in $\mathbf{FL}_c^!$.

Note that the above definition of Lawvere–Tierney topologies and cotopologies obviously generalises to ILL hyperdoctrines, i.e., just by replacing $\mathbf{FL}$ with $\mathbf{FL}_e^!$ in the above definition.

The exponential operation ! induces a Lawvere–Tierney cotopology on an ILL hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}_e^!$: that is, ! can naturally be considered a natural

transformation

$$!: P \to P$$

such that

$$!_C\,(a) =!\,a$$

for any $C \in \mathbf{C}$ and $a \in P(C)$. The naturality condition in the definition of Lawvere–Tierney cotopologies means that the following diagram commutes: given $f : C \to D$,

$$
\begin{array}{ccc}
P(D) & \xrightarrow{\;!_D\;} & P(D) \\
\downarrow{\scriptstyle P(f)} & & \downarrow{\scriptstyle P(f)} \\
P(C) & \xrightarrow[\;!_C\;]{} & P(C)
\end{array}
$$

The diagram above indeed commutes because the following hold:

$$!\,P(f)(x) = P(f)(!\,x),$$

where note that $P(f)$ is a homomorphism. In addition to this,

$$!_C : P(C) \to P(C)$$

is an interior operator on $P(C)$, and thus we have verified the following:

PROPOSITION 13. *The exponential* $!: P \to P$ *forms a Lawvere–Tierney cotopology on an* ILL *hyperdoctrine* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}_e^!$.

Another example of Lawvere–Tierney topology is the double negation topology. Just in the same way as above, the double negation operation $\neg\neg$ in Heyting algebras yields a Lawvere–Tierney topology $\neg\neg : P \to P$ on an intuitionistic hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$.

Given a Lawvere–Tierney (co)topology j on a hyperdoctrine, we can construct another functor P_j in the following manner:

PROPOSITION 14. *A Lawvere–Tierney (co)topology* j *on a given* $\mathbf{FL}$ *or* $\mathbf{FL}_e^!$ *hyperdoctrine* P *induces another functor* P_j *such that*

$$P_j(C) = \{j_C(x) \mid x \in P(C)\}$$

and

$$P_j(f : C \to D)(x) = P(f)(x),$$

which is well defined as a functor. In particular, $P(f)(x)$ *is an element of* $P_j(C)$.

PROOF. This follows from the naturality of the Lawvere–Tierney (co)topology j: there is some $y \in P(D)$ such that $x = j_D(y)$, and the naturality of transformation j gives us

$$P(f)(x) = P(f)(j_D(y)) = j_C(P(f)(y)),$$

and $j_C(P(f)(y))$ is an element of $P_j(C)$. ⊣

Note that P_j amounts to taking the fixpoints of j with respect to each C, since j_C is a closure or interior operator, and therefore idempotent in either case.

By the proposition above, the double negation Lawvere–Tierney topology $\neg\neg$ on an intuitionistic hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ induces a functor

$$P_{\neg\neg} : \ \mathbf{C}^{\mathrm{op}} \to \ \mathbf{BA},$$

where note that $P_{\neg\neg}(C)$ forms a Boolean algebra because the elements of $P_{\neg\neg}(C)$ are classical propositions, i.e., those elements that validate the double negation elimination. Likewise, the exponential Lawvere–Tierney cotopology ! on a substructural hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}_e^!$ induces a functor

$$P_! : \ \mathbf{C}^{\mathrm{op}} \to \ \mathbf{HA},$$

where note that $P_!(C)$ forms a Heyting algebra because the elements of $P_!(C)$ are structural propositions, i.e., those admitting the structural rules. Furthermore, in the following, we show that $P_{\neg\neg}$ and $P_!$ are actually intuitionistic and classical hyperdoctrines, respectively.

THEOREM 15. *Let* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}_e^!$ *be an* ILL *hyperdoctrine. Then, the following functor induced by the exponential Lawvere–Tierney cotopology* ! *on P*

$$P_! : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$$

forms an IL *hyperdoctrine*: *i.e., it has the quantifier structure as required.*

PROOF. Let us fix a projection $\pi : C \times D \to D$. We have to prove that

$$P_!(\pi) : P_!(D) \to P_!(C \times D)$$

has both right and left adjoints. By the adjoint functor theorem, it suffices to show that $P_!(\pi)$ preserves limits and colimits, namely infimums and supremums in the present case. Let us first prove that it preserves infimums. To show this, suppose that

$$\bigwedge\{x_i \mid i \in I\} \in P_!(D),$$

for $x_i \in P_!(D)$. It then holds that $!\,x_i = x_i$ and $!\bigwedge\{x_i \mid i \in I\} = \bigwedge\{x_i \mid i \in I\}$, since any element of $P_!(D)$ is a fixpoint of the exponential operation !. We then have

$$\begin{aligned}
P_!(\pi)(\bigwedge\{x_i \mid i \in I\}) &= P_!(\pi)(!\bigwedge\{x_i \mid i \in I\}) \\
&= P(\pi)(!\bigwedge\{x_i \mid i \in I\}) \\
&= !\bigwedge\{P(\pi)(x_i) \mid i \in I\} \\
&= !\bigwedge\{P_!(\pi)(x_i) \mid i \in I\} \\
&= \bigwedge\{P_!(\pi)(x_i) \mid i \in I\}.
\end{aligned}$$

The last equality holds because $!\bigwedge(\{P_!(\pi)(x_i) \mid i \in I\})$ is actually the infimum of $P_!(\pi)(x_i)$'s in $P_!(C \times D)$, which can be verified in the following way. We first have

$$!\bigwedge\{P_!(\pi)(x_i) \mid i \in I\} \leq P_!(\pi)(x_i),$$

for any $i \in I$, since $!\, x_i = x_i$ and $P_!(\pi)$ preserves $!$. Suppose that there is some $z \in P_!(C \times D)$ such that $z \leq P_!(\pi)(x_i)$ for any $i \in I$. Then we have got $!\, z = z$ and

$$z \leq \bigwedge \{ P_!(\pi)(x_i) \mid i \in I \},$$

which implies that

$$z =!\, z \leq!\, \bigwedge \{ P_!(\pi)(x_i) \mid i \in I \}.$$

This means that $!\, \bigwedge (\{ P_!(\pi)(x_i) \mid i \in I \})$ is the infimum of $P_!(\pi)(x_i)$'s. We have thus shown that $P_!(\pi)$ preserves limits. The case of colimits can be verified in the same way. ⊣

This is an analogue of the Girard translation formulated in terms of hyperdoctrines and Lawvere–Tierney (co)topology. The above theorem is more general than Girard's translation theorem in the sense that the latter corresponds to the case of syntactic hyperdoctrines in the former. Although in this paper we do not explicitly discuss substructural logics enriched with modalities and their hyperdoctrinal semantics, nevertheless, our methods work for them as well, yielding the corresponding soundness and completeness results in terms of hyperdoctrines with values in FL algebras with modalities; Girard's exponential $!$ is just a special case.

Now, the double negation translation version is as follows.

THEOREM 16. *Let* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ *be an* IL *hyperdoctrine. Then, the following functor induced by the double negation Lawvere–Tierney topology* $\neg\neg$ *on* P

$$P_{\neg\neg} : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$$

forms an CL *hyperdoctrine*: *i.e., it has the quantifier structure as required.*

PROOF. The argument to show this is mostly the same as above. Fix a projection $\pi : C \times D \to D$. We prove that

$$P_{\neg\neg}(\pi) : P_{\neg\neg}(D) \to P_{\neg\neg}(C \times D)$$

has both right and left adjoints. It suffices to show that $P_{\neg\neg}(\pi)$ preserves limits and colimits. To show that it preserves limits, suppose

$$\bigwedge \{ x_i \mid i \in I \} \in P_{\neg\neg}(D),$$

for $x_i \in P_{\neg\neg}(D)$. It then holds that $\neg\neg x_i = x_i$ and $\neg\neg \bigwedge \{ x_i \mid i \in I \} = \bigwedge \{ x_i \mid i \in I \}$. We then have

$$
\begin{aligned}
P_{\neg\neg}(\pi)(\bigwedge \{ x_i \mid i \in I \}) &= P(\pi)(\neg\neg \bigwedge \{ x_i \mid i \in I \}) \\
&= \neg\neg \bigwedge \{ P(\pi)(x_i) \mid i \in I \} \\
&= \bigwedge \{ P_{\neg\neg}(\pi)(x_i) \mid i \in I \}.
\end{aligned}
$$

The preservation of colimits can be shown in the same way. ⊣

While the double negation topology in topos theory is concerned with (theories over) higher-order logic, the above translation theorem is for (theories over) first-order logic in general; there is no similar result known for Heyting categories or

logoses, the first-order version of toposes. Note that intuitionistic hyperdoctrines and Heyting categories are linked with each other via the subobject hyperdoctrine functor and the partial equivalence category functor (which is the first-order version of the tripos-to-topos construction); the above theorem, thus, can be recast in terms of Heyting categories, though the hyperdoctrinal formulation appears more natural.

Let us finally remark on the naturality condition of Lawvere–Tierney topology on hyperdoctrines. From a logical point of view, the naturality condition means that the operator j commutes with substitution of terms for variables. Let us, for example, consider the syntactic hyperdoctrine for intuitionistic predicate logic

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}.$$

Recall that an object of $\mathbf{C}$ is a collection of typed variables, and an arrow of $\mathbf{C}$ is a term. Then, P maps a collection of variables $x_1, \dots, x_n$ to the Heyting algebra of formulas $\varphi(x_1, \dots, x_n)$ with those variables, and $P(t)$ is the substitution of t for variables concerned. The double negation Lawvere–Tierney topology on the hyperdoctrine P is indeed a natural transformation because substitution commutes with double negation:

$$(\neg\neg\varphi)[t/x] = \neg\neg(\varphi[t/x]).$$

Extending Lawvere hyperdoctrines to a substructural setting, thus, we can give a uniform treatment of different logical translations through the concept of Lawvere–Tierney topologies and cotopologies on hyperdoctrines.

§5. Concluding remarks. We have developed the hyperdoctrinal or fibred algebraic theory of substructural predicate logics, and given a uniform understanding of logical translation theorems on the basis of Lawvere–Tierney topologies and cotopologies on them. Let us finally comment upon further directions of research.

5.1. Fibred algebraic completions and cut elimination for substructural predicate logics. Unlike the case of substructural propositional logics, there has, so far, been no systematic theory of algebraic cut elimination for substructural predicate logics, since there has been no adequate concept of algebras of them available for that purpose. Hyperdoctrines or fibred algebras as we have introduced in this paper could serve as a framework in which to develop a systematic theory of algebraic cut elimination. By a systematic theory, we mean, in particular, the predicate logic counterpart of the propositional result that a logic enjoys the cut elimination if and only if the corresponding algebras are closed under algebraic completions (see Ciabattoni–Galatos–Terui [4, 5]). Extending the propositional correspondence between cut elimination and algebraic completion, we could show something like this: a predicate logic admits the cut elimination if and only if the corresponding fibred algebras are closed under fibred algebraic completions. Completions and canonical extensions of hyperdoctrines for coherent logic have been studied systematically by Coumans [6]; we could extend the methods so as to be applicable for substructural logics. It is technically similar to proving the translation theorems above to prove that hyperdoctrines or fibred algebras are closed under completions; this could be applied to give algebraic cut elimination proofs for

substructural predicate logics over FL in a uniform, systematic manner as achieved in the propositional substructural logic case.

5.2. Algebra-valued models of set theory and consistency problems on substructural set theory. Another potential application could be found in set theory, just as intuitionistic hyperdoctrines yield Heyting-valued models of set theory via the tripos-to-topos construction originally due to Hyland–Johnstone–Pitts [14]. We can reformulate it in the present context of FL hyperdoctrines. To do this, we work in the internal logic of FL hyperdoctrines $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$: i.e., we have types X and function symbols f corresponding to objects X and arrows f in $\mathbf{C}$ respectively, and also those predicate symbols R on a type $C \in \mathbf{C}$ that correspond to elements $R \in P(C)$.

We can then define a category $\mathbf{T}[P]$ as follows. An object of $\mathbf{T}[P]$ is a partial equivalence relation, i.e., a pair

$$(X, E_X)$$

such that X is an object in the base category $\mathbf{C}$, and E_X is an element of $P(X \times X)$ and is symmetric and transitive in the internal logic of P:

$$E_X(x, y) \vdash E_X(y, x) \, [x, y : X]$$

and

$$E_X(x, y), E_X(y, z) \vdash E_X(x, z) \, [x, y, z : X].$$

An arrow from (X, E_X) to (Y, E_Y) is then defined as $F \in P(X \times Y)$ such that (i) extensionality:

$$E_X(x_1, x_2), E_Y(y_1, y_2), F(x_1, y_1) \vdash F(x_2, y_2) \, [x_1, x_2 : X, y_1, y_2 : Y];$$

(ii) strictness:

$$F(x, y) \vdash E_X(x, x) \wedge E_Y(y, y) \, [x : X, y : Y];$$

(iii) single-valuedness:

$$F(x, y_1), F(x, y_2) \vdash E_Y(y_1, y_2) \, [x : X, y_1, y_2 : Y];$$

(iv) totality:

$$E_X(x, x) \vdash \exists y \, F(x, y) \, [x : X].$$

Such an F is called a functional relation.

Now, if Ω is a locale,

$$\mathbf{T}[\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)]$$

is the Higgs topos of Ω-valued sets, which is in turn equivalent to the category of sheaves on Ω; it has given a number of applications to consistency and independence problems in both classical and intuitionistic set theories. For a complete FL algebra Ω, which is a quantale with additional operations, $\mathbf{T}[\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)]$ may be seen as the category of Ω-valued sets. Quantale sets in the sense of Höhle-Kubiak [13] can be regarded as objects in $\mathbf{T}[\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)]$. We could interpret substructural set theories in the category of Ω-valued sets, and thereby might be able to solve consistency and independence problems in them. Presumably the most challenging problem

in this context is the consistency of Łukasiewicz predicate logic with the naive comprehension principle, which White [38] claimed to have proven, yet generally considered to be an open problem even now.

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