SOME INEQUALITIES FOR POLYNOMIALS AND RELATED ENTIRE FUNCTIONS II

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1. Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n. Then clearly

$$(1.1)\quad \int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta = 2\pi \sum_{\nu=0}^{n} |\nu a_\nu|^2 \le n^2 \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta ,$$

$$(1.2)\quad \sum_{\nu=0}^{n} \nu |a_\nu|^2 \le n \sum_{\nu=0}^{n} |a_\nu|^2 ,$$

and for $R > 1$

$$(1.3)\quad \int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta = 2\pi \sum_{\nu=0}^{n} |a_\nu|^2 R^{2\nu} \le R^{2n} 2\pi \sum_{\nu=0}^{n} |a_\nu|^2$$

$$= R^{2n} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta .$$

Note that if $w = p(z)$ maps $|z| < 1$ on a domain D of the w-plane then the area of D is given by $\pi \sum_{\nu=0}^{n} \nu |a_\nu|^2$.

For $p(z) \neq 0$ in $|z| < 1$, inequalities (1.1) and (1.3) have been replaced respectively by the following:

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$$(1.4)\qquad \int_0^{2\pi} |p'(e^{i\theta})|^2 d\theta \leq \frac{n^2}{2} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta ,$$

$$(1.5)\qquad \int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta \leq \frac{R^{2n}+1}{2} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta , \quad (R > 1) .$$

Inequality (1.4) was first proved by Lax ([3], pp. 512-513; for another proof see [2], Theorem 13) and (1.5) by the author himself ([5], Theorem 1). The extremal polynomial in each case is $p(z) = \alpha + \beta z^n$ where $|\alpha| = |\beta|$.

In this paper we shall generalize the above inequalities by considering polynomials $p(z) \neq 0$ in $|z| < K$ where K is an arbitrary positive number. We have not been able to solve the problem completely, e.g. we do not know the result corresponding to (1.4) when $p(z) \neq 0$ in $|z| < K$, where $K > 1$.

In this connection the following result is known [4].

THEOREM A. If $p(z) \neq 0$ in $|z| < K$ where $K \geq 1$, then for $R \geq K^2$

$$(1.6)\qquad \int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta \leq \frac{R^{2n}+K^{2n}}{1+K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta .$$

We prove

THEOREM 1. If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n such that $p(z) \neq 0$ for $|z| < K$, where $K \leq 1$, then

$$(1.7)\qquad \int_0^{2\pi} |p'(e^{i\theta})|^2 d\theta \leq \frac{n^2}{1+K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta ,$$

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$$(1.8)\qquad \sum_{\nu=0}^{n} \nu |a_\nu|^2 \leq \frac{n}{1+K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta ,$$

and for $R > 1$

$$(1.9)\qquad \int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leq \frac{R^{2n}+K^{2n}}{1+K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta .$$

In each case equality holds for $p(z) = \alpha z^n + \beta K^n$ *where* $|\alpha| = |\beta|$.

Now suppose that $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, $a_n \neq 0$ has all its zeros in $|z| \leq K$, where $K \geq 1$; then the polynomial $q(z) = z^n \overline{p(1/\bar{z})} = \sum_{\nu=0}^{n} \bar{a}_\nu z^{n-\nu}$ cannot vanish in $|z| < \frac{1}{K} \leq 1$, and if we apply (1.8) to $q(z)$ we shall get

$$\sum_{\nu=0}^{n} (n-\nu) |a_\nu|^2 \leq \frac{nK^{2n}}{1+K^{2n}} \sum_{\nu=0}^{n} |a_\nu|^2 ,$$

or

$$\sum_{\nu=0}^{n} \nu |a_\nu|^2 \geq \frac{n}{1+K^{2n}} \sum_{\nu=0}^{n} |a_\nu|^2 .$$

We can therefore state the following

THEOREM 2. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, $a_n \neq 0$ *has all its zeros in* $|z| \leq K$, *where* $K \geq 1$, *then*

$$(1.10)\qquad \sum_{\nu=0}^{n} \nu |a_\nu|^2 \geq \frac{n}{1+K^{2n}} \sum_{\nu=0}^{n} |a_\nu|^2 .$$

The case $K = 1$ of Theorem 2 was proposed by D. J. Newman as an advanced problem in the American Mathematical Monthly (vol. 69, 1962, problem No. 5040).

We can write (1.10) in the equivalent form

$$(1.11) \qquad |(p', p)| \geq \frac{n}{1+K^{2n}} \, ||p|| ,$$

where (p', p) the inner product of $p'(e^{i\theta})$ and $p(e^{i\theta})$ is equal to $\int_0^{2\pi} p'(e^{i\theta}) \overline{p(e^{i\theta})} d\theta$ and $||p|| = \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta$.

<u>Proof of (1.7).</u> The polynomial $P(z) = p(Kz)$ does not vanish for $|z| < 1$ and so the polynomial $Q(z) = z^n \overline{P(1/\bar{z})}$ $= z^n \overline{p(K/\bar{z})}$ has all its zeros in $|z| \leq 1$. Since $|Q(z)| = |P(z)|$ for $|z| = 1$ it follows that $|P(z)| \leq |Q(z)|$ for $|z| > 1$. From this we can conclude by a result of De Bruijn ([2], Theorem 2) that $|P'(z)| \leq |Q'(z)|$ for $|z| \geq 1$. In particular,

$$(1.12) \qquad |P'(\tfrac{1}{K} e^{i\theta})| \leq |Q'(\tfrac{1}{K} e^{i\theta})|$$

for $0 < K \leq 1$ and every θ such that $0 \leq \theta < 2\pi$. Now $P(z) = \sum_{\nu=0}^{n} a_\nu K^\nu z^\nu$, $Q(z) = \sum_{\nu=0}^{n} \bar{a}_\nu K^\nu z^{n-\nu}$; hence for $0 < K \leq 1$

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 d\theta = \frac{1}{K^2} \int_0^{2\pi} |P'(\tfrac{1}{K} e^{i\theta})|^2 d\theta$$

$$\leq \frac{1}{K^2(1+K^{2n})} \left\{ \int_0^{2\pi} |P'(\tfrac{1}{K} e^{i\theta})|^2 d\theta + K^{2n} \int_0^{2\pi} |Q'(\tfrac{1}{K} e^{i\theta})|^2 d\theta \right\}$$

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$$= \frac{2\pi}{K^2(1+K^{2n})} \{ \sum_{\nu=0}^{n} \nu^2 |a_\nu|^2 K^2 + K^{2n} \sum_{\nu=0}^{n} (n-\nu)^2 |a_\nu|^2 \frac{1}{K^{2n-4\nu-2}} \}$$

$$= \frac{2\pi}{1+K^{2n}} \sum_{\nu=0}^{n} \{ \nu^2 + (n-\nu)^2 K^{4\nu} \} |a_\nu|^2$$

$$(1.13) \leq \frac{2\pi}{1+K^{2n}} \sum_{\nu=0}^{n} \{ \nu^2 + (n-\nu)^2 \} |a_\nu|^2$$

$$\leq \frac{n^2}{1+K^{2n}} 2\pi \sum_{\nu=0}^{n} |a_\nu|^2$$

$$= \frac{n^2}{1+K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta .$$

Proof of (1.8). In order to prove (1.8) we start from (1.13). Thus we have

$$2\pi \sum_{\nu=0}^{n} \nu^2 |a_\nu|^2 \leq \frac{2\pi}{1+K^{2n}} \sum_{\nu=0}^{n} \{ \nu^2 + (n-\nu)^2 \} |a_\nu|^2$$

$$= \frac{2\pi}{1+K^{2n}} \sum_{\nu=0}^{n} (n^2 - 2n\nu + 2\nu^2) |a_\nu|^2 ,$$

or

$$\frac{2n}{1+K^{2n}} \sum_{\nu=0}^{n} \nu |a_\nu|^2 \leq \frac{n^2}{1+K^{2n}} \sum_{\nu=0}^{n} |a_\nu|^2 + \frac{1-K^{2n}}{1+K^{2n}} \sum_{\nu=0}^{n} \nu^2 |a_\nu|^2$$

$$\leq \frac{n^2}{1+K^{2n}} \sum_{\nu=0}^{n} |a_\nu|^2 + \frac{1-K^{2n}}{1+K^{2n}} \frac{n^2}{1+K^{2n}} \sum_{\nu=0}^{n} |a_\nu|^2$$

by (1.7). Hence

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$$2n \sum_{\nu=0}^{n} \nu |a_\nu|^2 \leq n^2 \sum_{\nu=0}^{n} |a_\nu|^2 + n^2 \frac{1-K^{2n}}{1+K^{2n}} \sum_{\nu=0}^{n} |a_\nu|^2$$

$$= \frac{2n^2}{1+K^{2n}} \sum_{\nu=0}^{n} |a_\nu|^2 .$$

Dividing both the sides by $2n$ we get (1.8).

Proof of (1.9). The polynomial $p'(z)$ is of degree $n-1$; therefore by (1.3) we have for every $r > 1$

$$\int_0^{2\pi} |p'(re^{i\theta})|^2 d\theta \leq r^{2n-2} \int_0^{2\pi} |p'(e^{i\theta})|^2 d\theta$$

$$\leq \frac{n^2}{1+K^{2n}} r^{2n-2} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta .$$

Multiplying both the sides by r and then integrating with respect to r from 1 to ρ we get

$$\sum_{\nu=0}^{n} \nu |a_\nu|^2 (\rho^{2\nu}-1) \leq \frac{n}{1+K^{2n}} (\rho^{2n}-1) \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta .$$

Hence for every $\rho > 1$ we have

$$\sum_{\nu=0}^{n} \nu |a_\nu|^2 (\rho^{\nu}-1) \leq \frac{n}{1+K^{2n}} (\rho^{n}-1) \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta ,$$

or

$$\sum_{\nu=0}^{n} \nu |a_\nu|^2 \rho^{\nu} \leq \frac{n}{1+K^{2n}} (\rho^{n}-1) \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta + \sum_{\nu=0}^{n} \nu |a_\nu|^2$$

$$\leq \frac{n}{1+K^{2n}} \rho^{n} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta$$

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by (1.8). Dividing both the sides of the above inequality by ρ and integrating with respect to ρ between the limits 1 and R we get

$$\sum_{\nu=0}^{n} |a_\nu|^2 (R^\nu - 1) \le \frac{R^n - 1}{1+K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta ,$$

or

$$\sum_{\nu=0}^{n} |a_\nu|^2 R^\nu \le \frac{R^n + K^{2n}}{1+K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta .$$

Finally, we replace R by R^2 to get (1.9). The proof of Theorem 1 is complete.

Now the question arises as to how far the restriction $K \le 1$ is essential for the validity of estimates (1.7) and (1.8). Let us consider the polynomial $p(z) = (z + K)^n$, where $K > 1$. Then for arbitrarily large K

$$\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta = 2\pi n^2 \{1 + ({}^{n-1}C_1 K)^2 + \ldots + ({}^{n-1}C_{n-2} K^{n-2})^2 + ({}^{n-1}C_{n-1} K^{n-1})^2\}$$

$$\sim 2\pi n^2 K^{2n-2} ,$$

whereas

$$\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta = 2\pi \{1 + ({}^{n}C_1 K)^2 + \ldots + ({}^{n}C_{n-1} K^{n-1})^2 + ({}^{n}C_n K^n)^2\}$$

$$\sim 2\pi K^{2n} .$$

Thus, for the case $K > 1$, we could at the most expect to have

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$$(1.14)\quad \int_0^{2\pi} |p'(e^{i\theta})|^2 d\theta \le \frac{n^2}{\phi(K)} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta ,$$

$$\phi(K) \sim K^2 \text{ as } K \to \infty .$$

Such an estimate in fact holds and the proof is trivial. We simply have to note that if the zeros $z_1, z_2, \ldots, z_n$ of the polynomial $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ all lie in $|z| \ge K$, then on expressing a_ν in terms of the zeros and comparing it with $a_o = z_1 z_2 \ldots z_n$ we shall get

$$|a_\nu| \le |a_o| \, {}^nC_{n-\nu} K^{-\nu}, \qquad \nu = 1, 2, \ldots, n .$$

Thus

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 d\theta = 2\pi \sum_{\nu=0}^{n} \nu^2 |a_\nu|^2 \le 2\pi |a_o|^2 \sum_{\nu=0}^{n} \nu^2 ({}^nC_{n-\nu})^2 K^{-2\nu}$$

$$\sim 2\pi |a_o|^2 n^2 K^{-2} \text{ as } K \to \infty ,$$

and (1.14) follows because

$$\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta = 2\pi \sum_{\nu=0}^{n} |a_\nu|^2 \ge 2\pi |a_o|^2 .$$

The hypothesis that the geometric mean of the moduli of the zeros is at least equal to K is much weaker than the assumption $p(z) \ne 0$ in $|z| < K$. However, in this case the problem under consideration can be completely solved.

THEOREM 3. Let the geometric mean of the moduli of the zeros of a polynomial $p(z)$ of degree n be K, where $K \le (n-1)^{-1/2n}$. Then for every $R > 1$,

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$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leq \frac{R^{2n} + K^{2n}}{1 + K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta .$$

If $K > (n-1)^{-1/2n}$ *then*

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leq \frac{R^{2n} + K^{2n}}{1 + K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta ,$$

or

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta < R^{2n-2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta$$

according as $R > \mathcal{R}$ *or* $1 < R < \mathcal{R}$, *where* $\mathcal{R}$ *is the only*† *root of*

$$R^{2n} - (1 + K^{2n}) R^{2n-2} + K^{2n} = 0 \tag{1.15}$$

in $(1, \infty)$.

Proof of Theorem 3. The hypothesis implies $|a_o| \geq K^n |a_n|$, so that

$$|a_n|^2 R^{2n} + |a_o|^2 \leq \frac{R^{2n} + K^{2n}}{1 + K^{2n}} (|a_n|^2 + |a_o|^2) .$$

for $R > 1$. If $1 \leq \nu \leq n-1$, then for $R > 1$

$$R^{2\nu} \leq R^{2n-2} ,$$

and so

† This fact follows, for example, from the Descartes' rule of signs.

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$$|a_\nu|^2 R^{2\nu} \le |a_\nu|^2 \frac{R^{2n}+K^{2n}}{1+K^{2n}}$$

if

$$(1.16) \qquad R^{2n-2} \le \frac{R^{2n}+K^{2n}}{1+K^{2n}} .$$

If $K \le (n-1)^{-1/2n}$ then (1.16) holds for every $R > 1$. Hence

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta = 2\pi\{(|a_n|^2 R^{2n} + |a_o|^2) + |a_{n-1}|^2 R^{2n-2} + \ldots + |a_2|^2 R^4 + |a_1|^2 R^2\}$$

$$\le \frac{R^{2n}+K^{2n}}{1+K^{2n}} 2\pi\{(|a_n|^2 + |a_o|^2) + |a_{n-1}|^2 + \ldots + |a_2|^2 + |a_1|^2\}$$

$$\le \frac{R^{2n}+K^{2n}}{1+K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta .$$

But if $K > (n-1)^{-1/2n}$, then for (1.16) to hold R should at least be equal to $\mathcal{R}$ where $\mathcal{R}$ is the root of (1.15) in $(1,\infty)$. If $1 < R < \mathcal{R}$ then R^{2n-2} is greater than $\frac{R^{2n}+K^{2n}}{1+K^{2n}}$ as well as $R^{2\nu}$, $\nu = 1, 2, \ldots, n-2$, and so we get

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta < R^{2n-2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta .$$

Given any $\varepsilon > 0$ we can clearly construct a polynomial $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ with $|a_o| \ge K^n |a_n|$, $K > (n-1)^{-1/2n}$ and

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$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta > (1-\varepsilon)R^{2n-2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta .$$

We can similarly prove the following two theorems.

THEOREM 4. *Let the geometric mean of the moduli of the zeros of a polynomial* $p(z)$ *of degree* n *be* K. *Then*

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta \leq \frac{n^2}{1+K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta$$

or

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta < (n-1)^2 \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta ,$$

according as $K \leq (2n-1)^{1/2n}(n-1)^{-1/n}$ *or* $K > (2n-1)^{1/2n}(n-1)^{-1/n}$ *respectively.*

THEOREM 5. *If the geometric mean of the moduli of the zeros of a polynomial* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *of degree* n *be* K *then*

$$\sum_{\nu=0}^{n} \nu |a_\nu|^2 \leq \frac{n}{1+K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta$$

or

$$\sum_{\nu=0}^{n} \nu |a_\nu|^2 < (n-1) \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta ,$$

according as $K \leq (n-1)^{-1/2n}$ *or* $K > (n-1)^{-1/2n}$ *respectively.*

2. Let $f(z)$ be an entire function of exponential type τ, periodic on the real axis with period 2π. Then it has the form ([1], p. 109)

$$f(z) = \sum_{\nu=-n}^{n} a_\nu e^{i\nu z}, \qquad n \leq \tau .$$

In addition, if $f(z)$ is $O(e^{\varepsilon|z|})$ on the positive imaginary axis for some ε less than 1, then we shall have

$$f(z) = \sum_{\nu=0}^{n} a_\nu e^{i\nu z}, \qquad n \leq \tau .$$

Hence Theorems A and 1 may be restated as follows. (We use $h_f(\theta)$ to denote the indicator of $f(z)$.)

THEOREM A'. *If* $f(z) \neq 0$ *for* $\operatorname{Im} z > K$ *where* $K < 0$, *and if* $h_f(\frac{\pi}{2}) < 1$, *then for* $y < 2K$

$$\int_{-\pi}^{\pi} |f(x+iy)|^2 \, dx \leq \frac{e^{2\tau|y|} + e^{2\tau|K|}}{1 + e^{2\tau|K|}} \int_{-\pi}^{\pi} |f(x)|^2 \, dx .$$

THEOREM 1'. *If* $f(z) \neq 0$ *for* $\operatorname{Im} z > K \geq 0$, *and if* $h_f(\frac{\pi}{2}) < 1$, *then*

$$\int_{-\pi}^{\pi} |f'(x)|^2 \, dx \leq \frac{\tau^2}{1+e^{-2K\tau}} \int_{-\pi}^{\pi} |f(x)|^2 \, dx ,$$

$$\left| \int_{-\pi}^{\pi} f'(x) \overline{f(x)} \, dx \right| \leq \frac{\tau}{1+e^{-2K\tau}} \int_{-\pi}^{\pi} |f(x)|^2 \, dx ,$$

and for $y < 0$

$$\int_{-\pi}^{\pi} |f(x+iy)|^2 \, dx \leq \frac{e^{2\tau|y|} + e^{-2\tau K}}{1 + e^{-2\tau K}} \int_{-\pi}^{\pi} |f(x)|^2 \, dx .$$

For the rest of this section let $f(z)$ be an entire function of exponential type τ belonging to L^2 on the real axis. We shall prove results for $f(z)$ which are analogous to Theorems A and 1 of the preceding section. According to a well known theorem of Paley and Wiener ([1], p. 103) an entire function $f(z)$ of exponential type τ belonging to L^2 on the real line has the representation

$$f(z) = \int_{-\tau}^{\tau} e^{izt} \phi(t)\, dt , \qquad \phi \in L^2 .$$

If (f', f) denotes the inner product of $f'(x)$ and $f(x)$ then

$$(f', f) = \int_{-\infty}^{\infty} f'(x) \overline{f(x)}\, dx = 2\pi i \int_{-\tau}^{\tau} t|\phi(t)|^2\, dt .$$

Analogously to (1.1), (1.2) and (1.3) we have

$$\text{(2.1)} \qquad \int_{-\infty}^{\infty} |f'(x)|^2\, dx \leq \tau^2 \int_{-\infty}^{\infty} |f(x)|^2\, dx ,$$

$$\text{(2.2)} \qquad |(f', f)| \leq \tau \int_{-\infty}^{\infty} |f(x)|^2\, dx ,$$

and for all y

$$\text{(2.3)} \qquad \int_{-\infty}^{\infty} |f(x+iy)|^2\, dx \leq e^{2\tau|y|} \int_{-\infty}^{\infty} |f(x)|^2\, dx .$$

These inequalities follow immediately from the above representation for $f(z)$ as a finite Fourier transform.

Corresponding to (1.4) and (1.5) we have ([5], Theorems 6 and 7).

THEOREM B. *If* $f(z) \neq 0$ *for* $\operatorname{Im} z > 0$, *and if* $h_f(\pi/2) = 0$, *then*

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$$(2.4) \qquad \int_{-\infty}^{\infty} |f'(x)|^2 dx \leq \frac{\tau^2}{2} \int_{-\infty}^{\infty} |f(x)|^2 dx ,$$

and for $y < 0$

$$(2.5) \qquad \int_{-\infty}^{\infty} |f(x+iy)|^2 dx \leq \frac{1}{2}(e^{2\tau|y|} + 1) \int_{-\infty}^{\infty} |f(x)|^2 dx .$$

Here we prove the following analogues of Theorem A and Theorem 1 respectively.

THEOREM 6. *If* $f(z) \neq 0$ *for* $\operatorname{Im} z > K$ *where* $K < 0$, *and if* $h_f(\pi/2) = 0$, *then for* $y < 2K$

$$(2.6) \qquad \int_{-\infty}^{\infty} |f(x+iy)|^2 dx \leq \frac{e^{2\tau|y|} + e^{2\tau|K|}}{1 + e^{2\tau|K|}} \int_{-\infty}^{\infty} |f(x)|^2 dx .$$

THEOREM 7. *If* $f(z) \neq 0$ *for* $\operatorname{Im} z > K \geq 0$, *and if* $h_f(\pi/2) = 0$, *then*

$$(2.7) \qquad \int_{-\infty}^{\infty} |f'(x)|^2 dx \leq \frac{\tau^2}{1 + e^{-2K\tau}} \int_{-\infty}^{\infty} |f(x)|^2 dx ,$$

$$(2.8) \qquad |(f', f)| \leq \frac{\tau}{1 + e^{-2K\tau}} \int_{-\infty}^{\infty} |f(x)|^2 dx ,$$

and for $y < 0$

$$(2.9) \qquad \int_{-\infty}^{\infty} |f(x+iy)|^2 dx \leq \frac{e^{2\tau|y|} + e^{-2\tau K}}{1 + e^{-2\tau K}} \int_{-\infty}^{\infty} |f(x)|^2 dx .$$

Let $f(z)$ be an entire function of order 1 and type τ such that $f(z)$ has all its zeros in $\operatorname{Im} z \geq K$ where $K \leq 0$. If further $h_f(\pi/2) = 0$ then the entire function $g(z) = e^{i\tau z}\overline{f(\bar{z})}$ has no zeros in $\operatorname{Im} z > -K \geq 0$ and $h_g(\pi/2) = 0$. If we apply (2.8) to $g(z)$

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we shall get

$$\int_0^{\tau} (\tau - t)\, |\emptyset(t)|^2\, dt \leq \frac{\tau e^{-2K\tau}}{1+e^{-2K\tau}} \int_0^{\tau} |\emptyset(t)|^2\, dt$$

or

$$\int_0^{\tau} t\, |\emptyset(t)|^2\, dt \geq \frac{\tau}{1+e^{-2K\tau}} \int_0^{\tau} |\emptyset(t)|^2\, dt\, .$$

We can therefore state the following analogue of Theorem 2.

THEOREM 3. *If* $f(z)$ *is an entire function of order* 1 *and type* τ *such that* $f(z)$ *has all its zeros in* $\operatorname{Im} z \geq K$ *where* $K \leq 0$ *and* $h_f(\pi/2) = 0$, *then*

$$|(f', f)| \geq \frac{\tau}{1+e^{-2K\tau}} \|f\| \tag{2.10}$$

where $\|f\| = \int_{-\infty}^{\infty} |f(x)|^2\, dx\, .$

Proof of Theorem 6. To start with we suppose that $f(z)$ has all its zeros on $\operatorname{Im} z = K < 0$. By Paley-Wiener theorem $f(z)$ has the representation

$$f(z) = \int_0^{\tau} e^{izt}\, \emptyset(t)\, dt\, , \qquad \emptyset \in L^2\, .$$

The function $F(z) = f(z+iK)$ as well as $\omega(z) = e^{i\tau z}\, \overline{f(\overline{z} + iK)}$ has all its zeros on the real axis. Besides $|F(x)| = |\omega(x)|$ for $-\infty < x < \infty$, hence for some γ in $0 \leq \gamma < 2\pi$ we have $F(z) = e^{i\gamma}\omega(z)$. From this it follows that

$$|\emptyset(\tau - t)| = e^{K(\tau - 2t)}\, |\emptyset(t)|\, , \qquad 0 \leq t \leq \tau\, .$$

Thus for $y < 0$,

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$$\int_{-\infty}^{\infty} |f(x+iy)|^2\,dx = 2\pi \int_0^{\tau} e^{-2yt}\,|\emptyset(t)|^2\,dt$$

$$= \pi \int_0^{\tau} [e^{-2yt}\,|\emptyset(t)|^2 + e^{-2y(\tau-t)}\,e^{2K(\tau-2t)}|\emptyset(t)|^2]\,dt$$

$$= \pi \int_0^{\tau} \frac{e^{2|K|(\tau-2t)}\,e^{2|y|t} + e^{2|y|(\tau-t)}}{1+e^{2|K|(\tau-2t)}}\,(|\emptyset(t)|^2$$

$$+ |\emptyset(\tau-t)|^2)\,dt\,.$$

Now

$$\frac{e^{2\tau|y|}+e^{2\tau|K|}}{1+e^{2\tau|K|}} - \frac{e^{2|K|(\tau-2t)}\,e^{2|y|t}+e^{2|y|(\tau-t)}}{1+e^{2|K|(\tau-2t)}}$$

$$= \frac{1}{(1+e^{2\tau|K|})(1+e^{2|K|(\tau-2t)})}\,\{(e^{2(\tau-t)|y|}-e^{4|K|(\tau-t)})(e^{2|y|t}-$$

$$+ e^{2|K|(\tau-2t)}\,(e^{2|y|(\tau-t)}-1)\,(e^{2|y|t}-e^{4|K|t})\}$$

$$\geq 0$$

if $y \leq 2K$. Therefore the greatest of the quantities

$$\frac{e^{2|K|(\tau-2t)}\,e^{2|y|t}+e^{2|y|(\tau-t)}}{1+e^{2|K|(\tau-2t)}}$$

for $0 \leq t \leq \tau$ is

$$\frac{e^{2\tau|y|}+e^{2\tau|K|}}{1+e^{2\tau|K|}}$$

if $y \leq 2K$. Hence for $y \leq 2K$ we have

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$$(2.6) \qquad \int_{-\infty}^{\infty} |f(x+iy)|^2 dx \leq \frac{e^{2\tau|y|} + e^{2\tau|K|}}{1+e^{2\tau|K|}} \int_{-\infty}^{\infty} |f(x)|^2 dx .$$

The theorem is proved for the special case when all the zeros of $f(z)$ lie on $\operatorname{Im} z = K < 0$.

Now let us consider the general case. If $\mathcal{F}(z) = e^{i\tau(z-ik)} \overline{f(\bar{z} + 2iK)}$ then $|\mathcal{F}(z)| = |f(z)|$ for $\operatorname{Im} z = K$. The function $G(z) = f(z + iK)e^{-i\tau z/2}$ has no zeros for $y > 0$, and $h_G(-\pi/2) = h_G(\pi/2) = \tau/2$. Therefore by a theorem of Levin ([1], p. 129) we have $|G(z)| \leq |G(\bar{z})|$ for $y < 0$, or $|G(z - iK)| \leq |G(\bar{z} + iK)|$ for $y < K$. Thus for $y < K$,

$$|f(z)| \leq |f(\bar{z} + 2iK)\, e^{-i\tau(\bar{z}+iK)/2}|\, |e^{i\tau(z-iK)/2}|$$

$$= |\overline{f(\bar{z} + 2iK)}\, e^{i\tau(z-iK)/2}|\, |e^{i\tau(z-iK)/2}|$$

$$= |\overline{f(\bar{z} + 2iK)}\, e^{i\tau(z-iK)}|$$

$$= |\mathcal{F}(z)| .$$

Since $|G(z - iK)| \leq |G(\bar{z} + iK)|$ for $y < K$ we have

$$|G(z + iK)| \geq |G(\bar{z} - iK)|$$

for $y > -K$ or

$$|G(z - iK| \geq |G(\bar{z} + iK)|$$

for $y > K$. From this we can deduce in the same way as above that $|\mathcal{F}(z)| \leq |f(z)|$ for $y \geq K$. Besides, it is easy to verify that for every η such that $0 \leq \eta < 2\pi$ the function $f(z) + e^{i\eta}\mathcal{F}(z)$ has all its zeros on $\operatorname{Im} z = K$ and we can apply the special case proved above to the function $f(z) + e^{i\eta}\mathcal{F}(z)$. Hence for $y \leq 2K$ we have

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$$\int_{-\infty}^{\infty} |f(x+iy)+e^{i\eta}\mathcal{F}(x+iy)|^2 dx \le \frac{e^{2\tau|y|}+e^{2\tau|K|}}{1+e^{2\tau|K|}} \int_{-\infty}^{\infty} |f(x)+e^{i\eta}\mathcal{F}(x)|^2 dx$$

Now integrate both the sides with respect to η from 0 to 2π. On inverting the order of integration and noting the above relations between $|f(z)|$ and $|\mathcal{F}(z)|$ for $y < K$ and $y \ge K$ we easily get

$$\int_0^{2\pi} |1+e^{i\eta}|^2 d\eta \int_{-\infty}^{\infty} |f(x+iy)|^2 dx$$

$$\le \frac{e^{2\tau|y|}+e^{2\tau|K|}}{1+e^{2\tau|K|}} \int_{-\infty}^{\infty} |f(x)|^2 dx \int_0^{2\pi} |1+e^{i\eta}|^2 dy .$$

This gives the desired result.

The above estimate for $\int_{-\infty}^{\infty} |f(x+iy)|^2 dx$ is valid only for $y \le 2K$. An estimate for $2K < y < 0$ can be obtained from the following consideration.

Since

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 dx = 2\pi \int_0^{\tau} e^{-2yt} |\phi(t)|^2 dt ,$$

it is easily verified that $\log \int_{-\infty}^{\infty} |f(x+iy)|^2 dx$ increases and is a downward convex function of y as $y \to -\infty$. Thus for $2K < y < 0$ we have

$$\{\log \int_{-\infty}^{\infty} |f(x+iy)|^2 dx - \log \int_{-\infty}^{\infty} |f(x)|^2 dx\} / |y|$$

$$\le \{\log \int_{-\infty}^{\infty} |f(x+2iK)|^2 dx - \log \int_{-\infty}^{\infty} |f(x+iy)|^2 dx\} / (2|K| - |y|)$$

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or

$$2|K|\log \int_{-\infty}^{\infty} |f(x+iy)|^2 dx - (2|K| - |y|) \log \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\leq |y| \log \int_{-\infty}^{\infty} |f(x+2iK)|^2 dx$$

$$\leq |y| \log\{\frac{e^{4\tau|K|} + e^{2\tau|K|}}{1 + e^{2\tau|K|}} \int_{-\infty}^{\infty} |f(x)|^2 dx\}$$

by (2.6). Hence

$$2|K|\log \int_{-\infty}^{\infty} |f(x+iy)|^2 dx \leq 2|K|\log \int_{-\infty}^{\infty} |f(x)|^2 dx + |y|\log e^{2\tau|K|},$$

or

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 dx \leq e^{\tau|y|} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

for $2K < y < 0$.

Proof of (2.7). Let $F(z) = f(z+iK)$ and consider $\omega(z) = e^{i\tau z}\overline{F(\bar{z})}$, which is an entire function of exponential type $\geq \tau$. Since $f(z)$ has no zeros for $\operatorname{Im} z > K$, $h_f(\pi/2) = 0$ and $h_f(-\pi/2) = \tau$, the function $\omega(z)$ has no zeros for $\operatorname{Im} z < 0$, $h_\omega(-\pi/2) = \tau$, and $h_\omega(\pi/2) = 0$. The function $e^{-i\tau z/2}\omega(z)$ therefore belongs to the class P discussed in ([1], p. 129) and by the theorem of Levin (loc. cit.)

$$|e^{-i\tau z/2}\omega(z)| \geq |e^{-i\tau\bar{z}/2}\omega(\bar{z})|$$

for $\operatorname{Im} z < 0$. Thus for $\operatorname{Im} z < 0$ we have

$$|F(z)| = |e^{i\tau z}\overline{\omega(\bar{z})}| \leq |\omega(z)| .$$

By another theorem of Levin ([1], p. 226) it follows that

$$|F'(z)| \leq |\omega'(z)|$$

for $\operatorname{Im} z \leq 0$. In particular, $|F'(x - iK)| \leq |\omega'(x - iK)|$. Since

$$\omega(z) = e^{i\tau z} \int_0^{\tau} e^{-K\tau} e^{-izt} \overline{\phi(t)}\, dt ,$$

we get

$$\int_{-\infty}^{\infty} |f'(x)|^2 dx = \int_{-\infty}^{\infty} |F'(x - iK)|^2 dx$$

$$\leq \frac{1}{1 + e^{-2K\tau}} \{\int_{-\infty}^{\infty} |F'(x-iK)|^2 dx + e^{-2K\tau} \int_{-\infty}^{\infty} |\omega'(x-iK)|^2 dx\}$$

$$= \frac{2\pi}{1 + e^{-2K\tau}} \{\int_0^{\tau} t^2 |\phi(t)|^2 dt + e^{-2K\tau} \int_0^{\tau} (\tau-t)^2 e^{2K(\tau-2t)} |\phi(t)|^2 dt\}$$

$$= \frac{2\pi}{1 + e^{-2K\tau}} \int_0^{\tau} \{t^2 + (\tau-t)^2 e^{-4Kt}\} |\phi(t)|^2 dt$$

$$(2.11) \leq \frac{2\pi}{1 + e^{-2K\tau}} \int_0^{\tau} \{t^2 + (\tau-t)^2\} |\phi(t)|^2 dt$$

$$\leq \frac{\tau^2}{1 + e^{-2K\tau}} 2\pi \int_0^{\tau} |\phi(t)|^2 dt$$

$$= \frac{\tau^2}{1 + e^{-2K\tau}} \int_{-\infty}^{\infty} |f(x)|^2 dx .$$

This is (2.7)

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Proof of (2.8). From (2.11) we have

$$2\pi \int_0^{\tau} t^2 \, |\phi(t)|^2 dt \le \frac{2\pi}{1 + e^{-2K\tau}} \int_0^{\tau} (\tau^2 - 2\tau t + 2t^2) \, |\phi(t)|^2 \, dt$$

or

$$\frac{2\tau}{1+e^{-2K\tau}} \int_0^{\tau} t \, |\phi(t)|^2 dt \le \frac{\tau^2}{1+e^{-2K\tau}} \int_0^{\tau} |\phi(t)|^2 dt$$

$$+ \frac{1-e^{-2K\tau}}{1+e^{-2K\tau}} \int_0^{\tau} t^2 \, |\phi(t)|^2 \, dt$$

$$\le \frac{\tau^2}{1+e^{-2K\tau}} \int_0^{\tau} |\phi(t)|^2 dt$$

$$+ \frac{1-e^{-2K\tau}}{1+e^{-2K\tau}} \, \frac{\tau^2}{1+e^{-2K\tau}} \int_0^{\tau} |\phi(t)|^2 dt$$

by (2.7). Hence

$$\int_0^{\tau} t \, |\phi(t)|^2 dt \le \frac{\tau}{1 + e^{-2K\tau}} \int_0^{\tau} |\phi(t)|^2 \, dt ,$$

and (2.8) follows.

Proof of (2.9). The function $f'(z)$ is of exponential type τ; therefore by (2.7) and (2.3) we have for every $\beta < 0$

$$2\pi \int_0^{\tau} t^2 \, e^{-2\beta t} \, |\phi(t)|^2 dt = \int_{-\infty}^{\infty} |f'(x+i\beta)|^2 dx$$

$$\le \frac{\tau^2}{1+e^{-2K\tau}} \, e^{-2\tau\beta} \int_{-\infty}^{\infty} |f(x)|^2 dx .$$

Integrating both the sides with respect to β from δ to 0 we get

$$2\pi \int_0^{\tau} t\, |\phi(t)|^2 (e^{-2\delta t} - 1)dt \leq \frac{\tau}{1+e^{-2K\tau}} (e^{-2\tau\delta} - 1) \int_{-\infty}^{\infty} |f(x)|^2 dx$$

or

$$2\pi \int_0^{\tau} t\, e^{-2\delta t} |\phi(t)|^2 dt \leq 2\pi \int_0^{\tau} t\, |\phi(t)|^2 dt$$

$$+ \frac{\tau}{1+e^{-2K\tau}} (e^{-2\tau\delta} - 1) \times \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\leq \frac{\tau}{1+e^{-2K\tau}} e^{-2\tau\delta} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

by (2. 8). Now integrating both the sides of the above inequality with respect to δ from y to 0 we get

$$2\pi \int_0^{\tau} (e^{-2yt} - 1)\, |\phi(t)|^2 dt \leq \frac{1}{1+e^{-2K\tau}} (e^{-2\tau y} - 1) \int_{-\infty}^{\infty} |f(x)|^2 dx$$

or

$$2\pi \int_0^{\tau} e^{-2yt} |\phi(t)|^2 dt \leq \frac{e^{-2\tau y} + e^{-2K\tau}}{1 + e^{-2K\tau}} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

which is the same as (2. 9).

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REFERENCES

1. R. P. Boas, Jr., Entire functions, Academic Press, New York 1954.

2. N. G. De Bruijn, Inequalities concerning polynomials in the complex domain, Neder. Akad. Wetensch., Proc., vol. 50 (1947), pp. 1265-1272.

3. P. D. Lax, Proof of a conjecture of P. Erdős on the derivative of a polynomial, Bull. Amer. Math. Soc., vol. 50 (1944), pp. 509-513.

4. M. A. Malik, An inequality for polynomials, Canadian Mathematical Bulletin, vol. 6 (1963), pp. 65-69.

5. Q. I. Rahman, Some inequalities for polynomials and related entire functions, Illinois J. Math., vol. 5 (1961), pp. 144-151.

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