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Non-standard oscillation theory for multiparameter eigenvalue problems

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An eigenvalue problem for k Sturm–Liouville equations coupled by k parameters $\lambda_1, \dots, \lambda_k$ is considered. In contrast to the standard case, for each r, the second-order derivative in the rth equation is multiplied by λ_r. This problem presents various interesting features. For example, the existence of eigenvalues with oscillation counts beyond a certain (computable) value is obtained without any of the restrictive definiteness conditions known from the standard case. Uniqueness is also analysed, and the results are given greater precision via eigencurve methods in the case of two equations coupled by two parameters.

1. Introduction

Multiparameter spectral theory is generally regarded as having started with Klein's famous oscillation theorem [13]. Separating variables in Laplace's equation via elliptic coordinates, Klein was led by geometric reasoning to a result where Lamé's equation had solutions with prescribed oscillation counts on each of two separate intervals. Klein's health deteriorated soon afterwards, but he made at least two further significant (indirect) contributions to multiparameter spectral theory. One was to interest his student Bôcher in the topic, and, on several occasions (see, for example, [6]), Bôcher expounded (and extended, for example, to more general equations and to k parameters) the results of (Sturm and) Klein, via analytical methods. Another major contribution of Klein was to interest Hilbert (whom he had appointed to Göttingen from his position in Königsberg) in the topic. Hilbert generalized Klein's problem to the form

$$Ay_1 + (\lambda_1 B + \lambda_2 C)y_1 = 0 \neq y_1 \quad \text{on } (a_1, b_1), \tag{1.1}$$

$$Dy_2 + (\lambda_1 E + \lambda_2 F)y_2 = 0 \neq y_1 \quad \text{on } (a_2, b_2), \tag{1.2}$$

where A and D are second differentiation operators; B, C, E and F are multiplication operators; and Dirichlet boundary conditions are imposed. He then derived a partial differential equation of the form

$$Ly = \lambda_1 Ry, \tag{1.3}$$

which was used to study, for example, completeness and expansion results.

Subsequent authors have generalized (1.1) and (1.2) to multiparameter eigenvalue problems of a 'standard' form consisting of k Sturm–Liouville (or more general abstract) equations

$$L_r y_r + \sum_{s=1}^{k} \lambda_s P_{rs} y_r = 0 \neq y_r, \quad r = 1, 2, \dots, k; \tag{1.4}$$

here we take $L_r y_r = -y_r'' + q_r(x_r) y_r$, subject to separated boundary conditions, and P_{rs} denotes multiplication by $p_{rs}(x_r)$, $x_r \in (a_r, b_r)$. If (1.4) admits a non-trivial solution y_r with n_r zeros in (a_r, b_r) for each r, then the eigenvalue $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$ has *oscillation count* $\boldsymbol{n} = (n_1, \dots, n_k)$. Hilbert did not prove oscillation theorems, but he gave this problem to Yoshikawa, who established an analogue [20] of Klein's theorem for the case where

$$B, C, E > 0 > F, \tag{1.5}$$

which forces

$$R = \begin{vmatrix} B & C \\ E & F \end{vmatrix}$$

to be of one sign in (1.3). This latter condition is nowadays known as 'right definiteness' (RD) and, for (1.4), it corresponds to the requirement that

$$\det P(x) \text{ should be of one sign for (almost all) } x_r, \tag{1.6}$$

where $P(x)_{rs} = p_{rs}(x_r)$. Richardson [15] was clearly aware of the existence and uniqueness of $\boldsymbol{\lambda}$ for any given $\boldsymbol{n}$ under RD for general k, although a complete proof does not seem to have been given until much later (see Volkmer [19] for a treatment involving more general conditions, and including a review of the more modern literature).

Hilbert used the assumptions $C > 0 > F$ in (1.1), (1.2), and, with definiteness of A, D (because $q_r = 0$ in L_r), these are sufficient for definiteness of the formal determinant

$$L = -\begin{vmatrix} A & C \\ D & F \end{vmatrix}.$$

This is 'left definiteness' (LD) of (1.3), and one can make this more precise by interpreting the determinant in the sense of tensor products (cf. Atkinson [1] in the context of matrices). Later authors (cf. [16] and references) have defined multiparameter LD by requiring that each L_r in (1.4) be (positive) definite and (after a rotation of $\boldsymbol{\lambda}$ axes) that the cofactors in the first column of $P(x)$ (see (1.6)) be of the same sign. These are Hilbert's conditions for the case of (1.1) and (1.2). For oscillation results under LD, see [19] and the references therein. If LD holds for (1.4) with continuous coefficients, then one can further transform the $\boldsymbol{\lambda}$ axes so that the cofactors in *each* column of $P(x)$ are of the same sign, and this will be referred to as full left definiteness (FLD). This condition was studied by Binding (see, for example, [3]) and leads to the existence and uniqueness, for each $\boldsymbol{n}$, of an eigenvalue $\boldsymbol{\lambda}$ in one of $\mathbb{R}^k_\pm$ if RD also holds. (Throughout, $\mathbb{R}^k_\pm$ will denote *open* orthants.) If, instead, $\det P(x)$ can take both signs, then, for each $\boldsymbol{n}$, there are two eigenvalues $\boldsymbol{\lambda}$, one in each $\mathbb{R}^k_\pm$, so in a sense the problem behaves as two cases that satisfy FLD and RD.

A different kind of oscillation theorem was studied by Richardson [14, 15] (for $k = 2, 3$) and Turyn [18] (for general k via geometrical methods). The result sought for (1.4) now concerns all 'large' oscillation counts (existence of N so that $n_r \geqslant N$ for all r guarantees the existence of corresponding $\boldsymbol{\lambda}$). We note that, even for $k = 2$, a non-trivial condition on the p_{rs} is necessary for this result [14, 18]. Faierman [8] and Sleeman [17] have used the cofactor part of LD to obtain such results. Sleeman has called this condition 'ellipticity', and Binding [3] has studied the version corresponding to FLD. This will be referred to as full ellipticity (FE) and, again, it guarantees that eigenvalues $\boldsymbol{\lambda}$ exist in the (open) orthants $\mathbb{R}^k_\pm$, but now for all oscillations counts which are 'large' in the above sense.

Our object here is to obtain analogues of some of the results in the previous two paragraphs for a 'non-standard' problem. A special case of this was introduced by Atkinson [2] in the form

$$\lambda_1 A y_1 + \lambda_2 C y_1 + q_1 y_1 = 0 \quad \text{on } (a_1, b_1), \tag{1.7}$$

$$\lambda_2 D y_2 + \lambda_1 F y_2 + q_2 y_2 = 0 \quad \text{on } (a_2, b_2), \tag{1.8}$$

where, as in (1.1) and (1.2), A and D represent double differentiation subject to Dirichlet boundary conditions, and again $C > 0 > F$. This problem does not fit into the earlier framework. Comparing the λ_1 and λ_2 terms with Hilbert's case, we see that the determinant

$$R = \begin{vmatrix} B & C \\ E & F \end{vmatrix}$$

has been replaced by the formal expression

$$\begin{vmatrix} A & C \\ F & D \end{vmatrix}.$$

As for Yoshikawa's problem, this is of one sign, but the operator entries now have different relative compactness and the problem turns out to have more in common with FLD and FE than with RD. Actually, Atkinson indicated the possibility of up to four eigenvalues per oscillation count $\boldsymbol{n}$, with finite accumulation as $\boldsymbol{n}$ varied, and he posed various completeness and expansion questions. In a memorial paper to Atkinson, Faierman and Mennicken [9] studied the existence and uniqueness of eigenvalues for (1.7) and (1.8) for each $\boldsymbol{n}$, using eigencurve methods. A homogeneous formulation is detailed in [9] as motivation for (1.7) and (1.8). Completeness and expansion results were also investigated in [9, 10].

We prove oscillation theorems for eigenvalue problems of the form

$$\lambda_r y_r'' + q_r(x_r) y_r + \sum_{s=1}^{k} \lambda_s p_{rs}(x_r) y_r = 0 \quad \text{for } r = 1, \dots, k, \tag{1.9}$$

subject to separated boundary conditions. This problem generalizes (1.7) and (1.8) and is of similar form to (1.4), but with the second-order derivative in the rth

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equation multiplied by λ_r. In § 2 we use degree theory to obtain the existence of eigenvalues with large oscillation counts. We impose no definiteness condition on the p_{rs}, and this seems to be the first result of such a nature: as indicated earlier, some condition on the p_{rs} is necessary in the case of (1.4). For example, if each q_r takes positive values on a set of positive measure, then (see theorem 2.3), for each large $\boldsymbol{n}$, there exists an eigenvalue $\boldsymbol{\lambda} \in \mathbb{R}_+^k$. There is a similar result for each (open) orthant, so the only condition imposed is that q_r should not vanish identically almost everywhere (a.e.) for every $r = 1, 2, \dots, k$. We remark that this condition is necessary for such a result for (1.9), even when $k = 1$ (note that q_r then behaves as a weight function after we divide by λ_1). In § 3 we require each q_r not to vanish (a.e.) on any interval. This condition (on the weight function) has been used implicitly or explicitly by various authors for the case $k = 1$ of (1.4) in the case of semidefinite weight (cf. [4, 7]), and elsewhere we shall show its relevance for cases with indefinite weight. Here we use it to obtain uniqueness of the eigenvalues $\boldsymbol{\lambda}$ obtained in § 2. In § 4 we show that these eigenvalues can be obtained as fixed points of contractions, thereby suggesting a method for their computation.

In § 5 we treat the case $k = 2$ via eigencurve methods. These allow us to sharpen the meaning of 'large' and to provide conditions guaranteeing any number of eigenvalues between 0 and 4 for a given oscillation count. When the operators corresponding to $-y_r'' + p_{rr} y_r$ are positive definite, as in Atkinson's problem (1.7), (1.8), our conditions become necessary and sufficient and generalize those of [9].

2. Existence of eigenvalues

We consider k differential equations

$$\lambda_r y_r'' + q_r(x_r) y_r + \sum_{s=1}^{k} \lambda_s p_{rs}(x_r) y_r = 0, \quad x_r \in [a_r, b_r],\ r = 1, 2, \dots, k, \tag{2.1}$$

coupled by k real parameters $\lambda_1, \dots, \lambda_k$. The prime indicates differentiation with respect to x_r. The equations (2.1) contain coefficient functions p_{rs} and q_r which we assume to be real-valued and integrable on $[a_r, b_r]$ for every $r, s = 1, 2, \dots, k$. For each equation we impose separated boundary conditions

$$\cos\alpha_r y_r(a_r) = \sin\alpha_r y_r'(a_r), \qquad \cos\beta_r y_r(b_r) = \sin\beta_r y_r'(b_r), \tag{2.2}$$

where, as usual, $\alpha_r \in [0, \pi)$ and $\beta_r \in (0, \pi]$. We call a tuple $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$ with $\lambda_r \neq 0$ for all r an *eigenvalue* of (2.1) and (2.2), if these equations admit a nontrivial solution y_r for each r. If y_r has n_r zeros in (a_r, b_r), we say that the eigenvalue has *oscillation count* $\boldsymbol{n} = (n_1, \dots, n_k)$.

LEMMA 2.1. *For every $\delta > 0$, there exists $N \in \mathbb{N}_0$ such that each eigenvalue $\boldsymbol{\lambda}$, with oscillation counts $n_r \geqslant N$ for all r, lies in $(-\delta, \delta)^k$.*

Proof. For given $\delta > 0$, set

$$v_r(x_r) := \delta^{-1} |q_r(x_r)| + \sum_{r \neq s = 1}^{k} |p_{rs}(x_r)|.$$

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Let $\mu_{r,m}$ be the eigenvalue of the right-definite Sturm–Liouville problem

$$-y_r'' - (p_{rr}(x_r) + v_r(x_r))y_r = \mu_r y_r \tag{2.3}$$

subject to the rth boundary condition in (2.2) with oscillation count m. The sequence $\mu_{r,m}$ converges to infinity as $m \to \infty$ for every r. We choose N so large that $\mu_{r,m} > 0$ for every $m \geqslant N$ and every r.

Let $\boldsymbol{\lambda}$ be an eigenvalue of (2.1) and (2.2) with oscillation counts satisfying $n_r \geqslant N$ for each r. We claim that $\boldsymbol{\lambda} \in (-\delta, \delta)^k$. If not, there exists $t \in \{1, 2, \dots, k\}$ such that $|\lambda_s| \leqslant |\lambda_t|$ for all s and $|\lambda_t| \geqslant \delta$. Let $(y_1, \dots, y_k)$ be eigenfunctions corresponding to $\boldsymbol{\lambda}$. Then

$$y_t'' + p_{tt}(x_t)y_t + g(x_t)y_t = 0, \tag{2.4}$$

where

$$g(x_t) = \lambda_t^{-1} q_t(x_t) + \sum_{t \neq s = 1}^{k} \frac{\lambda_s}{\lambda_t} p_{ts}(x_t).$$

By assumption, $g(x_t) \leqslant v_t(x_t)$ for all $x_t \in [a_t, b_t]$. Therefore, comparison of (2.3) with (2.4) yields $\mu_{t,n_t} \leqslant 0$, contrary to the choice of N. □

We will use later the fact that if we replace p_{rs} by τp_{rs} for $r \neq s$, then we can choose the same N for all $\tau \in [0, 1]$ in the statement of lemma 2.1. This follows with the same proof.

The following lemma concerning (indefinite) Sturm–Liouville problems will be applied in the proof of theorem 2.3. Results of a similar nature were given in [11,14] for continuous coefficients (see [5] for integrable coefficients).

LEMMA 2.2. *Consider the Sturm–Liouville problem*

$$-z'' + g(x)z = \lambda h(x)z, \quad x \in [a, b] \tag{2.5}$$

subject to

$$\cos\alpha z(a) = \sin\alpha z'(a), \qquad \cos\beta z(b) = \sin\beta z'(b), \tag{2.6}$$

where g and h are real-valued integrable functions on $[a, b]$ such that $h(x) > 0$ on a set of positive measure.

(i) *Let N_0 denote the (finite) number of non-positive eigenvalues ρ of the RD problem*

$$-z'' + g(x)z = \rho z$$

subject to (2.6). Then, for every $m \geqslant N_0$, there exists a positive eigenvalue λ of (2.5), (2.6) with oscillation count m.

(ii) *There exists $N_1 \in \mathbb{N}_0$ such that, for every $m \geqslant N_1$ and every eigenfunction z of (2.5), (2.6) with oscillation count m and belonging to a positive eigenvalue λ,*

$$\int_a^b (-z'' + gz)z = \lambda \int_a^b hz^2 > 0.$$

(iii) *If $N = \max(N_0, N_1)$, then, for every $m \geqslant N$, there exists a unique positive eigenvalue λ of (2.5), (2.6) with oscillation count m.*

THEOREM 2.3. *Suppose that, for each r, $q_r > 0$ on a set of positive measure. Then there exists $N \in \mathbb{N}_0$ such that, for every $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{N}_0^k$ with $n_r \geqslant N$ for each r, there exists an eigenvalue $\boldsymbol{\lambda} \in \mathbb{R}_+^k$ of (2.1), (2.2) with oscillation count $\boldsymbol{n}$.*

Proof. For given $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{N}_0^k$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}_+^k$ and $r = 1, 2, \dots, k$, we define $\mu_{r,n_r}(\boldsymbol{\lambda})$ as the eigenvalue of the RD problem

$$-\lambda_r y_r'' - q_r(x_r) y_r - \sum_{s=1}^{k} \lambda_s p_{rs}(x_r) y_r = \mu_r y_r$$

subject to (2.2) with oscillation count n_r. Then we introduce the continuous (actually analytic) map $M_{\boldsymbol{n}} \colon \mathbb{R}_+^k \to \mathbb{R}^k$ whose rth component function is $\boldsymbol{\lambda} \mapsto \mu_{r,n_r}(\boldsymbol{\lambda})$. The zeros of $M_{\boldsymbol{n}}$ are the eigenvalues we are looking for. We will apply Brouwer's degree of maps to prove the existence of such zeros for suitable $\boldsymbol{n}$.

We begin by choosing $\delta > 0$ so small that

$$\delta \sum_{r \neq s=1}^{k} \int_{a_r}^{b_r} |p_{rs}| < \int_{a_r}^{b_r} q_r^+, \quad r = 1, 2, \dots, k, \tag{2.7}$$

where q_r^+ denotes the positive part of q_r, and setting

$$w_r(x_r) := q_r(x_r) - \delta \sum_{r \neq s=1}^{k} |p_{rs}(x_r)|.$$

Then $w_r(x_r) > 0$ on a set of positive measure for every r.

We now use lemmas 2.1 and 2.2 to choose N so large that the following three statements hold.

1. Every eigenvalue $\boldsymbol{\lambda} \in \mathbb{R}_+^k$ with oscillation counts $n_r \geqslant N$ lies in $(0, \delta)^k$.

2. For every r and $n_r \geqslant N$ there exists a unique eigenvalue $\omega_{r,n_r} > 0$ of

$$-y_r'' - p_{rr}(x_r) y_r = \omega_r w_r(x_r) y_r$$

   subject to (2.2) with oscillation count n_r.

3. For every r and $n_r \geqslant N$ there exists a unique eigenvalue $\rho_{r,n_r} > 0$ of

$$-y_r'' - p_{rr}(x_r) y_r = \rho_r q_r(x_r) y_r$$

   subject to (2.2) with oscillation count n_r, and corresponding eigenfunctions y_r satisfy

$$\int_{a_r}^{b_r} q_r y_r^2 > 0.$$

We now prove that N has the property as in the statement of theorem 2.3. Consider a fixed $\boldsymbol{n} = (n_1, \dots, n_k)$ such that $n_r \geqslant N$ for each r. Choose $\epsilon \in (0, \delta)$ so small that $\epsilon < \omega_{r,n_r}^{-1}$ for each r. Suppose $\boldsymbol{\lambda} \in (0, \delta)^k$ satisfies $M_{\boldsymbol{n}}(\boldsymbol{\lambda}) = \boldsymbol{0}$. Since

$$q_r(x_r) + \sum_{r \neq s=1}^{k} \lambda_s p_{rs}(x_r) \geqslant w_r(x_r),$$

straightforward comparison of Prüfer angles shows that $\lambda_r^{-1} \leqslant \omega_{r,n_r}$. Therefore,

$$\epsilon < \omega_{r,n_r}^{-1} \leqslant \lambda_r.$$

It follows that all zeros of $M_{\boldsymbol{n}}$ lie in the set

$$Q = (\epsilon, \delta)^k.$$

In particular, Brouwer's degree $\deg(M_{\boldsymbol{n}}, Q, \mathbf{0})$ is well defined. In order to compute this degree, for each $\tau \in [0,1]$, we define the map $M_{\tau,\boldsymbol{n}}$ in the same manner as $M_{\boldsymbol{n}}$ but with p_{rs} replaced by τp_{rs} for $r \neq s$ (the functions q_r and p_{rr} remain unchanged). By repeating the above arguments with the same δ, N and ϵ, we find that all zeros of $M_{\tau,\boldsymbol{n}}$ lie in Q for every $\tau \in [0,1]$. It is easily seen that $\deg(M_{0,\boldsymbol{n}}, Q, \mathbf{0}) = 1$ since the only zero of $M_{0,\boldsymbol{n}}$ is $(\rho_{1,n_1}^{-1}, \dots, \rho_{k,n_k}^{-1}) \in Q$ and the derivative of $M_{0,\boldsymbol{n}}$ at this point is a diagonal matrix with positive entries. Therefore, by the homotopy invariance of the degree, we obtain $\deg(M_{\boldsymbol{n}}, Q, \mathbf{0}) = 1$, which implies that $M_{\boldsymbol{n}}$ has a zero in Q. The proof is complete. □

Theorem 2.3 can be extended as follows.

THEOREM 2.4. *Let $\sigma_1, \dots, \sigma_k \in \{0,1\}$. Suppose that $(-1)^{\sigma_r} q_r(x_r) > 0$ on sets of positive measure for each $r = 1, 2, \dots, k$. Then there exists N such that, for every $\boldsymbol{n} = (n_1, \dots, n_k)$ with $n_r \geqslant N$ for each r, there exists an eigenvalue $(\lambda_1, \dots, \lambda_k)$ with $(-1)^{\sigma_r}\lambda_r > 0$ of (2.1) and (2.2) with oscillation count $\boldsymbol{n}$.*

In particular, if $q_r(x_r) > 0$ as well as $q_r(x_r) < 0$ on sets of positive measure for each r, there exists N such that, for every $\boldsymbol{n} = (n_1, \dots, n_k)$ with $n_r \geqslant N$ for each r, there exist at least 2^k eigenvalues $\boldsymbol{\lambda}$ of (2.1), (2.2) with oscillation count $\boldsymbol{n}$.

Proof. We multiply the rth equation in (2.1) by $(-1)^{\sigma_r}$ and set $\tilde{\lambda}_r = (-1)^{\sigma_r}\lambda_r$. The desired statement then follows from theorem 2.3. □

3. Uniqueness of eigenvalues

In order to establish uniqueness of eigenvalues we start by treating just one equation in (2.1), say, the first. We consider the differential equation

$$\lambda_1 y'' + q(x)y + \sum_{s=1}^{k} \lambda_s p_s(x) y = 0, \quad x \in [a,b], \tag{3.1}$$

subject to boundary conditions

$$\cos\alpha y(a) = \sin\alpha y'(a), \qquad \cos\beta y(b) = \sin\beta y'(b). \tag{3.2}$$

We assume that $q, p_1, \dots, p_k$ are real-valued integrable functions on $[a,b]$ and $\alpha \in [0,\pi)$, $\beta \in (0,\pi]$.

To motivate the assumptions of the following lemma, recall from lemma 2.1 that $\boldsymbol{\lambda}$ tends to $\mathbf{0}$ as the oscillation counts tend to infinity.

LEMMA 3.1. *Suppose that $q(x)$ does not vanish a.e. on any subinterval of $[a,b]$ of positive length. For $m \in \mathbb{N}$, let $y = y_m$ be a non-trivial solution of (3.1) corresponding to $\lambda_1 = \lambda_{1m}, \dots, \lambda_k = \lambda_{km}$ satisfying the boundary conditions (3.2). Assume*

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that $\lambda_{1m} \neq 0$ *for all* m *and* $(\lambda_{1m}, \dots, \lambda_{km}) \to \mathbf{0}$ *as* $m \to \infty$*. Then, for sufficiently large* m*, we have*

$$\int_a^b (-y_m'' - p_1 y_m) y_m > 2 \sum_{s=2}^k \int_a^b |p_s| y_m^2. \tag{3.3}$$

Proof. Following the method of proof of [5, lemma 4.3], we see that there exists a constant K such that, for every m,

$$\int_a^b (-y_m'' - p_1 y_m) y_m - 2 \sum_{s=2}^k \int_a^b |p_s| y_m^2 \geqslant \frac{1}{2} \int_a^b y_m'^2 - K \int_a^b y_m^2. \tag{3.4}$$

Suppose (3.3) is false. By taking subsequences if necessary and normalizing y_m, (3.4) shows that we may assume that, for all m,

$$\int_a^b y_m^2 = 1, \qquad \int_a^b y_m'^2 \leqslant 2K. \tag{3.5}$$

By applying the Arzela–Ascoli theorem and again taking subsequences if necessary, we assume additionally that y_m converges uniformly to some continuous function y as $m \to \infty$. By (3.5), there is a sequence $x_m \in [a, b]$ such that $y_m'(x_m)$ is a bounded sequence and we may assume that x_m converges to some $u \in [a, b]$. Set

$$h_m(x) := q(x) + \sum_{s=1}^k \lambda_{sm} p_s(x).$$

By integrating (3.1), we find that

$$\lambda_{1m}(y_m'(x_m) - y_m'(x)) = \int_{x_m}^x h_m(t) y_m(t) \, \mathrm{d}t$$

and so

$$\int_a^b (y_m'(x_m) - y_m'(x))^2 \, \mathrm{d}x = \lambda_{1m}^{-2} \int_a^b \left(\int_{x_m}^x h_m(t) y_m(t) \, \mathrm{d}t \right)^2 \mathrm{d}x.$$

The left-hand side is a bounded sequence and $\lambda_{1m} \to 0$, so

$$\lim_{m \to \infty} \int_a^b \left(\int_{x_m}^x h_m(t) y_m(t) \, \mathrm{d}t \right)^2 \mathrm{d}x = 0. \tag{3.6}$$

Since $\boldsymbol{\lambda} \to \mathbf{0}$ and $x_m \to u$, we obtain

$$\lim_{m \to \infty} \int_{x_m}^x h_m(t) y_m(t) \, \mathrm{d}t = \int_u^x q(t) y(t) \, \mathrm{d}t \quad \text{for all } x \in [a, b]. \tag{3.7}$$

By Fatou's lemma, (3.6) and (3.7) yield

$$\int_u^x q(t) y(t) \, \mathrm{d}t = 0 \quad \text{for all } x \in [a, b].$$

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It follows that $q(x)y(x) = 0$ a.e. on $[a, b]$. By assumption, $q(x)$ does not vanish a.e. on any subinterval of $[a, b]$ of positive length, so $y(x) = 0$ for all $x \in [a, b]$. This contradicts

$$\int_a^b y^2 = 1$$

and so completes the proof. □

The following example shows that lemma 3.1 may not hold if we omit the assumption that q does not vanish a.e. on any interval of positive length.

For fixed $c > 0$, consider $[a, b] = [-c\pi, \pi]$, $k = 2$, $p_1(x) = 0$, $\alpha = 0$, $\beta = \pi$ and

$$p_2(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x \geqslant 0, \end{cases} \qquad q(x) = 1 - p_2(x).$$

We take $(\lambda_{1m}, \lambda_{2m}) = (m^{-2}, c^{-2}m^{-2})$ and

$$y_m(x) = \begin{cases} \sin \dfrac{x}{c} & \text{if } x < 0, \\ \dfrac{\sin(mx)}{cm} & \text{if } x \geqslant 0. \end{cases}$$

Then all the assumptions of lemma 3.1 are satisfied except the mentioned condition on q, while

$$\int_a^b y_m'^2 = \frac{(c+1)\pi}{2c^2}, \qquad \int_a^b |p_2| y_m^2 = \tfrac{1}{2} c\pi.$$

We return to the eigenvalue problem (2.1), (2.2).

THEOREM 3.2. *Suppose that, for each* $r = 1, 2, \ldots, k$, $q_r(x_r) > 0$ *on a set of positive measure and that* q_r *does not vanish a.e. on any subinterval of* $[a_r, b_r]$ *of positive length. Then there exists* $N \in \mathbb{N}_0$ *such that, for every* $\boldsymbol{n} = (n_1, \ldots, n_k)$ *with* $n_r \geqslant N$ *for all* r, *there exists a unique eigenvalue* $\boldsymbol{\lambda} \in \mathbb{R}_+^k$ *of (2.1), (2.2) with oscillation count* $\boldsymbol{n}$.

Proof. We choose δ and N as in the proof of theorem 2.3. It follows from lemmas 2.1 and 3.1 that there exists $N_1 \geqslant N$ such that

$$\int_{a_r}^{b_r} (-y_r'' - p_{rr} y_r) y_r > 2 \sum_{r \neq s = 1}^{k} \int_{a_r}^{b_r} |p_{rs}| y_r^2 \tag{3.8}$$

for all eigenfunctions $(y_1, \ldots, y_k)$ corresponding to eigenvalues $\boldsymbol{\lambda} \in (0, \infty)^k$ with oscillation counts $n_r \geqslant N_1$.

We claim that this N_1 has the property as stated in the theorem. To prove this, let $\boldsymbol{n} = (n_1, \ldots, n_k)$ with $n_r \geqslant N_1$ for each r. We define the map $M_{\boldsymbol{n}}$ and ϵ as in the proof of theorem 2.3. Then all eigenvalues $\boldsymbol{\lambda} \in (0, \infty)^k$ with oscillation count $\boldsymbol{n}$ lie in $Q = (\epsilon, \delta)^k$, and they are exactly the zeros of $M_{\boldsymbol{n}}$. Let $M_{\boldsymbol{n}}(\boldsymbol{\lambda}) = 0$, and let $(y_1, \ldots, y_k)$ be corresponding L^2-normalized eigenfunctions. The derivative $M_{\boldsymbol{n}}'(\boldsymbol{\lambda})$ is a $k \times k$ matrix with diagonal entries

$$\int_{a_r}^{b_r} (-y_r'' - p_{rr} y_r) y_r$$

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and off-diagonal entries

$$-\int_{a_r}^{b_r} p_{rs} y_r^2.$$

By (3.8), this matrix is diagonally dominant and so $\det M'_{\boldsymbol{n}}(\boldsymbol{\lambda}) > 0$. It follows that the zeros of $M_{\boldsymbol{n}}$ are isolated and their number is finite. The local degree of the zeros is 1 and so $\deg(M_{\boldsymbol{n}}, Q, \boldsymbol{0})$ equals the number of eigenvalues $\boldsymbol{\lambda} \in \mathbb{R}_+^k$ with oscillation count $\boldsymbol{n}$. We saw in the proof of theorem 2.3 that $\deg(M_{\boldsymbol{n}}, Q, \boldsymbol{0}) = 1$. This completes the proof. □

Theorem 3.2 may be extended to give uniqueness of eigenvalues $(\lambda_1, \dots, \lambda_k)$ of prescribed signs.

4. Eigenvalues as fixed points

We consider the differential equation (3.1) subject to boundary conditions (3.2). Let us assume that $q(x) > 0$ on a set of positive measure. We choose $\delta_0 > 0$ so small that

$$\delta_0 \sum_{s=2}^{k} \int_a^b |p_s| < \int_a^b q^+. \tag{4.1}$$

By lemma 2.2, there exists N_0 such that, for $\lambda_2, \dots, \lambda_k \in [0, \delta_0]$ and $n \geqslant N_0$, there exists $\lambda_1 > 0$ and a non-trivial solution y of (3.1), (3.2) with oscillation count n. We note that

$$0 < \lambda_1 \leqslant \omega_n^{-1}, \tag{4.2}$$

where ω_n is the eigenvalue with oscillation count n of the RD problem

$$-y'' - p_1(x) y = \omega w(x) y \tag{4.3}$$

subject to (3.2), where

$$w(x) := 1 + |q(x)| + \delta_0 \sum_{s=2}^{k} |p_s(x)|.$$

The following lemma allows us to consider λ_1 as a function of $\lambda_2, \dots, \lambda_k$.

Lemma 4.1. *Assume that $q(x) > 0$ on a set of positive measure, and $q(x)$ does not vanish on any subinterval of $[a, b]$ of positive length. Then there exists $\delta \in (0, \delta_0]$ and $N \geqslant N_0$ such that, for all $n \geqslant N$ and all $\lambda_2, \dots, \lambda_k \in [0, \delta]$, there exists a unique positive $\lambda_1 =: \Lambda_n(\lambda_2, \dots, \lambda_k)$ such that (3.1) and (3.2) admit a non-trivial solution y with oscillation count n. In addition, the function $\Lambda_n \colon [0, \delta]^{k-1} \to (0, \infty)$ is continuously differentiable and*

$$\sum_{s=2}^{k} \left| \frac{\partial \Lambda_n}{\partial \lambda_s} \right| \leqslant \frac{1}{2} \quad \textit{on } [0, \delta]^{k-1}. \tag{4.4}$$

Proof. Suppose there exist $\delta \in (0, \delta_0]$ and $N \geqslant N_0$ such that, for all $\lambda_2, \dots, \lambda_k \in [0, \delta]$, all $\lambda_1 > 0$ and all non-trivial solutions y of (3.1), (3.2) with oscillation count

$n \geqslant N$, we have

$$\int_a^b (-y'' - p_1 y)y > 2\sum_{s=2}^k \int_a^b |p_s| y^2. \tag{4.5}$$

If this claim were false, then, for every m, there would be $\lambda_{sm} \in [0, 1/m\delta_0]$ for $s = 2, \dots, k$, $\lambda_{1m} > 0$ and a non-trivial solution y_m of (3.1) with $\lambda_s = \lambda_{sm}$ satisfying (3.2) with oscillation count $n_m \geqslant m$ such that

$$\int_a^b (-y_m'' - p_1 y_m)y_m \leqslant 2\sum_{s=2}^k \int_a^b |p_s| y_m^2. \tag{4.6}$$

Since $\omega_{n_m} \to \infty$, it follows from (4.2) that $\lambda_{1m} \to 0$, so we contradict lemma 3.1. This establishes the claim.

We now show that δ, N have the stated properties. Let $\lambda_2, \dots, \lambda_k \in [0, \delta]$ and $n \geqslant N$. For every $\lambda_1 > 0$ and every solution y of (3.1), (3.2) with oscillation count n it follows from (4.5) that

$$\int_a^b (-y'' - p_1 y)y > 0.$$

According to lemma 2.2, this implies that λ_1 is uniquely determined by $\lambda_2, \dots, \lambda_k$ and n. We denote the corresponding function by $\lambda_1 = \Lambda_n(\lambda_2, \dots, \lambda_k)$. For $\lambda_1 > 0$ and $\lambda_2, \dots, \lambda_k \in \mathbb{R}$, let $\mu_n(\lambda_1, \dots, \lambda_k)$ denote the eigenvalue of the right-definite Sturm–Liouville problem

$$-\lambda_1 y'' - \left(q(x) + \sum_{s=1}^k \lambda_s p_s(x)\right)y = \mu y$$

subject to (3.2) with oscillation count n. It is known that μ_n is a continuously differentiable function. If y is a corresponding L^2-normalized eigenfunction, then

$$\frac{\partial \mu_n}{\partial \lambda_1} = \int_a^b (-y'' - p_1 y)y > 0$$

and

$$\frac{\partial \mu_n}{\partial \lambda_s} = -\int_a^b p_s y^2 \quad \text{for } s = 2, \dots, k.$$

Since

$$\mu_n(\Lambda_n(\lambda_2, \dots, \lambda_k), \lambda_2, \dots, \lambda_k) = 0,$$

the implicit function theorem implies that Λ_n is continuously differentiable and

$$\frac{\partial \Lambda_n}{\partial \lambda_s} = \left(\int_a^b (-y'' - p_1 y)y\right)^{-1} \int_a^b p_s y^2 \quad \text{for } s = 2, \dots, k.$$

Together with (4.5), this proves (4.4). The proof of the lemma is complete. □

We return to our coupled multiparameter problem (2.1), (2.2). We assume that, for each r, $\int q_r^+ > 0$ and $q_r(x_r)$ does not vanish a.e. on any subinterval of $[a_r, b_r]$ of

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positive length. For every $r = 1, \dots, k$, we apply lemma 4.1 to the rth equation in (2.1) (with λ_r playing the role of λ_1). We find $\delta > 0$ and $N \in \mathbb{N}_0$ such that, for all r, all $\lambda_1, \dots, \lambda_k \in [0, \delta]$ and all $n_r \geqslant N$, there is a unique $\Lambda_{rn_r} > 0$ such that the equation

$$\Lambda_{rn_r}(y_r'' + p_r(x_r)y_r) + \biggl(q_r(x_r) + \sum_{r \neq s=1}^{k} \lambda_s p_{rs}(x_r)\biggr) y_r = 0$$

has a non-trivial solution satisfying (2.2) and having oscillation count n_r. Note that Λ_{rn_r} does not depend on λ_r. By increasing N if necessary, we may assume that $\Lambda_{rn_r} \leqslant \delta$, and, by lemma 2.1, we may also assume that all eigenvalues $\boldsymbol{\lambda} \in \mathbb{R}_+^k$ with oscillation counts $n_r \geqslant N$ lie in $[0, \delta]^k$. For $\boldsymbol{n} = (n_1, \dots, n_k)$ with $n_r \geqslant N$, we obtain a map

$$\boldsymbol{\Lambda_n} \colon [0, \delta]^k \to [0, \delta]^k \tag{4.7}$$

whose components functions are Λ_{rn_r}. Note that the fixed points of $\boldsymbol{\Lambda_n}$ are exactly the eigenvalues of (2.1), (2.2) in $(0, \infty)^k$ with oscillation count $\boldsymbol{n}$. By lemma 4.1, $\boldsymbol{\Lambda}$ is continuously differentiable. Its derivative $\boldsymbol{\Lambda}_{\boldsymbol{n}}'(\boldsymbol{\lambda})$ is a $k \times k$ matrix with zeros on the main diagonal, and the sum of the absolute values of its entries in every row is at most $\frac{1}{2}$. Therefore, the matrix norm of $\boldsymbol{\Lambda}_{\boldsymbol{n}}'(\boldsymbol{\lambda})$ with respect to the max-norm is at most $\frac{1}{2}$. It follows that

$$\|\boldsymbol{\Lambda_n}(\boldsymbol{\lambda}) - \boldsymbol{\Lambda_n}(\boldsymbol{\mu})\|_\infty \leqslant \tfrac{1}{2}\|\boldsymbol{\lambda} - \boldsymbol{\mu}\|_\infty \quad \text{for all } \boldsymbol{\lambda}, \boldsymbol{\mu} \in [0, \delta]^k.$$

Banach's fixed-point theorem shows that $\boldsymbol{\Lambda_n}$ has exactly one fixed point.

We have proved the following.

THEOREM 4.2. *Suppose that, for each $r = 1, 2, \dots, k$, $q_r(x_r) > 0$ on a set of positive measure and that q_r does not vanish a.e. on any subinterval of $[a_r, b_r]$ of positive length. Then there exist $\delta > 0$ and $N \in \mathbb{N}_0$ such that, for every $\boldsymbol{n} = (n_1, \dots, n_k)$ with $n_r \geqslant N$ for all r, the map $\boldsymbol{\Lambda_n} \colon [0, \delta]^k \to [0, \delta]^k$ of (4.7) is a contraction. The unique fixed point of $\boldsymbol{\Lambda_n}$ is the unique eigenvalue $\boldsymbol{\lambda} \in \mathbb{R}_+^k$ of (2.1), (2.2) with oscillation count $\boldsymbol{n}$.*

5. The two-parameter case

In this section we consider the problem (2.1), (2.2) of § 2 for the case where $k = 2$, under the additional assumption

$$p_{12}(x_1) > 0 \quad \text{a.e. on } [a_1, b_1], \qquad p_{21}(x_2) < 0 \quad \text{a.e. on } [a_2, b_2]. \tag{5.1}$$

For any integer $m \geqslant 0$, we denote by φ_m the eigenvalue with oscillation count m for the RD problem

$$-y_1'' - p_{11}y_1 = \varphi p_{12} y_1, \tag{5.2}$$

subject to (2.2) for $r = 1$. Also, for $\lambda_1 > 0$, we let $\lambda_2 = f_m(\lambda_1)$ denote the eigenvalue with oscillation count m for the RD problem (2.1), (2.2) with eigenparameter λ_2 for $r = 1$.

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LEMMA 5.1. *The function f_m is continuous and satisfies*

$$\lim_{\lambda_1\to 0+} f_m(\lambda_1) = f_0 := -\operatorname{ess\,sup}\left\{\frac{q_1(x_1)}{p_{12}(x_1)} : x_1 \in [a_1, b_1]\right\} \tag{5.3}$$

and

$$\lim_{\lambda_1\to\infty} \frac{f_m(\lambda_1)}{\lambda_1} = \varphi_m. \tag{5.4}$$

Proof. For $\lambda_1 > 0$, we can rewrite (2.1) in the form

$$-y_1'' - p_{11}y_1 = (\rho q_1 + \sigma p_{12})y_1, \tag{5.5}$$

where $\rho = 1/\lambda_1$ and $\sigma = \lambda_2/\lambda_1$. Considered with the boundary condition (2.2) for $r = 1$, (5.5) defines the mth eigenvalue $\sigma = \sigma_m$ as a continuous (even analytic) function of ρ (cf. [4]), so f_m is continuous, and

$$\lim_{\lambda_1\to\infty} \frac{f_m(\lambda_1)}{\lambda_1} = \lim_{\rho\to 0} \sigma_m(\rho) = \sigma_m(0), \tag{5.6}$$

which equals φ_m by definition. Finally,

$$\lim_{\lambda_1\to 0+} f_m(\lambda_1) = \lim_{\rho\to\infty} \frac{\sigma_m(\rho)}{\rho}, \tag{5.7}$$

which equals f_0 by [4, theorem 3.1]. □

Next, for any integer $n \geqslant 0$, we denote by γ_n the eigenvalue with oscillation count n for the RD problem

$$-y_2'' - p_{22}y_2 = \gamma p_{21}y_2, \tag{5.8}$$

subject to (2.2) for $r = 2$, and for $\lambda_2 > 0$ we let $\lambda_1 = g_n(\lambda_2)$ denote the unique positive eigenvalue of the RD problem (2.1), (2.2) with eigenparameter λ_1 for $r = 2$.

For lemma 5.1, we obtain the following.

LEMMA 5.2. *The function g_n is continuous and satisfies*

$$\lim_{\lambda_2\to 0+} g_n(\lambda_2) = g^0 := \operatorname{ess\,sup}\left\{\frac{q_2(x_2)}{-p_{21}(x_2)} : x_2 \in [a_2, b_2]\right\} \tag{5.9}$$

and

$$\lim_{\lambda_2\to\infty} \frac{g_n(\lambda_2)}{\lambda_2} = \gamma_n. \tag{5.10}$$

It may be helpful to interpret these lemmas geometrically. The graph of f_m is the mth eigencurve of (2.1), (2.2) for $r = 1$, and emanates from the point $(0, f_0)$ into the half-plane $\lambda_1 > 0$, with polar angle approaching $\tan^{-1}\varphi_m$ far from the origin. The f_m increase (pointwise) to ∞ with m and their polar angles approach $\frac{1}{2}\pi$ from below. Similarly, the graphs of the g_n (which are functions of λ_2) emanate from $(g^0, 0)$ into the half-plane $\lambda_2 > 0$, and their polar angles approach π from below as $n \to \infty$ (note that $\gamma_n \to -\infty$).

To formulate the existence theorem, we need more notation. We write $\sigma_{mn} < 0$ if $1 < \varphi_m\gamma_n$, and $\sigma_{mn} > 0$ if $1 > \varphi_m\gamma_n$ or $\varphi_m \leqslant 0$ or $\gamma_n \leqslant 0$. Also,

$$f_m^0 := f_m(g^0)$$

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is defined via (5.9) if $g^0 > 0$ (which will be implicitly assumed when conditions on f_m^0 appear, as in (i) and (ii) below). Similarly, we define $g_{n0} := g_n(f_0)$ via (5.3) on the assumption that $f_0 > 0$. We note that it is possible for g^0 to be infinite, and then we interpret the conditions on f_m^0 via appropriate limits.

THEOREM 5.3. *Consider the eigenvalue problem (2.1), (2.2) of § 2 for the case* $k = 2$ *under assumption (5.1). Then an eigenvalue* (λ_1, λ_2) *with oscillation count* (m, n) *exists in the open quadrant* $\mathbb{R}_+^2$ *under any of the following three conditions:*

(i) $\sigma_{mn} f_m^0 > 0$, *and* $f_0 \leqslant 0$ *if* $\sigma_{mn} > 0$*;*

(ii) $f_m^0 g_{n0} > 0$, *and* $\sigma_{mn} > 0$ *if* $f_m^0 > 0$*;*

(iii) $\sigma_{mn} g_{n0} > 0$, *and* $g^0 \leqslant 0$ *if* $\sigma_{mn} > 0$.

Proof. Geometrically, each of (i)–(iii) forces the graphs of f_m and g_n to intersect in $\mathbb{R}_+^2$, and any such intersection point (λ_1, λ_2) is an eigenvalue with oscillation count (m, n). We shall give an analytic argument in the case that will be the most important in what follows; the other cases involve similar reasoning.

Suppose that (i) holds with $\sigma_{mn} > 0$, so $f_m^0, g^0 > 0 \geqslant f_0$, and write

$$h(t) = g_n(f_m(t)) - t.$$

Consider first the case when $f_m(t)$ vanishes for some $t > g^0$ (for example, when $\varphi_m < 0$, which forces $g^0 < \infty$), and let $I = (u, v)$ be the maximal interval containing g^0 where f_m is positive. Then h is a continuous function on I

$$\lim_{t \to u+} h(t) = g^0 - u > 0 \tag{5.11}$$

and

$$\lim_{t \to v-} h(t) = g^0 - v < 0. \tag{5.12}$$

We turn next to the case when $\varphi_m = 0$. By the above, we can assume that $f_m(t) > 0$, and hence $h(t)$ is defined and continuous for all $t > u$. Moreover, $\sigma_m(0) = 0$ in (5.6), so since σ_m is an analytic function, we see that

$$c := \lim_{\lambda_1 \to \infty} f_m(\lambda_1) = \lim_{\rho \to 0} \frac{\sigma_m(\rho)}{\rho} = \sigma_m'(0)$$

exists $c \geqslant 0$ since $f_m(t) > 0$ for all $t > u$. Thus $f_m(t)$ is bounded above for $t > u$. Extending $g_n(\lambda_2)$ by continuity to $\lambda_2 = 0$ if $c = 0, g^0 < \infty$, and noting that $b = f_m^0 > 0$ if $g^0 = \infty$, we easily see that

$$h(w) < 0 \tag{5.13}$$

for sufficiently large w.

Finally, if $\varphi_m > 0$, then $f_m(t) \to \infty$ as $t \to \infty$ so we can use (5.4), (5.10) and $\sigma_{mn} > 0$, which gives $\gamma_n - 1/\varphi_m < 0$, to conclude that

$$\frac{h(t)}{f_m(t)} = \frac{g_n(f_m(t))}{f_m(t)} - \frac{t}{f_m(t)} < 0,$$

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and hence

$$h(t) < 0 \tag{5.14}$$

for sufficiently large t. In each case, then, (5.11)–(5.14) and the intermediate value theorem give existence of λ_1 so that $h(\lambda_1) = 0$, and this implies that $\lambda_2 := f_m(\lambda_1)$ gives $\lambda_1 = g_n(\lambda_2)$, as required. □

We remark that conditions (i)–(iii) are not exhaustive, but they allow more possibilities than in theorem 2.3. Of course, the latter result is aimed at large oscillation counts, and we now show that theorem 5.3 is more precise in this respect. Let $m = M_1$, $m = M_2$ and $n = N_1$ be the minimal oscillation counts such that $\varphi_m > 0$, $f_m^0 > 0$ and $\gamma_n < 0$, respectively. From theorem 5.3(i), we obtain the following.

Corollary 5.4. *If $f_0 \leqslant 0 < g^0$, $m \geqslant \max\{M_1, M_2\}$ and $n \geqslant N_1$, then the problem of theorem 5.3 has an eigenvalue $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ with oscillation count (m, n).*

One may extend the definitions of f_m and g_n to negative arguments. The resulting properties are similar to those in lemmas 5.1 and 5.2, and, in particular, we obtain the following.

Lemma 5.5. *We have*

$$\lim_{\lambda_1 \to 0-} f_m(\lambda_1) = f^0 := -\operatorname{ess\,inf}\left\{\frac{q_1(x_1)}{p_{12}(x_1)} : x_1 \in [a_1, b_1]\right\} \tag{5.15}$$

and

$$\lim_{\lambda_2 \to 0-} g_n(\lambda_2) = g_0 := \operatorname{ess\,inf}\left\{\frac{q_2(x_2)}{-p_{21}(x_2)} : x_2 \in [a_2, b_2]\right\}. \tag{5.16}$$

The corresponding limits at $-\infty$ remain unchanged. With these facts, one may proceed to an analogue of theorem 2.4 for the problem of theorem 5.3. The details will be left to the reader, but we remark that they give sufficient conditions for eigenvalues with the same oscillation count in any of the four open quadrants. Instead, we shall use the above ideas to explore a special case, which includes the assumptions of [2, 9], and which yields sharper results. We shall assume from now on that each of (5.2) and (5.8), with the relevant boundary conditions from (2.2), is LD. For brevity we will refer to this as the LD case, although it differs from the multiparameter LD condition in § 1.

One consequence of LD is that $\varphi_m > 0$ for each m, and $f_m(\lambda_1)$ is strictly increasing in λ_1 (this can be seen, for example, from the variational characterization of $\lambda_2 = f_m(\lambda_1)$, or from an expression which is positive for $f_m'(\lambda_1)$). Similarly, $\gamma_n < 0$ (so $\sigma_{mn} > 0$) and g_n is strictly decreasing for each n. The conditions of theorem 5.3(iii) cannot then be satisfied, while those of (i) and (ii) simplify considerably, and also become necessary, as follows.

Corollary 5.6. *Under LD, the problem of theorem 5.3 has an eigenvalue $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ with oscillation count (m, n) if and only if $g^0 > 0$ and either*

(i) $f_0 \leqslant 0$ *and* $f_m^0 > 0$, *or*

(ii) $f_0 > 0$ *and* $g_{n0} > 0$.

If such an eigenvalue exists, it must be unique.

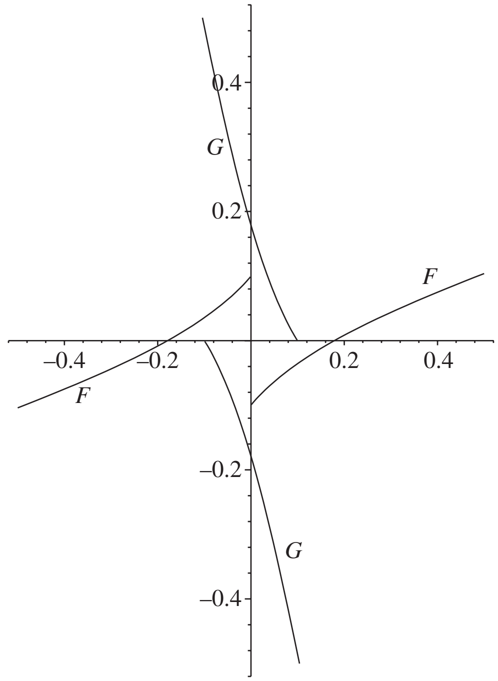

Figure 1. Eigencurves without intersection.

This follows from the above remarks and inspection of the eigencurves. Using lemma 5.5 as well, we can give conditions for the number of eigenvalues (which must be 0 or 1) in each (open) quadrant. The following result, which also uses $f_{m0} := f_m(g_0)$ (see (5.16)) gives most cases modulo rotation of the axes.

Corollary 5.7.

(i) *If $f^0 \leqslant 0 \leqslant g_0$ (i.e. both $q_r \geqslant 0$ a.e.) and $f_0 < 0$ (i.e. also $q_1 \neq 0$), then the number of eigenvalues with oscillation count (m, n) is 2 if f_m^0 and f_{m0} take opposite signs, 1 if they have the same sign (or one vanishes), and 0 if both vanish (so $g^0 = g_0$, i.e. q_2/p_{21} is constant).*

(ii) *If $f_0 < 0 < f^0$ (i.e. q_1 is indefinite) and $g_0 \geqslant 0$ (i.e. $q_2 \geqslant 0$ a.e.), then the number of eigenvalues with oscillation count (m, n) is at most 3, and any integer between 0 and 3 can be realized.*

(iii) *If $f_0, g_0 < 0 < f^0, g^0$ (i.e. both q_r are indefinite), then the number of eigenvalues with oscillation count (m, n) is at most 4, and any integer between 0 and 4 can be realized.*

This again follows from inspection of the eigencurves. The situation with no eigenvalues was not considered in [9], and we shall give an example from part (iii) above with no eigenvalues for $m = n = 0$.

Example 5.8. We take $[a_1, b_1] = [a_2, b_2] = [-1, 1]$ with $p_1(x) = -p_2(x) = 10$ as constant functions, and $q_1 = q_2$ as the step function with value -1 on $[-1, 0)$ and value 1 on $(0, 1]$. Let $(n_1, n_2) = (0, 0)$.

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The pairs (λ_1, λ_2) for which (2.1) and (2.2) admit a non-trivial solution with oscillation count 0 form curves labelled by F and G in figure 1, respectively. There are no intersections, so no eigenvalue with oscillation count $(0, 0)$ exists.

Turning again to large oscillation counts, we may use the fact that f_m and g_n tend to $+\infty$ and $-\infty$, respectively, as m and n become large to conclude the following modification of corollary 5.7. For simplicity we use strict inequalities on f_0, etc.

Corollary 5.9. *For any sufficiently large oscillation count* (m, n)*,*

(i) *if* $f^0 < 0 < g_0$*, then there is exactly one eigenvalue, and it lies in* $\mathbb{R}^2_+$*,*

(ii) *if* $f_0 < 0 < f^0$ *and* $g_0 > 0$*, then there are exactly two eigenvalues, one in each of the first two open quadrants,*

(iii) *if* $f_0, g_0 < 0 < f^0, g^0$*, then there are exactly four eigenvalues, one in each open quadrant.*

The above results have a certain similarity to those under the condition labelled FE in § 1. Actually, we can mimic the FLD situation more closely (although for different reasons) with some degenerate cases as follows.

Corollary 5.10. *For any oscillation count* (m, n)*,*

(i) *if* $f_0 = f^0 = 0 = g_0 = g^0$*, i.e. both* q_r *vanish a.e., then there are no eigenvalues,*

(ii) *if* $f_0 = f^0 = 0 < g_0$*, then there is one eigenvalue, and it lies in* $\mathbb{R}^2_+$*,*

(iii) *if* $f_0 = f^0 = 0$ *and* $g_0 < 0 < g^0$*, then there are two eigenvalues, one in each of* $\mathbb{R}^2_\pm$*.*

References

1 F. V. Atkinson. *Multiparameter eigenvalue problems: matrices and compact operators*, Mathematics in Science and Engineering, vol. 82, part 1 (New York: Academic Press, 1972).

2 F. V. Atkinson. Some questions on Klein's oscillation theorem. Lecture at the 4th International Workshop on Multiparameter Spectral Theory, Calgary (1989).

3 P. Binding. Abstract oscillation theorems for multiparameter eigenvalue problems. *J. Diff. Eqns* **49** (1983), 331–343.

4 P. Binding and P. Browne. Asymptotics of eigencurves for second order ordinary differential equations. I. *J. Diff. Eqns* **88** (1990), 30–45.

5 P. Binding and H. Volkmer. Oscillation theory for Sturm–Liouville problems with indefinite coefficients. *Proc. R. Soc. Edinb.* A **131** (2001), 989–1002.

6 M. Bôcher. The theorems of oscillation of Sturm and Klein (second paper). *Bull. Am. Math. Soc.* **4** (1898), 365–376.

7 W. Everitt, M. Kwong and A. Zettl. Oscillation of eigenfunctions of weighted regular Sturm–Liouville problems. *J. Lond. Math. Soc.* **27** (1983), 106–120.

8 M. Faierman. An oscillation theorem for a two-parameter system of differential equations. *Quaestiones Math.* **3** (1979), 313–321.

9 M. Faierman and R. Mennicken. A non-standard multiparameter eigenvalue problem in ordinary differential equations. *Math. Nachr.* **278** (2005), 1550–1560.

10 M. Faierman, M. Möller and B. Watson. Completeness theorems for a non-standard two-parameter eigenvalue problem. *Integ. Eqns Operat. Theory* **60** (2008), 37–52.

11 O. Haupt. Über eine Methode zum Beweise von Oszillationstheoremen. *Math. Ann.* **76** (1915), 67–104.

12 D. Hilbert. *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* (Leipzig: Teubner, 1912).

13 F. Klein. Über Körper, welche von confocalen Flächen zweiten Grades begrenzt sind. *Math. Annalen* **18** (1881), 410–427.

14 R. Richardson. Theorems of oscillation for two linear differential equations of the second order with two parameters. *Trans. Am. Math. Soc.* **13** (1912), 22–34.

15 R. Richardson. Über die notwendig und hinreichenden Bedingungen fur das Bestehen eines Kleinschen Oszillationstheorems. *Math. Ann.* **73** (1913), 289–304.

16 B. Sleeman. *Multiparameter spectral theory in Hilbert space*, Research Notes in Mathematics, vol. 22 (Pitman, 1978).

17 B. Sleeman. Klein oscillation theorems for multiparameter eigenvalue problems in ordinary differential equations. *Nieuw Arch. Wisk.* **27** (1979), 341–362.

18 L. Turyn. Sturm–Liouville problems with several parameters. *J. Diff. Eqns* **38** (1980), 239–259.

19 H. Volkmer. *Eigenvalue problems and expansion theorems*, Lecture Notes in Mathematics, vol. 1356 (Springer, 1988).

20 J. Yoshikawa. Ein zweiparametriges Oszilationstheorem. *Göttingen Nachr.* **95** (1910), 586–594.

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