Sine, cosine and exponential integrals

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The complete sine integral: first method

In this article, we explore the integrals, over appropriate intervals, of $\frac{\sin t}{t}$, $\frac{\cos t}{t}$ and $\frac{e^t}{t}$. We will present and compare various methods for dealing with them; some are quite well known, others rather less so. We start with the 'complete sine integral':

Theorem 1:

$$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}. \tag{1}$$

Note first that there is no problem of convergence at 0, because $\frac{\sin t}{t} \to 1$ as $t \to 0$.

A very quick and neat proof of (1) (to be seen, for example, in [1]) lies to hand if we assume the following well-known series identity: for $x \neq k\pi$,

$$\frac{1}{\sin x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x + n\pi}. \tag{2}$$

One proof of (2) [2, pp. 17-18] is by taking $x = 0$ in the Fourier series for $\cos ax$ on $[-\pi, \pi]$.

To derive (1), note first that, since $\frac{\sin t}{t}$ is an even function,

$$\int_{-\infty}^\infty \frac{\sin t}{t}\,dt = 2\int_0^\infty \frac{\sin t}{t}\,dt.$$

Denote this by I. The substitution $t = x + n\pi$ gives

$$\int_{n\pi}^{(n+1)\pi} \frac{\sin t}{t}\,dt = (-1)^n \int_0^\pi \frac{\sin x}{x + n\pi}\,dx.$$

Assuming that termwise integration of the series is valid, we add these identities for all integers n to obtain at once

$$I = \int_0^\pi \sin x \frac{1}{\sin x}\,dx = \pi.$$

The termwise integration (for any readers who care) is easily justified by uniform convergence, as follows. By combining the terms for n and $-n$ and multiplying by $\sin x$, we can rewrite the series (2) as

$$\frac{\sin x}{x} + 2x\sin x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2\pi^2} = 1.$$

For $0 < x < \pi$ and $n \geqslant 2$,

$$\left|\frac{2x\sin x}{x^2 - n^2\pi^2}\right| \leqslant \frac{2\pi}{(n^2 - 1)\pi^2}.$$

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Since $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$ is convergent, it follows, by Weierstrass's 'M-test', that the series converges uniformly on the open interval $(0, \pi)$: this is all we need.

We note some immediate variants and consequences of (1). First, for any $a > 0$, the substitution $at = u$ gives

$$\int_0^\infty \frac{\sin at}{t}\,dt = \int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{2}.$$

In particular,

$$\int_0^\infty \frac{\sin t \cos t}{t}\,dt = \frac{1}{2}\int_0^\infty \frac{\sin 2t}{t}\,dt = \frac{\pi}{4}. \tag{3}$$

We will use this several times later.

Next, we can derive the following integral:

$$\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}. \tag{4}$$

To do this, take $0 < \delta < R$ and integrate by parts on $[\delta, R]$:

$$\int_\delta^R \frac{\sin^2 t}{t^2}\,dt = \left[-\frac{\sin^2 t}{t}\right]_\delta^R = \int_\delta^R \frac{2\sin t \cos t}{t}\,dt.$$

Now $\frac{\sin^2 R}{R} \to 0$ as $R \to \infty$ and $\frac{\sin^2 \delta}{\delta} \to 0$ as $\delta \to 0^+$. Taking limits and applying (3), we obtain (4). This argument is reversible, so (4) equally implies (1). This is a viable alternative, because one can prove (4) in a similar way to (1), using the series

$$\frac{1}{\sin^2 x} = \sum_{n=-\infty}^{\infty} \frac{1}{(x - n\pi)^2};$$

this method is followed in [2, pp. 186-187].

One can develop this process further to evaluate the integrals of $\sin^n x / x^m$ for various m and n: see [3].

The incomplete sine integral

The 'incomplete' sine integral is the function

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt.$$

First, some simple facts about it. Since $\frac{\sin t}{t} \leqslant 1$ for $t > 0$, we have $\mathrm{Si}(x) \leqslant x$ for all $x > 0$. Also, by the fundamental theorem of calculus, the derivative $\mathrm{Si}'(x)$ is $\frac{\sin x}{x}$. Hence $\mathrm{Si}(x)$ is increasing on intervals $[2n\pi, (2n + 1)\pi]$ and decreasing on intervals $[(2n - 1)\pi, 2n\pi]$, so it has

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maxima at the points $(2n + 1)\pi$ and minima at the points $2n\pi$. Now write, temporarily, $A_n = \int_{n\pi}^{(n+2)\pi} \frac{\sin t}{t}\, dt$. By substituting $t + \pi = u$ on $[n\pi, (n + 1)\pi]$, we see that

$$A_n = \int_{n\pi}^{(n+1)\pi} \left(\frac{1}{t} - \frac{1}{t + \pi}\right) \sin t\, dt$$

in which $\frac{1}{t} - \frac{1}{t+\pi} > 0$. If n is even, then $\sin t \geqslant 0$ on $[n\pi, (n+1)\pi]$, so $A_n \geqslant 0$ and $\mathrm{Si}[(n+2)\pi] \geqslant \mathrm{Si}(n\pi)$. Hence $\mathrm{Si}(2n\pi) \geqslant \dots \geqslant \mathrm{Si}(2\pi) \geqslant \mathrm{Si}(0) = 0$ for all n, so in fact $\mathrm{Si}(x) \geqslant 0$ for all $x \geqslant 0$. Meanwhile, if n is odd, then $A_n \leqslant 0$, so that $\mathrm{Si}(\pi) \geqslant \mathrm{Si}(3\pi) \geqslant \dots$; hence the greatest value of $\mathrm{Si}(x)$ is $\mathrm{Si}(\pi)$. Of course, (1) says that $\mathrm{Si}(x) \to \frac{\pi}{2}$ as $x \to \infty$.

By integrating the series

$$\frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n + 1)!},$$

we obtain the explicit series expression

$$\mathrm{Si}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n + 1)!\,(2n + 1)} = x - \frac{x^3}{3!\,3} + \frac{x^5}{5!\,5} - \dots ,$$

from which, in principle, $\mathrm{Si}(x)$ can be calculated, though in practice the calculation is only pleasant for fairly small x. One finds, for example, $\mathrm{Si}(\pi) \approx 1.85194$ (recall that this is the greatest value) and $\mathrm{Si}(2\pi) \approx 1.41816$.

The complementary sine and cosine integrals, and analogues of (1) for $\cos t$

We cannot simply replace $\sin t$ by $\cos t$ in (1), or in the definition of $\mathrm{Si}(x)$, because the resulting integral would be divergent at 0. To formulate results that make sense for both $\sin t$ and $\cos t$, we consider instead the complementary integrals

$$S(x) = \int_x^{\infty} \frac{\sin t}{t}\, dt, \qquad C(x) = \int_x^{\infty} \frac{\cos t}{t}\, dt.$$

(Here I am departing from the established notation, which is $\mathrm{si}(x)$ and $\mathrm{ci}(x)$ where we have $-S(x)$ and $-C(x)$).

By (1), we have $S(0) = \frac{\pi}{2}$ and $S(x) = \frac{\pi}{2} - \mathrm{Si}(x)$. By the remarks above, $S(x)$ has maxima at $2n\pi$ and minima at $(2n - 1)\pi$, with greatest value $\frac{\pi}{2}$ and least value $S(\pi)$. Also, $S(\pi) \approx -0.28114$ and $S(2\pi) \approx 0.15264$.

Meanwhile, $C(x)$ is defined for $x > 0$, but not at $x = 0$. It has maxima at $(2n - \frac{1}{2})\pi$ and minima at $(2n + \frac{1}{2})\pi$, with overall least value at $\frac{\pi}{2}$.

The next result gives pleasantly simple approximations to $S(x)$ and $C(x)$ for large x (it also incorporates the proof that the integrals defining them converge in the first place).

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Proposition 1: We have

$$S(x) = \frac{\cos x}{x} + q_1(x), \qquad C(x) = -\frac{\sin x}{x} + r_1(x), \tag{5}$$

where $|q_1(x)|$ and $|r_1(x)|$ are not greater than $\frac{2}{x^2}$. Hence $xS(x) - \cos x$ and $xC(x) + \sin x$ tend to 0 as $x \to \infty$.

Proof: Integrating by parts twice, we obtain

$$\begin{aligned} S(x) &= \left[-\frac{\cos t}{t}\right]_x^\infty - \int_x^\infty \frac{\cos t}{t^2}\,dt \\ &= \frac{\cos x}{x} - \left[\frac{\sin t}{t^2}\right]_x^\infty + q_2(x) \\ &= \frac{\cos x}{x} + \frac{\sin x}{x^2} + q_2(x), \end{aligned}$$

where

$$q_2(x) = \int_x^\infty \frac{2\sin t}{t^3}\,dt.$$

Now $|q_2(x)| \leqslant \int_x^\infty \frac{2}{t^3}\,dt = \frac{1}{x^2}$. The stated expression for $S(x)$ follows. The proof for $C(x)$ is similar: we leave the details to the reader.

Of course, this also shows that $S(x)$ and $C(x)$ tend to 0 as $x \to \infty$. The process can be repeated to deliver increasingly accurate asymptotic expressions, and inequalities, for $S(x)$ and $C(x)$. By developing this approach, the following inequality for $S(x)$ was established in [4]:

$$|S(x)| \leqslant \frac{\pi}{2} - \tan^{-1} x.$$

Can we find a formula that enables us to calculate $C(x)$, and that opens the way to some kind of analogue of (1)? The key is to introduce the function

$$C^*(x) = \int_0^x \frac{1 - \cos t}{t}\,dt.$$

(This function is sometimes denoted by $\mathrm{Cin}(x)$.) It is elementary that $0 \leqslant 1 - \cos t \leqslant \frac{1}{2}t^2$, so that $0 \leqslant \frac{1 - \cos t}{t} \leqslant \frac{1}{2}t$, for $t > 0$. Hence there is no problem of convergence of the integral at 0, and we have $0 \leqslant C^*(x) \leqslant \frac{1}{4}x^2$ for all $x > 0$. Inserting the series for $\cos t$ and integrating, we obtain the power series expression

$$C^*(x) = \sum_{n=1}^\infty (-1)^{n-1} \frac{x^{2n}}{(2n)!\,(2n)} = \frac{x^2}{2!\,2} - \frac{x^4}{4!\,4} + \dots .$$

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We now relate $C^*(x)$ and $C(x)$. We have

$$C^*(x) - C^*(1) = \int_1^x \frac{1 - \cos t}{t}\, dt = \ln x - \int_1^x \frac{\cos t}{t}\, dt = \ln x - C(1) + C(x),$$

so

$$C(x) = C^*(x) - \ln x + c, \tag{6}$$

where c is constant, in fact $c = C(1) - C^*(1)$.

Even without knowing c, we can draw some conclusions from (6). One, which we will use later, is $xC(x) \to 0$ as $x \to 0^+$ (since $\lim_{x \to 0^+} (x \ln x) = 0$). Another is the following integral, which can be regarded as one kind of analogue of (1). It is a special case of the 'Frullani integral': see [5, pp. 133-135] or [3], where it is used in the evaluation of the integral of $\sin^n x / x^m$.

Proposition 2: For $a, b > 0$,

$$\int_0^\infty \frac{\cos at - \cos bt}{t}\, dt = \ln b - \ln a.$$

Proof: The substitution $at = u$ gives

$$\int_0^x \frac{1 - \cos at}{t}\, dt = \int_0^{ax} \frac{1 - \cos u}{u}\, du = C^*(ax).$$

Hence

$$\begin{aligned}\int_0^x \frac{\cos at - \cos bt}{t}\, dt &= C^*(bx) - C^*(ax) \\ &= C(bx) - C(ax) + \ln bx - \ln ax \\ &= C(bx) - C(ax) + \ln b - \ln a \\ &\to \ln b - \ln a \text{ as } x \to \infty.\end{aligned}$$

However, for a fully satisfactory version of (6), and for the calculation of $C(x)$, of course we need to know the value of c. The answer turns out to be that $c = -\gamma$, where γ is Euler's constant (yet another appearance of this famous constant!). Let us state this fact as a theorem:

Theorem 2:

$$C(x) = C^*(x) - \ln x - \gamma. \tag{7}$$

Surprisingly, this result is not mentioned in the comprehensive article [6] on Euler's constant. It can be seen stated without proof in compilations of formulae, such as [7]. However, it is not easy to find accessible references with a proof; part of the rationale for the present article is to supply one. At the same time, the method will also give a second proof of

Theorem 1. Later, we describe an alternative route to both theorems using contour integration.

The starting point for the proof will be the following expression for c derived from (6): since $\lim_{x \to \infty} C(x) = 0$, we have

$$-c = \lim_{x \to \infty} \left[C^*(x) - \ln x \right]. \tag{8}$$

Similarly, since $C^*(x) \to 0$ as $x \to 0^+$, it will follow from (7), once proved, that $C(x) + \ln x \to -\gamma$ as $x \to 0^+$. This describes the nature of $C(x)$ near 0, so can be regarded as the true analogue of (1).

Also, (7), together with the series for $C^*(x)$, enables us to calculate $C(x)$. We find, for example, $C\left(\frac{\pi}{2}\right) \approx -0.47200$ (recall that this is the least value) and $C(\pi) \approx -0.07367$.

Second proof of Theorem 1 and proof of Theorem 2

We will use the following elementary version of the Riemann-Lebesgue Lemma, which is easily proved by integration by parts:

If f is continuous on $[a, b]$ and has a continuous derivative on (a, b), then

$$\int_a^b f(t) \sin nt \, dt \to 0 \text{ as } n \to \infty,$$

and similarly with $\sin nt$ replaced by $\cos nt$. We also use:

Lemma 1: Let

$$h(t) = \frac{1}{t} - \frac{1}{\sin t}.$$

Then $h(t) \to 0$ as $t \to 0$.

Proof: By the series for $\sin t$, and the continuity of power series functions, we have

$$h(t) = \frac{t - \sin t}{t \sin t} = \frac{t^3/3! - t^5/5! + \dots}{t^2 - t^4/3! + \dots}$$
$$= \frac{t/3! - t^3/5! + \dots}{1 - t^2/3! + \dots}$$
$$\to 0 \text{ as } t \to 0.$$

Second proof of Theorem 1: It is sufficient to show that $I_n \to \frac{\pi}{2}$ as $n \to \infty$, where

$$I_n = \int_0^{(n + \frac{1}{2})\pi} \frac{\sin t}{t} \, dt.$$

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Substituting $t = (2n + 1)u$ (and then writing t for u), we have

$$I_n = \int_0^{\pi/2} \frac{\sin(2n + 1)t}{t}\, dt.$$

Let $D_n(t) = 1 + 2\sum_{r=1}^{n} \cos 2rt$ (applied to $\frac{1}{2}t$, this is the *Dirichlet kernel*). Note that $\int_0^{\pi/2} \cos 2rt\, dt = 0$ for non-zero integers r, so $\int_0^{\pi/2} D_n(t)dt = \frac{\pi}{2}$.

Since $\sin(a + b) - \sin(a - b) = 2\cos a \sin b$, we have

$$\sin(2r + 1)t - \sin(2r - 1)t = 2\cos 2rt \sin t.$$

Adding for $1 \leqslant r \leqslant n$, we obtain

$$\sin(2n + 1)t - \sin t = 2\sin t \sum_{r=1}^{n} \cos 2rt,$$

hence

$$D_n(t) = \frac{\sin(2n + 1)t}{\sin t}.$$

So we have

$$I_n - \frac{\pi}{2} = I_n - \int_0^{\pi/2} D_n(t) = \int_0^{\pi/2} h(t)\sin(2n + 1)t\, dt.$$

By Lemma 1, $h(t)$ becomes continuous on $\left[0, \frac{\pi}{2}\right]$ if assigned the value 0 at 0. So the Riemann-Lebesgue Lemma applies to show that $I_n - \frac{\pi}{2} \to 0$ as $n \to \infty$.

Though this proof of (1) is not quite as neat as our first one, it is more self-contained because it does not depend on the series (2). It appears in numerous books, e.g. [8, pp. 42-43]. For readers familiar with it, we mention that the *Fejér kernel* can be used in a similar way to prove (4) instead of (1).

Proof of Theorem 2: Recall from (8) that $-c = \lim_{x\to\infty} \left[C^*(x) - \ln x\right]$. Let $J_n = C^*\left[\left(n + \frac{1}{2}\right)\pi\right]$. We will show that $J_n - \ln n\pi \to \gamma$ as $n \to \infty$, with n restricted to even values. Since $\ln\left(n + \frac{1}{2}\right)\pi - \ln n\pi \to 0$ as $n \to \infty$, this will imply that $c = -\gamma$.

Substituting $t = (2n + 1)u$ (and then writing t for u), we have

$$J_n = \int_0^{\pi/2} \frac{1 - \cos(2n + 1)t}{t}\, dt.$$

Let $\tilde{D}_n(t) = 2\sum_{r=1}^{n} \sin 2rt$ (applied to $\frac{1}{2}t$, this is the *conjugate Dirichlet kernel*). Since $\cos(a - b) - \cos(a + b) = 2\sin a \sin b$, we have

$$\cos(2r - 1)t - \cos(2r + 1)t = 2\sin 2rt \sin t,$$

hence by addition

$$\tilde{D}_n(t) = \frac{\cos t - \cos(2n + 1)t}{\sin t},$$

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so that

$$\frac{1 - \cos(2n + 1)t}{t} = \tilde{D}_n(t) + \frac{1}{t} - \frac{\cos t}{\sin t} - h(t)\cos(2n + 1)t. \quad (9)$$

The integral of $\tilde{D}_n(t)$, unlike the integral of $D_n(t)$, needs a bit of work. Observe that

$$\int_0^{\pi/2} \sin 2rt \, dt = \frac{1}{2r}(1 - \cos r\pi) = \begin{cases} 0 & \text{for } r \text{ even,} \\ \frac{1}{r} & \text{for } r \text{ odd.} \end{cases}$$

For even n, the odd numbers less than n can be listed as $2r - 1$ for $1 \leqslant r \leqslant \frac{n}{2}$, so

$$\int_0^{\pi/2} \tilde{D}_n(t)\,dt = \sum_{r=1}^{n/2} \frac{2}{2r - 1}.$$

Lemma 2:

$$\sum_{r=1}^{k} \frac{2}{2r - 1} = \ln k + 2\ln 2 + \gamma + \rho_k.$$

where $\rho_k \to 0$ as $k \to \infty$.

Proof: Write $H_k = \sum_{r=1}^{k} \frac{1}{r}$. Then

$$\sum_{r=1}^{k} \frac{2}{2r - 1} = 2H_{2k} - \sum_{r=1}^{k} \frac{2}{2r} = 2H_{2k} - H_k.$$

Now $H_k = \ln k + \gamma + q_k$, where $q_k \to 0$ as $k \to \infty$. So

$$\begin{aligned} 2H_{2k} - H_k &= 2\ln 2k + 2\gamma + 2q_{2k} - \ln k - \gamma - q_k \\ &= \ln k + 2\ln 2 + \gamma + \rho_k, \end{aligned}$$

where $\rho_k = 2q_{2k} - q_k \to 0$ as $k \to \infty$.

Applying this with $k = \frac{n}{2}$, we have

$$\int_0^{\pi/2} \tilde{D}_n(t)\,dt = \ln n + \ln 2 + \gamma + \rho_{n/2}. \quad (10)$$

Completion of the proof of Theorem 2: As before, by the Riemann-Lebesgue lemma, $\int_0^{\pi/2} h(t)\cos(2n + 1)t\,dt \to 0$ as $n \to \infty$: denote this by σ_n. Now

$$\int_0^{\pi/2} \left(\frac{1}{t} - \frac{\cos t}{\sin t}\right) dt = \lim_{\delta \to 0^+} [\ln t - \ln \sin t]_\delta^{\pi/2} = \ln\frac{\pi}{2} + \lim_{\delta \to 0^+} \ln\frac{\sin\delta}{\delta} = \ln\frac{\pi}{2}.$$

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Inserting this and (10) into (9), we obtain

$$J_n = \ln n + \ln 2 + \gamma + \rho_{n/2} + \ln \frac{\pi}{2} - \sigma_n = \ln n\pi + \gamma + r_n,$$

where $r_n \to 0$ as $n \to \infty$.

A minor variation is to work with $2 \sum_{r=1}^{n} \sin(2r - 1)t$. This avoids the adjustment from k to $\frac{n}{2}$, but requires the evaluation $\int_0^{\pi/2} \left(\frac{1}{t} - \frac{1}{\sin t}\right) dt = \ln \frac{\pi}{4}$, which is slightly harder.

Some integrals involving $S(x)$ and $C(x)$

We apply our results to some integrals involving $S(x)$ and $C(x)$ (most of which can be seen stated without proof in [7]). These applications will actually use Theorem 1, Proposition 1 and (6), but not Theorem 2.

By the fundamental theorem of calculus, we have $S'(x) = -\sin x/x$ and $C'(x) = -\cos x/x$. Hence $\frac{d}{dx}\left[xS(x)\right] = S(x) - \sin x$ and $\frac{d}{dx}\left[xC(x)\right] = C(x) - \cos x$, so antiderivatives of $S(x)$ and $C(x)$ are as follows:

$$\int S(x)\, dx = xS(x) - \cos x, \qquad \int C(x)\, dx = xC(x) + \sin x.$$

By (5), $xS(x) - \cos x \to 0$ as $x \to \infty$, so we deduce at once

$$\int_0^\infty S(x)\, dx = \Big[xS(x) - \cos x\Big]_0^\infty = 1.$$

Recall from (6) that $xC(x) \to 0$ as $x \to 0^+$. So we have similarly

$$\int_0^\infty C(x)\, dx = \Big[xC(x) + \sin x\Big]_0^\infty = 0.$$

Next, we consider the integrals of $S(x)\sin x$ and $C(x)\cos x$. Integrating by parts and using (3), together with $\lim_{x\to\infty} S(x) = 0$, we find

$$\int_0^\infty S(x)\sin x\, dx = \Big[-S(x)\cos x\Big]_0^\infty - \int_0^\infty \frac{\sin x}{x}\cos x\, dx = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \quad (11)$$

Now $C(x)\sin x \to 0$ as $x \to 0^+$, since $\frac{\sin x}{x} \to 1$, so we have similarly

$$\int_0^\infty C(x)\cos x\, dx = \Big[C(x)\sin x\Big]_0^\infty + \int_0^\infty \frac{\cos x}{x}\sin x\, dx = 0 + \frac{\pi}{4} = \frac{\pi}{4}. \quad (12)$$

However, similar reasoning shows that $\int_0^\infty S(x)\cos x\, dx$ is divergent, since $\int_0^\infty \frac{\sin^2 x}{x} dx$ is divergent.

Of course, the integrals in (11) and (12) are really double integrals. Formal reversal of the double integrals duly delivers the stated values. However, the conditions for reversal of improper integrals are not satisfied,

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and one should really consider the integral on $[0, R]$ of $\int_x^R \frac{\sin t}{t} dt = S(x) - S(R)$. This simply leads, rather less directly, to the same limiting process that we considered above.

Since $\frac{d}{dt} S(t)^2 = 2S(t) S'(t) = -2S(t) \frac{\sin t}{t}$, we can express $S(x)^2$ as an integral:

$$S(x)^2 = 2 \int_x^\infty \frac{S(t) \sin t}{t} dt,$$

so in particular we have

$$\int_0^\infty \frac{S(t) \sin t}{t} dt = \tfrac{1}{2} S(0)^2 = \frac{\pi^2}{8}.$$

Finally, without using this, we establish

$$\int_0^\infty S(x)^2 dx = \int_0^\infty C(x)^2 dx = \frac{\pi}{2}.$$

Integrate by parts:

$$\int_0^\infty 1.\, S(x)^2 dx = \left[xS(x)^2 \right]_0^\infty + 2 \int_0^\infty xS(x) \frac{\sin x}{x} dx = 2 \int_0^\infty S(x) \sin x \, dx = \frac{\pi}{2},$$

in which we used (11) and $\lim_{x \to \infty} \left[xS(x)^2 \right] = 0$. The integral of $C(x)^2$ is similar, with the additional remark that $\lim_{x \to 0} \left[xC(x)^2 \right] = 0$.

Exponential integrals

Define

$$E(x) = \int_x^\infty \frac{e^{-t}}{t} dt, \qquad E^*(x) = \int_0^x \frac{1 - e^{-t}}{t} dt.$$

$E(x)$, as well as its various mutations, is known as the 'exponential integral'; the notation sometimes used is $E_1(x)$ for our $E(x)$ and $\mathrm{Ein}(x)$ for $E^*(x)$. We copy our treatment of $C(x)$ and $C^*(x)$, rather more briefly. Clearly, $E(x)$ is positive and satisfies the simple inequality

$$E(x) \leqslant \frac{1}{x} \int_x^\infty e^{-t} dt = \frac{e^{-x}}{x},$$

which can be refined by integrating by parts as in Proposition 1. Since $0 < 1 - e^{-t} \leqslant t$ for $t > 0$, we have $0 < E^*(x) \leqslant x$ for all $x > 0$. Using the series for e^{-t}, we can derive the power series expression

$$E^*(x) = x - \frac{x^2}{2!\,2} + \frac{x^3}{3!\,3} + \dots,$$

from which we find, for example, $E^*(1) \approx 0.79660$.

Exactly as for $C(x)$, we have

$$E(x) = E^*(x) - \ln x + c', \tag{13}$$

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where $c' = E(1) - E^*(1)$. As before, $-c' = \lim_{x \to \infty} \left[E(x) + \ln x\right]$ and $c' = \lim_{x \to 0^+} \left[E(x) + \ln x\right]$. However, unlike c, we can equate c' to an interesting single integral. Integration by parts gives

$$E(1) = \left[e^{-t} \ln t\right]_1^\infty + \int_1^\infty e^{-t} \ln t \, dt = \int_1^\infty e^{-t} \ln t \, dt,$$

and in the same way, since $\left(1 - e^{-t}\right) \ln t \to 0$ as $t \to 0^+$, we find $E^*(1) = -\int_0^1 e^{-t} \ln t$. Together, these identities give

$$c' = \int_0^\infty e^{-t} \ln t \, dt. \tag{14}$$

The integral in (14) is also often called the 'exponential integral'. Readers familiar with the gamma function will recognise that it equates to $\Gamma'(1)$.

It is a well-known fact that c', like c, equals $-\gamma$. (Is this a coincidence? Read on!) Versions of the proof can be seen in numerous books and articles, e.g. [9, pp. 176-177] and [1]. Here we sketch a version adapted to the way we have arrived at the problem.

Theorem 3: We have $c' = -\gamma$, hence

$$E(x) = E^*(x) - \ln x - \gamma, \tag{15}$$

$$\int_0^\infty e^{-t} \ln t \, dt = -\gamma. \tag{16}$$

Proof: We show that $E^*(n) - \ln n \to \gamma$ as $n \to \infty$. Since $e^{-t} = \lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n$, it seems at least plausible that $E^*(n)$ is approximated by K_n, where

$$\begin{aligned} K_n &= \int_0^n \frac{1}{t}\left[1 - \left(1 - \frac{t}{n}\right)^n\right] dt \\ &= \int_0^1 \frac{1}{u}\left[1 - (1 - u)^n\right] du \\ &= \int_0^1 \frac{1 - v^n}{1 - v} \, dv \\ &= \int_0^1 \left(1 + v + \ldots + v^{n-1}\right) dv \\ &= 1 + \frac{1}{2} + \ldots + \frac{1}{n}, \end{aligned}$$

so that $K_n - \ln x \to \gamma$ as $n \to \infty$.

The approximation step really does need to be justified, but fortunately this is not hard. It is elementary that $1 + x \leqslant e^x \leqslant \frac{1}{1+x}$ for $0 < x < 1$,

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and hence that $(1 - x^2)e^{-x} \leqslant 1 - x \leqslant e^{-x}$. After substituting $x = \frac{t}{n}$ and using $(1 - a)^n \geqslant 1 - na$, we deduce that

$$\left(1 - \frac{t^2}{n}\right)e^{-t} \leqslant \left(1 - \frac{t}{n}\right)^n \leqslant e^{-t}$$

for $0 \leqslant t \leqslant n$. It follows that $E^*(n) \leqslant K_n \leqslant E^*(n) + \Delta_n$, where

$$\Delta_n = \frac{1}{n}\int_0^n te^{-t}dt < \frac{1}{n}.$$

Hence, for example, $E(1) = E^*(1) - \gamma \approx 0.21938$.

Numerous other integrals can be derived from (16). We mention two. The substitution $at = u$ (where $a > 0$) gives

$$\int_0^\infty e^{-at}\ln t\,dt = \int_0^\infty e^{-u}(\ln u - \ln a)\tfrac{1}{a}du = -\tfrac{1}{a}(\gamma + \ln a).$$

The substitution $e^{-t} = u$ gives

$$\int_0^1 \ln\ln\frac{1}{u}\,du = -\gamma.$$

By (15), we have $\gamma = E^*(x) - E(x) - \ln x$. This has been used for the calculation of γ to great degrees of accuracy. For a suitably chosen x, one can estimate $E(x)$ by the method of Proposition 1, and also calculate $E^*(x)$ and $\ln x$. For a survey of this topic, see [10].

Exactly as in Proposition 2, we find

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}dt = \ln b - \ln a. \tag{17}$$

The reader might like to establish some of the following integrals, either by copying our earlier methods or by reversing double integrals (and in one case using (17)):

$$\int_0^\infty E(x)\,dx = 1, \qquad \int_0^\infty x^n E(x)\,dx = \frac{n!}{n+1},$$

$$\int_0^\infty e^{-x}E(x)\,dx = \ln 2, \qquad \int_0^\infty E(x)^2\,dx = 2\ln 2.$$

A contour integral that unites the results

Finally, we describe a well-known contour integral method that provides a third proof of Theorem 1, and at the same time relates $C^*(x)$ to $E^*(x)$ in such a way that either of Theorem 2 and Theorem 3 can be deduced from the other.

Let C_R be the circular arc of radius R in the positive quadrant, represented by $z = \text{R}e^{i\theta}$ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$. Denote by Γ the closed contour

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consisting of C_R together with the real interval $[0, R]$, and the imaginary axis from iR to 0. Let

$$f(z) = \frac{1 - e^{iz}}{z}.$$

Then f has no pole at 0, since $f(z) = -i + \frac{1}{2}z + \dots$. By Cauchy's integral theorem, $\int_\Gamma f(z)\,dz = 0$. The contribution of the real axis is

$$\int_0^R \frac{1 - e^{-it}}{t}\,dt = C^*(R) - i\,\mathrm{Si}(R).$$

The contribution of the imaginary axis, taken towards the origin, is

$$-\int_0^R \frac{1 - e^{-t}}{t}\,dt = -E^*(R).$$

Now consider C_R. Here the contribution of the term $\frac{1}{z}$ is, of course, $\frac{\pi}{2}i$. The contribution of the other term is

$$I_R = -\int_{C_R} \frac{e^{iz}}{z}\,dz = -\int_0^{\pi/2} ie^{iRe^{i\theta}}\,d\theta.$$

Its magnitude is estimated by the following Lemma.

Lemma 3: We have

$$0 \leqslant \int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \leqslant \frac{\pi}{2R}.$$

Proof: The function $\sin\theta$ is concave on $[0, \frac{\pi}{2}]$, since its derivative $\cos\theta$ is decreasing. This means that its graph lies above the straight line connecting its values at 0 and $\frac{\pi}{2}$, so $\sin\theta \geqslant \frac{2\theta}{\pi}$ on $[0, \frac{\pi}{2}]$. Hence

$$\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \leqslant \int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta = \left[-\frac{\pi}{2R}e^{-2R\theta/\pi}\right]_0^{\pi/2} = \frac{\pi}{2R}\left(1 - e^{-R}\right).$$

Since $\left|e^{iRe^{i\theta}}\right| = e^{-R\sin\theta}$, it follows that $|I_R| \leqslant \frac{\pi}{2R}$.

Considering first the imaginary part, and writing x for R for consistency with our earlier results, we obtain $\left|\mathrm{Si}(x) - \frac{\pi}{2}\right| \leqslant \frac{\pi}{2x}$. This is a third proof of Theorem 1, enhanced by the stated estimate for $\left|\mathrm{Si}(x) - \frac{\pi}{2}\right| = |S(x)|$. The method can be seen, for example, in [11, p. 123]. However, as already mentioned, a stronger inequality for $S(x)$ was given in [4].

Meanwhile, consideration of the real part shows that

$$\left|C^*(x) - E^*(x)\right| \leqslant \frac{\pi}{2x},$$

so that $C^*(x) - E^*(x) \to 0$ as $x \to \infty$. This, of course, does not evaluate either integral, but with (6) and (13) it implies that $c = c'$, thereby enabling us to deduce Theorem 2 from Theorem 3 or conversely. I hope that some readers will share my view that this is a rather neat conclusion to our journey.

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References

1. Nick Lord, Intriguing integrals: an Euler-inspired odyssey, *Math. Gaz.* **91** (November 2007) pp. 415-427.
2. P. L. Walker, *The theory of Fourier series and integrals*, John Wiley (1986).
3. J. Trainin, Integrating expressions of the form $\dfrac{\sin^n x}{x^m}$ and others, *Math. Gaz.* **94** (July 2010) pp. 216-223.
4. Graham Jameson, Nick Lord and James McKee, An inequality for Si (x), *Math. Gaz.* **99** (March 2015) pp. 133-139.
5. W. L. Ferrar, *Integral calculus*, Oxford University Press (1958).
6. Jeffrey C. Lagarias, Euler's constant: Euler's work and modern developments, *Bull. American Math. Soc.* **50** (2013) pp. 527-628.
7. N. M. Temme, Exponential, logarithmic, sine and cosine integrals, (2014), available at http://dlmf.nist.gov
8. E. C. Titchmarsh, *The theory of functions*, Oxford University Press (1939).
9. George Boros and Victor H. Moll, *Irresistible integrals*, Cambridge University Press (2004).
10. Xavier Gourdon and Pascal Sebah, The Euler constant γ, (2003), available at http://numbers.computation.free.fr/Constants/constants.html
11. R. B. Burckel, *An Introduction to classical complex analysis*, vol. 1, Academic Press (1979).

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