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The free mobility of a parallel manipulator

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SUMMARY

Singularities of a manipulator have been addressed repeatedly. However, the singularities and the degree(s) of freedom, as a matter of fact, are two different aspects of the mobility of a manipulator. Consequently, this paper dedicates to discussing the mobility properties through *mobility space*, which synchronously define the type, number and direction characteristics of the independent motions that the manipulator should execute. The *mobility space* of a manipulator can be obtained with reciprocal screws of the manipulator via singular value decomposition, which instantaneously depicts the singularity and mobility problems of the manipulators. Application example demonstrates that this methodology can investigate the all-sided mobility properties of parallel manipulators.

KEYWORDS: Parallel manipulator; Mobility; Kinematic chain; Reciprocal screw theory; Singular value decomposition.

I. INTRODUCTION

Singularity of a mechanism has been addressed repeatedly by many scholars and thought to be the position or pose where the mechanism gains or loses one or more degrees of freedom (DoF).[1–11] The DoF of a mechanism has been discussed by Grübler and Kutzbach.[12–14] Hunt,[13] Phillips,[14,15] and many other scholars and in our former articles.[16–17]

Gosselin and Wang[1] split the singularities of mechanisms into three types. The first type occurs in a situation when the mechanism loses one or more DoFs. The second type occurs in a situation when the mechanism gains one or more DoFs. The last one occurs when the positioning equations degenerate, which is also referred to as architecture singularity. Ficher[2] investigated the singular configurations of the manipulator and pointed out that the singular configurations are positions where the end-effector gains one or more DoFs. Gosselin and Angeles[3] presented an analysis of the different kinds of singularities encountered in closed-loop kinematic chains. They classified these singularities into three main groups, which are based on the properties of the Jacobian matrices of a chain. Merlet[4] proposed a method based on Grassman line geometry to determine the singular configurations. Agrawal and Roth[5] addressed a method to choose the active and passive joints to avoid many singularities. The approach to study this problem is based on instantaneous properties of series chains as derived from the theory of screw systems.[12–15,18] Daniali *et al*.[6] divided the singularities of planar parallel manipulators into three groups. The classification scheme relies on the properties of the Jacobian matrices of the manipulator. Park and Kim[8] proposed a geometric classification method and classified the closed chain singularities into three basic types: *configuration space singularities*, *actuator singularities* and *end-effector singularities*. Simaan and Shoham[11] proposed *linear complex singularities*, *linear congruence singularities*, *plane singularities*, *flat pencil singularities* and *point singularities*.

The former researchers' efforts mainly focused on searching the singular solutions and grouping the singularities. Yang *et al*.[19] proposed three types of singularities, the *forward singularity*, the *inverse singularity* and the *combined singularity*. The forward singularity is also called *architecture singularity* and can be avoided in the process of concept design; the inverse singularity occurs when the active velocities and passive velocities will not be determined from the mobile end-effector velocity; the combined singularity occurs when both forward and inverse singularities are simultaneously occur, which is also called multi-singularity. Fang and Tsai.[9] Sen and Mruthyunjaya,[20] Wang and Liu,[21] Zlatanov *et al*.,[22] investigated the classification problems of singularities. Burdick,[23] Legnani *et al*.,[24] Ranganath *et al*.,[25] Bandyopadhyay and Ghosal,[26] Wolf *et al*.,[27] Bhattacharya *et al*.[28] researched on the singular solutions and the classification problems.

The criterion mathematic expressions of singularities are derived either from velocity equations or from constraint and geometric conditions. Gosselin and Angeles,[3] Daniali *et al*.,[6] Zlatanov *et al*.,[22] Zlatanov *et al*.,[25] Hernández *et al*.,[29] Nokleby and Podhorodeski[30] obtained velocity-degenerate (singular) configurations of joint-redundant manipulators. Gosselin and Wang,[1] Ficher,[2] Merlet,[4] Ranganath *et al*.,[25] Wolf *et al*.,[27] Bhattacharya *et al*.,[28] Basu and Ghosal[31] presented a geometric condition for singularities.

However, the singularities and the DoFs, as a matter of fact, are two different aspects of the mobility of a manipulator. Consequently, this paper dedicates discussing the mobility of a manipulator based on the singular value decomposition (SVD) of reciprocal screws. According to the reciprocal screws, we investigate the constraint spaces spanned by the reciprocal screws with algebra methods,[32–34] then the free mobility space(s) of the end-effector and its DoF with SVD.

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II. BASIC CONCEPTS AND PRIMARY FORMULATIONS TO INVESTIGATE THE MOBILITY OF A MANIPULATOR

Singularity of a manipulator is often defined as[1,3,10,26,31]:

The position or pose where the DoF of the mechanism with an end-effector will change.

In reference [16], we have discussed the computation of DoF of the platforms (end-effector); we will further exploit the *numeral, type* and *direction* properties of DoF of the end-effector via SVD in this paper.

The formula for calculating the DoF of the end-effector in spatial parallel mechanism[16] is given as follows:

$$M = 6 - d \tag{1}$$

where M denotes the DoF of the end-effector, d denotes the number of dimensions that all the reciprocal screws can be spanned in the normal linear spaces.

According to reciprocal screw theory, we can get the following equation:

$$\$_{\mathrm{i}} \circ \$_{\mathrm{i}}^{\mathrm{r}} = 0 \tag{2}$$

where "$\circ$" represents the reciprocal production of two screws.

The reciprocal screws $\$_i^r$ can be solved with linear algebra methods. As a result, we can search the constraint spaces that the reciprocal screws of the system should be spanned in the six-dimension mobility spaces. If all of the reciprocal screws of each kinematic chain have been solved, the base and the dimension of the constraint space can be gained.[16]

Following, we will firstly introduce the preliminary reciprocal screw theory.[12–15,18,32,33]

A unit screw \$ is defined by a straight line with an associated pitch h and is conveniently denoted by six screw coordinates:

$$\$ = (\mathbf{s}\ \mathbf{s}_0 + h\mathbf{s}). \tag{3}$$

where $\mathbf{s} = (\,L\ \ M\ \ N\,)$ is a unit vector pointing in the direction of the screw axis, $\mathbf{s}_0 + h\mathbf{s} = \mathbf{r} \times \mathbf{s} + h\mathbf{s} = (\,P\ \ Q\ \ R\,)$ defines the moment of the screw axis about the origin of the coordinate system, $\mathbf{r}$ is the position vector of any point on the screw axis with respect to the coordinate system.

If the pitch of a screw is equal to zero, the screw coordinates reduce to:

$$\$ = (\,\mathbf{s}\ \ \mathbf{s}_0\,). \tag{4}$$

If a screw passes through the origin of the coordinate system, the screw coordinates will be:

$$\$ = (\,\mathbf{s}\ \ h\mathbf{s}\,). \tag{5}$$

On the other hand, if the pitch of a screw is infinite, the unit screw is defined as:

$$\$ = (\,\mathbf{0}\ \ \mathbf{s}\,). \tag{6}$$

According to the above definition, the unit screw associated with a revolute joint is a screw of zero pith pointing along the joint axis. The unit screw associated with a prismatic joint is a screw of infinite pitch pointing in the direction of the joint axis.

The kinematic screw is often defined as:

$$\$ = (\,L\ \ M\ \ N\ \ P\ \ Q\ \ R\,) \tag{7}$$

where the first three components denote the angular velocity, the last three components denote the linear velocity of a point in the rigid body that is instantaneously coincident with the origin of the coordinate system.

Similarly, $\$^r$ is defined as:

$$\$^r = (\,L^r\ \ M^r\ \ N^r\ \ P^r\ \ Q^r\ \ R^r\,) \tag{8}$$

where the first three components denote the resultant force, the last three components denote the resultant moment about the origin of the coordinate system.

Two screws, \$ and $\$^r$, are called to be reciprocal if they satisfy the equation:

$$LP^r + MQ^r + NR^r + PL^r + QM^r + RN^r = 0. \tag{9}$$

Equation (9) is often rewritten in the form of equation (2).

Similarly, $\$^r$ can also be denoted as

$$\$^r = (\,\mathbf{s}\ \ \mathbf{s}_0 + h\mathbf{s}\,) \tag{10}$$

where $\mathbf{s} = (\,L^r\ \ M^r\ \ N^r\,)$ is a unit vector pointing in the direction of the screw axis, $\mathbf{s}_0 + h\mathbf{s} = \mathbf{r} \times \mathbf{s} + h\mathbf{s} = (\,P^r\ \ Q^r\ \ R^r\,)$. The motion constrained by $\$^r$ can be obtained:

$$\$_M = \left(\mathbf{s}\ \ \mathbf{s}_0 + \frac{\mathbf{s}}{h}\right). \tag{11}$$

According to the definition of the reciprocal screws, the reciprocal screws of kinematic screws are a set of general forces. The meaning of formula (9) is that the work done at any instant by the alien forces to a stable rigid body should always be zero.

Suppose a reciprocal screw acted on point A at a rigid body, which is shown in figure 1, has the form below:

$$\$_A^r = (\,\mathbf{s}\ \ \mathbf{s}_{A0} + h\mathbf{s}\,) \tag{12}$$

where $\mathbf{s}_{A0} = \mathbf{r}_A \times \mathbf{s}$.

We now investigate the equivalent expressions of a screw at different points, points A and B for example. According to the definition of a screw, if the point is changed from point A

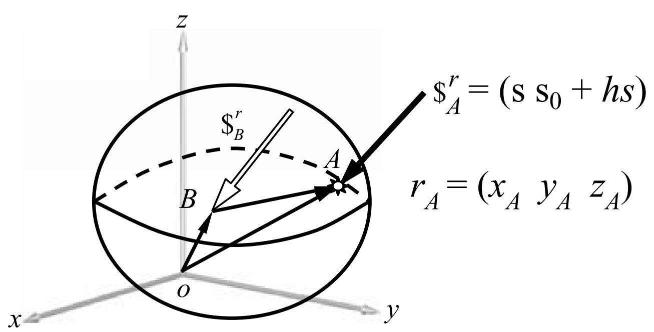

Fig. 1. Transformation of reciprocal screws.

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to point B, $\$^r_A$ denoted by equation (12) will be equivalently expressed at point B:

$$\$^r_A = (\mathbf{s}\ \ \mathbf{s}_{B0} - \mathbf{r}_{AB} \times \mathbf{s} + h\mathbf{s}) \tag{13}$$

where $\mathbf{s}_{B0} = \mathrm{r}_B \times \mathbf{s}$, $\mathbf{r}_{AB} = \mathbf{r}_B - \mathbf{r}_A$.

If B is superimposed with the origin of the coordinate system, equation (13) will be:

$$\$^r_A = (\mathbf{s}\ \mathbf{s}_{O0} + \mathbf{r}_A \times \mathbf{s} + h\mathbf{s}) \tag{14}$$

where $\mathbf{s}_{O0} = \mathbf{r}_O \times \mathbf{s} = \mathbf{0}$.

Next, we will investigate the stability of a rigid body under a set of reciprocal screws.

If the set of reciprocal screws do not meet at one point, we should transform them to one point, say, the equivalent expressions of the screws can be denoted at the origin of the coordinate system by equation (13). Assume that a set of reciprocal screws passing through one same point are denoted by C, where

$$C = \begin{bmatrix} \$_1 \\ \$_2 \\ \vdots \\ \$_m \end{bmatrix}.$$

Now, we can analyze the constraint spaces that can be spanned by these reciprocal screws.

According to algebra [34], the *singular value decomposition* (SVD) of an $m \times n$ matrix C has the following form:

$$C = U\Sigma V^T \tag{15}$$

where U denotes an $m \times m$ orthogonal matrix, V denotes an $n \times n$ orthogonal matrix, Σ denotes an $m \times n$ diagonal matrix, with

$$\sigma_{ij} = \begin{cases} 0 & i \neq j \\ \sigma_i \geq 0 & i = j. \end{cases} \tag{16}$$

The diagonal entries σ_i are called the singular values of C and are usually ordered so that $\sigma_i \geq \sigma_{i+1}$, $i = 1, 2, \ldots, n-1$. The columns u_i of U and v_i of V are the corresponding left and right singular vectors.

If $C = U\Sigma V^T$, then the columns of U corresponding to nonzero singular values form an orthonomal basis of $span\{C\}$, and the remaining columns of U form an orthonomal basis of its orthogonal complement, denoted by $span\{C\}^{\perp}$. Similarly, the columns of V corresponding to zero singular values form an orthonomal basis for the null space of C, $\{x \in \Re^n\colon Cx = 0\}$, and the remaining columns of V form an orthonomal basis for the orthogonal complement of the null space.

The set of reciprocal screws applied to the manipulator have the following form:

$$\$^r_M = \begin{bmatrix} L_1 & M_1 & N_1 & P_1 & Q_1 & R_1 \\ L_2 & M_2 & N_2 & P_2 & Q_2 & R_2 \\ & & & \vdots & & \\ L_m & M_m & N_m & P_m & Q_m & R_m \end{bmatrix}_{m\times 6}, (m = 1, 2, \cdots). \tag{17}$$

Therefore, $\$^r_M$ is an $m \times 6$ matrix and V should be a 6×6 orthonomal matrix. For $\$^r_M$ is a set of reciprocal screws (constraints) applied to the manipulator, therefore, the rows of V^T corresponding to zero singular values form an orthonomal basis of the null space of $\$^r_M$, and the remaining rows of V^T form an orthonomal basis of the orthogonal complement of the null space, which is a basis of the constraint space, $\$^r_M$.

Therefore, the free mobility space of the manipulator can be directly obtained from the rows of V^T corresponding to zero singular values with equation (11).

III. EXAMPLE AND DISCUSSION

We now analyze the DoF and singularity of the parallel manipulator shown in figure 2, which is made up of 3-PUU (1 Prismatic Joint and 2 Universal Joints) kinematic chains.

Firstly, we will create an absolute coordinates $oxyz$ as figure 2 shows. The plane xoy is perpendicular to the three vertical guides and x-axis is parallel to the midline of triangle $B_1B_2B_3$. Assuming that the radius of the circumcircle of triangle $B_1B_2B_3$ is R, the coordinates of the three sliders are as follows:

$$P_1(R\quad 0\quad z_1),\quad P_2\left(-\frac{1}{2}R\quad -\frac{\sqrt{3}}{2}R\quad z_2\right),$$
$$P_3\left(-\frac{1}{2}R\quad \frac{\sqrt{3}}{2}R\quad z_3\right).$$

Assume that the geometric center of manipulator $M_1M_2M_3$ is C, the local coordinate system $o_cx_cy_cz_c$ is shown in figure 2, whose origin is superimposed with point C. The three vertexes of the manipulator are denoted by M_i, ($i = 1$, 2, 3), and the sliders are denoted by P_i ($i = 1, 2, 3$).

Because all the limbs connecting the manipulator with the base are identical, we only need to select one of them to study the reciprocal screws of the limbs, for example, we select limb P_2M_2 as the analyzing object shown in figure 3.

The Plücker coordinates of slider P_2 are:

$$\$_1 = (0\quad 0\quad 0\quad 0\quad 0\quad 1).$$

The universal joint P_2 can be decomposed as two orthogonal revolute pairs.

The individual Plücker coordinates are:

$$\$_2 = (0\quad -\cos\alpha_2\quad \sin\alpha_2\quad 0\quad 0\quad 0),$$
$$\$_3 = (1\quad 0\quad 0\quad 0\quad 0\quad 0).$$

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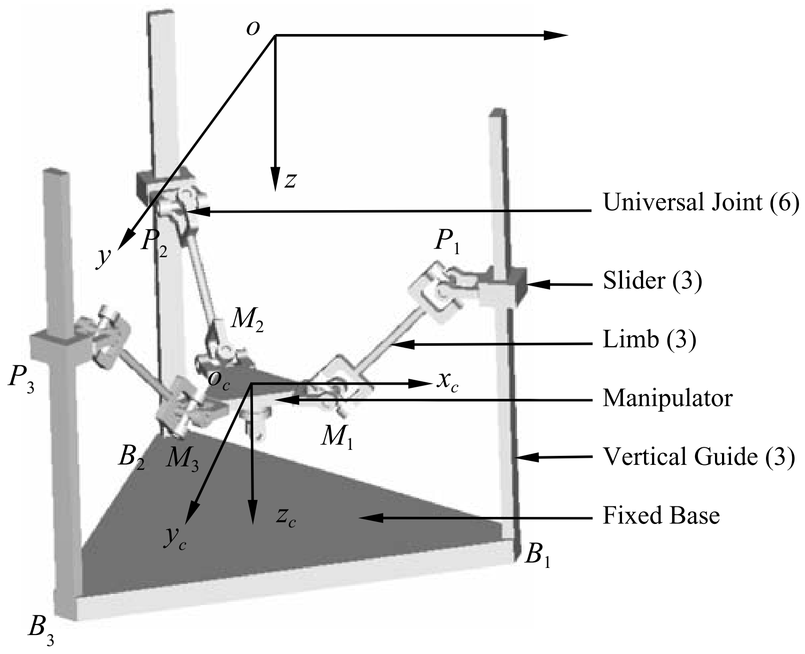

Fig. 2. A spatial parallel mechanism with 3-PUU.

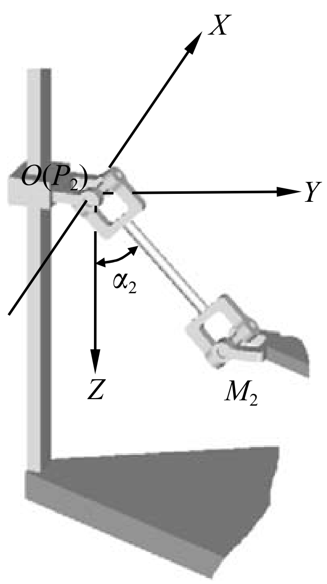

Fig. 3. The coordinate system for limb P_2M_2.

The same is true for universal joint M_2. The coordinates of M_2 are:

$$M_2(0 \quad l\sin\alpha_2 \quad l\cos\alpha_2).$$

The Plücker coordinates can be denoted as:

$$\$_4 = \left(\$_4^1 \quad \$_4^0\right), \quad \$_5 = \left(\$_5^1 \quad \$_5^0\right).$$

where

$$\$_4^1 = (0 \quad -\cos\alpha_2 \quad \sin\alpha_2), \quad \$_5^1 = (1 \quad 0 \quad 0),$$
$$\$_4^0 = \mathbf{r}_{M_2} \times \$_4^1 = (l \quad 0 \quad 0),$$
$$\$_5^0 = \mathbf{r}_{M_2} \times \$_5^1 = (0 \quad l\cos\alpha_2 \quad -l\sin\alpha_2).$$

$$\therefore \ \$_4 = (0 \quad -\cos\alpha_2 \quad \sin\alpha_2 \quad l \quad 0 \quad 0),$$
$$\$_5 = (1 \quad 0 \quad 0 \quad 0 \quad l\cos\alpha_2 \quad -l\sin\alpha_2).$$

$\therefore$ Therefore, the kinematic screws of the limb P_2M_2 can be expressed as:

$$\$_{P_1B_1} = \begin{bmatrix} \$_1 \\ \$_2 \\ \$_3 \\ \$_4 \\ \$_5 \end{bmatrix}. \tag{18}$$

The reciprocal screws of branch P_2B_2 can be gained:

$$\$^r_{P_2B_2} = (0 \quad 0 \quad 0 \quad 0 \quad \sin\alpha_2 \quad \cos\alpha_2). \tag{19}$$

According to the physical meaning of the reciprocal screws of kinematic screws, $\$^r_{P_2B_2}$ denotes a pure moment that is perpendicular to the universal joint plane.

The same are true for the rest two limbs of the manipulator. So, the reciprocal screws exerted to the manipulator are 3 pure moments of couples shown in figure 4.

According to equation (13), we can transform the 3 pure moments of couples to the origin of the absolute coordinate system shown in figure 4. If we presume the angle from the normal vector of the universal joint plane of the *ith*, ($i = 1$, 2, 3) limb to *xoy* plane is denoted by β_i and the angle from the projection line of the normal vector to x-axis is denoted by θ_i, the reciprocal screws can be rewritten as follows:

$$\$^{r_n}_{P_iB_i} = (0 \quad 0 \quad 0 \quad \cos\beta_i\cos\theta_i \quad \cos\beta_i\sin\theta_i \quad \sin\beta_i),$$
$$(i = 1, 2, 3). \tag{20}$$

Considering the geometric characteristics of the mechanism shown in figure 2, we know $0 \leq \beta_i \leq \frac{\pi}{2}$ and

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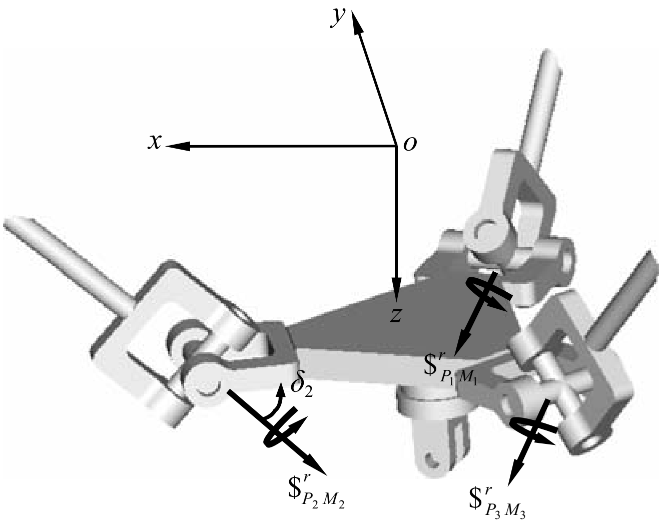

Fig. 4. Reciprocal screws of the manipulator.

can obtain:

$$\theta_i = \frac{2(i-1)\pi}{3}, \quad (i = 1, 2, 3) \tag{21}$$

Therefore, (20) can be simplified:

$$\$^r = \begin{bmatrix} \$^{r_n}_{P_1B_1} \\ \$^{r_n}_{P_2B_2} \\ \$^{r_n}_{P_3B_3} \end{bmatrix} \tag{22}$$

where

$$\$^{r_n}_{P_1B_1} = (0 \quad 0 \quad 0 \quad \cos\beta_1 \quad 0 \quad \sin\beta_1),$$

$$\$^{r_n}_{P_2B_2} = \left(0 \quad 0 \quad 0 \quad -\frac{1}{2}\cos\beta_2 \quad \frac{\sqrt{3}}{2}\cos\beta_2 \quad \sin\beta_2\right),$$

$$\$^{r_n}_{P_3B_3} = \left(0 \quad 0 \quad 0 \quad -\frac{1}{2}\cos\beta_3 \quad -\frac{\sqrt{3}}{2}\cos\beta_3 \quad \sin\beta_3\right).$$

Assume

$$K = \begin{bmatrix} \cos\beta_1 & 0 & \sin\beta_1 \\ -\frac{1}{2}\cos\beta_2 & \frac{\sqrt{3}}{2}\cos\beta_2 & \sin\beta_2 \\ -\frac{1}{2}\cos\beta_3 & -\frac{\sqrt{3}}{2}\cos\beta_3 & \sin\beta_3 \end{bmatrix} \tag{23}$$

and let

$$|K| = 0 \tag{24}$$

which yields

$$\frac{\sqrt{3}}{2}(\sin\beta_1\cos\beta_2\cos\beta_3 + \sin\beta_2\cos\beta_1\cos\beta_3 + \sin\beta_3\cos\beta_1\cos\beta_2) = 0 \tag{25}$$

The solutions of equation (25) are the singularities of the mechanism. We will discuss the solutions in the following two cases:

1 When $\beta_i \neq \frac{\pi}{2}$, $(i = 1, 2, 3)$, equation (25) can be simplified as:

$$\tan\beta_1 + \tan\beta_2 + \tan\beta_3 = 0. \tag{26}$$

Because $0 \leq \beta_i \leq \frac{\pi}{2}$, $(i = 1, 2, 3)$, the solutions of (26) are:

$$\beta_i = 0, (i = 1, 2, 3). \tag{27}$$

2 When one of β_1, β_2 and β_3 equals $\frac{\pi}{2}$, equation (25) can be simplified. To simplify the discussion, we might as well assume $\beta_1 = \frac{\pi}{2}$. As a result, equation (25) will be degenerated to:

$$\cos\beta_2\cos\beta_3 = 0. \tag{28}$$

The solutions of (28) are:

$$\beta_2 = \frac{\pi}{2} \quad \text{or} \quad \beta_3 = \frac{\pi}{2}.$$

Considering the circulate symmetry of β_i, $i = 1, 2, 3$, we can obtain the solutions for this case are that any two of the following three equations hold:

$$\beta_1 = \frac{\pi}{2}, \quad \beta_2 = \frac{\pi}{2} \quad \text{and} \quad \beta_3 = \frac{\pi}{2}.$$

Therefore, at ordinary positions, the rank of $\$^r$ is:

$$R(\$^r) = 3. \tag{29}$$

When $\beta_i = 0, (i = 1, 2, 3)$, equation (22) has the form below:

$$\$^r = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{bmatrix}. \tag{30}$$

The singular value decomposition can be denoted as:

$$\$^r = U\sum V^T$$

where

$$U = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \end{bmatrix} \tag{31}$$

$$\sum = \begin{bmatrix} \frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{32}$$

$$V^T = \begin{bmatrix} \$^r_1 \\ \$^r_2 \\ \$^r_3 \\ \$^r_4 \\ \$^r_5 \\ \$^r_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{33}$$

Therefore, the last four rows of V^T corresponding to zero singular values form an orthonomal basis for the null space

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of $\$^r_M$ denoted by $N(\$^r_M)$, and the remaining first two rows of V^T form the constraint space of $\$^r_M$, denoted by $C(\$^r_M)$, that is,

$$C\left(\$^r_M\right) = \begin{bmatrix} \$^r_1 \\ \$^r_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (34)$$

where $\$^r_1$ denotes a toque about x-axis in the absolute coordinate system shown in figure 4, $\$^r_2$ denotes a toque about y-axis in the absolute coordinate system shown in figure 4.

$$d = \dim C\left(\$^r_M\right) = 2 \quad (35)$$

$$N\left(\$^r_M\right) = \begin{bmatrix} \$^r_3 \\ \$^r_4 \\ \$^r_5 \\ \$^r_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (36)$$

The free mobility space, denoted by S_F, can be obtained from $N(\$^r_M)$ according to equation (11):

$$S_F = \begin{bmatrix} S^1_F \\ S^2_F \\ S^3_F \\ S^4_F \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \quad (37)$$

According to equation (37), the instantaneous DoF of the manipulator at this configuration is:

$M = \dim S_F = 6 - d = 6 - 2 = 4$ (One Rotational DoF about z-axis denoted by S^4_F + Three Translational DoFs denoted by S^1_F, S^2_F and S^3_F).

Therefore, when $\beta_i = 0$, ($i = 1, 2, 3$), the manipulator and the three sliders keep coplanar, and the mechanism instantaneously becomes a planar $3 - P_zRR$ parallel mechanism shown in figure 5, which is one singular case of the manipulator and is impossible for the spatial parallel manipulator in reality.

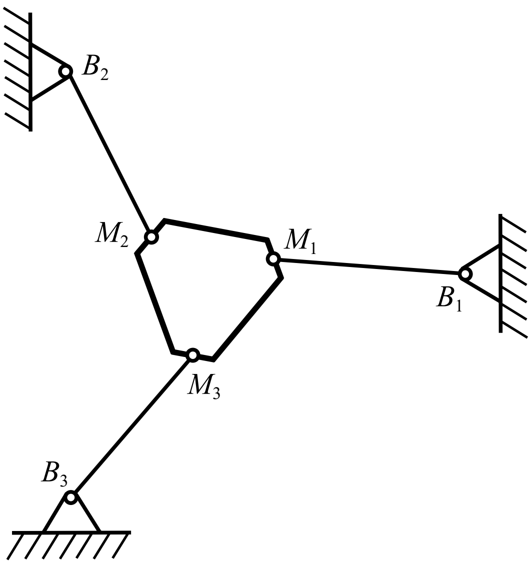

Fig. 5. One singular case for the manipulator when $\beta_i = 0$, $i = 1, 2, 3$.

Similarly, when $\beta_i = \frac{\pi}{2}$, ($i = 1, 2, 3$), the three limbs are parallel to each other and are perpendicular to the manipulator synchronously, which is another singular case and can be avoided by adjusting the structure parameters of the mechanism.

When $0 < \beta_i < \frac{\pi}{2}$, ($i = 1, 2, 3$), $\dim C(\$^r_M) = 3$ and the reciprocal screws of the three limbs can prevent the manipulator from rolling about the x, y and z axes. The mobility spaces of the manipulator can be similarly obtained:

$$S_F = \begin{bmatrix} S^1_F \\ S^2_F \\ S^3_F \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (38)$$

Therefore, the DoF of the manipulator is:

$M = \dim S_F = 3$ (Three Translational DoFs).

where S^1_F denotes one translational motion along x-axis, S^2_F denotes one translational motion along y-axis, S^3_F denotes one translational motion along z-axis, These three free motions the end-effector can make are depicted fully by equation (38).

IV. CONCLUSION

Through singular value decomposition of reciprocal screws, this paper addresses the *mobility space*, which defines the *type, number* and *direction* characteristics of the independent motions that the manipulator should execute. We firstly investigate the reciprocal screws applied to the end-effector and obtain the spaces spanned by the reciprocal screws of the kinematic chains via SVD, and then the free mobility spaces of the end-effector and its DoF. With this method, we can both obtain the numeral properties of the DoF of the end-effectors of spatial parallel manipulators and find their type and direction properties, which will be widely used to design and synthesize new parallel manipulators. Application example demonstrates that this methodology can be utilized to exploit the all-sided mobility characteristics of parallel manipulators, including the singularity and DoF.

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