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On the dimension of points which escape to infinity at given rate under exponential iteration

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Abstract. We prove a number of results concerning the Hausdorff and packing dimension of sets of points which escape (at least in average) to infinity at a given rate under non-autonomous iteration of exponential maps. In particular, we generalize the results proved by Sixsmith in 2016 and answer his question on annular itineraries for exponential maps.

Key words: exponential map, iteration, Hausdorff dimension, escaping set
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1. *Introduction*

In this paper we study the iteration of *exponential maps*

$$E_\lambda(z) = \lambda e^z, \quad z \in \mathbb{C}, \ \lambda \in \mathbb{C} \setminus \{0\}$$

and, more generally, the *non-autonomous exponential iteration*

$$\cdots \circ E_{\lambda_n} \circ \cdots \circ E_{\lambda_1},$$

where $\lambda_1, \lambda_2, \ldots \in \mathbb{C} \setminus \{0\}$. We study the dimension of sets of points $z \in \mathbb{C}$ which escape to infinity (at least in average) at a prescribed speed, meaning that $a_n \le |E_{\lambda_n} \circ \cdots \circ E_{\lambda_1}(z)| \le b_n$ for given sequences a_n, b_n.

For a transcendental entire map $f : \mathbb{C} \to \mathbb{C}$ the *escaping set* $I(f)$ is defined as

$$I(f) = \{z \in \mathbb{C} : |f^n(z)| \to \infty \text{ as } n \to \infty\},$$

while the *Julia set* $J(f)$ is the set of points $z \in \mathbb{C}$, where the iterates f^n do not form a normal family in any neighbourhood of z. There is a close relationship between the Julia set and escaping set—the set $J(f)$ is equal to the boundary of $I(f)$, as proved by Eremenko in [Ere89]. Furthermore, Eremenko and Lyubich showed in [EL92] that for functions f in the class

$$\mathcal{B} = \{\text{transcendental entire maps with a bounded set of critical and asymptotic values}\},$$

in particular for exponential maps, the escaping set is contained in the Julia set, so $J(f) = \overline{I(f)}$.

The dimension of the Julia sets of transcendental entire functions was first considered by McMullen in [McM87], who proved that all Julia sets of exponential maps have Hausdorff dimension ($\dim_H$) equal to 2. Since then, the question of the size of the Julia and escaping sets and their dynamically defined subsets has attracted a lot of attention (see the references mentioned in this section).

In fact, in [McM87] it was shown that $\dim_H(A(E_\lambda)) = 2$, where

$$A(f) = \{z \in I(f) : |f^{n+l}(z)| \geq M_f^n(R), n \in \mathbb{N}, \text{ for some } l \geq 0\}$$

is the *fast escaping set* of f, introduced by Bergweiler and Hinkkanen in [BH99] and then studied by Rippon and Stallard in [RS12]. Here $R > 0$ is a large fixed number, $M_f(r) = \max_{|z|=r} |f(z)|$ for $r > 0$ and M_f^n denotes the nth iterate of $M_f(\cdot)$. In fact, results by Bergweiler, Karpińska and Stallard [BKS09] and Rippon and Stallard [RS14] imply that $\dim_H A(f) = 2$ for all transcendental entire $f \in \mathcal{B}$ of finite order or 'not too large' infinite order. It is then a natural question to determine the dimension of subsets of $J(f) \cap I(f)$ consisting of points escaping to infinity at a slower rate, or other dynamically defined subsets of $J(f) \cap I(f) \setminus A(f)$. A number of such sets, including *slow escaping set*

$$L(f) = \left\{z \in I(f) : \limsup_{n\to\infty} \frac{1}{n} \log |f^n(z)| < \infty\right\}$$

and *moderately slow escaping set*

$$M(f) = \left\{z \in I(f) : \limsup_{n\to\infty} \frac{1}{n} \log\log |f^n(z)| < \infty\right\},$$

have been defined and studied in recent years (see e.g. [RS14, RS11]).

Remark 1.1. (Topological structure) It is well known (see e.g. [DK84, DT86, AO93, SZ03]) that escaping sets of exponential maps contain disjoint hairs (simple curves converging to ∞ with some special properties). For exponential maps with an attracting fixed point and, more generally, for maps of finite order from the class $\mathcal{B}$ with a unique Fatou component, the Julia set is the union of hairs together with their endpoints (see [Kar99b, Bar07, RRRS11]). In [RRS10], Rempe, Rippon and Stallard showed that for all transcendental entire $f \in \mathcal{B}$ of finite order, the hairs without endpoints are

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contained in $A(f)$. Therefore, for exponential maps with an attracting fixed point, the set $I(E_\lambda) \setminus A(E_\lambda)$ is contained in the union of endpoints of the hairs.

Remark 1.2. (Points with bounded trajectories) Let $J_{bd}(f)$ denote the set of points in the Julia set of f with bounded trajectories. In [**Kar99a**] it is proved that the Hausdorff dimension of $J_{bd}(E_\lambda)$ is larger than 1. Furthermore, in [**UZ03**] it is shown that $\dim_H(J(E_\lambda) \setminus I(E_\lambda)) \in (1, 2)$ for all hyperbolic exponential maps E_λ. More generally, $\dim_H(J_{bd}(f)) > 1$ for every transcendental entire map in the class $\mathcal{B}$ (see [**BKZ09**]) and $\dim_H(J(f) \setminus (I(f) \cup J_{bd}(f)) > 1$ for every transcendental entire map f in the class $\mathcal{B}$ (see [**OS16**]).

To conduct a refined analysis of the sets of points with given escape rate, for a transcendental entire map f and sequences $\underline{a} = (a_n)_{n=1}^\infty$, $\underline{b} = (b_n)_{n=1}^\infty$ with $0 < a_n \le b_n$, let

$$I_{\underline{a}}^{\underline{b}}(f) = \{z \in \mathbb{C} : a_n \le |f^n(z)| \le b_n \text{ for every sufficiently large } n \in \mathbb{N}\},$$

$$I^{\underline{b}}(f) = \{z \in \mathbb{C} : |f^n(z)| \le b_n \text{ for every sufficiently large } n \in \mathbb{N}\}.$$

To guarantee that the sets $I_{\underline{a}}^{\underline{b}}(f)$ are not empty, one usually assumes that the sequence $\underline{a}$ is *admissible*, which roughly means $a_{n+1} < M_f(a_n)$ (with a precise definition depending on the context).

Surprisingly, the natural question of the dimension of the sets $I_{\underline{a}}^{\underline{b}}(f)$ has not been answered completely, even for the well-known exponential family. Let us summarize what is known about the size of the sets $I_{\underline{a}}^{\underline{b}}(E_\lambda)$ and, more generally, the sets $I_{\underline{a}}^{\underline{b}}(f)$ for $f \in \mathcal{B}$. In [**Rem06**] Rempe proved that $I_{\underline{a}}^{\underline{b}}(E_\lambda) \ne \emptyset$ for every admissible sequence $\underline{a} = (a_n)_{n=1}^\infty$ with $a_n \to \infty$ and $b_n = ca_n$, $c > 1$. This result was generalized by Rippon and Stallard in [**RS11**] to the case of arbitrary transcendental entire (or meromorphic) maps f. Moreover, they showed that if $b_n \to \infty$, then $I^{\underline{b}}(f) \cap I(f) \ne \emptyset$. In [**BP13**] Bergweiler and Peter proved that $\dim_H(I(f) \cap I^{\underline{b}}(f)) \ge 1$ for every transcendental entire map f in the class $\mathcal{B}$, provided $b_n \to \infty$.

In [**KU06**] Karpińska and Urbański, considering a related topic, studied the Hausdorff dimension of subsets of the escaping set for exponential maps consisting of points whose symbolic itineraries (describing the imaginary part of $E_\lambda^n(z)$) grow in modulus to infinity at a given rate. They found that the Hausdorff dimension of these sets could achieve any number in the interval $[1, 2]$. As noted in [**Six16**], the subsets of $I(E_\lambda)$ considered in [**KU06**] are contained in the fast escaping set $A(E_\lambda)$.

A motivation for our work was the paper [**Six16**] by Sixsmith, who proved several results on the dimension of the sets $I_{\underline{a}}^{\underline{b}}(E_\lambda)$. In particular, he showed that $\dim_H I_{\underline{a}}^{\underline{b}}(E_\lambda) = 1$ in the following cases:

(a) $a_n = c_1 R^n$, $b_n = c_2 R^n$ for $c_1, c_2 > 0$, $R > 1$;

(b) $a_n = n^{(\log^+)^p(n)}$, $b_n = R^n$ for $p \in \mathbb{N}$, $R > 1$, where $(\log^+)^p$ denotes the pth iterate of $\log^+ = \max(\log, 0)$;

(c) $a_n = \exp(n \log^{+p}(n))$, $b_n = \exp(e^{pn})$ for $p \in \mathbb{N}$;

(d) $\lim_{n\to\infty} a_n = \infty$, $a_{n+1} < R^{a_n^{(1/\log R)}}$, $\lim_{n\to\infty}(\log a_{n+1}/\log(a_1 \cdots a_n)) = 0$ and $b_n = Ra_n$ for large n, where $R > 1$ is a sufficiently large constant.

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Note that in cases (a) and (b) the sets $I_{\underline{a}}^{b}(E_\lambda)$ are contained in the slow escaping set $L(E_\lambda)$, while in cases (c) and (d) they are subsets of the moderately slow escaping set $M(E_\lambda)$.

In [**Six16**, Remark 2] the author posed a question: whether the condition in (d) could be weakened. In this paper we answer this question, extending the results described in (a)–(d) and proving a number of facts concerning the dimension of points with given escape rate. The results are presented in a more general setting of non-autonomous iteration

$$E_{\underline{\lambda}} = (E_{\lambda_n} \circ \cdots \circ E_{\lambda_1})_{n=1}^{\infty}$$

of exponential maps, with an arbitrary choice of $\lambda_n \in \mathbb{C} \setminus \{0\}$. Furthermore, the points under consideration are not necessarily escaping. Generally, we only assume that $(a_n)_{n=1}^{\infty}$ is admissible, $a_n > a$ for large a, $(a_1 \cdots a_n)^{1/n} \to \infty$ as $n \to \infty$, and $b_n \geq ca_n$ for $c > 1$.

Let us summarize the main results of the paper. The exact formulations are contained in § 2.

- In Theorem 2.1 we present a general condition which implies that the Hausdorff dimension of $I_{\underline{a}}^{b}(E_{\underline{\lambda}})$ is at most 1.
- In Theorem 2.5 and Corollary 2.6 we provide basic estimates for the Hausdorff and packing dimensions of $I_{\underline{a}}^{b}(E_{\underline{\lambda}})$ in terms of the growth of moduli of the annuli $\{z \in \mathbb{C} : \{a_n \leq |z| \leq b_n\}$ compared to the mean geometric growth of the sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$.
- Corollary 2.8 provides conditions under which the dimensions achieve extremal values of 1 or 2.
- In Theorem 2.11, generalizing the results of [**Six16**], we show that the sets $I_{\underline{a}}^{b}(E_{\underline{\lambda}})$ with moderately slow escape rate have Hausdorff dimension 1.
- Theorem 2.14 shows the same for the sets of points with any given exact growth rate.
- In Theorem 2.15 we provide exact formulas for the Hausdorff and packing dimensions of $I_{\underline{a}}^{b}(E_{\underline{\lambda}})$ in the case where $\sup_n |\lambda_n| < \infty$, $(\log b_n / \log a_n) \to 1$.
- In Theorem 2.16 we show that the packing dimension of $I_{\underline{a}}^{b}(E_{\underline{\lambda}})$ can achieve any value in the interval $[1, 2]$, with the Hausdorff dimension being equal to 1.

At the end of §2 we pose a question, which we find interesting: whether there exists a set $I_{\underline{a}}^{b}(E_{\underline{\lambda}})$ with Hausdorff dimension between 1 and 2.

In [**Six16**], the result (d) mentioned above was described in the language of *annular itineraries*, which are the sequences of non-negative integers s_n defined by the condition $f^n(z) \in \mathcal{A}_{s_n}$ for a partition of the plane by a sequence of concentric annuli $\mathcal{A}_s$, $s \geq 0$, with radii growing to infinity as $s \to \infty$. In [**RS15**] Rippon and Stallard proved that for all transcendental entire maps f there exist escaping points with any given admissible annular itinerary. In §3 of this paper we also take up this approach, determining the dimension of sets of points sharing a given annular itinerary under non-autonomous exponential iteration for various sequences of annuli $\mathcal{A}_s$ (Theorems 3.1 and 3.2).

2. *Results*

2.1. *Preliminaries.*

We consider a non-autonomous exponential iteration

$$E_{\underline{\lambda}} = (E_{\lambda_n} \circ \cdots \circ E_{\lambda_1})_{n=1}^{\infty}$$

for $\underline{\lambda} = (\lambda_n)_{n=1}^{\infty}$, where $\lambda_n \in \mathbb{C} \setminus \{0\}$. We extend the definition of the sets $I_{\underline{a}}^{\underline{b}}(f)$ to the non-autonomous setup, setting

$$I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = \{z \in \mathbb{C} : a_n \leq |E_{\lambda_n} \circ \cdots \circ E_{\lambda_1}(z)| \leq b_n \text{ for every sufficiently large } n \in \mathbb{N}\},$$

for $\underline{a} = (a_n)_{n=1}^{\infty}$, $\underline{b} = (b_n)_{n=1}^{\infty}$ with $0 < a_n < b_n$. Note that, in general, the sequences a_n and b_n need not be increasing and need not tend to infinity. We denote by Δ_n the (suitably normalized) modulus of the annulus $\{z \in \mathbb{C} : a_n \leq |z| \leq b_n\}$, that is,

$$\Delta_n = \log \frac{b_n}{a_n}.$$

Our results concern the Hausdorff and packing dimensions (see e.g. [**Fal03**, **Mat95**] for definitions), which are denoted, respectively, by $\dim_H$ and $\dim_P$. Recall that

$$\dim_H \leq \dim_P.$$

2.2. *General estimates.* Our first result provides an upper estimate of the Hausdorff dimension of the sets $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$. Geometrically, it states that $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$ can be larger than 1 only if the moduli Δ_n grow quickly enough compared with the mean geometric growth of the sequence $\underline{a}$. The proof is contained in §5.

THEOREM 2.1. *Let* $\underline{a} = (a_n)_{n=1}^{\infty}$, $\underline{b} = (b_n)_{n=1}^{\infty}$ *be such that* $\inf_{n\in\mathbb{N}} a_n > 0$ *and*

$$\liminf_{n\to\infty} \left(\frac{\Delta_{n+1}}{a_1 \cdots a_n}\right)^{(1/n)} = 0. \tag{1}$$

Then $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \leq 1$.

Remark 2.2. It is straightforward to check that (1) holds provided

$$\lim_{n\to\infty} (a_1 \cdots a_n)^{(1/n)} = \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{\log \Delta_{n+1}}{\log(a_1 \cdots a_n)} < 1,$$

or

$$\limsup_{n\to\infty} (a_1 \cdots a_n)^{(1/n)} = \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log \Delta_{n+1}}{\log(a_1 \cdots a_n)} < 1.$$

Before formulating the next results, we introduce the notion of admissibility used in our context. Recall that this condition, bounding the growth of the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, is introduced to ensure that the sets $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$ under consideration are non-empty.

Definition 2.3. We say that sequences $\underline{a} = (a_n)_{n=1}^{\infty}$, $\underline{b} = (b_n)_{n=1}^{\infty}$ are *admissible* if, for sufficiently large n, we have

$$a_{n+1} \leq |\lambda_{n+1}| e^{q a_n}, \quad b_{n+1} \geq |\lambda_{n+1}| e^{-q a_n}$$

for a constant $0 < q < 1$.

From now on, our general assumptions will be the following.

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Assumptions.

(a) The sequences $\underline{a} = (a_n)_{n=1}^{\infty}$, $\underline{b} = (b_n)_{n=1}^{\infty}$ are admissible.

(b) $(a_1 \cdots a_n)^{1/n} \to \infty$ as $n \to \infty$.

(c) $\Delta_n > \Delta > 0$ for $n \in \mathbb{N}$.

(d) $\liminf_{n\to\infty} a_n > a$, where a is a sufficiently large constant, depending on Δ and q from Definition 2.3.

Remark 2.4. Note that if $a_n \to \infty$ and the sequence $|\lambda_n|$ is bounded away from 0 and ∞, then the assumptions reduce to $\Delta_n > \Delta > 0$ and $a_{n+1} \le e^{qa_n}$, $0 < q < 1$, for large n.

The next result provides general lower and upper estimates of the Hausdorff and packing dimensions of the sets $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$ in terms of the growth of the moduli Δ_n compared with the growth of the sequences $\underline{a}$ and $\underline{b}$. The proof is contained in §§ 6–8.

THEOREM 2.5. *Suppose that assumptions (a)–(d) are satisfied. Then*

$$1 + \inf_x \liminf_{n\to\infty} \phi_n(x) \le \dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \le 1 + \sup_x \liminf_{n\to\infty} \phi_n(x),$$

$$1 + \inf_x \limsup_{n\to\infty} \psi_n(x) \le \dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \le 1 + \sup_x \limsup_{n\to\infty} \psi_n(x),$$

where $x = (x_1, x_2, \ldots) \in [a_1, b_1] \times [a_2, b_2] \times \cdots$ *and*

$$\phi_n(x) = \frac{\log(\min(\Delta_2, x_1) \cdots \min(\Delta_n, x_{n-1}))}{\log(x_1 \cdots x_n) - \log \min(\Delta_{n+1}, x_n)},$$

$$\psi_n(x) = \frac{\log(\min(\Delta_2, x_1) \cdots \min(\Delta_{n+1}, x_n))}{\log(x_1 \cdots x_n)}.$$

Theorems 2.1 and 2.5 imply a number of corollaries, presented below. The first one shows, in particular, that the Hausdorff dimension of the considered sets $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$ is at least 1.

COROLLARY 2.6. *Under the assumptions of Theorem 2.5,*

$$1 \le \dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \le 1 + \liminf_{n\to\infty} \frac{\log(\Delta_1 \cdots \Delta_n)}{\log(a_1 \cdots a_{n-1}) + \log^+(a_n/\Delta_{n+1})},$$

$$1 \le \dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \le 1 + \limsup_{n\to\infty} \frac{\log(\Delta_1 \cdots \Delta_{n+1})}{\log(a_1 \cdots a_n)}.$$

If, in addition,

$$\sup_{n\in\mathbb{N}} \frac{\Delta_{n+1}}{a_n} < \infty, \tag{2}$$

then

$$\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1, \quad \dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \ge 1 + \limsup_{n\to\infty} \frac{\log(\Delta_1 \cdots \Delta_{n+1})}{\log(b_1 \cdots b_n)}.$$

Proof. Note first that by the assumptions, $\log(a_1 \cdots a_{n-1}) > 0$ and $\min(\Delta_n, a_{n-1}) \ge c$ for large n and some constant $c > 0$. Hence, the numerator in the expression for ϕ_n in

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Theorem 2.5 is larger than Cn for a constant $C \in \mathbb{R}$, while the denominator is not smaller than $\log(a_1 \cdots a_{n-1})$, which is positive. Thus,

$$\phi_n \geq -\frac{|C|n}{\log(a_1 \cdots a_{n-1})} \xrightarrow[n\to\infty]{} 0$$

since $(a_1 \cdots a_n)^{1/n} \to \infty$. Hence, $\liminf_{n\to\infty} \phi_n \geq 0$, so $\dim_H I_{\underline{a}}^{b}(E_{\underline{\lambda}}) \geq 1$. The remaining assertions follow from Theorem 2.1, Remark 2.2 and Theorem 2.5 in a straightforward way. □

Remark 2.7. If $\sup_{n\in\mathbb{N}} |\lambda_n| < \infty$, then condition (2) holds provided $\sup_{n\in\mathbb{N}}(\log b_n/\log a_n) < \infty$.

Proof. If $\sup_{n\in\mathbb{N}}(\log b_n/\log a_n) < \infty$, then $\Delta_{n+1} \leq c \log a_{n+1}$ for a constant $c > 0$. This, together with the admissibility, implies

$$\frac{\Delta_{n+1}}{a_n} \leq c\frac{\log a_{n+1}}{a_n} \leq c\bigg(q + \frac{\log|\lambda_{n+1}|}{a_n}\bigg) \leq c\bigg(q + \frac{\log^+ \sup_n |\lambda_n|}{a}\bigg)$$

for large n. □

The following fact provides conditions under which the Hausdorff and packing dimensions of $I_{\underline{a}}^{b}(E_{\underline{\lambda}})$ achieve extremal values 1 or 2. Note that assertion (d) is a refinement of the McMullen result from [**McM87**].

COROLLARY 2.8. *Under the assumptions of Theorem 2.5:*

(a) *if* $\limsup_{n\to\infty}(\log \Delta_{n+1}/\log a_n) \leq 0$, *then* $\dim_H I_{\underline{a}}^{b}(E_{\underline{\lambda}}) = \dim_P I_{\underline{a}}^{b}(E_{\underline{\lambda}}) = 1$;

(b) *if* $\liminf_{n\to\infty}(\log \Delta_{n+1}/\log a_n) < 1$, *then* $\dim_H I_{\underline{a}}^{b}(E_{\underline{\lambda}}) = 1$;

(c) *if* $\liminf_{n\to\infty}(\log \Delta_{n+1}/\log b_n) \geq 1$, *then* $\dim_P I_{\underline{a}}^{b}(E_{\underline{\lambda}}) = 2$;

(d) *if* $\liminf_{n\to\infty}(\log \Delta_{n+1}/\log b_n) > 1$, *then* $\dim_H I_{\underline{a}}^{b}(E_{\underline{\lambda}}) = \dim_P I_{\underline{a}}^{b}(E_{\underline{\lambda}}) = 2$.

Remark 2.9. Assertion (b) holds also under the weaker assumption $\liminf_{n\to\infty} (\log \Delta_{n+1}/\log(a_1 \cdots a_n)) < 1$, while (d) holds also under the weaker assumption $\inf_{n\in\mathbb{N}}(\Delta_{n+1}/b_n) > 0$.

Proof of Corollary 2.8 and Remark 2.9. To prove assertion (a), note that by assumption, for any $\varepsilon > 0$ there exists $n_0 > 0$ such that $\log \Delta_{n+1} < \varepsilon \log a_n$ for $n \geq n_0$, which gives

$$\frac{\log(\Delta_1 \cdots \Delta_{n+1})}{\log(a_1 \cdots a_n)} < \frac{\log(\Delta_1 \cdots \Delta_{n_0})}{\log(a_1 \cdots a_n)} + \varepsilon \xrightarrow[n\to\infty]{} \varepsilon$$

since $(a_1 \cdots a_n)^{1/n} \to \infty$. Hence, assertion (a) follows from Corollary 2.6. Assertion (b) under the weaker assumption from Remark 2.9 holds by Theorem 2.1, Remark 2.2 and Corollary 2.6. To show (c), take a small $\varepsilon > 0$ and note that by assumption, there exists $n_0 > 0$ such that

$$\log \Delta_{n+1} \geq (1 - \varepsilon) \log b_n$$

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for $n \geq n_0$; so, for $x_n \in [a_n, b_n]$, $n \geq n_0$, we have

$$\log(\min(\Delta_{n+1}, x_n)) \geq \min((1-\varepsilon)\log b_n, \log x_n) \geq (1-\varepsilon)\log x_n.$$

Hence, there exists a constant $C \in \mathbb{R}$ such that for ψ_n from Theorem 2.5,

$$\psi_n(x_1, x_2, \ldots) \geq \frac{C + (1-\varepsilon)(\log x_{n_0} + \cdots + \log x_n)}{\log x_1 + \cdots + \log x_n} \xrightarrow[n\to\infty]{} 1$$

since $x_n \geq a_n$ and $(a_1 \cdots a_n)^{1/n} \to \infty$. This implies $\limsup_{n\to\infty} \psi_n \geq 1$, so (c) holds by Theorem 2.5.

To show assertion (d) under the weaker assumption from Remark 2.9, note that if $\inf_{n\in\mathbb{N}} \Delta_{n+1}/b_n > 0$, then there exist $n_0, c > 0$ such that

$$\log \Delta_{n+1} \geq \log b_n + \log c$$

for $n \geq n_0$; so for $x_n \in [a_n, b_n]$ we have

$$\log(\min(\Delta_{n+1}, x_n)) \geq \min(\log b_n + \log c, \log x_n) \geq \log x_n - |\log c|.$$

Hence, for ϕ_n from Theorem 2.5 and a constant $C \in \mathbb{R}$,

$$\phi_n(x_1, x_2, \ldots) \geq \frac{C - |\log c| n + \log x_{n_0} + \cdots + \log x_{n-1}}{\log x_1 + \cdots + \log x_{n-1} - |\log c|} \xrightarrow[n\to\infty]{} 1,$$

as $x_n \geq a_n$ and $(a_1 \cdots a_n)^{1/n} \to \infty$. This gives $\limsup_{n\to\infty} \phi_n \geq 1$, and (d) holds by Theorem 2.5. Note that $\inf_{n\in\mathbb{N}}(\log \Delta_{n+1}/\log b_n) > 0$ is indeed a weaker assumption, since the condition $\liminf_{n\to\infty}(\log \Delta_{n+1}/\log b_n) > 1$ implies

$$\frac{\Delta_{n+1}}{b_n} > b_n^c \geq a_n^c \geq a^c$$

for large n and a constant $c > 0$. □

2.3. *Moderately slow escaping points.* We extend the notion of the moderately slow escaping set to the non-autonomous setting.

Definition 2.10. Let

$$M(E_{\underline{\lambda}}) = \left\{ z \in I(E_{\underline{\lambda}}) : \limsup_{n\to\infty} \frac{1}{n} \log\log |E_{\lambda_n} \circ \cdots \circ E_{\lambda_1}(z)| < \infty \right\}$$

be the *moderately slow escaping set* of E_λ.

The following result shows that if the considered set $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$ is contained in the moderately slow escaping set, then its Hausdorff dimension is equal to 1. In particular, this generalizes results (a)–(d) from [**Six16**] mentioned in the introduction, since the sets considered in [**Six16**] are contained in the moderately slow escaping set.

THEOREM 2.11. *Under the assumptions of Theorem 2.5, if* $\inf_{n\in\mathbb{N}}(\log^+ b_n)^{(1/n)} < \infty$, *then*

$$\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1.$$

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In particular, this holds if $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$ *is contained in the moderately slow escaping set* $M(E_{\underline{\lambda}})$ *and the assumptions of Theorem 2.5 are satisfied.*

Remark 2.12. By Theorem 2.1, the fact $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \le 1$ holds under weaker assumptions $\inf_{n\in\mathbb{N}} a_n > 0$, $\lim_{n\to\infty}(a_1 \cdots a_n)^{(1/n)} = \infty$ and $\inf_{n\in\mathbb{N}}(\log^+ b_n)^{(1/n)} < \infty$.

Proof of Theorem 2.11. The first assertion follows from Theorem 2.1 and Corollary 2.6 in a straightforward way. To show the second one, note that if $z \in I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \subset M(E_{\underline{\lambda}})$, then $a_n \le |z| \le \min(b_n, e^{e^{cn}})$ for large n and a constant $c > 1$. Hence, if $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \subset M(E_{\underline{\lambda}})$, then $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$ is contained in a countable union of sets of the form $I_{\underline{a}}^{\underline{b}'}(E_{\underline{\lambda}})$, where

$$\underline{b}' = (b_n')_{n=1}^\infty \quad \text{for } b_n' = \min(b_n, e^{e^{cn}}),$$

for some $c > 1$. Since $(\log^+ b_n')^{1/n} \le e^c$, Theorem 2.1 implies $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \le 1$. The opposite inequality follows from Theorem 2.5. □

2.4. *Points with exact growth rate.*

Our results enable us to determine the Hausdorff and packing dimensions of the set of points which share the same growth rate under iteration of $E_{\underline{\lambda}}$.

Definition 2.13. We say that the iterations of a point $z \in \mathbb{C}$ under $E_{\underline{\lambda}}$ *have growth rate* $\underline{a}$ for a sequence $\underline{a} = (a_n)_{n=1}^\infty$ if $a_n/c \le |E_{\lambda_n} \circ \cdots \circ E_{\lambda_1}(z)| \le c a_n$ for large n and some constant $c > 1$, that is, $z \in I_{\underline{a}/c}^{c\underline{a}}(E_{\underline{\lambda}})$.

Corollary 2.8 immediately implies the following.

THEOREM 2.14. *If* $a_{n+1} \le |\lambda_{n+1}| e^{q a_n}$ *for large* n *and some constant* $0 < q < 1$, $(a_1 \cdots a_n)^{1/n} \to \infty$ *as* $n \to \infty$ *and* $\liminf_{n\to\infty} a_n > a$*, where* a *is a sufficiently large constant depending on* q*, then the set of points with growth rate* $\underline{a}$ *has Hausdorff dimension* 1*. If* $a_{n+1} \le |\lambda_{n+1}| e^{q a_n}$ *for large* n *and* $a_n \to \infty$ *as* $n \to \infty$*, then the set of points with growth rate* $\underline{a}$ *has Hausdorff and packing dimensions* 1.

2.5. *Precise dimension formulas.*

In the case $\sup_{n\in\mathbb{N}} |\lambda_n| < \infty$, $\lim_{n\to\infty}(\log b_n/\log a_n) = 1$, we can exactly determine the Hausdorff and packing dimensions of $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$.

THEOREM 2.15. *Under the assumptions of Theorem 2.5, if* $\sup_{n\in\mathbb{N}}|\lambda_n| < \infty$ *and* $\lim_{n\to\infty}(\log b_n/\log a_n) = 1$*, then*

$$\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1, \quad \dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1 + \limsup_{n\to\infty} \frac{\log(\Delta_1 \cdots \Delta_{n+1})}{\log(a_1 \cdots a_n)}.$$

Proof. If $\lim_{n\to\infty}(\log b_n/\log a_n) = 1$, then

$$\lim_{n\to\infty} \frac{\log(b_1 \cdots b_n)}{\log(a_1 \cdots a_n)} = 1$$

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by the Stolz–Cesàro theorem. Therefore, the theorem follows directly from Corollary 2.6 and Remark 2.7. □

The following result provides examples of sets $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$ with packing dimension equal to any given value in the interval [1, 2].

THEOREM 2.16. *For every* $D \in [1, 2]$ *and every sequence* $(\lambda_n)_{n=1}^{\infty}$ *with* $\lambda_n \in \mathbb{C} \setminus \{0\}$*, such that* $\sup_{n\in\mathbb{N}} |\lambda_n| < \infty$*, there exist admissible sequences* $\underline{a} = (a_n)_{n=1}^{\infty}$*,* $\underline{b} = (b_n)_{n=1}^{\infty}$ *with* $a_n \to \infty$*,* $\inf_{n\in\mathbb{N}} \Delta_n > 0$ *and* $\lim_{n\to\infty}(\log b_n/\log a_n) = 1$*, such that*

$$\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1, \quad \dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = D.$$

Theorem 2.16 is implied by the following corollary, which is a direct consequence of Theorem 2.15 and the Stolz–Cesàro theorem.

COROLLARY 2.17. *Under the assumptions of Theorem 2.5, if* $\sup_{n\in\mathbb{N}} |\lambda_n| < \infty$*,* $\lim_{n\to\infty}(\log b_n/\log a_n) = 1$ *and* $\lim_{n\to\infty}(\log \Delta_{n+1}/\log a_n) = d$ *for* $d \in [0, 1]$*, then* $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1$ *and* $\dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1 + d$*.*

The following example shows that the assumptions of Corollary 2.17 are actually satisfied for some sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, which proves Theorem 2.16.

Example 2.18. For any sequence $(\lambda_n)_{n=1}^{\infty}$ with $\sup_{n\in\mathbb{N}} |\lambda_n| < \infty$, $a_{n+1} = \exp(n a_n^d)$ for $d \in [0, 1)$ and $b_n = a_n^{1+(1/n)}$, then $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1$, $\dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1 + d$. If $a_{n+1} = \exp(n a_n^{(n-1)/n})$, $b_n = a_n^{1+(1/n)}$, then $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1$, $\dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 2$.

Proof. It is a direct calculation to check that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ satisfy the assumptions of Corollary 2.17. □

We end this section by stating a question, which we find interesting to answer.

Question. Does there exist a set $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$ with $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \in (1, 2)$?

3. *Annular itineraries*

Sets of the form $I_{\underline{a}}^{\underline{b}}(f)$ appear naturally in the study of *annular itineraries* $\underline{s}(z) = (s_n)_{n=0}^{\infty}$ of points $z \in \mathbb{C}$ under a map $f : \mathbb{C} \to \mathbb{C}$, defined by

$$f^n(z) \in \mathcal{A}_{s_n}, \quad n \geq 0, \quad \text{where } \mathcal{A}_s = \{z \in \mathbb{C} : R_s \leq |z| < R_{s+1}\}, \quad s \geq 0,$$

for some sequence $0 = R_0 < R_1 < R_2 < \cdots$, with $R_s \to \infty$ as $s \to \infty$. Such annular itineraries, for $R_s = M_f^{s-1}(R_1)$, were studied by Rippon and Stallard in [RS15]. In [Six16], Sixsmith, considering exponential maps, used the annuli defined by $R_s = R^s$ for a large $R > 1$. He proved that if $s_n \to \infty$, then the set of points sharing the itinerary $\underline{s}(z) = (s_n)_{n=0}^{\infty}$ has Hausdorff dimension at most 1, while the dimension is equal to 1 if, in addition, R is sufficiently large, $\underline{s}$ is slowly-growing, that is, $((s_{n+1})/(s_1 + \cdots + s_n)) \to 0$ and $\underline{s}$ is admissible, in the sense that $s_{n+1} < e^{s_n}$. Here we extend the results, answering a question from [Six16] and showing that the assumption of the slow growth can be omitted. We also analyse annular itineraries defined by another partition of the plane, given by

$R_s = R^{s^\kappa}$ for $\kappa > 1$. In this case one can find examples of the sets of points sharing the same itinerary, with packing dimension larger than 1.

We extend the notion of annular itineraries to the non-autonomous setup, setting

$$\underline{s}(z) = (s_n)_{n=0}^\infty \quad \text{where } E_{\lambda_n} \circ \cdots \circ E_{\lambda_1}(z) \in \mathcal{A}_{s_n}.$$

We assume $s_n > 0$ for $n \geq 0$. For given symbolic sequence $\underline{s} = (s_n)_{n=0}^\infty$, let

$$\mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = \{z \in \mathbb{C} : \underline{s}(z) = \underline{s}.\}$$

Note that

$$\mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \quad \text{for } a_n = R_{s_n},\ b_n = R_{s_n+1}.$$

We say that a sequence $\underline{s} = (s_n)_{n=0}^\infty$ is *admissible* if the sequences $\underline{a} = (a_n)_{n=1}^\infty$, $\underline{b} = (a_n)_{n=1}^\infty$ for $a_n = R_{s_n}$, $b_n = R_{s_n+1}$ are admissible.

3.1. *Case* $R_s = R^s$. Consider annular itineraries $\underline{s} = (s_n)_{n=0}^\infty$ of points under non-autonomous iteration $E_{\underline{\lambda}}$ with respect to the annuli

$$\mathcal{A}_s = \{z \in \mathbb{C} : R^s \leq |z| < R^{s+1}\},$$

for $s \geq 0$ and $R > 1$.

THEOREM 3.1. *The following statements hold.*

(a) *If* $\limsup_{n\to\infty}((s_1 + \cdots + s_n)/n) = \infty$, *then* $\dim_H \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) \leq 1$.

(b) *If* $\underline{s}$ *is admissible,* $\lim_{n\to\infty}((s_1 + \cdots + s_n)/n) = \infty$ *and* R *is sufficiently large, then*

$$\dim_H \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = \dim_P \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1.$$

Proof. As noted above, we have $\dim \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = \dim I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$ for

$$a_n = R^{s_n}, \quad b_n = R^{s_n+1}.$$

In particular, $a_n \geq R$, $a_1 \cdots a_n = R^{s_1+\cdots+s_n}$ and $\Delta_n = \log R$. Hence, the assertions follow immediately from Theorem 2.1 and Corollary 2.6. □

Note that the admissibility condition according to Definition 2.3 has the form

$$s_{n+1} \leq \frac{qR^{s_n} + \log|\lambda_{n+1}|}{\log R}$$

for $0 < q < 1$, and in the non-autonomous case it is satisfied provided $s_{n+1} \leq R^{qs_n}$, if R is sufficiently large.

3.2. *Case* $R_s = R^{s^\kappa}$. Consider now annular itineraries $\underline{s} = (s_n)_{n=0}^\infty$ with respect to the annuli

$$\mathcal{A}_s = \{z \in \mathbb{C} : R^{s^\kappa} \leq |z| < R^{(s+1)^\kappa}\},$$

for $s \geq 0$ and $R > 1$, $\kappa > 1$.

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THEOREM 3.2. *Suppose* $\sup_{n\in\mathbb{N}} |\lambda_n| < \infty$, $\underline{s}$ *is admissible and* R *is sufficiently large. Then the following statements hold.*

(a) *If* $\limsup\limits_{n\to\infty}((s_1^\kappa + \cdots + s_n^\kappa)/n) = \infty$, *then* $\dim_H \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) \le 1$.

(b) *If* $\lim\limits_{n\to\infty} s_n = \infty$, *then:*

$$\dim_H \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1;$$
$$\dim_P \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1 + \frac{\kappa - 1}{\log R} \limsup_{n\to\infty} \frac{\log s_{n+1}}{s_1^\kappa + \cdots + s_n^\kappa};$$
$$\dim_P \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) < 2 - \frac{1}{\kappa}.$$

Proof. In this case we have $\dim \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = \dim I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$ for

$$a_n = R^{s_n^\kappa}, \quad b_n = R^{(s_n+1)^\kappa}.$$

In particular, $a_n \ge R^\kappa$ and $a_1 \cdots a_n = R^{s_1^\kappa + \cdots + s_n^\kappa}$. By the assumption $\sup_{n\in\mathbb{N}} |\lambda_n| < \infty$, the admissibility condition is equivalent to

$$s_{n+1} \le \left(\frac{q}{\log R}\right)^{1/\kappa} R^{s_n^\kappa/\kappa} \tag{3}$$

for large n and a constant $0 < q < 1$. Moreover,

$$\frac{\log b_n}{\log a_n} = \left(1 + \frac{1}{s_n}\right)^\kappa,$$

and

$$\Delta_n = ((s_n + 1)^\kappa - s_n^\kappa) \log R \ge (\kappa s_n^{\kappa-1}) \log R \ge \kappa \log R.$$

By the mean value theorem,

$$(\kappa - 1) \log s_{n+1} - c_1 \le \log \Delta_{n+1} \le (\kappa - 1) \log s_{n+1} + c_1 \tag{4}$$

for a constant $c_1 > 0$ and, by (3),

$$\log s_{n+1} \le \frac{s_n^\kappa}{\kappa} \log R + c_2 \tag{5}$$

for a constant $c_2 > 0$. Furthermore, (4) and (5) imply

$$\limsup_{n\to\infty} \frac{\log \Delta_{n+1}}{\log(a_1 \cdots a_n)} \le \frac{\kappa - 1}{\kappa} \limsup_{n\to\infty} \frac{s_n^\kappa + (\kappa/(\kappa - 1))((c_1 + c_2(\kappa - 1))/\log R)}{s_1^\kappa + \cdots + s_n^\kappa} \le \frac{\kappa - 1}{\kappa} < 1,$$

which proves (a) by Theorem 2.1, since $\limsup_{n\to\infty}(a_1 \cdots a_n)^{1/n} = \infty$ by the assumptions.

The first assertion of (b) follows from (a) and Corollary 2.6. To prove the other ones, note that if $s_n \to \infty$, then $(\log b_n/\log a_n) \to 1$, so by Theorem 2.15 and (4),

$$\dim_P \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1 + \frac{\kappa - 1}{\log R} \limsup_{n\to\infty} \frac{\log s_1 + \cdots + \log s_{n+1}}{s_1^\kappa + \cdots + s_n^\kappa}.$$

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Since

$$\frac{\log s_1 + \cdots + \log s_n}{s_1^\kappa + \cdots + s_n^\kappa} \to 0,$$

we have

$$\dim_P \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1 + \frac{\kappa - 1}{\log R} \limsup_{n\to\infty} \frac{\log s_{n+1}}{s_1^\kappa + \cdots + s_n^\kappa},$$

and, by (5), $\dim_P \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) < 2 - (1/\kappa)$, which proves the second and third assertions of (b). □

Finally, we provide examples of sets $\mathcal{I}_{\underline{s}}(E_{\underline{\lambda}})$ with packing dimension larger than 1.

COROLLARY 3.3. *Suppose* $\sup_{n\in\mathbb{N}} |\lambda_n| < \infty$, $\lim_{n\to\infty}((\log s_{n+1})/s_n^\kappa) = ((d \log R)/(\kappa - 1))$ *for* $d \in [0, 1 - (1/\kappa))$ *and* R *is sufficiently large. Then* $\underline{s}$ *is admissible and* $\dim_H \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1$, $\dim_P \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1 + d$.

Proof. The proof follows directly from assertion (b) of Theorem 3.2 and (3). □

The conditions of Corollary 3.3 are actually satisfied for some sequences $(s_n)_{n=0}^\infty$, as shown in the following example.

Example 3.4. If $s_{n+1} = R^{(d/(\kappa-1))s_n^\kappa}$ for $d \in [0, 1 - (1/\kappa))$, then $\dim_H \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1$ and $\dim_P \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1 + d$.

4. *Proofs of Theorems 2.1 and 2.5—preliminaries*

We use the notation

$$\operatorname{diam} X = \sup\{|x - y| : x, y \in X\}$$

and

$$\operatorname{dist}(z, X) = \inf\{|z - x| : x \in X\}, \quad \operatorname{dist}(X, Y) = \inf\{|x - y| : x \in X, y \in Y\}$$

for $z \in \mathbb{C}$; $X, Y \subset \mathbb{C}$.

Let

$$J_N = \{z \in \mathbb{C} : a_{N+n} \le |E_{\lambda_{N+n}} \circ \cdots \circ E_{\lambda_N}(z)| \le b_{N+n} \text{ for every } n \ge 0\}$$

for $N \in \mathbb{N}$. By definition,

$$I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = J_1 \cup \bigcup_{N=2}^{\infty} (E_{\lambda_{N-1}} \circ \cdots \circ E_{\lambda_1})^{-1}(J_N)$$

and

$$J_{N_1} \subset (E_{\lambda_{N_2-1}} \circ \cdots \circ E_{\lambda_{N_1}})^{-1}(J_{N_2})$$

for every $1 \le N_1 < N_2$. As E_{λ_n} are non-constant holomorphic maps, we have $\dim J_{N_1} \le \dim J_{N_2}$ for $N_1 < N_2$ and

$$\dim I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = \dim\left(\bigcup_{N=1}^{\infty} J_N\right) = \sup_{N\in\mathbb{N}} \dim J_N = \lim_{N\to\infty} \dim J_N, \tag{6}$$

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where dim denotes the Hausdorff or packing dimension. Therefore, to estimate the dimensions of the sets $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$, it is sufficient to bound the suitable dimensions of J_N for large N.

From now on, we fix a large N and write J for J_N. For $n \geq 0$, let

$$A_n = \log \frac{a_{N+n}}{|\lambda_{N+n}|}, \quad B_n = \log \frac{b_{N+n}}{|\lambda_{N+n}|}$$

and

$$S_n = \{z \in \mathbb{C} : A_n \leq \operatorname{Re}(z) \leq B_n\}$$

for $n \in \mathbb{N}$. Recall that

$$B_n - A_n = \Delta_{N+n} = \log \frac{b_{N+n}}{a_{N+n}}.$$

Note that

$$z \in S_n \iff a_{N+n} \leq |E_{\lambda_{N+n}}(z)| \leq b_{N+n},$$

so

$$J = \{z \in \mathbb{C} : z \in S_0,\ E_{\lambda_{N+n}} \circ \cdots \circ E_{\lambda_N}(z) \in S_{n+1} \text{ for every } n \geq 0\}. \tag{7}$$

For a small $\delta > 0$ and $j, k \in \mathbb{Z}$, let

$$V_j^{(n)} = \{z \in \mathbb{C} : j\delta - \log|\lambda_{N+n}| \leq \operatorname{Re}(z) < (j+1)\delta - \log|\lambda_{N+n}|\},$$
$$H_k^{(n)} = \{z \in \mathbb{C} : k\pi - \operatorname{Arg}(\lambda_{N+n}) \leq \operatorname{Im}(z) < (k+1)\pi - \operatorname{Arg}(\lambda_{N+n})\}$$

with $\operatorname{Arg}(\lambda_{N+n}) \in [0, 2\pi)$. Set

$$K_{j,k}^{(n)} = V_j^{(n)} \cap H_k^{(n)}$$

(see Figure 1). Note that

$$\text{if } K_{j,k}^{(n)} \cap S_n \neq \emptyset \quad \text{then } \frac{\log a_{N+n}}{\delta} - 1 < j < \frac{\log b_{N+n}}{\delta}, \text{ so } e^{-\delta} a_{N+n} < e^{j\delta} < b_{N+n}. \tag{8}$$

We have

$$E_{\lambda_{N+n}}(K_{j,k}^{(n)}) = U_{j,k}$$

for

$$U_{j,k} = \{z \in \mathbb{C} : e^{j\delta} \leq |z| < e^{(j+1)\delta},\ k\pi \leq \operatorname{Arg}(z) < (k+1)\pi \bmod 2\pi\}.$$

Note that

$$U_{j,k+2} = U_{j,k}.$$

Set

$$\mathcal{K}^{(n)} = \{K_{j,k}^{(n)} : j, k \in \mathbb{Z}\}.$$

for $n \geq 0$, and

$$\mathcal{K}_{j,k}^{(n)} = \{K \in \mathcal{K}^{(n)} : K \cap U_{j,k} \cap S_n \neq \emptyset\},$$
$$\tilde{\mathcal{K}}_{j,k}^{(n)} = \{K \in \mathcal{K}^{(n)} : K \subset U_{j,k} \cap S_n\}$$

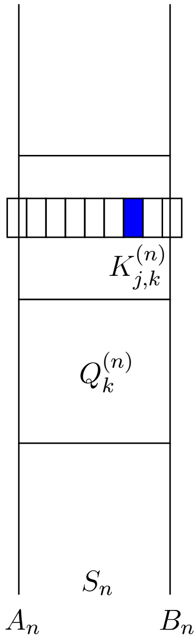

FIGURE 1. The sets S_n, $Q_k^{(n)}$ and $K_{j,k}^{(n)}$.

for $n \geq 1$. Obviously,

$$\tilde{\mathcal{K}}_{j,k}^{(n)} \subset \mathcal{K}_{j,k}^{(n)}.$$

Let

$$Q_k^{(n)} = \{z \in \mathbb{C} : z \in S_n,\ \Delta_{N+n} k \leq \operatorname{Im}(z) \leq \Delta_{N+n}(k+1)\}$$

for $n \geq 0$, $k \in \mathbb{Z}$, and

$$\mathcal{Q}_{j,k}^{(n)} = \{Q_l^{(n)} : Q_l^{(n)} \cap U_{j,k} \neq \emptyset,\ l \in \mathbb{Z}\}$$

for $n \geq 0$; $k, j \in \mathbb{Z}$ (see Figure 1). Finally, let

$$U_k = \bigcup_{j \in \mathbb{Z}} U_{j,k} = \{z \in \mathbb{C} \setminus \{0\} : k\pi \leq \operatorname{Arg}(z) < (k+1)\pi\}$$

and

$$g_k^{(n)} : U_k \to H_k^{(n)} \tag{9}$$

for $k \in \mathbb{Z}$ be inverse branches of $E_{\lambda_{N+n}}$ on U_k. Note that $g_k^{(n)}$ can be extended to any simply connected domain in $\mathbb{C} \setminus \{0\}$ containing U_k.

5. *Proof of Theorem 2.1*

Fix $j_0, k_0 \in \mathbb{Z}$, and take $j_1, \ldots, j_n \in \mathbb{Z}$, $k_1, \ldots, k_n \in \mathbb{Z}$ and $l \in \mathbb{Z}$ such that

$$K_{j_1,k_1}^{(1)} \in \mathcal{K}_{j_0,k_0}^{(1)}, \ldots, K_{j_n,k_n}^{(n)} \in \mathcal{K}_{j_{n-1},k_{n-1}}^{(n)},\ Q_l^{(n+1)} \in \mathcal{Q}_{j_n,k_n}^{(n+1)}$$

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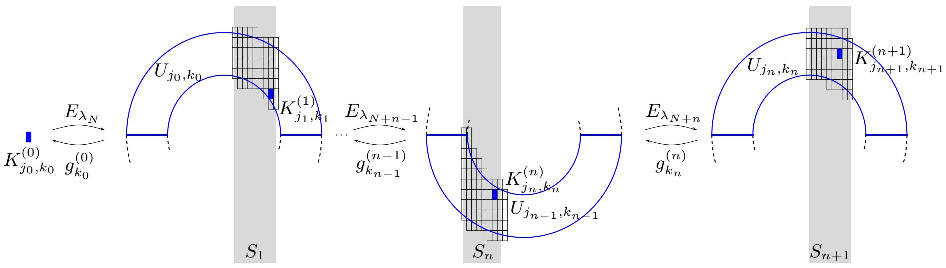

FIGURE 2. Successive images of the sets $K^{(n)}_{j,k}$.

(see Figure 2). Define inductively

$$Q_{j_n,k_n,l} = g^{(n)}_{k_n}(Q^{(n+1)}_l \cap U_{j_n,k_n}),$$
$$Q_{j_{m-1},k_{m-1},\ldots,j_n,k_n,l} = g^{(m-1)}_{k_{m-1}}(Q_{j_m,k_m,\ldots,j_n,k_n,l} \cap U_{j_{m-1},k_{m-1}}) \qquad \text{for } m = n, \ldots, 1,$$

and let

$$\mathcal{E}^{(n)} = \{Q_{j_0,k_0,\ldots,j_n,k_n,l} : K^{(1)}_{j_1,k_1} \in \mathcal{K}^{(1)}_{j_0,k_0}, \ldots, K^{(n)}_{j_n,k_n} \in \mathcal{K}^{(n)}_{j_{n-1},k_{n-1}}, Q^{(n+1)}_l \in \mathcal{Q}^{(n+1)}_{j_n,k_n},$$
$$j_1, \ldots, j_n \in \mathbb{Z},\ k_1, \ldots, k_n \in \mathbb{Z},\ l \in \mathbb{Z}\}$$

for $n \in \mathbb{N}$.

LEMMA 5.1. *For every $n \in \mathbb{N}$, the family $\mathcal{E}^{(n)}$ is a cover of $J \cap K^{(0)}_{j_0,k_0}$.*

Proof. Take $z \in J \cap K^{(0)}_{j_0,k_0}$. By (7), for every $m \geq 1$, $E_{\lambda_{N+m-1}} \circ \cdots \circ E_{\lambda_N}(z) \in K^{(m)}_{j_m,k_m} \cap S_m$ for some $j_m, k_m \in \mathbb{Z}$. Hence, for given $n \in \mathbb{N}$, $E_{\lambda_{N+m-1}} \circ \cdots \circ E_{\lambda_N}(z) \in K^{(m)}_{j_m,k_m} \cap U_{j_{m-1},k_{m-1}} \cap S_m$ for $m = 1, \ldots, n$, and $E_{\lambda_{N+n}} \circ \cdots \circ E_{\lambda_N}(z) \in Q^{(n+1)}_l \cap U_{j_n,k_n}$ for some $l \in \mathbb{Z}$. Therefore, $K^{(m)}_{j_m,k_m} \in \mathcal{K}^{(m)}_{j_{m-1},k_{m-1}}$ for $m = 1, \ldots, n$ and $Q^{(n+1)}_l \in \mathcal{Q}^{(n+1)}_{j_n,k_n}$. By induction,

$$g^{(m)}_{k_m}(E_{\lambda_{N+m}} \circ \cdots \circ E_{\lambda_N}(z)) = E_{\lambda_{N+m-1}} \circ \cdots \circ E_{\lambda_N}(z) \in Q_{j_m,k_m,\ldots,j_n,k_n,l} \cap U_{j_{m-1},k_{m-1}}$$

for $m = n, \ldots, 1$, and

$$g^{(0)}_{k_0}(E_{\lambda_N}(z)) = z \in Q_{j_0,k_0,\ldots,j_n,k_n,l} \in \mathcal{E}^{(n)}. \qquad \square$$

By (8) we can write

$$\mathcal{E}^{(n)} \subset \bigcup_{(\log a_{N+1}/\delta)-1<j_1<\log b_{N+1}/\delta} \cdots \bigcup_{(\log a_{N+n}/\delta)-1<j_n<\log b_{N+n}/\delta} \mathcal{E}^{(n)}_{j_1,\ldots,j_n}, \tag{10}$$

where

$$\mathcal{E}^{(n)}_{j_1,\ldots,j_n} = \{Q_{j_0,k_0,\ldots,j_n,k_n,l} \in \mathcal{E}^{(n)} : k_1, \ldots, k_n \in \mathbb{Z},\ l \in \mathbb{Z}\}.$$

Take $j_1, \ldots, j_n$ as in (10). If $Q_{j_0,k_0,\ldots,j_n,k_n,l} \in \mathcal{E}^{(n)}_{j_1,\ldots,j_n}$, then $K^{(m)}_{j_m,k_m} \cap U^{(m-1)}_{j_{m-1},k_{m-1}} \neq \emptyset$ for $m = 1, \ldots, n$, so

$$K^{(m)}_{j_m,k_m} \subset \{z \in \mathbb{C} : |\mathrm{Im}(z)| < e^{(j_{m-1}+1)\delta}\}.$$

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Moreover, by (8) and the assumption $\inf a_n > 0$, we can assume $e^{j_{m-1}\delta} > a$ for a constant $a > 0$. This implies

$$\#\{k_m \in \mathbb{Z} : Q_{j_0,k_0,\ldots,j_n,k_n,l} \in \mathcal{E}^{(n)}_{j_1,\ldots,j_n}\} \le c_1 e^{j_{m-1}\delta}$$

for every $k_0, \ldots, k_{m-1}, k_{m+1}, \ldots, k_n, l \in \mathbb{Z}$ and some constant $c_1 > 0$, so

$$\#\{(k_1, \ldots, k_n) \in \mathbb{Z}^n : Q_{j_0,k_0,\ldots,j_n,k_n,l} \in \mathcal{E}^{(n)}_{j_1,\ldots,j_n}\} \le c_1^n e^{(j_0+\cdots+j_{n-1})\delta}$$

for every $l \in \mathbb{Z}$. Similarly, $Q_l^{(n+1)} \in \mathcal{Q}^{(n+1)}_{j_n,k_n}$, so $Q_l^{(n+1)} \cap U^{(n)}_{j_n,k_n} \ne \emptyset$ and

$$Q_l^{(n+1)} \subset \{z \in \mathbb{C} : |\mathrm{Im}(z)| < e^{j_n\delta} + \Delta_{N+n+1}\},$$

which gives

$$\#\{l \in \mathbb{Z} : Q_{j_0,k_0,\ldots,j_n,k_n,l} \in \mathcal{E}^{(n)}_{j_1,\ldots,j_n}\} < \frac{e^{j_n\delta}}{\Delta_{N+n+1}} + 2$$

for every $k_0, \ldots, k_n \in \mathbb{Z}$. We conclude that

$$\#\mathcal{E}^{(n)}_{j_1,\ldots,j_n} \le c_1^n e^{(j_0+\cdots+j_n)\delta}\bigg(\frac{1}{\Delta_{N+n+1}} + \frac{2}{e^{j_n\delta}}\bigg). \tag{11}$$

Now we estimate the diameter of the sets $Q_{j_0,k_0,\ldots,j_n,k_n,l} \in \mathcal{E}^{(n)}_{j_1,\ldots,j_n}$. We have

$$\operatorname{diam} Q_{j_n,k_n,l} \le \sup_{U_{j_n,k_n}} |(g^{(n)}_{k_n})'| \operatorname{diam}(Q_l^{(n+1)} \cap U_{j_n,k_n}) \le \frac{1}{e^{j_n\delta}} \min(\sqrt{2}\Delta_{N+n+1}, e^{(j_n+1)\delta}).$$

Note also that any two points z_1, z_2 in $U_{j,k}$, for $j, k \in \mathbb{Z}$, can be joined within $U_{j,k}$ by a circle arc of length at most $2\pi|z_1 - z_2|$. Hence,

$$\begin{aligned}\operatorname{diam} Q_{j_{m-1},k_{m-1},\ldots,j_n,k_n,l} &\le 2\pi \sup_{U_{j_{m-1},k_{m-1}}} |(g^{(m-1)}_{k_{m-1}})'| \operatorname{diam} Q_{j_m,k_m,\ldots,j_n,k_n,l}\\ &= \frac{2\pi}{e^{j_{m-1}\delta}} \operatorname{diam} Q_{j_m,k_m,\ldots,j_n,k_n,l}\end{aligned}$$

for $m = 1, \ldots, n$, which implies

$$\operatorname{diam} Q_{j_0,k_0,\ldots,j_n,k_n,l} \le (2\pi)^n \frac{\min(\sqrt{2}\Delta_{N+n+1}, e^{(j_n+1)\delta})}{e^{(j_0+\cdots+j_n)\delta}}. \tag{12}$$

Fix $D > 1$ and let

$$P^{(n)}_{j_1,\ldots,j_n} = \sum_{Q\in\mathcal{E}^{(n)}_{j_1,\ldots,j_n}} (\operatorname{diam} Q)^D, \quad P^{(n)} = \sum_{Q\in\mathcal{E}^{(n)}} (\operatorname{diam} Q)^D$$

for $n \in \mathbb{N}$. By (11) and (12),

$$P^{(n)}_{j_1,\ldots,j_n} \le c_2^n \frac{(1/\Delta_{N+n+1} + 2/e^{j_n\delta})(\min(\sqrt{2}\Delta_{N+n+1}, e^{(j_n+1)\delta})^D}{e^{(j_0+\cdots+j_n)\delta(D-1)}} \le c_3^n\bigg(\frac{\Delta_{N+n+1}}{e^{(j_0+\cdots+j_n)\delta}}\bigg)^{D-1}$$

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for some constants $c_2, c_3 > 0$ (the latter estimate is by a straightforward calculation). Hence, by (10),

$$P^{(n)} \le c_3^n \sum_{(\log a_{N+1}/\delta)-1<j_1<\log b_{N+1}/\delta} \cdots \sum_{(\log a_{N+n}/\delta)-1<j_n<\log b_{N+n}/\delta} \left(\frac{\Delta_{N+n+1}}{e^{(j_0+\cdots+j_n)\delta}}\right)^{D-1}$$
$$\le c_4^n \left(\frac{\Delta_{N+n+1}}{a_N\cdots a_{N+n}}\right)^{D-1}$$

for some constant $c_4 > 0$. By assumption, for infinitely many n we have

$$\frac{\Delta_{N+n+1}}{a_N\cdots a_{N+n}} < \varepsilon_n^n,$$

where $\varepsilon_n > 0$, $\varepsilon_n \to 0$; therefore,

$$P^{(n)} \le (c_4\varepsilon_n^{D-1})^n < \frac{1}{2^n}$$

for infinitely many n, so $\liminf_{n\to\infty} P^{(n)} = 0$. Recall that by Lemma 5.1, $\mathcal{E}^{(n)}$ is a sequence of covers of $J \cap K_{j_0,k_0}^{(0)}$. Hence, by the definition of the Hausdorff measure we have $\dim_H(J \cap K_{j_0,k_0}^{(0)}) \le D$ for any $j_0, k_0 \in \mathbb{Z}$ and $D > 1$, so in fact $\dim_H J \le 1$. By (6), $\dim_H I_{\underline{a}}^{b}(E_{\underline{\lambda}}) \le 1$, which proves Theorem 2.1.

6. *Proof of Theorem 2.5—preliminaries*

Observe first that if N is chosen large enough, then the assumptions of Theorem 2.5 can be written as

$$|\lambda_{N+n+1}|e^{-qa_{N+n}} \le a_{N+n+1} \le |\lambda_{N+n+1}|e^{qa_{N+n}}, \tag{13}$$
$$\lim_{n\to\infty} (a_N\cdots a_{N+n})^{1/n} = \infty, \tag{14}$$
$$a_{N+n} > a, \tag{15}$$
$$\Delta_{N+n} > \Delta \tag{16}$$

for $n \ge 0$ and some constants $0 < q < 1$, $a > 0$, $\Delta > 0$, where a is sufficiently large depending on q and Δ (to be specified later). We fix δ, used in the definition of the sets $K_{j,k}^{(n)}$, to be a positive number such that

$$\delta < \min(\Delta/4, 1), \quad \sqrt{q}e^{\delta} < 1. \tag{17}$$

For $n \ge 0$, $j \in \mathbb{Z}$, let

$$D_j^{(n)} = \min(B_n, e^{(j+1)\delta}) - \max(A_n, -e^{(j+1)\delta}).$$

The following lemma estimates the size of sets $U_{j,k} \cap S_n$.

LEMMA 6.1. *There exist $c_1, c_2, c_3 > 0$ such that for every $n \ge 1$, $j, k \in \mathbb{Z}$:*

(a) *$U_{j,k} \cap S_n$ is contained in a rectangle of width $D_j^{(n)}$ and height $e^{(j+1)\delta}$;*

(b) *if $K_{j,k}^{(n-1)} \cap S_{n-1} \ne \emptyset$, then $U_{j,k} \cap S_n$ contains a rectangle of width $c_1 D_j^{(n)}$ and height $c_1 e^{j\delta}$; moreover, $\mathcal{K}_{j,k}^{(n)}$ is non-empty and contains a set $K_{j',k'}^{(n)}$ with $j', k' \in 2\mathbb{Z}$;*

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(c) *if* $K_{j,k}^{(n-1)} \cap S_{n-1} \neq \emptyset$, *then*

$$c_3 \leq \frac{1}{c_2} \min(\Delta_{N+n}, e^{j\delta}) \leq D_j^{(n)} \leq c_2 \min(\Delta_{N+n}, e^{j\delta}) \leq c_2 e^{j\delta}.$$

Proof. Let

$$\tilde{A}_n = \max(A_n, -e^{(j+1)\delta}), \quad \tilde{B}_n = \min(B_n, e^{(j+1)\delta})$$

for $n \geq 0$, $j \in \mathbb{Z}$. By the definition of $D_j^{(n)}$,

$$D_j^{(n)} = \tilde{B}_n - \tilde{A}_n, \quad \{z \in \mathbb{C} : \mathrm{Re}(z) \in [\tilde{A}_n, \tilde{B}_n]\} \subset S_n \tag{18}$$

and

$$U_{j,k} \cap S_n \subset \begin{cases} \{z \in \mathbb{C} : \mathrm{Re}(z) \in [\tilde{A}_n, \tilde{B}_n],\ \mathrm{Im}(z) \in [0, e^{(j+1)\delta}]\} & \text{if } k \text{ is even,} \\ \{z \in \mathbb{C} : \mathrm{Re}(z) \in [\tilde{A}_n, \tilde{B}_n],\ \mathrm{Im}(z) \in [-e^{(j+1)\delta}, 0]\} & \text{if } k \text{ is odd,} \end{cases}$$

which, together with (8), (15) and (16), gives assertion (a). Note also that by (13) and (17), we have

$$\tilde{A}_n < qe^{(j+1)\delta} < \sqrt{q}e^{j\delta} < e^{j\delta}, \quad \tilde{B}_n > -qe^{(j+1)\delta} > -\sqrt{q}e^{j\delta} > -e^{j\delta}. \tag{19}$$

This, together with (8), (15) and (16), gives assertion (c). Moreover, (19) implies that the vertical line $\{z \in \mathbb{C} : \mathrm{Re}(z) = (\tilde{A}_n + \tilde{B}_n)/2\}$ intersects the circle $\partial\mathbb{D}(0, ((e^\delta + 1)/2)e^{j\delta})$ at some point z_0. Then the upper (respectively lower) half of the disc $\mathbb{D}(z_0, ((e^\delta - 1)/2)e^{j\delta})$ is contained in $U_{j,k}$ for even (respectively odd) k. It follows that $U_{j,k} \cap S_n$ contains a rectangle of width $\min(D_j^{(n)}, \sqrt{2}((e^\delta - 1)/2)e^{j\delta})$ and height $\frac{\sqrt{2}}{2}((e^\delta - 1)/2)e^{j\delta}$. This, together with assertion (c), proves the first part of (b). To show the second part of (b), it is enough to notice that by (8), (15), (16), (17), (19) and the definition of $D_j^{(n)}$,

$$\min\left(D_j^{(n)}, \sqrt{2}\frac{e^\delta - 1}{2}e^{j\delta}\right) \geq \min\left(\Delta_{N+n}, \sqrt{2}\frac{e^\delta - 1}{2}e^{j\delta}\right) > 4\delta, \quad \frac{\sqrt{2}}{2}\frac{e^\delta - 1}{2}e^{j\delta} > 4\pi,$$

if a is chosen sufficiently large. □

We will also need the following technical lemma.

LEMMA 6.2. *Suppose* $K_{j,k}^{(n-1)} \cap S_{n-1} \neq \emptyset$ *for some* $n \geq 1$, $j, k \in \mathbb{Z}$ *and*

$$|A_n + e^{j\delta}| > \varepsilon e^{j\delta}, \quad |B_n - e^{j\delta}| > \varepsilon e^{j\delta} \tag{20}$$

for some constant $\varepsilon > 0$. *Then for every* $z \in U_{j,k} \cap S_n$ *there exists a right triangle* $T \subset U_{j,k} \cap S_n$, *with one of its vertices at* z, *a horizontal leg of length* $c_1 D_j^{(n)}$ *and a vertical leg of length* $c_2 e^{j\delta}$, *containing at least one element of* $\tilde{\mathcal{K}}_{j,k}^{(n)}$, *where the constants* $c_1, c_2 > 0$ *depend only on* a, ε *and* q.

The proof of Lemma 6.2, using (19) and (20), is an elementary but a bit tedious exercise and is left to the reader.

The next lemma provides basic estimates of the derivative of the inverse branches of $E_{\lambda_{N+n}} \circ \cdots \circ E_{\lambda_N}$.

LEMMA 6.3. *For every $n \in \mathbb{N}$ and $j_0, \ldots, j_n \in \mathbb{Z}$, $k_0, \ldots, k_n \in \mathbb{Z}$, such that $K^{(0)}_{j_0,k_0} \cap S_0 \neq \emptyset$, $K^{(1)}_{j_1,k_1} \in \mathcal{K}^{(1)}_{j_0,k_0}, \ldots, K^{(n)}_{j_n,k_n} \in \mathcal{K}^{(n)}_{j_{n-1},k_{n-1}}$, the branch*

$$g^{(0)}_{k_0} \circ \cdots \circ g^{(n-1)}_{k_{n-1}}$$

is defined on $K^{(n)}_{j_n,k_n}$, for some extensions of the branches from (9), with the distortion bounded by a constant independent of n and $j_0, \ldots, j_n$, $k_0, \ldots, k_n$. Moreover,

$$\frac{c^{-n}}{e^{\delta(j_0+\cdots+j_{n-1})}} < \left|(g^{(0)}_{k_0} \circ \cdots \circ g^{(n-1)}_{k_{n-1}})'|_{K^{(n)}_{j_n,k_n}}\right| < \frac{c^n}{e^{\delta(j_0+\cdots+j_{n-1})}} < \frac{1}{2^n}$$

for some constant $c > 0$.

Proof. Take $j_0, \ldots, j_n, k_0, \ldots, k_n$ as in the lemma. By assumption,

$$K^{(m-1)}_{j_{m-1},k_{m-1}} \cap S_{m-1} \neq \emptyset, \quad K^{(m)}_{j_m,k_m} \cap U_{j_{m-1},k_{m-1}} \neq \emptyset \tag{21}$$

for $m = 1, \ldots, n$. Let

$$d_0 = \sqrt{\pi^2 + \delta^2}$$

be the diameter of the sets $K \in \bigcup_{n=0}^{\infty} \mathcal{K}^{(n)}$. The first assertion of (21), together with (8) and (15), implies

$$e^{j_{m-1}\delta} \geq e^{-\delta} a_{N+m-1} \geq e^{-\delta} a > 2d_0 + 2, \tag{22}$$

if a is chosen sufficiently large. Hence,

$$U_{j_{m-1},k_{m-1}} \subset \{z \in \mathbb{C} : |z| \geq e^{-\delta} a\}, \tag{23}$$

and the branch $g^{(m-1)}_{k_{m-1}}$ on $U_{j_{m-1},k_{m-1}}$ can be extended to

$$\hat{U}_{j_{m-1},k_{m-1}} = \{z \in \mathbb{C} : \operatorname{dist}(z, U_{j_{m-1},k_{m-1}}) < 2d_0\}.$$

Let

$$V_{m,m} = K^{(m)}_{j_m,k_m}, \quad V_{m,s} = g^{(m)}_{k_m} \circ \cdots \circ g^{(s-1)}_{k_{s-1}}(K^{(s)}_{j_s,k_s})$$

for $m = 0, \ldots, n$, $s = m+1, \ldots, n$. Now we show, by backward induction on m, that

$$\begin{aligned} &V_{m,s} \text{ are well defined} && \text{for } s = m, \ldots, n, \\ &\operatorname{diam} V_{m,s} \leq \frac{d_0}{2^{s-m}} && \text{for } s = m, \ldots, n, \\ &V_{m,s} \subset \hat{U}_{j_{m-1},k_{m-1}} && \text{for } s = m, \ldots, n, \\ &V_{m,s} \cap V_{m,s+1} \neq \emptyset && \text{for } s = m, \ldots, n-1. \end{aligned} \tag{24}$$

For $m = n$, (24) follows from (21). Suppose, by induction, that (24) holds for some $1 \leq m \leq n$. Then $V_{m-1,s} = g^{(m-1)}_{k_{m-1}}(V_{m,s})$ for $s = m, \ldots, n$ are well defined. Take $s \in \{m-1, \ldots, n\}$. By (22) and the fourth assertion of (24),

$$V_{m,s} \subset \left\{z \in \mathbb{C} : |z| \geq e^{-\delta} a - d_0\left(1 + \cdots + \frac{1}{2^m}\right)\right\} \subset \{z \in \mathbb{C} : |z| \geq 2\},$$

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so

$$\operatorname{diam} V_{m-1,s} \le \sup_{V_{m,s}} |(g_{k_{m-1}}^{(m-1)})'| \operatorname{diam} V_{m,s} \le \frac{\operatorname{diam} V_{m,s}}{2} < \frac{d_0}{2^{s-m+1}}. \tag{25}$$

By (21) and the fourth assertion of (24), $V_{m-1,s} \cap V_{m-1,s+1} \neq \emptyset$ for $s = m-1, \ldots, n-1$. Hence, by (21), to have $V_{m-1,s} \subset \hat{U}_{j_{m-2},k_{m-2}}$ for $s = m-1, \ldots, n$, it is enough to check that

$$\operatorname{diam} V_{m-1,m-1} + \cdots + \operatorname{diam} V_{m-1,n} < 2d_0,$$

which follows from (25). This ends the inductive proof of (24).

By (24), for $m = 0$, $s = n$, we conclude that the branch $g_{k_0}^{(0)} \circ \cdots \circ g_{k_{n-1}}^{(n-1)}$ is defined on $K_{j_n,k_n}^{(n)}$. The distortion of the branch is estimated in a standard way. By (22), (23) and (24), for $z_1, z_2 \in V_{m,n}$ we have

$$\frac{|(g_{k_{m-1}}^{(m-1)})'(z_1)|}{|(g_{k_{m-1}}^{(m-1)})'(z_2)|} = \frac{|z_2|}{|z_1|} \le 1 + \frac{|z_1 - z_2|}{|z_1|} \le 1 + \frac{\operatorname{diam} V_{m,n}}{e^{-\delta}a - 2d_0} \le 1 + \frac{d_0}{2^{n-m+1}}.$$

Hence, for $z_1, z_2 \in V_{m,n}$,

$$\frac{|(g_{k_0}^{(0)} \circ \cdots \circ g_{k_{n-1}}^{(n-1)})'(z_1)|}{|(g_{k_0}^{(0)} \circ \cdots \circ g_{k_{n-1}}^{(n-1)})'(z_2)|} \le \prod_{m=1}^{n} \left(1 + \frac{d_0}{2^{m+1}}\right) \le \exp\left(\sum_{m=1}^{n} \frac{d_0}{2^{m+1}}\right) < e^{d_0/2},$$

so the distortion of the branch is universally bounded. Finally, (22) and the third assertion of (24) give

$$\begin{aligned} \frac{c^{-n}}{e^{(j_0+\cdots+j_{n-1})\delta}} \le \prod_{m=1}^{n} \frac{1}{e^{(j_{m-1}+1)\delta} + 2d_0} &< \left| (g_{k_0}^{(0)} \circ \cdots \circ g_{k_{n-1}}^{(n-1)})'|_{K_{j_n,k_n}^{(n)}} \right| \\ &< \prod_{m=1}^{n} \frac{1}{e^{j_{m-1}\delta} - d_0} \le \frac{c^n}{e^{(j_0+\cdots+j_{n-1})\delta}} < \frac{1}{Q^n} \end{aligned}$$

for $c = \max(e^{\delta}(1 + 2d_0/a), 1/(1 - 2d_0 e^{\delta}/a))$, $Q = e^{-\delta}a/c$. Choosing a sufficiently large, we can assume $Q \ge 2$. □

7. *Proof of Theorem 2.5—estimate from above*

In this section, we prove the upper estimate in Theorem 2.5. First, we do this under an additional technical assumption:

$$|A_n + e^{j\delta}| > \varepsilon e^{j\delta}, \quad |B_n - e^{j\delta}| > \varepsilon e^{j\delta} \tag{26}$$

for every $n \ge 0$, $j \in \mathbb{Z}$ and some constant $\varepsilon > 0$. In the last subsection we show how to reduce the general situation to this case.

7.1. *Construction of the measure* $\boldsymbol{\mu}$.

Take $j_0, k_0 \in \mathbb{Z}$ such that $J \cap K_{j_0,k_0}^{(0)} \neq \emptyset$. In particular, we have $K_{j_0,k_0}^{(0)} \cap S_0 \neq \emptyset$. By Lemma 6.3, we can define families $\mathcal{F}^{(n)}$, $n \ge 0$, setting

$$\mathcal{F}^{(0)} = \{K_{j_0,k_0}\} \quad \text{for } K_{j_0,k_0} = K_{j_0,k_0}^{(0)},$$

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and

$$\mathcal{F}^{(n)} = \{K_{j_0,k_0,\ldots,j_n,k_n} = g^{(0)}_{k_0} \circ \cdots \circ g^{(n-1)}_{k_{n-1}}(K^{(n)}_{j_n,k_n}) : \\ K^{(1)}_{j_1,k_1} \in \mathcal{K}^{(1)}_{j_0,k_0}, \ldots, K^{(n)}_{j_n,k_n} \in \mathcal{K}^{(n)}_{j_{n-1},k_{n-1}},\ j_1,\ldots,j_n \in \mathbb{Z},\ k_1,\ldots,k_n \in \mathbb{Z}\}$$

for $n \in \mathbb{N}$.

Since, for given n, the sets $K^{(n)}_{j,k} \in \mathcal{K}^{(n)}$ are pairwise disjoint, the sets $K_{j_0,k_0,\ldots,j_n,k_n} \in \mathcal{F}^{(n)}$ are also pairwise disjoint. Moreover, for every set $K_{j_0,k_0,\ldots,j_n,k_n} \in \mathcal{F}^{(n)}$ and $j_{n+1}, k_{n+1} \in \mathbb{Z}$, we have

$$K_{j_0,k_0,\ldots,j_{n+1},k_{n+1}} \in \mathcal{F}^{(n+1)} \iff K^{(n+1)}_{j_{n+1},k_{n+1}} \in \mathcal{K}^{(n+1)}_{j_n,k_n} \\ \iff K_{j_0,k_0,\ldots,j_{n+1},k_{n+1}} \cap K_{j_0,k_0,\ldots,j_n,k_n} \neq \emptyset. \tag{27}$$

For $K_{j_0,k_0,\ldots,j_n,k_n} \in \mathcal{F}^{(n)}$, let

$$N_{j_0,k_0,\ldots,j_n,k_n} = \#\{(j_{n+1},k_{n+1}) : K_{j_0,k_0,\ldots,j_{n+1},k_{n+1}} \in \mathcal{F}^{(n+1)}\} = \#\mathcal{K}^{(n+1)}_{j_n,k_n}.$$

By (8), (15) and Lemma 6.1,

$$0 < N_{j_0,k_0,\ldots,j_n,k_n} \le c_1 D^{(n+1)}_{j_n} e^{j_n \delta} \tag{28}$$

for some constant $c_1 > 0$, and, if $K_{j_0,k_0,\ldots,j_{n+1},k_{n+1}} \in \mathcal{F}^{(n+1)}$, then

$$\frac{\operatorname{diam} K_{j_0,k_0,\ldots,j_{n+1},k_{n+1}}}{\operatorname{diam} K_{j_0,k_0,\ldots,j_n,k_n}} \le c \sup_{U_{j_n,k_n}} |(g^{(n)}_{k_n})'| < \frac{ce^{\delta}}{a} < \frac{1}{2} \tag{29}$$

for a constant $c > 0$, provided a is chosen sufficiently large.

Let

$$K_\infty = \overline{\bigcap_{n=0}^{\infty} \bigcup \mathcal{F}^{(n)}}.$$

In the same way as for Lemma 5.1, we show

$$J \cap K^{(0)}_{j_0,k_0} \subset K_\infty. \tag{30}$$

For every $n \ge 0$ and $K_{j_0,k_0,\ldots,j_n,k_n} \in \mathcal{F}^{(n)}$, choose a point

$$z_{j_0,k_0,\ldots,j_n,k_n} \in K_{j_0,k_0,\ldots,j_n,k_n},$$

and note that by (27) and (29), if $K_{j_0,k_0,\ldots,j_m,k_m} \in \mathcal{F}^{(m)}$ for some $m > n$, then

$$|z_{j_0,k_0,\ldots,j_m,k_m} - z_{j_0,k_0,\ldots,j_n,k_n}| < \left(1 + \cdots + \frac{1}{2^{m-n}}\right) \operatorname{diam} K_{j_0,k_0,\ldots,j_n,k_n} < \frac{d_0}{2^{n-1}}. \tag{31}$$

Define a sequence of Borel probability measures μ_n, $n \ge 0$, setting

$$\mu_0 = \nu_{z_{j_0,k_0}},$$

$$\mu_{n+1} = \sum_{K_{j_0,k_0,\ldots,j_n,k_n} \in \mathcal{F}^{(n)}} \ \sum_{(j_{n+1},k_{n+1}) : K_{j_0,k_0,\ldots,j_{n+1},k_{n+1}} \in \mathcal{F}^{(n+1)}} \frac{\nu_{z_{j_0,k_0,\ldots,j_{n+1},k_{n+1}}}}{N_{j_0,k_0} \cdots N_{j_0,k_0,\ldots,j_n,k_n}},$$

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where ν_z denotes the Dirac measure at z. By definition, for every $K_{j_0,k_0,\ldots,j_n,k_n} \in \mathcal{F}^{(n)}$,

$$\mu_{n+1}(\{z_{j_0,k_0,\ldots,j_n,k_n,j_{n+1},k_{n+1}} : K_{j_0,k_0,\ldots,j_n,k_n,j_{n+1},k_{n+1}} \in \mathcal{F}^{(n+1)}\}) = \mu_n(\{z_{j_0,k_0,\ldots,j_n,k_n}\}),$$

so by induction, using (31), we obtain

$$\begin{aligned}\mu_m(\mathbb{D}(z_{j_0,k_0,\ldots,j_n,k_n}, 2\operatorname{diam} K_{j_0,k_0,\ldots,j_n,k_n})) &\geq \mu_n(\{z_{j_0,k_0,\ldots,j_n,k_n}\})\\ &= \frac{1}{N_{j_0,k_0}\cdots N_{j_0,k_0,\ldots,j_{n-1},k_{n-1}}}\end{aligned} \tag{32}$$

for every $m \geq n$. By (31),

$$\operatorname{supp} \mu_n = \{z_{j_0,k_0,\ldots,j_n,k_n} : K_{j_0,k_0,\ldots,j_n,k_n} \in \mathcal{F}^{(n)}\} \subset \mathbb{D}(z_{j_0,k_0}, 2d_0).$$

Hence, the sequence μ_n converges weakly along a subsequence to a Borel probability measure μ with support in $\overline{\mathbb{D}(z_{j_0,k_0}, 2d_0)}$.

Take $K_{j_0,k_0,\ldots,j_n,k_n} \in \mathcal{F}^{(n)}$. By (28) and (32),

$$\mu(\hat{K}_{j_0,k_0,\ldots,j_n,k_n}) \geq \frac{c_1^{-n}}{D_{j_0}^{(1)}\cdots D_{j_{n-1}}^{(n)} e^{(j_0+\cdots+j_{n-1})\delta}} \tag{33}$$

for

$$\hat{K}_{j_0,k_0,\ldots,j_n,k_n} = \{z \in \mathbb{C} : \operatorname{dist}(z, K_{j_0,k_0,\ldots,j_n,k_n}) \leq 2\operatorname{diam} K_{j_0,k_0,\ldots,j_n,k_n}\}.$$

7.2. *Estimate of the local dimension of* $\boldsymbol{\mu}$. Since every point in the support of μ is a limit of points from $\operatorname{supp} \mu_{n_s}$ for some $n_s \to \infty$, taking a suitable subsequence and using (31) we obtain

$$\begin{aligned}\operatorname{supp} \mu \subset \{z \in \mathbb{C} : z = \lim_{n\to\infty} z_{j_0,k_0,\ldots,j_n,k_n}, \text{ where } j_1, k_1, j_2, k_2, \ldots \in \mathbb{Z}\\ \text{and } K_{j_0,k_0,\ldots,j_n,k_n} \in \mathcal{F}^{(n)} \text{ for every } n \geq 0\}.\end{aligned}$$

The same argument shows

$$K_\infty \subset \operatorname{supp} \mu. \tag{34}$$

Take $z = \lim_{n\to\infty} z_{j_0,k_0,\ldots,j_n,k_n} \in \operatorname{supp} \mu$, where $K_{j_0,k_0,\ldots,j_n,k_n} \in \mathcal{F}^{(n)}$ for every $n \geq 0$. For simplicity, denote

$$d_n = \operatorname{diam} K_{j_0,k_0,\ldots,j_n,k_n}, \quad z_n = z_{j_0,k_0,\ldots,j_n,k_n}.$$

By (31), we have

$$|z - z_n| \leq 2d_n. \tag{35}$$

Let

$$r_n = Cd_n$$

for a large constant $C > 0$. Note that by (29), the sequence r_n is strictly decreasing to 0.

Now we estimate $\mu(\mathbb{D}(z, r))$ for a small r. Let n be such that

$$r_{n+1} \leq r < r_n,$$

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and let

$$R = \frac{r}{\sqrt{C} d_{n+1}}.$$

Note that if r varies in $[r_{n+1}, r_n)$, then R varies in $[R_-^{(n)}, R_+^{(n)})$ for

$$R_-^{(n)} = \sqrt{C}, \quad R_+^{(n)} = \sqrt{C}\frac{d_n}{d_{n+1}}.$$

By Lemma 6.3,

$$\frac{\sqrt{C}}{c_2^{n+1}} \frac{R}{e^{(j_0+\cdots+j_n)\delta}} < r < c_2^{n+1}\sqrt{C}\frac{R}{e^{(j_0+\cdots+j_n)\delta}} \tag{36}$$

and

$$\frac{\sqrt{C}}{c_2} e^{j_n\delta} \le R_+^{(n)} \le c_2\sqrt{C} e^{j_n\delta} \tag{37}$$

for some constant $c_2 > 0$. Enlarging also C, by Lemma 6.1 and (37) we can assume

$$\frac{c_3}{\sqrt{C}} R_-^{(n)} < D_{j_n}^{(n+1)} < R_+^{(n)} \tag{38}$$

for some constant $c_3 > 0$.

Let

$$w = E_{\lambda_{N+n}} \circ \cdots \circ E_{\lambda_N}(z_{n+1}).$$

By definition, $w \in K_{j_{n+1},k_{n+1}}^{(n+1)} \in \mathcal{K}_{j_n,k_n}^{(n+1)}$. Take $K_{j'_{n+1},k'_{n+1}}^{(n+1)} \in \mathcal{K}_{j_n,k_n}^{(n+1)}$ such that $K_{j'_{n+1},k'_{n+1}}^{(n+1)} \subset \mathbb{D}(w, R)$. By (27), $K_{j_0,k_0,\ldots,j_n,k_n,j'_{n+1},k'_{n+1}} \in \mathcal{F}^{(n+1)}$, and by Lemma 6.3, there exists a constant $c_4 > 0$ such that

$$|z_{j_0,k_0,\ldots,j_n,k_n,j'_{n+1},k'_{n+1}} - z_{n+1}| < c_4 R d_{n+1}, \quad \operatorname{diam} K_{j_0,k_0,\ldots,j_n,k_n,j'_{n+1},k'_{n+1}} < c_4 d_{n+1}.$$

Using this together with (35) we obtain

$$\begin{aligned}
\hat{K}_{j_0,k_0,\ldots,j_n,k_n,j'_{n+1},k'_{n+1}} &\subset \mathbb{D}(z, (c_4 R + c_4 + 2) d_{n+1}) \\
&= \mathbb{D}\bigg(z, \frac{c_4}{\sqrt{C}} r + \frac{c_4+2}{C} r_{n+1}\bigg) \\
&\subset \mathbb{D}\bigg(z, \bigg(\frac{c_4}{\sqrt{C}} + \frac{c_4+2}{C}\bigg) r\bigg) \subset \mathbb{D}(z, r)
\end{aligned} \tag{39}$$

if C is chosen sufficiently large.

By (26) and Lemma 6.1, there exist $u \in K_{j_{n+1},k_{n+1}}^{(n+1)} \cap U_{j_n,k_n} \cap S_{n+1}$ and a right triangle $T \subset U_{j_n,k_n} \cap S_{n+1}$, with one of its vertices at u, a horizontal leg of length $cD_{j_n}^{(n+1)}$ and a vertical leg of length $c' e^{j_n\delta}$, for some constants $c, c' > 0$, containing at least one element of $\mathcal{K}_{j_n,k_n}^{(n+1)}$. Note also that Lemma 6.3 implies that if $K_{j'_{n+1},k'_{n+1}}^{(n+1)}, K_{j''_{n+1},k''_{n+1}}^{(n+1)} \in \mathcal{K}_{j_n,k_n}^{(n+1)}$ and $\operatorname{dist}(K_{j'_{n+1},k'_{n+1}}^{(n+1)}, K_{j''_{n+1},k''_{n+1}}^{(n+1)}) > c_5$ for a sufficiently large constant $c_5 > 0$, then $\hat{K}_{j_0,k_0,\ldots,j_n,k_n,j'_{n+1},k'_{n+1}}$ and $\hat{K}_{j_0,k_0,\ldots,j_n,k_n,j''_{n+1},k''_{n+1}}$ are disjoint. Using these facts and noting that $R \ge \sqrt{C}$ for a large C, we show by elementary geometry considerations that $\mathbb{D}(w, R)$

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contains at least M sets $K^{(n+1)}_{j'_{n+1},k'_{n+1}} \in \mathcal{K}^{(n+1)}_{j_n,k_n}$, such that $\hat{K}_{j_0,k_0,\ldots,j_n,k_n,j'_{n+1},k'_{n+1}}$ are pairwise disjoint, where

$$M = \begin{cases} c_6 R^2 & \text{if } R \le D^{(n+1)}_{j_n}, \\ c_6 D^{(n+1)}_{j_n} R & \text{if } R > D^{(n+1)}_{j_n}, \end{cases}$$

for some constant $c_6 > 0$. By (33) and (39),

$$\mu(\mathbb{D}(z,r)) \ge \begin{cases} \dfrac{c_6 c_1^{-(n+1)} R^2}{D^{(1)}_{j_0} \cdots D^{(n+1)}_{j_n} e^{(j_0+\cdots+j_n)\delta}} & \text{if } R \le D^{(n+1)}_{j_n}, \\ \dfrac{c_6 c_1^{-(n+1)} R}{D^{(1)}_{j_0} \cdots D^{(n)}_{j_{n-1}} e^{(j_0+\cdots+j_n)\delta}} & \text{if } R > D^{(n+1)}_{j_n}, \end{cases}$$

so by (36),

$$\frac{\log \mu(\mathbb{D}(z,r))}{\log r} \le 1 + h_n(R), \tag{40}$$

where

$$h_n(x) = \begin{cases} \dfrac{\log(D^{(1)}_{j_0} \cdots D^{(n+1)}_{j_n}) - \log x + c_7 n}{(j_0+\cdots+j_n)\delta - \log x - c_7 n} & \text{if } x \le D^{(n+1)}_{j_n}, \\ \dfrac{\log(D^{(1)}_{j_0} \cdots D^{(n)}_{j_{n-1}}) + c_7 n}{(j_0+\cdots+j_n)\delta - \log x - c_7 n} & \text{if } x > D^{(n+1)}_{j_n}, \end{cases} \tag{41}$$

for $x \in [R^{(n)}_-, R^{(n)}_+)$ and some constant $c_7 > 0$, which can be chosen arbitrarily large. Note that by (8) and (14), we have

$$\frac{j_0+\cdots+j_n}{n} \to \infty \quad \text{as } n \to \infty. \tag{42}$$

Together with (37), this implies that the denominators in (41) are positive for large n, so h_n is well defined.

Now we estimate the infimum and supremum of the function h_n.

LEMMA 7.1. *We have*

$$\lim_{n\to\infty} \left| \inf_{[R^{(n)}_-, R^{(n)}_+)} h_n - \frac{\log(D^{(1)}_{j_0} \cdots D^{(n)}_{j_{n-1}})}{(j_0+\cdots+j_n)\delta - \log D^{(n+1)}_{j_n}} \right| = 0,$$

$$\lim_{n\to\infty} \left| \sup_{[R^{(n)}_-, R^{(n)}_+)} h_n - \max\left(\frac{\log(D^{(1)}_{j_0} \cdots D^{(n)}_{j_{n-1}})}{(j_0+\cdots+j_{n-1})\delta}, \frac{\log(D^{(1)}_{j_0} \cdots D^{(n+1)}_{j_n})}{(j_0+\cdots+j_n)\delta} \right) \right| = 0.$$

Proof. We can write

$$h_n(x) = \begin{cases} h^{(n)}_1(x) + h^{(n)}_2(x) & \text{if } x \le D^{(n+1)}_{j_n}, \\ h^{(n)}_3(x) & \text{if } x > D^{(n+1)}_{j_n}, \end{cases}$$

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where

$$h_1^{(n)}(x) = 1 + \frac{\log(D_{j_0}^{(1)} \cdots D_{j_n}^{(n+1)}) - (j_0 + \cdots + j_n)\delta - c_8 n}{(j_0 + \cdots + j_n)\delta - \log x - c_7 n},$$

$$h_2^{(n)}(x) = \frac{(2c_7 + c_8)n}{(j_0 + \cdots + j_n)\delta - \log x - c_7 n},$$

$$h_3^{(n)}(x) = \frac{\log(D_{j_0}^{(1)} \cdots D_{j_{n-1}}^{(n)}) + c_7 n}{(j_0 + \cdots + j_n)\delta - \log x - c_7 n}$$

for $x \in [R_-^{(n)}, R_+^{(n)})$ and a large constant $c_8 > 0$. Let

$$\varepsilon_n = \sup_{[R_-^{(n)}, R_+^{(n)})} |h_2^{(n)}|,$$

and note that by (37) and (42), we have $\varepsilon_n \to 0$ as $n \to \infty$. By Lemma 6.1, $h_1^{(n)}$ is decreasing and $h_3^{(n)}$ is increasing, if c_7 and c_8 are chosen sufficiently large. This together with (38) implies that if $D_{j_n}^{(n+1)} > R_-^{(n)}$, then

$$\left| \inf_{[R_-^{(n)}, R_+^{(n)})} h_n - h_1^{(n)}(D_{j_n}^{(n+1)}) \right| \le \varepsilon_n, \quad \left| \sup_{[R_-^{(n)}, R_+^{(n)})} h_n - \max(h_1^{(n)}(R_-^{(n)}), h_3^{(n)}(R_+^{(n)})) \right| \le \varepsilon_n,$$

and if $D_{j_n}^{(n+1)} < R_-^{(n)}$, then

$$\left| \inf_{[R_-^{(n)}, R_+^{(n)})} h_n - h_1^{(n)}(R_-^{(n)}) \right| \le \varepsilon_n, \quad \left| \sup_{[R_-^{(n)}, R_+^{(n)})} h_n - h_3^{(n)}(R_+^{(n)}) \right| \le \varepsilon_n.$$

Furthermore, using (37), (38) and (42), we obtain

$$\left| h_1^{(n)}(D_{j_n}^{(n+1)}) - \frac{\log(D_{j_0}^{(1)} \cdots D_{j_{n-1}}^{(n)})}{(j_0 + \cdots + j_n)\delta - \log D_{j_n}^{(n+1)}} \right| \to 0,$$

$$\left| h_1^{(n)}(R_-^{(n)}) - \frac{\log(D_{j_0}^{(1)} \cdots D_{j_n}^{(n+1)})}{(j_0 + \cdots + j_n)\delta} \right| \to 0,$$

$$\left| h_3^{(n)}(R_+^{(n)}) - \frac{\log(D_{j_0}^{(1)} \cdots D_{j_{n-1}}^{(n)})}{(j_0 + \cdots + j_{n-1})\delta} \right| \to 0$$

and

$$|h_1^{(n)}(D_{j_n}^{(n+1)}) - h_1^{(n)}(R_-^{(n)})| \to 0 \quad \text{if } D_{j_n}^{(n+1)} < R_-^{(n)}$$

as $n \to \infty$. This proves the lemma. □

7.3. *Conclusion.* By (30), (34), (40) and Lemma 7.1, for every $j_0, k_0 \in \mathbb{Z}$ such that $J \cap K_{j_0,k_0}^{(0)} \neq \emptyset$ and every $z \in J \cap K_{j_0,k_0}^{(0)}$, there exist $j_1, k_1, j_2, k_2, \ldots \in \mathbb{Z}$ with

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$z = \lim_{n\to\infty} z_{j_0,k_0,\ldots,j_n,k_n}$ and

$$\liminf_{r\to 0} \frac{\log \mu(\mathbb{D}(z,r))}{\log r} \le 1 + \liminf_{n\to\infty} \frac{\log(D^{(1)}_{j_0} \cdots D^{(n)}_{j_{n-1}})}{(j_0 + \cdots + j_n)\delta - \log D^{(n+1)}_{j_n}},$$

$$\limsup_{r\to 0} \frac{\log \mu(\mathbb{D}(z,r))}{\log r} \le 1 + \limsup_{n\to\infty} \frac{\log(D^{(1)}_{j_0} \cdots D^{(n+1)}_{j_n})}{(j_0 + \cdots + j_n)\delta}.$$

This, together with Lemma 6.1, (8) and (42), implies

$$\liminf_{r\to 0} \frac{\log \mu(\mathbb{D}(z,r))}{\log r} \le 1 + \liminf_{n\to\infty} \Phi_n(\delta j_0, \ldots, \delta j_n),$$

$$\limsup_{r\to 0} \frac{\log \mu(\mathbb{D}(z,r))}{\log r} \le 1 + \limsup_{n\to\infty} \Psi_n(\delta j_0, \ldots, \delta j_n), \tag{43}$$

where

$$\Phi_n(x_0, \ldots, x_n) = \frac{\min(\log \Delta_{N+1}, x_0) + \cdots + \min(\log \Delta_{N+n}, x_{n-1})}{x_0 + \cdots + x_n - \min(\log \Delta_{N+n+1}, x_n)},$$

$$\Psi_n(x_0, \ldots, x_n) = \frac{\min(\log \Delta_{N+1}, x_0) + \cdots + \min(\log \Delta_{N+n+1}, x_n)}{x_0 + \cdots + x_n} \tag{44}$$

for $x_0 \in [\log a_N, \log b_N]$, $x_1 \in [\log a_{N+1}, \log b_{N+1}], \ldots$. By the standard dimension estimates (see e.g. [**Mat95**, **PU10**]), (43) gives

$$\dim_H J \le 1 + \sup_x \liminf_{n\to\infty} \Phi_n(x_0, \ldots, x_n),$$

$$\dim_P J \le 1 + \sup_x \limsup_{n\to\infty} \Psi_n(x_0, \ldots, x_n)$$

for $x = (x_0, x_1, \ldots) \in [\log a_N, \log b_N] \times [\log a_{N+1}, \log b_{N+1}] \times \cdots$. Together with (6), this proves the upper estimate in Theorem 2.5.

7.4. *General case.* Suppose now that the assumption (26) does not hold. For $n \ge 0$, if $A_n < 0$, then let $\alpha_n \in \mathbb{Z}$ be such that

$$-e^{(\alpha_n+1)\delta} \le A_n < -e^{\alpha_n \delta}.$$

Similarly, if $B_n > 0$, then let $\beta_n \in \mathbb{Z}$ be such that

$$e^{\beta_n \delta} \le B_n < e^{(\beta_n+1)\delta}.$$

Set $a'_m = a_m$, $b'_m = b_m$ for $1 \le m < N$, and

$$a'_{N+n} = |\lambda_{N+n}| e^{A'_n}, \quad b'_{N+n} = |\lambda_{N+n}| e^{B'_n}$$

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for $n \geq 0$, where

$$A'_n = \begin{cases} -e^{(\alpha_n+3/2)\delta} & \text{if } -e^{(\alpha_n+1)\delta} \leq A_n < -e^{\alpha_n\delta}, \\ A_n & \text{if } A_n \geq 0, \end{cases}$$

$$B'_n = \begin{cases} e^{(\beta_n+3/2)\delta} & \text{if } e^{\beta_n\delta} \leq B_n < e^{(\beta_n+1)\delta}, \\ B_n & \text{if } B_n \leq 0. \end{cases}$$

By definition,

$$-\frac{A'_n}{e^{j\delta}} \leq e^{-\delta/2} \quad \text{or} \quad -\frac{A'_n}{e^{j\delta}} \geq e^{\delta/2}, \quad \frac{B'_n}{e^{j\delta}} \leq e^{-\delta/2} \quad \text{or} \quad \frac{B'_n}{e^{j\delta}} \geq e^{\delta/2}$$

for every $n \geq 0$, $j \in \mathbb{Z}$, so condition (26) is satisfied for A'_n, B'_n instead of A_n, B_n. Therefore, we can repeat the proof contained in this section, replacing a_n by a'_n and b_n by b'_n. Since

$$e^{-3\delta/2} A_n \leq A'_n \leq A_n, \quad B_n \leq B'_n \leq e^{3\delta/2} B_n$$

for every $n \geq 0$, this replacement does not spoil the assumptions of Theorem 2.5. Moreover, the values of $\log D_{j_n}^{(n+1)}$, $n \geq 0$, change at most by an additive constant. Hence, using (42), we see that the right-hand sides of the inequalities in (43) do not change, so the upper estimates of the Hausdorff and packing dimensions of $I_{\underline{a}'}^{\underline{b}'}(E_{\underline{\lambda}})$ for $\underline{a}' = (a'_n)_{n=1}^\infty$, $\underline{b}' = (b'_n)_{n=1}^\infty$ are the same as those for $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$. But since $a'_n \leq a_n$ and $b'_n \geq b_n$, we have $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \subset I_{\underline{a}'}^{\underline{b}'}(E_{\underline{\lambda}})$, so the estimates are also valid for $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$.

8. *Proof of Theorem 2.5—estimate from below*

8.1. *Construction of the measure $\tilde{\boldsymbol{\mu}}$.* By (17), we can find $j_0, k_0 \in 2\mathbb{Z}$ such that $K_{j_0,k_0}^{(0)} \subset S_0$. Define families $\tilde{\mathcal{F}}^{(n)}$, $n \geq 0$, by

$$\tilde{\mathcal{F}}^{(0)} = \{\tilde{K}_{j_0,k_0}\} \quad \text{for } \tilde{K}_{j_0,k_0} = K_{j_0,k_0}^{(0)},$$

and

$$\tilde{\mathcal{F}}^{(n)} = \{\tilde{K}_{j_0,k_0,\ldots,j_n,k_n} = g_{k_0}^{(0)} \circ \cdots \circ g_{k_{n-1}}^{(n-1)}(K_{j_n,k_n}^{(n)}) : \\ K_{j_1,k_1}^{(1)} \in \tilde{\mathcal{K}}_{j_0}^{(1)}, \ldots, K_{j_n,k_n}^{(n)} \in \tilde{\mathcal{K}}_{j_{n-1}}^{(n)}, \ j_1, \ldots, j_n \in 2\mathbb{Z}, \ k_1, \ldots, k_n \in 2\mathbb{Z}\}$$

for $n \geq 1$. Note that here we consider only even values of $j_0, k_0, j_1, k_1, \ldots$. Obviously, for every $\tilde{K}_{j_0,k_0,\ldots,j_n,k_n} \in \tilde{\mathcal{F}}^{(n)}$ and $j_{n+1}, k_{n+1} \in 2\mathbb{Z}$,

$$\begin{aligned} &\text{if} \quad \tilde{K}_{j_0,k_0,\ldots,j_{n+1},k_{n+1}} \in \tilde{\mathcal{F}}^{(n+1)}, \\ &\text{then} \quad K_{j_{n+1},k_{n+1}}^{(n+1)} \in \tilde{\mathcal{K}}_{j_n,k_n}^{(n+1)} \quad \text{and} \quad \tilde{K}_{j_0,k_0,\ldots,j_{n+1},k_{n+1}} \subset \tilde{K}_{j_0,k_0,\ldots,j_n,k_n}. \end{aligned} \tag{45}$$

Moreover, the sets $\overline{\tilde{K}_{j_0,k_0,\ldots,j_n,k_n}}$ are pairwise disjoint for given n. Let

$$\tilde{K}_\infty = \bigcap_{n=0}^{\infty} \overline{\bigcup \tilde{\mathcal{F}}^{(n)}} = \bigcap_{n=0}^{\infty} \bigcup \{\overline{\tilde{K}_{j_0,k_0,\ldots,j_n,k_n}} : \tilde{K}_{j_0,k_0,\ldots,j_n,k_n} \in \tilde{\mathcal{F}}^{(n)}\}.$$

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By definition, we have

$$\tilde{K}_\infty \subset \overline{J \cap K^{(0)}_{j_0,k_0}} = J \cap \overline{K^{(0)}_{j_0,k_0}}, \tag{46}$$

since J is closed.

For $\tilde{K}_{j_0,k_0,\dots,j_n,k_n} \in \tilde{\mathcal{F}}^{(n)}$, let

$$\tilde{N}_{j_0,k_0,\dots,j_n,k_n} = \#\{(j_{n+1}, k_{n+1}) : \tilde{K}_{j_0,k_0,\dots,j_{n+1},k_{n+1}} \in \tilde{\mathcal{F}}^{(n+1)}\}.$$

By (8), (15) and Lemma 6.1,

$$\tilde{N}_{j_0,k_0,\dots,j_n,k_n} \ge \tilde{c}_1 D^{(n+1)}_{j_n} e^{j_n\delta} > 0 \tag{47}$$

for a constant $\tilde{c}_1 > 0$.

For every $n \ge 0$ and $\tilde{K}_{j_0,k_0,\dots,j_n,k_n} \in \tilde{\mathcal{F}}^{(n)}$, choose a point

$$\tilde{z}_{j_0,k_0,\dots,j_n,k_n} \in \tilde{K}_{j_0,k_0,\dots,j_n,k_n}$$

and define a sequence of Borel probability measures $\tilde{\mu}_n$, $n \ge 0$, setting

$$\tilde{\mu}_0 = \nu_{\tilde{z}_{j_0,k_0}},$$

$$\tilde{\mu}_{n+1} = \sum_{\tilde{K}_{j_0,k_0,\dots,j_n,k_n} \in \tilde{\mathcal{F}}^{(n)}} \; \sum_{(j_{n+1},k_{n+1}) : \tilde{K}_{j_0,k_0,\dots,j_{n+1},k_{n+1}} \in \tilde{\mathcal{F}}^{(n+1)}} \frac{\nu_{\tilde{z}_{j_0,k_0,\dots,j_{n+1},k_{n+1}}}}{\tilde{N}_{j_0,k_0} \cdots \tilde{N}_{j_0,k_0,\dots,j_n,k_n}}.$$

By definition, if $\tilde{K}_{j_0,k_0,\dots,j_n,k_n} \in \tilde{\mathcal{F}}^{(n)}$, then

$$\tilde{\mu}_m(\tilde{K}_{j_0,k_0,\dots,j_n,k_n}) = \tilde{\mu}_n(\tilde{K}_{j_0,k_0,\dots,j_n,k_n}) = \frac{1}{\tilde{N}_{j_0,k_0} \cdots \tilde{N}_{j_0,k_0,\dots,j_{n-1},k_{n-1}}} \tag{48}$$

for every $m \ge n$. Hence, taking a weak limit along a subsequent of $\tilde{\mu}_n$, we find a Borel probability measure $\tilde{\mu}$ such that

$$\operatorname{supp} \tilde{\mu} \subset \tilde{K}_\infty \tag{49}$$

and

$$\tilde{\mu}(\tilde{K}_{j_0,k_0,\dots,j_n,k_n}) \le \tilde{\mu}_n(\tilde{K}_{j_0,k_0,\dots,j_n,k_n})$$

for $\tilde{K}_{j_0,k_0,\dots,j_n,k_n} \in \tilde{\mathcal{F}}^{(n)}$, so by (47) and (48),

$$\tilde{\mu}(\tilde{K}_{j_0,k_0,\dots,j_n,k_n}) \le \frac{\tilde{c}_1^{-n}}{D^{(1)}_{j_0} \cdots D^{(n)}_{j_{n-1}} e^{(j_0+\dots+j_{n-1})\delta}}. \tag{50}$$

8.2. *Estimate of the local dimension of* $\tilde{\boldsymbol{\mu}}$. Take a point $z \in \tilde{K}_\infty$. Then there exist $j_1, k_1, j_2, k_2, \dots \in 2\mathbb{Z}$ such that $\tilde{K}_{j_0,k_0,\dots,j_n,k_n} \in \tilde{\mathcal{F}}^{(n)}$ for every $n \ge 0$,

$$\tilde{K}_{j_0,k_0,\dots,j_n,k_n} \subset \dots \subset \tilde{K}_{j_0,k_0}.$$

Set

$$\tilde{d}_n = \operatorname{diam} \tilde{K}_{j_0,k_0,\dots,j_n,k_n}, \quad \tilde{z}_n = \tilde{z}_{j_0,k_0,\dots,j_n,k_n}.$$

In the same way as for (29), we show

$$\tilde{d}_{n+1} < \frac{\tilde{d}_n}{Q}, \tag{51}$$

where $Q > 0$ is a constant, which can be chosen arbitrarily large, provided a is big enough. In particular, this implies that z is the unique point of $\bigcap_{n=0}^{\infty} \tilde{K}_{j_0,k_0,\dots,j_n,k_n}$. Since $z, \tilde{z}_n \in \tilde{K}_{j_0,k_0,\dots,j_n,k_n}$, we have

$$|z - \tilde{z}_n| \le \tilde{d}_n. \tag{52}$$

Let

$$\tilde{r}_n = \frac{\tilde{d}_n}{\tilde{C}}$$

for a large constant $\tilde{C} > 0$. By (51), the sequence $\tilde{r}_n$ is strictly decreasing to 0. To estimate $\tilde{\mu}(\mathbb{D}(z, r))$ for a small r, take n such that

$$\tilde{r}_{n+1} \le r < \tilde{r}_n,$$

let

$$\tilde{R} = \frac{\sqrt{\tilde{C}}r}{\tilde{d}_{n+1}}$$

and note that if r varies in $[\tilde{r}_{n+1}, \tilde{r}_n)$, then $\tilde{R}$ varies in $[\tilde{R}_-^{(n)}, \tilde{R}_+^{(n)})$ for

$$\tilde{R}_-^{(n)} = \frac{1}{\sqrt{\tilde{C}}}, \qquad \tilde{R}_+^{(n)} = \frac{1}{\sqrt{\tilde{C}}}\frac{\tilde{d}_n}{\tilde{d}_{n+1}}.$$

By Lemma 6.3, we have

$$\frac{1}{\tilde{c}_2^{n+1}\sqrt{\tilde{C}}}\frac{\tilde{R}}{e^{(j_0+\cdots+j_n)\delta}} < r < \frac{\tilde{c}_2^{n+1}}{\sqrt{\tilde{C}}}\frac{\tilde{R}}{e^{(j_0+\cdots+j_n)\delta}} \tag{53}$$

and

$$\frac{e^{j_n\delta}}{\tilde{c}_2\sqrt{\tilde{C}}} \le \tilde{R}_+^{(n)} \le \frac{\tilde{c}_2}{\sqrt{\tilde{C}}}e^{j_n\delta} \tag{54}$$

for some constant $\tilde{c}_2 > 0$. Enlarging also $\tilde{C}$, by Lemma 6.1 and (54) we can assume

$$\tilde{R}_-^{(n)} < D_{j_n}^{(n+1)} < \tilde{c}_3\sqrt{\tilde{C}}\tilde{R}_+^{(n)} \tag{55}$$

for some constant $\tilde{c}_3 > 0$.

Let

$$\tilde{w} = E_{\lambda_{N+n}} \circ \cdots \circ E_{\lambda_N}(\tilde{z}_{n+1}).$$

Then $\tilde{w} \in K_{j_{n+1},k_{n+1}}^{(n+1)} \in \tilde{\mathcal{K}}_{j_n,k_n}^{(n+1)}$. Take $j_1', \dots, j_{n+1}' \in 2\mathbb{Z}$, $k_1', \dots, k_{n+1}' \in 2\mathbb{Z}$ such that $\tilde{K}_{j_0,k_0,j_1',k_1',\dots,j_{n+1}',k_{n+1}'} \in \tilde{\mathcal{F}}^{(n+1)}$ and $(j_1', k_1', \dots, j_{n+1}', k_{n+1}') \ne (j_1, k_1, \dots, j_{n+1}, k_{n+1})$. Let

$$m = \min(\{s \in [1, n+1] : (j_s', k_s') \ne (j_s, k_s)\}).$$

We have $\operatorname{dist}(K_{j_m,k_m}^{(m)}, K_{j_m',k_m'}^{(m)}) = \delta$, so, by Lemma 6.3 and (45),

$$\operatorname{dist}(\tilde{z}_{n+1}, \tilde{K}_{j_0,k_0,j_1',k_1',\dots,j_{n+1}',k_{n+1}'}) \ge \operatorname{dist}(\tilde{K}_{j_0,k_0,j_1,k_1,\dots,j_m,k_m}, \tilde{K}_{j_0,k_0,j_1',k_1',\dots,j_m',k_m'}) > \tilde{c}_4\tilde{d}_m$$

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for some constant $\tilde{c}_4 > 0$. Hence, if $m \le n$, then by (51) and (52),

$$\operatorname{dist}(z, \tilde{K}_{j_0,k_0,j_1',k_1',\ldots,j_{n+1}',k_{n+1}'}) \ge \left(\tilde{c}_4 - \frac{1}{Q}\right)\tilde{d}_n > \frac{\tilde{c}_4 \tilde{C} r}{2} > r,$$

if $\tilde{C}$ and Q are chosen sufficiently large. Consequently, if $\tilde{K}_{j_0,k_0,j_1',k_1',\ldots,j_{n+1}',k_{n+1}'}$ intersects $\mathbb{D}(\tilde{z}, r)$, then $(j_1', k_1', \ldots, j_n', k_n') = (j_1, k_1, \ldots, j_n, k_n)$. Furthermore, if $\mathbb{D}(\tilde{w}, \tilde{R})$ does not intersect $K^{(n+1)}_{j_{n+1}',k_{n+1}'}$, then by Lemma 6.3,

$$\operatorname{dist}(\tilde{z}_{n+1}, \tilde{K}_{j_0,k_0,j_1',k_1',\ldots,j_{n+1}',k_{n+1}'}) \ge \tilde{c}_5 \tilde{R} \tilde{d}_{n+1}$$

for some constant $\tilde{c}_5 > 0$, so by (52),

$$\operatorname{dist}(z, \tilde{K}_{j_0,k_0,j_1',k_1',\ldots,j_{n+1}',k_{n+1}'}) \ge (\tilde{c}_5 \tilde{R} - 1)\tilde{d}_{n+1} = \tilde{c}_5 \sqrt{\tilde{C}} r - \frac{\tilde{r}_{n+1}}{\tilde{C}} \ge \left(c_5 \sqrt{\tilde{C}} - \frac{1}{\tilde{C}}\right) r > r$$

provided $\tilde{C}$ is chosen sufficiently large. We conclude that if $\tilde{K}_{j_0,k_0,j_1',k_1',\ldots,j_{n+1}',k_{n+1}'} \in \tilde{\mathcal{F}}^{(n+1)}$ and $\tilde{K}_{j_0,k_0,j_1',k_1',\ldots,j_{n+1}',k_{n+1}'}$ intersects $\mathbb{D}(\tilde{z}, r)$, then $(j_1', k_1', \ldots, j_n', k_n') = (j_1, k_1, \ldots, j_n, k_n)$ and $\mathbb{D}(\tilde{w}, \tilde{R})$ intersects $K^{(n+1)}_{j_{n+1}',k_{n+1}'}$. Note also that in this case we have $K^{(n+1)}_{j_{n+1}',k_{n+1}'} \in \tilde{\mathcal{K}}^{(n+1)}_{j_n,k_n}$, which follows from (54), if $\tilde{C}$ is chosen sufficiently large. Since by Lemma 6.1, the set $\bigcup \tilde{\mathcal{K}}^{(n+1)}_{j_n,k_n}$ is contained in a vertical strip of width $D^{(n+1)}_{j_n}$ passing through $\tilde{w}$, the disc $\mathbb{D}(\tilde{w}, \tilde{R})$ intersects at most $\tilde{M}$ sets $K^{(n+1)}_{j_{n+1}',k_{n+1}'} \in \tilde{\mathcal{K}}^{(n+1)}_{j_n,k_n}$, where

$$\tilde{M} = \begin{cases} \tilde{c}_6 \tilde{R}^2 & \text{if } \tilde{R} \le D^{(n+1)}_{j_n}, \\ \tilde{c}_6 D^{(n+1)}_{j_n} \tilde{R} & \text{if } \tilde{R} > D^{(n+1)}_{j_n}, \end{cases}$$

for some constant $\tilde{c}_6 > 0$. By (50),

$$\tilde{\mu}(\mathbb{D}(\tilde{z}, r)) \ge \begin{cases} \dfrac{\tilde{c}_6 \tilde{c}_1^{-(n+1)} \tilde{R}^2}{D^{(1)}_{j_0} \cdots D^{(n+1)}_{j_n} e^{(j_0+\cdots+j_n)\delta}} & \text{if } \tilde{R} \le D^{(n+1)}_{j_n}, \\ \dfrac{\tilde{c}_6 c_1^{-(n+1)} \tilde{R}}{D^{(1)}_{j_0} \cdots D^{(n)}_{j_{n-1}} e^{(j_0+\cdots+j_n)\delta}} & \text{if } \tilde{R} > D^{(n+1)}_{j_n}, \end{cases}$$

so, by (53),

$$\frac{\log \tilde{\mu}(\mathbb{D}(z, r))}{\log r} \le 1 + \tilde{h}_n(\tilde{R}), \tag{56}$$

where

$$\tilde{h}_n(x) = \begin{cases} \dfrac{\log(D^{(1)}_{j_0} \cdots D^{(n+1)}_{j_n}) - \log x + \tilde{c}_7 n}{(j_0 + \cdots + j_n)\delta - \log x - \tilde{c}_7 n} & \text{if } x \le D^{(n+1)}_{j_n}, \\ \dfrac{\log(D^{(1)}_{j_0} \cdots D^{(n)}_{j_{n-1}}) + \tilde{c}_7 n}{(j_0 + \cdots + j_n)\delta - \log x - \tilde{c}_7 n} & \text{if } x > D^{(n+1)}_{j_n}, \end{cases}$$

for $x \in [\tilde{R}^{(n)}_-, \tilde{R}^{(n)}_+)$ and some constant $\tilde{c}_7 > 0$. Note that $j_0, j_1, \ldots$ satisfy (42). In the same way as for Lemma 7.1, using (55) instead of (38), we prove the following.

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LEMMA 8.1. *We have*

$$\lim_{n\to\infty}\left|\inf_{[R_-^{(n)},R_+^{(n)})}\tilde{h}_n - \frac{\log(D_{j_0}^{(1)}\cdots D_{j_{n-1}}^{(n)})}{(j_0+\cdots+j_n)\delta - \log D_{j_n}^{(n+1)}}\right| = 0,$$

$$\lim_{n\to\infty}\left|\sup_{[R_-^{(n)},R_+^{(n)})}\tilde{h}_n - \max\left(\frac{\log(D_{j_0}^{(1)}\cdots D_{j_{n-1}}^{(n)})}{(j_0+\cdots+j_{n-1})\delta}, \frac{\log(D_{j_0}^{(1)}\cdots D_{j_n}^{(n+1)})}{(j_0+\cdots+j_n)\delta}\right)\right| = 0.$$

8.3. *Conclusion.* By (46), (49), (56) and Lemma 8.1, we can find $j_0, k_0 \in 2\mathbb{Z}$ such that for $\tilde{\mu}$-almost every $z \in J \cap K_{j_0,k_0}^{(0)}$ there exist $j_1, k_1, j_2, k_2, \ldots \in 2\mathbb{Z}$ with $z = \lim_{n\to\infty} z_{j_0,k_0,\ldots,j_n,k_n}$, and

$$\liminf_{r\to 0}\frac{\log\tilde{\mu}(\mathbb{D}(z,r))}{\log r} \ge 1 + \liminf_{n\to\infty}\frac{\log(D_{j_0}^{(1)}\cdots D_{j_{n-1}}^{(n)})}{(j_0+\cdots+j_n)\delta - \log D_{j_n}^{(n+1)}},$$

$$\limsup_{r\to 0}\frac{\log\tilde{\mu}(\mathbb{D}(z,r))}{\log r} \ge 1 + \limsup_{n\to\infty}\frac{\log(D_{j_0}^{(1)}\cdots D_{j_n}^{(n+1)})}{(j_0+\cdots+j_n)\delta}.$$

This, together with Lemma 6.1, (8) and (42), implies

$$\begin{aligned}\liminf_{r\to 0}\frac{\log\tilde{\mu}(\mathbb{D}(z,r))}{\log r} &\ge 1 + \liminf_{n\to\infty}\Phi_n(\delta j_0,\ldots,\delta j_n),\\ \limsup_{r\to 0}\frac{\log\tilde{\mu}(\mathbb{D}(z,r))}{\log r} &\ge 1 + \limsup_{n\to\infty}\Psi_n(\delta j_0,\ldots,\delta j_n),\end{aligned} \tag{57}$$

for Φ, Ψ defined in (44). Again, by the standard dimension estimates, (57) shows that

$$\dim_H J \ge 1 + \inf_x \liminf_{n\to\infty}\Phi_n(x_0,\ldots,x_n),$$

$$\dim_P J \ge 1 + \inf_x \limsup_{n\to\infty}\Psi_n(x_0,\ldots,x_n),$$

for $x = (x_0, x_1, \ldots) \in [\log a_N, \log b_N] \times [\log a_{N+1}, \log b_{N+1}] \times \cdots$. Together with (6), this proves the lower estimate in Theorem 2.5.

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