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On the Continued Fraction Expansion of Fixed Period in Finite Fields

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Abstract. The Chowla conjecture states that if t is any given positive integer, there are infinitely many prime positive integers N such that $\operatorname{Per}(\sqrt{N}) = t$, where $\operatorname{Per}(\sqrt{N})$ is the period length of the continued fraction expansion for $\sqrt{N}$. C. Friesen proved that, for any $k \in \mathbb{N}$, there are infinitely many square-free integers N, where the continued fraction expansion of $\sqrt{N}$ has a fixed period. In this paper, we describe all polynomials $Q \in \mathbb{F}_q[X]$ for which the continued fraction expansion of $\sqrt{Q}$ has a fixed period. We also give a lower bound of the number of monic, non-squares polynomials Q such that $\deg Q = 2d$ and $Per\sqrt{Q} = t$.

1 Introduction

Let $\mathbb{F}_q$ be the finite field of odd characteristic with q elements, and denote by $\mathbb{F}_q((X^{-1}))$ the field of formal Laurent series in X^{-1} over $\mathbb{F}_q$ given by

$$\mathbb{F}_q((X^{-1})) = \Big\{ \sum_{i \geq n} w_i X^{-i},\ w_i \in \mathbb{F}_q,\ n \in \mathbb{Z} \Big\}.$$

We have the inclusions $\mathbb{F}_q[X] \subset \mathbb{F}_q(X) \subset \mathbb{F}_q((X^{-1}))$. Elements in $\mathbb{F}_q(X)$ are called *rational*, and those which lie in $\mathbb{F}_q((X^{-1}))$ but not in $\mathbb{F}_q(X)$ are called *irrational*. We define a norm on $\mathbb{F}_q((X^{-1}))$ as follows: If $w \in \mathbb{F}_q((X^{-1}))$ is non-zero, then we can write $w = \sum_{i \geq n} w_i X^{-i}$, where $w_n \neq 0$. In this case we define $|w| = q^{-n}$. If $w = 0$ we define $|w| = 0$. Observe that if $w = P/Q$ is rational Laurent series with $P, Q \in \mathbb{F}_q[X]$ then $|w| = q^{\deg P - \deg Q}$. We denote by $\overline{\mathbb{F}_q((X^{-1}))}$ an algebraic closure of $\mathbb{F}_q((X^{-1}))$. We note that the absolute value has a unique extension to $\overline{\mathbb{F}_q((X^{-1}))}$. To denote this extended absolute value, we also use the symbol $|\cdot|$.The notation $[\cdot]$ will be used to denote both the polynomial part of an element of $\mathbb{F}_q((X^{-1}))$ as well as the integer part of a real number.

For more information about formal power series, see [2, 10].

It is easy to verify that a continued fraction theory exists for the field $\mathbb{F}_q((X^{-1}))$ (see [1, 5]), in particular, any irrational Laurent series $w \in \mathbb{F}_q((X^{-1}))$ has a unique infinite continued fraction expansion

$$w = A_0 + \cfrac{1}{A_1 + \frac{1}{A_2 + \frac{1}{\ddots}}}, \tag{1.1}$$

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where $A_j \in \mathbb{F}_q[X]$, with $\deg A_j \geq 1$ for $j \geq 1$. As a shorthand for (1.1), we write

$$w = [A_0; A_1, A_2, \dots],$$

and, as usual, we refer to $\frac{H_n}{K_n} = [A_0; A_1, A_2, \dots, A_n]$ $(n \geq 0)$ as the n-th convergent to w and call the polynomials A_j $(j \geq 0)$ the partial quotients of w. We also have $H_0 = A_0$, $H_1 = A_0 A_1 + 1$, $K_0 = 1$, $K_1 = A_1$, and in general

$$H_n = A_n H_{n-1} + H_{n-2} \quad \text{and} \quad K_n = A_n K_{n-1} + K_{n-2} \quad (n \geq 2). \tag{1.2}$$

Readers interested in an overview of basic results concerning continued fractions over $\mathbb{F}_q((X^{-1}))$ are referred to [1, 8, 9, 11].

2 Main Results

The purpose of this paper is to describe all polynomials $Q \in \mathbb{F}_q[X]$ for which the continued fraction expansion of $\sqrt{Q}$ has a fixed period. Then our main result is stated as follows.

Theorem 2.1 *The equation $\sqrt{Q} = \big[[\sqrt{Q}]; \overline{A_1, \dots, A_{t-1}, 2[\sqrt{Q}]}\big]$ has, for any symmetric $(t-1)$–tuple $(A_1, \dots, A_{t-1})$ (i.e., $(A_1, \dots, A_{t-1}) = (A_{t-1}, \dots, A_1)$) of positive degree polynomials, infinitely many non-squares solutions Q.*

We get the following corollary immediately.

Corollary 2.2 *For any positive integer t there exist infinitely many non-squares polynomials Q with $\sqrt{Q}$ having a continued fraction expansion of period t.*

In the real case, Friesen [4] shows that given any symmetric $(t-1)$-tuple of positive integers, $(a_1, \dots, a_{t-1})$, if $Q_{-1} = 0$, $Q_0 = 1$ and $Q_n = a_n Q_{n-1} + Q_{n-2}$, for $n = 1, 2, \dots, t-1$, then the equation $\sqrt{N} = \big[[\sqrt{N}]; \overline{a_1, \dots, a_{t-1}, 2[\sqrt{N}]}\big]$ has infinitely many square-free solutions N whenever either Q_{t-2} or $(Q_{t-2}^2 - (-1)^t)/Q_{t-1}$ is even. If both quantities are odd, then there are no solutions N even if the square-free condition is dropped.

We will be concerned with computing the number of non-squares polynomials such that $\deg Q = 2d$ and $Per\sqrt{Q} = t$. Let

$$\theta(d, t) = \sharp\big\{ Q \in \mathbb{F}_q[X] : Q \text{ monic}, \deg Q = 2d \text{ and } \operatorname{Per}(\sqrt{Q}) = t \big\};$$

then we have the following theorem.

Theorem 2.3 (i) *Let $t \geq 3$ and $d \geq t-2$. Then*

$$\theta(d, t) \geq \begin{cases} (q-1)^p q^d \sum_{n=p}^{[d/2]} \binom{n-1}{p-1} q^{-n} & \text{if } t = 2p+1, \\ (q-1)^p q^d \sum_{k=1}^{d-2p-2} \sum_{n=p-1}^{[(d-k)/2]} \binom{n-1}{p-2} q^{-n} & \text{if } t = 2p. \end{cases}$$

(ii) *$\theta(d, 1) = q^d$ and $\theta(d, 2) = (dq - d - 1)q^d$, for $d \geq 1$.*

The remainder of the paper is organised in the following way. Section 3 will be devoted to explaining basic algebraic properties in the field of formal Laurent series,

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some definitions, theorems and lemmas are given in this section. Some elementary properties of periodic continued fractions are also given. In Section 4, Theorems 2.1 and 2.3 are established.

3 Formal Power Series

3.1 Algebraic Properties

Theorem 3.1 *For $n \geq 2$ with $\gcd(n, q) = 1$, let Q be a monic polynomial $\in \mathbb{F}_q[X]$ that is not an n-th power and $\deg Q \equiv 0 \pmod n$. If*

$$P(Y) = Y^n - Q,$$

then P has unique root $f \in \mathbb{F}_q((X^{-1}))$ such that $[f] = T$, where $Q = T^n - S$, and $T(\text{monic})$, $S \in \mathbb{F}_q[X]$ such that $0 \leq \deg S < (n-1)\deg T$.

The proof of Theorem 3.1 uses following lemmas.

Lemma 3.2 (See [7]) *Let $P(Y) = A_d Y^d + \cdots + A_0$, with $A_i \in \mathbb{F}_q[X]$ and $|A_{d-1}| > \max_{i \neq d-1} |A_i|$. Then P has only one root $w \in \mathbb{F}_q((X^{-1}))$ satisfying $|w| > 1$. Moreover, $[w] = -[\frac{A_{d-1}}{A_d}]$, and all conjugates of w in $\overline{\mathbb{F}_q((X^{-1}))}$ have an absolute value strictly smaller than 1.*

Lemma 3.3 *Let $Q \in \mathbb{F}_q[X]$, $n \geq 2$, such that $\gcd(n, q) = 1$ and $\deg Q \equiv 0 \pmod n$. Then the polynomial Q is uniquely expressible as $\alpha T^n - S$, where $T(\text{monic})$, $S \in \mathbb{F}_q[X]$ such that $\deg S < (n-1)\deg T$ and $\alpha \in \mathbb{F}_q \setminus \{0\}$.*

Proof We may assume without loss of generality that Q is a monic polynomial such that $\deg Q = dn$,

$$Q = \sum_{i=0}^{dn} \alpha_i X^i \quad \text{and} \quad T = \sum_{i=0}^{d} \beta_i X^i, \quad \beta_d = 1$$

We must, in order to have $\deg(Q - T^n) < (n-1)\deg T$, require that for all $k \in \{(n-1)d, \ldots, nd\}$,

$$\alpha_k = \sum_{\substack{i_1 + \cdots + i_n = k \\ 0 \leq i_j \leq d}} \beta_{i_1} \cdots \beta_{i_n}. \tag{3.1}$$

We establish the lemma by resolving the system (3.1). Assuming that all β_i are known for $s < i \leq d$, then from (3.1),

$$\alpha_{(n-1)d+s} = \sum_{\substack{i_1 + \cdots + i_n = (n-1)d+s \\ 0 \leq i_j \leq d}} \beta_{i_1} \cdots \beta_{i_n} = n\beta_s + \sum_{\substack{i_1 + \cdots + i_n = (n-1)d+s \\ 0 \leq i_j \leq d, i_j \neq s}} \beta_{i_1} \cdots \beta_{i_n},$$

hence

$$\beta_s = \frac{1}{n}\alpha_{(n-1)d+s} - \frac{1}{n}\sum_{\substack{i_1+\cdots+i_n=(n-1)d+s\\ 0\le i_j\le d,\, i_j\neq s}} \beta_{i_1}\cdots\beta_{i_n}.$$

This completes the proof of the lemma. ∎

Corollary 3.4 *Let Q be a monic polynomial $\in \mathbb{F}_q[X]$, $n \ge 2$ such that* $\gcd(n,q) = 1$ *and* $\deg Q \equiv 0 \pmod n$*. Then Q is not n-th power if and only if $Q = T^n - S$, where T*(monic)*, $S \in \mathbb{F}_q[X] \setminus \{0\}$ such that* $\deg S < (n-1)\deg T$.

Now, we are prepared to give the proof of Theorem 3.1.

Proof Put $Y = T + \frac{1}{Z}$; then

$$P(Y) = 0 \tag{3.2}$$

$$\Updownarrow$$

$$SZ^n + nT^{n-1}Z^{n-1} + \cdots + \binom{n}{n-k}T^kZ^k + \cdots + 1 = 0 \tag{3.3}$$

From Lemma 3.2, the equation (3.3) has a unique root $g \in \mathbb{F}_q((X^{-1}))$ such that $[g] = -n[\frac{T^{n-1}}{S}]$; consequently, $f = T + \frac{1}{g}$ is the unique root in $\mathbb{F}_q((X^{-1}))$ of (3.2) with $[f] = T$. ∎

We shall use the notation $\sqrt[n]{Q}$ to designate the unique root in $\mathbb{F}_q((X^{-1}))$ of equation (3.2) with $[\sqrt[n]{Q}] = T$, where $Q = T^n - S$, $0 \le \deg S < (n-1)\deg T$.

Corollary 3.5 *Let Q be a monic and non-square polynomial in $\mathbb{F}_q[X]$ with even degree. Then there exist a monic polynomial T and $S \in \mathbb{F}_q[X] \setminus \{0\}$ such that*

$$Q = T^2 - S \quad \textit{with} \quad \deg S < \deg T$$

as well as algebraic equations

$$\sqrt{Q} = T + \frac{1}{g_Q} \quad \textit{and} \quad Sg_Q^2 + 2Tg_Q + 1 = 0,$$

where g_Q is a quadratic formal power series such that $|g_Q| > 1$.

3.2 Periodic Continued Fractions

We say that a regular continued fraction is periodic or ultimately periodic if it consists of an initial block of length n followed by a repeating block of length m; *i.e.,* if it is of the form

$$[A_0; A_1, \dots, A_n, \overline{A_{n+1}, \dots, A_{n+m}}\,],$$

where $[A_0; A_1, \dots, A_n, \overline{A_{n+1}, \dots, A_{n+m}}\,]$ means that $A_{n+1+km} = A_{n+1}, \dots, A_{n+(k+1)m} = A_{n+m}$, for every $k \ge 1$. Moreover, no block of length shorter than m has this property, and the initial block does not end with a copy of the repeating block.

If the initial block has length 0, we say that the continued fraction is purely periodic.

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We recall also the definition of quadratic irrational Laurent series. A Laurent series w is called *quadratic irrational* if it is a root of a polynomial AY^2+BY+C with $A, B, C \in \mathbb{F}_q[X]$, $A \neq 0$, and $B^2 - 4AC$ is not a perfect square.

There is a classical result (the analogue of Lagrange' Theorem).

Proposition 3.6 *Let w be an algebraic element over $\mathbb{F}_q(X)$. Then α is quadratic if and only if the continued fraction expansion of w is ultimately periodic.*

One can prove this result by following the proof in the real case as in [8]. Note the following characterization of purely periodic power series, which is the analogue of Galois' Theorem.

Proposition 3.7 (See [7]) *A quadratic formal power series w of non zero integral part is purely periodic if and only if w satisfies an equation*

$$Aw^2 + Bw + C = 0,$$

with $A, B, C \in \mathbb{F}_q[X] \smallsetminus \{0\}$, $\deg B > \max(\deg A, \deg C)$.

Furthermore, if $w = [\overline{A_1, A_2, \dots, A_t}]$, then the algebraic conjugate of w is

$$-\Big(w + \frac{B}{A}\Big) = [0; \overline{-A_t, -A_{t-1}, \dots, -A_1}].$$

Before giving the proof of Theorem 2.1, we establish a few basic facts about the continued fraction expansion of $\sqrt{Q}$.

Lemma 3.8 *Let Q be a monic and non-square polynomial in $\mathbb{F}_q[X]$ with even degree; then the period of the continued fraction expansion of $\sqrt{Q}$ starts with the second term. Furthermore, if the period consists of the t terms $A_0, \dots, A_{t-1}$, then $A_{t-1} = 2[\sqrt{Q}]$, and the sequence $A_0, \dots, A_{t-2}$ is symmetric.*

Proof By Corollary 3.5, we have $Q = T^2 - S$ with T and $S \in \mathbb{F}_q[X] \smallsetminus \{0\}$ and $\deg S < \deg T$, as well as

$$\sqrt{Q} = T + \frac{1}{g_Q} \quad \text{and} \quad S g_Q^2 + 2T g_Q + 1 = 0, \quad |g_Q| > 1. \tag{3.4}$$

Let $w = 2T + \frac{1}{g_Q}$. Then $w^2 - 2Tw - S = 0$, so by Proposition 3.7, continued fraction expansions of g_Q and w are purely periodic, and furthermore, if $g_Q = [\overline{A_0, A_1, \dots, A_{t-1}}]$, then

$$\sqrt{Q} = [T; \overline{A_0, A_1, \dots, A_{t-1}}] \quad \text{and} \quad w = [\overline{2T, A_0, A_1, \dots, A_{t-2}}]. \tag{3.5}$$

Consequently,

$$A_{t-1} = 2T = 2\big[\sqrt{Q}\big]. \tag{3.6}$$

On the other hand, from (3.4) and by trivial computation, we obtain that

$$w = 2T + \frac{1}{g_Q} = \frac{1}{g_Q + \frac{2T}{S}}, \tag{3.7}$$

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so by Proposition 3.7 and (3.6),

$$g_Q + \frac{2T}{S} = \left[\, 0; \overline{2T, A_{t-2}, \dots, A_0}\, \right],$$

and hence

$$\left(g_Q + \frac{2T}{S}\right)^{-1} = \left[\, \overline{2T, A_{t-2}, \dots, A_0}\, \right]. \tag{3.8}$$

From (3.5), (3.7), and (3.8), we have

$$\left[\, \overline{2T, A_0, A_1, \dots, A_{t-2}}\, \right] = \left[\, \overline{2T, A_{t-2}, A_{t-3}, \dots, A_0}\, \right],$$

so by identifying coefficients, we obtain that $A_0 = A_{t-2}, \dots, A_k = A_{t-k-2}$. ∎

Lemma 3.9 *Set $g_Q = [\overline{A_0, A_1, \dots, A_{t-1}}]$, $t \geq 3$ and let $\frac{H_i}{K_i}$ be the i-th convergent of g_Q. Then $H_{t-3} = K_{t-2}$ and $K_{t-1} = -SH_{t-2}$.*

Proof First, we have that

$$\frac{H_{t-2}}{H_{t-3}} = [A_{t-2}; A_{t-3}, \dots, A_0].$$

Since

$$\frac{H_{t-2}}{K_{t-2}} = [A_0; A_1, \dots, A_{t-2}]$$

and $A_0, A_1, \dots, A_{t-2}$ is symmetric, we have

On the other hand, from the identity

$$g_Q = \frac{H_{t-1} g_Q + H_{t-2}}{K_{t-1} g_Q + K_{t-2}}$$

we get

$$K_{t-1} g_Q^2 + (K_{t-2} - H_{t-1}) g_Q - H_{t-2} = 0. \tag{3.9}$$

From (3.4) and (3.9), we deduce that $K_{t-1} = -SH_{t-2}$. ∎

4 Proofs of the Main Results

Proof of Theorem 2.1 In the sequel, we assume that $t \geq 3$. By Lemma 3.9 and (1.2), we have

$$2TK_{t-2} + K_{t-3} = K_{t-1} = -SH_{t-2}, \tag{4.1}$$

$$K_{t-2} = H_{t-3}. \tag{4.2}$$

Equations (4.1) and (4.2) give us a necessary and sufficient condition for $Q = T^2 - S$ to be a polynomial solution of the equation

$$\sqrt{Q} = \left[\left[\sqrt{Q}\right]; \overline{A_0; \dots, A_{t-2}, 2\left[\sqrt{Q}\right]}\right]. \tag{4.3}$$

In fact, we have the identity

$$H_{t-3} K_{t-2} - H_{t-2} K_{t-3} = (-1)^t, \tag{4.4}$$

and the equation (4.1) yields

$$2TK_{t-2} + SH_{t-2} = -K_{t-3}. \tag{4.5}$$ ∎

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Example 4.1 (i) In $\mathbb{F}_3$, let $t = 3$ and let $(A_0 = X, A_1 = X)$ the symmetric pair. It is clear that

$$K_{t-3} = K_0 = 1, \quad K_{t-2} = K_1 = X, \quad H_{t-3} = H_0 = X,$$
$$H_{t-2} = H_1 = X^2 + 1, \quad 2T = (X^2 + 1)P + X, \quad S = -(XP + 1).$$

Then the only polynomial solutions of (4.3) are $(X^2 + 1)^2 P^2 + 2X^3 P + X^2 + 1$, where P is any polynomial in $\mathbb{F}_q[X]$, and

$$\sqrt{(X^2+1)^2P^2 + 2X^3P + X^2 + 1} = \big[2(X^2+1)P + 2X; \overline{X, X, (X^2+1)P + X}\,\big].$$

(ii) In $\mathbb{F}_3$, let $t = 4$ and let $(A_0 = X, A_1 = X + 1, A_2 = X)$ be the symmetric triple. It is clear that

$$K_{t-3} = K_1 = X + 1, \quad K_{t-2} = K_2 = X^2 + X + 1,$$
$$H_{t-3} = H_1 = X^2 + X + 1, \quad H_{t-2} = H_2 = X^3 + X^2 + 2X,$$
$$2T = (X^3 + X^2 - X)P - X^3 + X^2 + X - 1, \quad S = -(X^2 + X + 1)P + X^2 - X + 1.$$

Then the only polynomial solutions of (4.3) are $(X^3 + X^2 - X)^2P^2 + (X^6 + X^3 + 1)P + X^6 + X^5 + 2X^4 + X^3 + X^2 - X$, where P is any polynomial in $\mathbb{F}_q[X]$, and

$$\begin{aligned}((X^3 + X^2 - X)^2P^2 + (X^6 + X^3 + 1)P + X^6 + X^5 + 2X^4 + X^3 + X^2 - X)^{\frac{1}{2}} = \\ \big[2(X^3 + X^2 - X)P + X^3 + 2X^2 + 2X + 1; \\ \overline{X, X + 1, X, (X^3 + X^2 - X)P - X^3 + X^2 + X - 1}\,\big].\end{aligned}$$

Remark 4.2 For the special cases $t = 1$ and $t = 2$, we have the following cases.

Case $t = 1$. All polynomial solutions of (4.3) are of the form $Q = T^2 + 1$, where T is a monic polynomial in $\mathbb{F}_q[X] \setminus \mathbb{F}_q$ and $\sqrt{Q} = [T; \overline{2T}\,]$.

Case $t = 2$. All polynomial solutions of (4.3) are of the form $Q = T^2 + H$, where T is a monic polynomial in $\mathbb{F}_q[X] \setminus \mathbb{F}_q$, H is a divisor of T with $H \neq 1$, $\deg H < \deg T$ and $\sqrt{Q} = [T; \overline{2T/H, 2T}\,]$.

Proof of Theorem 2.3 Let $t \geq 3$, $d \geq t - 2$, $(A_0, A_1, A_2, \ldots, A_{t-2})$ be any symmetric $(t-1)$–tuple of positive degree polynomials, let $r = [A_0; A_1, A_2, \ldots, A_{t-2}]$, and let $\frac{H_i}{K_i}$ be the i-th convergent of r. It is clear by (1.2) that

$$\deg H_i = \sum_{k=0}^{i} \deg A_k. \tag{4.6}$$

Our goal is to give a lower bound for $\theta(d, t)$ to the number of monic polynomials Q of degree $2d$ such that $\mathrm{Per}(\sqrt{Q}) = t$; in other words, $\theta(d, t)$ is the number of monic polynomials T with degree d satisfying (4.5). Note that if

$$\Phi_0(d, t) = \sharp\Big\{\, T(\text{monic}) \in \mathbb{F}_q[X],\ \text{satisfying (4.5) and } \deg H_{t-2} = \sum_{k=0}^{t-2} \deg A_k \leq d \Big\}$$

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and

$$\Phi(d,t)=\sharp\Big\{\,T\in\mathbb{F}_q[X],\ \text{satisfying (4.5) and } \deg H_{t-2}=\sum_{k=0}^{t-2}\deg A_k\le d\Big\},$$

then $\theta(d,t)\ge\Phi_0(d,t)$ and $\Phi_0(d,t)=\frac{1}{q-1}\Phi(d,t)$.

Next, we will concentrate on the estimation of $\Phi(d,t)$. By the division algorithm, there are $\alpha, R\in\mathbb{F}_q[X]$ such that

$$(-1)^tK_{t-3}H_{t-3}=\alpha H_{t-2}+R \quad\text{and}\quad \deg R<\deg H_{t-2}. \tag{4.7}$$

Choosing $\beta\in\mathbb{F}_q[X]$ such that $\deg\beta=d-\deg H_{t-2}$ and $2T=\beta H_{t-2}-R$, then from (4.7), we have

$$2T=(\alpha+\beta)H_{t-2}-(-1)^tH_{t-3}K_{t-3}. \tag{4.8}$$

Now let

$$S=-(\alpha+\beta)K_{t-2}+(-1)^tK_{t-3}^2. \tag{4.9}$$

It is clear that $\deg(2T)=d$. By combining (4.8), (4.9), and (4.4), we obtain (4.5); then $\deg S<\deg T$.

Now, if $\deg H_{t-2}\le d=\deg 2T$, then by (4.6), we derive that $\deg A_k<d=\deg 2T$ for all $k=0,\dots,t-2$; therefore, the period $\overline{A_0,\dots,A_{t-2},2T}$ contains no repeating period of length l ($l\ne t$) such that $l|t$. Consequently, $\Phi(d,t)$ is the number of t−tuple $(A_0,\cdots,A_{t-2},\beta)$, where $(A_0,\dots,A_{t-2})$ is a symmetric $(t-1)$-tuple of positive degree polynomials, $\beta\in\mathbb{F}_q[X]$ with $\sum_{k=0}^{t-2}\deg A_k\le d$ and $\deg\beta=d-\sum_{k=0}^{t-2}\deg A_k$.

Now, we discuss the value of $\Phi(d,t)$ with respect to the parity of t.

Case $t=2p+1$. The total number of ways of breaking n into an ordered sum of p positive integers is given by the binomial coefficient $\binom{n-1}{p-1}$, and there are exactly $(q-1)q^\delta$ polynomials of degree δ. It follows that there are $\binom{n-1}{p-1}(q-1)^{p+1}q^{d-n}$ $(p+1)$-tuple $(A_0,\dots,A_{p-1},\beta)$ such that

$$2n=\sum_{k=0}^{t-2}\deg A_k=2\sum_{k=0}^{p-1}\deg A_k \quad\text{and}\quad \deg\beta=d-\sum_{k=0}^{t-2}\deg A_k=d-2n.$$

Then by summing over all possible values of n ($p\le n\le d/2$), we have

$$\Phi(d,t)=(q-1)^{p+1}q^d\sum_{n=p}^{[d/2]}\binom{n-1}{p-1}q^{-n}.$$

Case $t=2p$. Then there are $\binom{n-1}{p-2}(q-1)^{p+1}q^{d-n}$ $(p+1)$-tuples $(A_0,\dots,A_{p-1},\beta)$ such that

$$k=\deg A_{p-1},$$
$$2n+k=\sum_{k=0}^{t-2}\deg A_k=2\sum_{k=0}^{p-2}\deg A_k+\deg A_{p-1},$$
$$\deg\beta=d-\sum_{k=0}^{t-2}\deg A_k=d-(2n+k).$$

Then by summing over all possible values of n and k ($1 \le k \le d - 2(p+1)$ and $p - 1 \le n \le (d-k)/2$), we have

$$\Phi(d,t) = (q-1)^{p+1} q^d \sum_{k=1}^{d-2p-2} \sum_{n=p-1}^{[(d-k)/2]} \binom{n-1}{p-2} q^{-n}.$$

From Remark 4.2, $\theta(d,1) = q^d$ and

$$\begin{aligned}\theta(d,2) &= \sum_{\substack{T \text{ monic}, H \\ H \neq 1, H|T}} \sum_{\substack{\deg T = d \\ \deg H < d}} 1 = \sum_{\substack{T \text{ monic}, H \\ H|T}} \sum_{\substack{\deg T = d \\ \deg H < d}} 1 - \sum_{\substack{T \text{ monic} \\ \deg T = d}} 1 \\ &= \frac{1}{q-1} \sum_{\substack{T,H \\ H|T}} \sum_{\substack{\deg T = d \\ \deg H < d}} 1 - q^d = \frac{1}{q-1} \sum_{H,K} \sum_{\substack{\deg HK = d \\ \deg H < d}} 1 - q^d \\ &= \frac{1}{q-1} \sum_{h=0}^{d-1} \sum_{\deg H = h} (q-1) q^{d-h} - q^d = (dq - d - 1) q^d. \qquad \blacksquare\end{aligned}$$

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