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ACYCLIC MODELS

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ABSTRACT. Acyclic models is a powerful technique in algebraic topology and homological algebra in which facts about homology theories are verified by first verifying them on "models" (on which the homology theory is trivial) and then showing that there are enough models to present arbitrary objects. One version of the theorem allows one to conclude that two chain complex functors are naturally homotopic and another that two such functors are object-wise homologous. Neither is entirely satisfactory. The purpose of this paper is to provide a uniform account of these two, fixing what is unsatisfactory and also finding intermediate forms of the theorem.

1. **Introduction.** Categorical versions of acyclic models have a long history [Eilenberg, MacLane, 1953], [Appelgate, 1965]. Two somewhat similar versions were proved in [Barr, Beck, 1965] and [André, 1967]. Here are statements of those two theorems. These are not the original versions, but have been put into a form to emphasize their similarities as well as their differences.

Suppose $\mathcal{X}$ is a category, $G: \mathcal{X} \longrightarrow \mathcal{X}$ a functor and $\epsilon: G \longrightarrow \mathrm{Id}$ a natural transformation to the identity functor. Then for any abelian category $\mathcal{A}$ and functor $F: \mathcal{X} \longrightarrow \mathcal{A}$ there is a functor we call $FG^\bullet$ that goes from $\mathcal{X}$ to the category of chain complexes over $\mathcal{A}$ whose n-th term is FG^{n+1} and whose n-th boundary operator is $\sum_{i=0}^{n}(-1)^i FG^i \epsilon G^{n-i}$. This chain complex is augmented over F via $F\epsilon: FG \longrightarrow F$. If $K = \{K_n \parallel n \geq 0\}$, together with a boundary operator d is a chain complex functor from $\mathcal{X}$ to $\mathcal{A}$, then $KG^\bullet$ is a double complex functor. We say that K is *ϵ-presentable* if for all $n \geq 0$, the augmented chain complex $K_n G^\bullet \longrightarrow K_n \longrightarrow 0$ is contractible and that K is *weakly ϵ-presentable* if for each $n \geq 0$, $K_n G^\bullet \longrightarrow K_n \longrightarrow 0$ is acyclic. If $L \longrightarrow L_{-1} \longrightarrow 0$ is a chain complex functor, we say that L is G-contractible if the chain complex functor $LG \longrightarrow L_{-1}G \longrightarrow 0$ is contractible and G-acyclic if $LG \longrightarrow L_{-1}G \longrightarrow 0$ is acyclic. Then the Barr, Beck theorem states:

THEOREM 1.1. *Let $K \longrightarrow K_{-1} \longrightarrow 0$ and $L \longrightarrow L_{-1} \longrightarrow 0$ be augmented chain complex functors such that K is ϵ-presentable and $L \longrightarrow L_{-1} \longrightarrow 0$ is G-contractible. Then any natural transformation $f_{-1}: K_{-1} \longrightarrow L_{-1}$ extends to a natural chain transformation $f: K \longrightarrow L$ and any two extensions of f_{-1} are naturally homotopic.*

André's theorem is:

THEOREM 1.2. *Let $K \longrightarrow K_{-1} \longrightarrow 0$ and $L \longrightarrow L_{-1} \longrightarrow 0$ be augmented chain complex functors such that both K and L are weakly ϵ-presentable and both $K \longrightarrow K_{-1} \longrightarrow 0$ and*

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$L \longrightarrow L_{-1} \longrightarrow 0$ are G-acyclic. If $K_{-1} \cong L_{-1}$, then for each object X of $\mathcal{X}$ and any $n \geq 0$, $H_n(KX) \cong H_n(LX)$.

Theorem 1.1 is usually used to show that when $K \longrightarrow K_{-1} \longrightarrow 0$ and $L \longrightarrow L_{-1} \longrightarrow 0$ satisfy both conditions and $K_{-1} \cong L_{-1}$, then K and L are naturally chain homotopic. This conclusion is quite strong when it holds; unfortunately, there are many situations in which it fails. As long ago as 1944, Eilenberg proved that (his new definition of) singular chain groups gave homology and cohomology groups on simplicial complexes that were isomorphic to those given by the oriented simplicial theory. More precisely, let C^{sing}, C^{ori} and C^{ord} denote the singular, oriented simplicial and unoriented (called ordered) chain group functors, resp. There are evidently natural transformations

$$C^{\text{sing}} \longleftarrow C^{\text{ord}} \longrightarrow C^{\text{ori}}$$

which Eilenberg showed to give homotopy equivalences when applied to any simplicial complex. Let us call a natural transformation between chain complex functors a *quasi-homotopy equivalence* if it is a homotopy equivalence on each object. A quasi-natural equivalence induces a natural equivalence of homology (and dually, of cohomology) groups. Applied to the simplicial complexes, this results in a homology equivalence, even a natural equivalence, between oriented simplicial homology and singular homology, without exhibiting, as Eilenberg remarks, a map between them in either direction that induces the isomorphism. See [Eilenberg, 1944], especially the discussion on pp. 246–247. (In fact, it is possible to find a natural transformation $C^{\text{ori}} \longrightarrow C^{\text{sing}}$ that induces the isomorphism.)

Thus in practice, we often resort to the weaker Theorem 1.2. This is certainly easier to use, but its conclusions are too weak for what is wanted. It has three main flaws. First, the conclusion is isomorphism, not natural isomorphism. Second, it is valid only for deriving isomorphisms. The third flaw is that the isomorphism is not induced by anything, even when you begin with a map $K \longrightarrow L$ that induces the given isomorphism in degree 0.

In this paper we describe a new version of acyclic models that gives Theorems 1.1 and 1.2 as special cases, but the version of the latter it gives repairs the three difficulties just mentioned. Moreover, another instance of the same theorem gives the same kind of quasi-homotopy that Eilenberg observed in 1944. The basic idea is to parametrize the theorem in terms of a special class of "trivial" chain complex functors. This class could be the contractible complexes, the acyclic ones or somewhere in between, such as the quasi-contractible complexes (that is, those that are quasi-homotopic to the 0 complex). The main theorem is stated and proved in terms of this class.

As an application of the theorem proved here, we show that on a manifold of class C^p the inclusion of the chain group based on chains of class C^p into those of class C^q, for any $q \leq p$ induces a quasi-homotopy equivalence. In particular, this is true when $p = \infty$ and $q = 0$ so that the homology based on C^∞ chains is equivalent to the ordinary singular homology.

I would like to thank Rob Milson for many discussions that led to this paper. It was he who pointed out the difficulties in extracting naturality from an isomorphism derived

from a double complex, which was the main motivation behind the results. As well, it was he who observed the difficulty in getting even an unnatural transformation between homology theories in the case that the maps in the acyclic models theorem were not themselves natural. Finally, he was helpful in pinning down the details of the homology of manifolds.

1.1. *Notation.* We usually use "id" to denote an identity map (and "Id" for the identity functor), but sometimes call it 1, especially during extended computations. Similarly, we usually denote the composite of g and f by $f \circ g$, but sometimes omit the $\circ$ when it cannot cause confusion.

We denote by $\mathcal{A}$ a fixed abelian category and by $\mathcal{X}$ another category. $\mathcal{B}$ and $\mathcal{C}$ denote the categories of graded objects and chain complexes, resp., in the functor category **Fun**$(\mathcal{X}, \mathcal{A})$. The complexes will all be chain complexes and bounded below, usually by 0, unless otherwise noted. Similarly, the double complexes will have uniform lower bounds in both directions. Again that is usually 0, unless something else is specified.

1.2. *Double complexes.* A double complex $C = C_{\bullet\bullet}$ is understood to have two boundary operators, say $d\colon C_{mn} \longrightarrow C_{m\,n-1}$ and $\partial\colon C_{mn} \longrightarrow C_{m-1\,n}$ satisfying $d \circ \partial = -\partial \circ d$. The total complex, Tot$(C)$, is defined as the complex that has in degree n the direct sum $\sum_{i=n_0}^{n-n_0} C_{i,n-i}$ where n_0 is the lower bound. The boundary operator has the matrix

$$\begin{pmatrix} d & \partial & 0 & 0 & \cdots & 0 & 0 \\ 0 & d & \partial & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \partial & 0 \\ 0 & 0 & 0 & 0 & \cdots & d & \partial \end{pmatrix}$$

whose square is readily shown to be 0.

Denote by $U\colon \mathcal{C} \longrightarrow \mathcal{B}$ the forgetful functor. An exact sequence $0 \longrightarrow L \longrightarrow C \longrightarrow K \longrightarrow 0$ of objects and arrows of $\mathcal{C}$ will be called U-split if $0 \longrightarrow UL \longrightarrow UC \longrightarrow UK \longrightarrow 0$ is split in $\mathcal{B}$. Here and throughout most of this paper, the boundary operator in all chain complexes will be denoted d, relying on context to disambiguate the use.

If $K = \{K_n \parallel n \geq n_0\}$ is a chain complex we let SK be the chain complex defined by $(SK)_n = K_{n-1}$ with boundary operator $-d$. SK is called the *suspension* of K. Suspension is an automorphism on the category of simplicial complexes.

2. Acyclic classes.

2.1. *Acyclic classes.* A class Γ of objects of $\mathcal{C}$ will be called an *acyclic class* provided:

AC-1. The 0 complex is in Γ.

AC-2. The complex C belongs to Γ if and only if SC does.

AC-3. If the complexes K and L are homotopic and $K \in \Gamma$, then $L \in \Gamma$.

AC-4. Every complex in Γ is acyclic.

AC-5. If C is a double complex, all of whose rows are in Γ, then the total complex of C belongs to Γ.

2.2. *Mapping cones.* Suppose that $f: K \longrightarrow L$ is a map in $\mathcal{C}$. We define a complex $C = C_f$ by letting $C_n = L_n \oplus K_{n-1}$ with boundary operator $\begin{pmatrix} d & f \\ 0 & -d \end{pmatrix}$. Then C is a chain complex and we have an exact sequence

$$0 \longrightarrow L \longrightarrow C \longrightarrow SK \longrightarrow 0$$

It is almost as easy to see that the connecting homomorphism $H_n(SK) = H_{n-1}(K) \longrightarrow H_{n-1}(L)$ is $H_{n-1}(f)$. C is called the *mapping cone* of f.

We note that the exact sequence $0 \longrightarrow L \longrightarrow C \longrightarrow SK \longrightarrow 0$ is U-split. This turns out to characterize mapping cone sequences.

PROPOSITION 2.3. *A U-split exact sequence*

$$0 \longrightarrow L \longrightarrow C \longrightarrow K \longrightarrow 0$$

is isomorphic to the mapping cone of a unique map $S^{-1}K \longrightarrow L$.

See [Barr, to appear], Proposition 6.2, for a proof.

THEOREM 2.4. *A map of complexes is a homology equivalence if and only if its mapping cone is acyclic and it is a homotopy equivalence if and only if its mapping cone is contractible.*

PROOF. For homology, both directions are immediate consequences of the exactness of the homology triangle and the fact that an object in a homology triangle is 0 if and only if the map opposite is an isomorphism.

Next we look at homotopy. If $f: K \longrightarrow L$ is a homotopy equivalence, then there is a map $g: K \longrightarrow L$ and maps $s: K \longrightarrow K$ and $t: L \longrightarrow L$ such that $1 - fg = td + dt$ and $1 - gf = sd + ds$. If Z is any object of $\mathcal{A}$, and $F: \mathcal{X} \longrightarrow \mathcal{A}$ is any functor, then there is a functor we denote $\mathrm{Hom}(Z, F): \mathcal{X} \longrightarrow \mathcal{A}$ defined by $\mathrm{Hom}(Z, F)(X) = \mathrm{Hom}(Z, FX)$. Similarly, we let $F = (\{F_n\}, d)$ be any chain complex functor to get a chain complex functor $\mathrm{Hom}(Z, F)$ whose n-th term is $\mathrm{Hom}(Z, F_n)$ and boundary operator is $\mathrm{hom}(Z, d)$. We can carry out this construction successively with $F = K$, $F = L$ and $F = C$, the mapping cone. The sequence of chain complex functors into the category of abelian groups

$$(*) \qquad 0 \longrightarrow \mathrm{Hom}(Z, L) \longrightarrow \mathrm{Hom}(Z, C) \longrightarrow \mathrm{Hom}(Z, K) \longrightarrow 0$$

can easily be seen to be the mapping cone sequence of $\mathrm{Hom}(Z, f)$ and the latter map is a homotopy equivalence using $\mathrm{Hom}(Z, g)$, $\mathrm{Hom}(Z, s)$ and $\mathrm{Hom}(Z, t)$. A homotopy equivalence is certainly a homology isomorphism, so that the exactness of the homology triangle of $(*)$ implies that the complex $\mathrm{Hom}(Z, C)$ is exact. We apply this to $Z_n = \ker d: C_n \longrightarrow C_{n-1}$, the object of cycles of degree n, with $i_n: Z_n \longrightarrow C_n$ the inclusion map. Since $di_n = 0$, i_n is a cycle in the complex $\mathrm{Hom}(Z_n, C)$. Since that complex is exact, i_n is also a boundary, so that there is a map $z_n: Z_n \longrightarrow C_{n+1}$ such that $dz_n = i_n$. Now the image of $d\colon C_n \longrightarrow C_{n-1}$ is included in (actually equal to) Z_{n-1} so that, by abuse of notation, we can form the composite $z_{n-1}d$ and one sees immediately that $dz_{n-1}d = d$ so that $d(1 - z_n d) = 0$. We can then compose $1 - z_{n-1}d$ with z_n and $dz_n(1 - z_{n-1}d) = 1 - z_{n-1}d$. Let $s_n = z_n(1 - z_{n-1}d)$. We calculate

$$s_{n-1}d + ds_n = z_{n-1}(1 - z_{n-2}d)d + dz_n(1 - z_{n-1}d) = z_{n-1}d + (1 - z_{n-1}d) = 1$$

which shows that the s_n are a contracting homotopy in C. For the converse, suppose that C is contractible. Let the contracting homotopy u have matrix $\begin{pmatrix} t & r \\ g & -s \end{pmatrix}$. Then the matrix of $du + ud$ is calculated to be

$$\begin{pmatrix} dt + fg + td & dr - fs + tf - rd \\ -dg + gd & ds + gf + sd \end{pmatrix}$$

If we set this equal to the identity, we conclude that $dt + fg + td = 1$, $-dg + gd = 0$ and $ds + gf + sd = 1$ from which we see that g is a chain map and homotopy inverse to f. ■

PROPOSITION 2.5. *If $0 \longrightarrow L \xrightarrow{f} C \xrightarrow{g} K \longrightarrow 0$ is a U-split exact sequence of chain complexes, then K is homotopic to the mapping cone C_f and L is homotopic to SC_g.*

PROOF. Except for an unavoidable arbitrariness whether to suspend one or desuspend the other term in a mapping cone, the two parts are dual; we need prove only one. Let $u\colon UC \longrightarrow UL$ and $v\colon UK \longrightarrow UC$ be such that $uf = 1$, $gv = 1$, $fu + vg = 1$ and $uv = 0$. The last equation actually follows from the first three. I claim that $\begin{pmatrix} v \\ -udv \end{pmatrix} : K \longrightarrow C_f$ is a chain map. In fact

$$\begin{aligned}
\begin{pmatrix} d & f \\ 0 & -d \end{pmatrix}\begin{pmatrix} v \\ -udv \end{pmatrix} &= \begin{pmatrix} dv - fudv \\ dudv \end{pmatrix} = \begin{pmatrix} dv - (1 - vg)dv \\ ufdudv \end{pmatrix} = \begin{pmatrix} vgdv \\ udfudv \end{pmatrix} \\
&= \begin{pmatrix} vdgv \\ ud(1 - vg)dv \end{pmatrix} = \begin{pmatrix} vd \\ -udvgdv \end{pmatrix} \\
&= \begin{pmatrix} vd \\ -udvdgv \end{pmatrix} = \begin{pmatrix} vd \\ -udvd \end{pmatrix} = \begin{pmatrix} v \\ -udv \end{pmatrix} d
\end{aligned}$$

It is clear that $(g \quad 0)\begin{pmatrix} v \\ -udv \end{pmatrix} = 1$. The other composite is $\begin{pmatrix} v \\ -udv \end{pmatrix}(g \quad 0) = \begin{pmatrix} vg & 0 \\ -udvg & 0 \end{pmatrix}$. We have

$$\begin{aligned}
\begin{pmatrix} d & f \\ 0 & -d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}\begin{pmatrix} d & f \\ 0 & -d \end{pmatrix} &= \begin{pmatrix} fu & 0 \\ -du & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ ud & uf \end{pmatrix} \\
&= \begin{pmatrix} fu & 0 \\ -du + ud & uf \end{pmatrix} = \begin{pmatrix} 1 - vg & 0 \\ ud - ufdu & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 - vg & 0 \\ ud - udfu & 1 \end{pmatrix} = \begin{pmatrix} 1 - vg & 0 \\ udvg & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} vg & 0 \\ -udvg & 0 \end{pmatrix}
\end{aligned}$$

■

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3. Properties of acyclic classes.

PROPOSITION 3.1. *If $0 \longrightarrow L \longrightarrow C \longrightarrow K \longrightarrow 0$ is a U-split exact sequence of objects of $\mathcal{C}$ and if any two belong to Γ, then so does the third.*

PROOF. Suppose that L and K, and hence $S^{-1}K$ belong to Γ. We know that C is the mapping cone of a map $f: S^{-1}K \longrightarrow L$. We can think of this as a double complex as in the following diagram. In this diagram, we use d for the boundary operator in K so that $-d$ is the boundary operator in SK and the squares commute as shown.

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{-d} & K_{n+1} & \xrightarrow{-d} & K_n & \xrightarrow{-d} & \cdots & \xrightarrow{-d} & K_1 & \xrightarrow{-d} & K_0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle f} & & \\
\cdots & \xrightarrow[d]{} & L_n & \xrightarrow[d]{} & L_{n-1} & \xrightarrow[d]{} & \cdots & \xrightarrow[d]{} & L_0 & \longrightarrow & 0
\end{array}$$

If we replace the $-d$ in the upper row by d, the squares will anticommute and the resultant diagram can be considered as a double complex in which all rows belong to Γ. From AC-5 the total complex also belongs to Γ, but that is just the mapping cone of f which is isomorphic to C and hence belongs to Γ.

Now suppose that L and C belong to Γ. We have just seen that the mapping cone of $L \longrightarrow C$ is in Γ. It then follows from Proposition 2.5 and AC-3 that $K \in \Gamma$. Dually, if C and K are in Γ, so is L. ∎

3.1. *Arrows determined by an acyclic class.* Given an acyclic class Γ, let Σ denote the class of arrows f whose mapping cone is in Γ. It follows from AC-3 and 4 and the preceding proposition that this class lies between the class of homotopy equivalences and that of homology equivalences.

PROPOSITION 3.3. *Σ is closed under composition.*

PROOF. Suppose that $f: K \longrightarrow L$ and $g: L \longrightarrow M$ are each in Σ. Then C_f and C_g are in Γ. Let $h: S^{-1}C_g \longrightarrow C_f$ have matrix $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. The n-th term of $S^{-1}C_g$ is $M_{n+1} \oplus L_n$ while that of C_f is $L_n \oplus K_{n-1}$ so this makes sense. Thus there is an exact sequence $0 \longrightarrow C_f \longrightarrow C_h \longrightarrow C_g \longrightarrow 0$ and it follows from Proposition 3.1 that $C_h \in \Gamma$. The n-th term of C_h is $L_n \oplus K_{n-1} \oplus M_n \oplus L_{n-1}$ and the matrix of the boundary operator is

$$\begin{pmatrix} d & f & 0 & -1 \\ 0 & -d & 0 & 0 \\ 0 & 0 & d & g \\ 0 & 0 & 0 & -d \end{pmatrix}$$

Let $C_{-\mathrm{id}}$ be the mapping cone of the negative of the identity of L. Thus $(C_{-\mathrm{id}})_n = L_n \oplus L_{n-1}$ and the boundary operator is $\begin{pmatrix} d & -1 \\ 0 & -d \end{pmatrix}$. The mapping cone of gf has $M_n \oplus K_{n-1}$ in degree n and boundary operator $\begin{pmatrix} d & gf \\ 0 & -d \end{pmatrix}$. I claim there is an exact sequence

$$0 \longrightarrow C_{gf} \xrightarrow{i} C_h \xrightarrow{j} C_{-\mathrm{id}} \longrightarrow 0.$$

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In fact, let i and j be the maps given by the matrixes

$$i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & f \end{pmatrix}, \quad j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -f & 0 & 1 \end{pmatrix}$$

Matrix multiplication shows that these are chain maps. The sequences are U-split exact; for example, $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ splits i and it follows from Theorem 3.1 that $C_{gf} \in \Gamma$ and hence $gf \in \Sigma$. ∎

THEOREM 3.4. *Suppose $C = C_{\bullet\bullet} = \{C_{mn} \parallel m \geq 0, n \geq 0\}$ is a double complex that is augmented over the single complex $C_{-1\bullet}$ and such that for each $n \geq 0$, the complex*

$$\cdots \longrightarrow C_{mn} \longrightarrow C_{m-1\,n} \longrightarrow \cdots \longrightarrow C_{0n} \longrightarrow C_{-1\,n} \longrightarrow 0$$

belongs to Γ. Then the induced map $\mathrm{Tot}(C) \longrightarrow C_{-1\bullet}$ *is in Σ.*

PROOF. The mapping cone of the induced map is just the double complex including the augmentation term. From AC-5 it follows that that total double complex is in Γ since each row is. Thus the induced map is in Σ. ∎

4. **Examples.** We will be interested in three main examples of acyclic classes, although many others are possible by varying the contractibility conditions depending on what the contraction is supposed to preserve.

4.1. *Acyclic complexes.* Let Γ consist of the acyclic complexes, in which case Σ consists of homology isomorphisms. AC-1, 2, 3, and 4 are obvious. AC-5 is readily proved using spectral sequences, but it is also easy to give a direct proof. If C is a double complex, each of whose rows is acyclic, let $R^p(C)$ and $F^p(C)$ denote the p-th row and the truncation above the p-th row, resp. Then there is an exact sequence of double complexes

$$0 \longrightarrow F^{p-1}(C) \longrightarrow F^p(C) \longrightarrow R^p(C) \longrightarrow 0$$

that splits as a sequence of bigraded objects. It follows that the sequence of associated single complexes has the same property. Moreover, $F^0(C) = R^0(C)$ and then by induction each $F^p(C)$ belongs to Γ. The n-th homology depends only on the fragment $C_{n+1} \longrightarrow C_n \longrightarrow C_{n-1}$, which is constant after $F^{n+1}(C)$ so that the inclusion $F^{n+1}(C) \longrightarrow C$ induces an isomorphism on n-th homology. Since $F^{n+1}(C)$ is acyclic, its n-th homology is trivial and therefore $H_n(C) = 0$. Since this is true for all n, it follows that C is acyclic.

4.2. *Contractible complexes.* Let Γ consist of the contractible complexes. AC-1, 2 and 4 are obvious. To see AC-3, suppose that $f\colon K \longrightarrow L$ and $g\colon L \longrightarrow K$ are chain maps and $s\colon K \longrightarrow K$ and $t\colon L \longrightarrow L$ are maps such that $1 = ds + sd$ and $1 - fg = dt + td$. (So we are making an assumption weaker than that K is homotopic to L; more like that L is a homotopy retract of K.) Then I claim that $t + fsg$ is a contracting homotopy in L. In fact,

$$d(t + fsg) + (t + fsg)d = dt + td + f(ds + sd)g = 1 - fg + fg = 1$$

To prove AC-5, suppose that we are given a double complex K_{mn}, defined for $m \geq 0$ and $n \geq -1$, but that $K_{mn} = 0$ for $n = -1$. This makes no real difference, but it avoids there being a special case at the lowest dimension. We use the terminology and notation from 1.4. Thus one boundary operator is $d: K_{mn} \longrightarrow K_{m\,n-1}$ and the other is $\partial : K_{mn} \longrightarrow K_{m-1\,n}$ with $d\partial = -\partial d$. Suppose further that for each m and n, there is a map $s: K_{mn} \longrightarrow K_{m\,n+1}$ that satisfies $ds + sd = 1$. The total complex has, in degree n, the direct sum $L_n = \sum_{i=0}^{n} K_{i\,n-i}$ and is 0 when $n = -1$ and the boundary operator $D: L_n \longrightarrow L_{n-1}$ has the matrix given in 1.4. For the rest of this proof, we will not use S for suspension, but for a contracting homotopy in the double complex, which we now define in degree n as a map $S: L_n \longrightarrow L_{n+1}$ with the matrix

$$\begin{pmatrix} s & -s\partial s & s\partial s\partial s & \cdots & (-1)^n s(\partial s)^n \\ 0 & s & -s\partial s & \cdots & (-1)^{n-1} s(\partial s)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Direct matrix multiplication shows that $SD + DS$ is upper triangular and has $sd + ds = 1$ in each diagonal entry (including the last, since in that case the $sd = 0$ so that $ds = 1$). In carrying that out, it is helpful to block D into an upper triangular matrix and a single column and S into an upper triangular matrix and a single row of zeroes. In order to see that $SD + DS = 1$, we must show that the above diagonal entries vanish. First we claim that for $i > 0$, $ds(\partial s)^i = (\partial s)^i + (s\partial)^i ds$. In fact, for $i = 1$,

$$ds\partial s = (1 - sd)\partial s = \partial s - sd\partial s = \partial s + s\partial ds$$

Assuming that the conclusion is true for $i - 1$,

$$\begin{aligned} ds(\partial s)^i &= \big((\partial s)^{i-1} + (s\partial)^{i-1} ds\big)\partial s = (\partial s)^i + (s\partial)^{i-1}(1 - sd)\partial s \\ &= (\partial s)^i + (s\partial)^{i-1}\partial s - (s\partial)^{i-1} sd\partial s = (\partial s)^i + (s\partial)^{i-1} s\partial ds \\ &= (\partial s)^i + (s\partial)^i ds \end{aligned}$$

Now suppose we choose indices $i < j$. The i, j-th entry of SD is

$$\begin{pmatrix} 0 & \cdots & 0 & s & \cdots & (-1)^{j-i-1} s(\partial s)^{j-i-1} & (-1)^{j-i} s(\partial s)^{j-i} & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \partial \\ d \\ 0 \\ \vdots \end{pmatrix}$$

$$\begin{aligned} &= (-1)^{j-i-1} s(\partial s)^{j-i-1}\partial + (-1)^{j-i} s(\partial s)^{j-i} d = (-1)^{j-i-1}\big((s\partial)^{j-i} - (s\partial)^{j-i} sd\big) \\ &= (-1)^{j-i-1}\big((s\partial)^{j-i} - (s\partial)^{j-i}(1 - ds)\big) = (-1)^{j-i-1}(s\partial)^{j-i} ds \end{aligned}$$

and the i,j-th entry of DS is

$$\begin{aligned}
\begin{pmatrix} 0 & \cdots & 0 & d & \partial & 0 & \cdots \end{pmatrix}
\begin{pmatrix} (-1)^j s(\partial s)^j \\ \vdots \\ (-1)^{j-i} s(\partial s)^{j-i} \\ (-1)^{j-i-1} s(\partial s)^{j-i-1} \\ \vdots \end{pmatrix}
&= (-1)^{j-i} ds(\partial s)^{j-i} + (-1)^{j-i-1} \partial s(\partial s)^{j-i-1} \\
&= (-1)^{j-i}\big(ds(\partial s)^{j-i} - (\partial s)^{j-i}\big) \\
&= (-1)^{j-i}\big((\partial s)^{j-i} + (s\partial)^{j-i} ds - (\partial s)^{j-i}\big) \\
&= (-1)^{j-i}(s\partial)^{j-i} ds
\end{aligned}$$

so that the terms cancel and $SD + DS = 1$.

4.3. *Quasi-contractible complexes.* Until now the fact that $\mathcal{C}$ was a functor category did not play much of a role. The third example depends on that fact. Say that a chain complex is *quasi-contractible* if for each object X of $\mathcal{X}$, the complex CX is contractible. Each of the previous results on contractible complexes carries over to these quasi-contractible ones, except that in each case the conclusion is object by object. Recall that a map f of chain complexes is a quasi-homotopy equivalence if at each object X, fX is a is a homotopy equivalence. It is clear that f is a quasi-homotopy equivalence if and only if its mapping cone is quasi-contractible. The earlier material on contractible complexes implies that the quasi-contractible complexes constitute an acyclic class.

5. **The main theorem.** Now let us suppose we are given an acyclic class Γ on $\mathcal{C}$ and that Σ is the associated class of maps. We denote by $\Sigma^{-1}\mathcal{C}$ the category of fractions gotten by inverting all the arrows in Σ. This category is characterized by the fact that there is a functor $T\colon \mathcal{C} \longrightarrow \Sigma^{-1}\mathcal{C}$ such that $\sigma \in \Sigma$ implies that $T(\sigma)$ is an isomorphism and if $S\colon \mathcal{C} \longrightarrow \mathcal{D}$ is any functor such that $S(\sigma)$ is an isomorphism for all $\sigma \in \Sigma$, then there is a unique $\hat{S}\colon \Sigma^{-1}\mathcal{C} \longrightarrow \mathcal{D}$ such that $\hat{S} \circ T = S$. From AC-4 and Theorem 2.4 it follows that the homology inverts all arrows of Σ and hence that homology factors through $\Sigma^{-1}\mathcal{C}$ as described. In particular, any map in $\Sigma^{-1}\mathcal{C}$ induces a map in homology.

Suppose that $G\colon \mathcal{X} \longrightarrow \mathcal{X}$ is an endofunctor and that $\epsilon\colon G \longrightarrow \mathrm{Id}$ is a natural transformation. Recall that for any functor $F\colon \mathcal{X} \longrightarrow \mathcal{A}$, the augmented chain complex functor $FG^\bullet \longrightarrow F$ is defined to have the functor FG^{n+1} in degree n, $n \geq -1$ with boundary operator $d = \sum_{i=0}^{n}(-1)^i FG^i\epsilon G^{n-i}\colon FG^{n+1} \longrightarrow FG^n$. If $K \longrightarrow K_{-1}$ is an augmented chain complex functor, then we denote by $KG^\bullet$ the double complex that has $K_m G^{n+1}$ in bidegree m, n. This is augmented in both directions, once, using ϵ, over the single complex K and second, using the augmentation of K, over the complex $K_{-1}G^\bullet$. We say that K is *ϵ-presentable with respect to* Γ if for each $n \geq 0$, the augmented chain complex $K_n G^\bullet \longrightarrow K_n \longrightarrow 0$ belongs to Γ. We say that K is *G-acyclic with respect to* Γ if the augmented complex $0 \longrightarrow KG \longrightarrow K_{-1}G \longrightarrow 0$ belongs to Γ.

THEOREM 5.1. *Suppose $\alpha: K \longrightarrow K_{-1}$ and $\beta: L \longrightarrow L_{-1}$ are augmented chain complex functors. Suppose G is an endofunctor on X and $\epsilon: G \longrightarrow \mathrm{Id}$ a natural transformation for which K is ϵ-presentable and $L \longrightarrow L_{-1} \longrightarrow 0$ is G-acyclic, both with respect to Γ. Then given any natural transformation $f_{-1}: K_{-1} \longrightarrow L_{-1}$ there is, in $\Sigma^{-1}C$, a unique arrow $f: K \longrightarrow L$ that extends f_{-1}.*

PROOF. For all $m \geq 0$, the augmented complex $K_m G^\bullet \longrightarrow K_m \longrightarrow 0$ belongs to Γ and hence, by AC-5, the total augmented complex $KG^\bullet \longrightarrow K \longrightarrow 0$ belongs to Γ whence by Theorem 2.4 the arrow $K\epsilon: KG^\bullet \longrightarrow K$ is in Σ. The same reasoning implies that $\beta G^\bullet: LG^\bullet \longrightarrow L_{-1}G^\bullet$ is also in Σ. We can summarize the situation in the diagram

$$\begin{array}{ccccc}
K_{-1}G^\bullet & \xleftarrow{\alpha G^\bullet} & KG^\bullet & \xrightarrow{K\epsilon} & K \\
{\scriptstyle f_{-1}G^\bullet}\downarrow & & & & \\
L_{-1}G^\bullet & \xleftarrow[\beta G^\bullet]{} & LG^\bullet & \xrightarrow[L\epsilon]{} & L
\end{array}$$

with $K\epsilon$ and $\beta G^\bullet$ in Σ. We now invert Σ to get the map

$$f = L\epsilon \circ (\beta G^\bullet)^{-1} \circ f_{-1}G^\bullet \circ \alpha G^\bullet \circ (K\epsilon)^{-1}: K \longrightarrow L$$

I claim that this map extends f_{-1} in the sense that $f_{-1} \circ \alpha = \beta \circ f$ and that f is unique with this property. Begin by observing that naturality of α and β imply that $\alpha \circ K\epsilon = K_{-1}\epsilon \circ \alpha G^\bullet$ and $\beta \circ L\epsilon = L_{-1}\epsilon \circ \beta G^\bullet$. Then the first claim follows from the diagram

$$\begin{array}{ccccccccc}
K & \xrightarrow{(K\epsilon)^{-1}} & KG^\bullet & \xrightarrow{\alpha G^\bullet} & K_{-1}G^\bullet & \xrightarrow{f_{-1}G^\bullet} & L_{-1}G^\bullet & \overset{(\beta G^\bullet)^{-1}}{\underset{\beta G^\bullet}{\rightleftarrows}} & LG^\bullet \\
& {\scriptstyle =}\searrow & \downarrow{\scriptstyle K\epsilon} & & \downarrow{\scriptstyle K_{-1}\epsilon} & & \downarrow{\scriptstyle L_{-1}\epsilon} & & \downarrow{\scriptstyle L\epsilon} \\
& & K & \xrightarrow[\alpha]{} & K_{-1} & \xrightarrow[f_{-1}]{} & L_{-1} & \xleftarrow[\beta]{} & L
\end{array}$$

Now suppose that $g: K \longrightarrow L$ is another arrow in $\Sigma^{-1}C$ for which $f_{-1} \circ \alpha = \beta \circ g$. Then $f_{-1}G^\bullet \circ \alpha G^\bullet = \beta G^\bullet \circ gG^\bullet$, which implies that $(\beta G^\bullet)^{-1} \circ f_{-1}G^\bullet \circ \alpha G^\bullet = gG^\bullet$ and then

$$L\epsilon \circ (\beta G^\bullet)^{-1} \circ f_{-1}G^\bullet \circ \alpha G^\bullet = L\epsilon \circ gG^\bullet = g \circ K\epsilon$$

from which we conclude that

$$g = L\epsilon \circ (\beta G^\bullet)^{-1} \circ f_{-1}G^\bullet \circ \alpha G^\bullet \circ (K\epsilon)^{-1} = f$$

■

COROLLARY 5.2. *Suppose that K and L are each ϵ-presentable and G-acyclic on models with respect to Γ. Then any natural isomorphism $f_{-1}: K_{-1} \longrightarrow L_{-1}$ extends to a unique isomorphism $f: K \longrightarrow L$ in $\Sigma^{-1}C$. Moreover if $g: K \longrightarrow L$ is a natural transformation for which $\beta \circ g_0 = f_{-1} \circ \alpha$, then $g = f$ in $\Sigma^{-1}C$.*

PROOF. If f_{-1} is an isomorphism with inverse g_{-1}, then there is a map $f: K \longrightarrow L$ that extends f and $g: L \longrightarrow K$ that extends g_{-1}. Then $g \circ f$ extends $g_{-1} \circ f_{-1} = \mathrm{id}$, as does the identity so that by the uniqueness of the preceding, we see that in $\Sigma^{-1}\mathcal{C}$, $g \circ f = \mathrm{id}$. Similarly, $f \circ g = \mathrm{id}$ in the fraction category. This shows that $K \cong L$. The second claim is obvious. ∎

In order to recover the form of the acyclic models theorem from [Barr, Beck, 1966], we require:

THEOREM 5.3. *Suppose $G: \mathcal{X} \longrightarrow \mathcal{X}$ is a functor and $\epsilon: G \longrightarrow \mathrm{Id}$ is a natural transformation. Then for any functor $C: \mathcal{X} \longrightarrow \mathcal{A}$, $CG^\bullet \longrightarrow C \longrightarrow 0$ is contractible if and only if $C\epsilon$ splits.*

PROOF. The necessity of the condition is obvious. If $C\epsilon$ splits, let $\theta: C \longrightarrow C\epsilon$ such that $C\epsilon \circ \theta = \mathrm{id}$. Let $s = \theta G^n: CG^n \longrightarrow CG^{n+1}$. Then for $d^i = G^i\epsilon G^{n-i}$, we have $d^0 s = C\epsilon G^n \circ \theta G^n = \mathrm{id}$ and for $i > 0$,

$$\begin{aligned} d^i s &= CG^i\epsilon \circ G^{n-i} \circ \theta G^n = (CG^i\epsilon \circ \theta G^i)G^{n-i} \\ &= (\theta G^{i-1} \circ CG^{i-1}\epsilon)G^{n-i} = \theta G^{n-1} \circ CG^{i-1}\epsilon G^{n-i} = sd^{i-1} \end{aligned}$$

using naturality of θ. Then with $d = \sum_{i=0}^{n}(-1)^i d^i$,

$$\begin{aligned} ds + sd &= \sum_{i=0}^{n}(-1)^i d^i s + \sum_{i=0}^{n-1} sd^i \\ &= \mathrm{id} + \sum_{i=1}^{n}(-1)^i sd^{i-1} + \sum_{i=0}^{n-1}(-1)^i sd^i = 1 \end{aligned}$$

∎

COROLLARY 5.4. *Let $K \longrightarrow K_{-1} \longrightarrow 0$ and $L \longrightarrow L_{-1} \longrightarrow 0$ be augmented chain complex functors such that $GK_n \longrightarrow K_n$ is split epi for all $n \geq 0$ and $L \longrightarrow L_{-1} \longrightarrow 0$ is G-contractible. Then any natural transformation $K_{-1} \longrightarrow L_{-1}$ extends to a natural chain transformation $f: K \longrightarrow L$ and any two extensions of f are naturally homotopic.* ∎

6. **Calculuses of fractions.** We fix an acyclic class Γ and let Σ be the corresponding class of arrows.

THEOREM 6.1. *In $\Sigma^{-1}\mathcal{C}$ every map factors as $f \circ \sigma^{-1}$ where $f \in \mathcal{C}$ and $\sigma \in \Sigma$. Dually, every map factors as $\tau^{-1} \circ g$, where $g \in \mathcal{C}$ and $\tau \in \Sigma$. This is not shown by verifying that there are calculuses of right and left fractions, but rather directly, as shown below.*

THEOREM 6.2. *Suppose $L \xrightarrow{f} N \xleftarrow{\sigma} M$ are maps of chain complexes with $\sigma \in \Sigma$. Let K denote the chain complex whose n-th term is $K_n = L_n \oplus M_n \oplus N_{n+1}$, with boundary operator given by the matrix $D = \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ -f & \sigma & -d \end{pmatrix}$. Let $\tau = (1 \quad 0 \quad 0): K \longrightarrow L$, $g = (0 \quad 1 \quad 0): K \longrightarrow M$, and $h = (0 \quad 0 \quad 1): UK \longrightarrow UN$. Then g and τ are chain maps, τ belongs to Σ, and h defines a homotopy between σg and $f\tau$.*

PROOF. The proofs that D is a boundary operator and that g are simple matrix computations and are left to the reader. We compute that

$$\begin{aligned} dh+hd &= d(0\quad 0\quad 1)+(0\quad 0\quad 1)\begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ -f & \sigma & -d \end{pmatrix} \\ &= (0\quad 0\quad d)+(-f\quad \sigma\quad -d)=(-f\quad \sigma\quad 0) \\ &= \sigma(0\quad 1\quad 0)-f(1\quad 0\quad 0)=\sigma g-f\tau \end{aligned}$$

We still have to show that $\tau \in \Sigma$. But

$$0 \longrightarrow S^{-1}C_\sigma \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}} K \xrightarrow{\tau} L \longrightarrow 0$$

is a U-split exact sequence of chain maps so that it follows from Proposition 2.5 that $S^{-1}C_\tau$ is homotopic to C_σ. Since $C_\sigma \in \Gamma$, AC-3 implies that $C_\tau \in \Gamma$. ∎

PROPOSITION 6.3. *Homotopic maps become equal in* $\Sigma^{-1}C$.

PROOF. We apply the previous theorem to $L \xrightarrow{1} L \xleftarrow{1} L$ to give the homotopy commutative square

$$\begin{array}{ccc} K & \xrightarrow{\rho} & L \\ \phi\downarrow & & \downarrow 1 \\ L & \xrightarrow[1]{} & L \end{array}$$

Moreover, the map $h: L \longrightarrow K$ with matrix $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is a chain map such that $\rho \circ h = \phi \circ h = \mathrm{id}$. In $\Sigma^{-1}C$, ρ and ϕ are invertible, whence $h = \rho^{-1} = \phi^{-1}$ so that $\rho = \phi$. Given two chain maps $p, q: C \longrightarrow L$ and a map $s: C \longrightarrow L$ such that $q - p = sd + ds$, the map $u = \begin{pmatrix} p \\ q \\ s \end{pmatrix}: C \longrightarrow K$ is a chain map for which $\rho \circ u = p$ and $\phi \circ u = q$, so that in $\Sigma^{-1}C$, we have $p = q$. ∎

COROLLARY 6.4. *Every map in* $\Sigma^{-1}C$ *has the form* $f \circ \sigma^{-1}$ *where* $f \in C$ *and* $\sigma \in \Sigma$. ∎

COROLLARY 6.5. *Every map in* $\Sigma^{-1}C$ *has the form* $\sigma^{-1} \circ f$ *where* $f \in C$ *and* $\sigma \in \Sigma$. ∎

These facts hold despite the fact that there is no calculus of left—or dually of right—fractions in this case. For example, in the proof of Proposition 6.3, the homotopy equivalence h equalizes ρ and ϕ, but only the 0 map coequalizes them and that is a homotopy equivalence if and only if N is contractible.

7. Application to homology on manifolds. Consider a manifold M of class C^p. For $q \leq p$, we can form the group $C^q_n(M)$ of singular n simplexes of class C^q in M. Intuitively, we feel that the resultant chain complex should not depend, up to homology, on q. We would expect a process analogous to simplicial approximation to allow us to smooth a simplex of class C^q to obtain a homologous simplex of class C^p. It is not hard to give a proof (using a double complex) that the homology of $C^q(M)$ is isomorphic to that of $C^p(M)$, but these proofs do not demonstrate that the isomorphism is induced by the inclusion of $C^p \subseteq C^q$ and they do not address the question of naturality. But we can use the theory described here to do both. By choosing $p = \infty$, $q = 0$ and using Stokes' theorem, this also demonstrates that singular cohomology is naturally isomorphic to de Rham cohomology on C^∞ manifolds and the isomorphism is induced by the restriction of singular cochains to the C^∞ cochains.

We define a special category $\mathcal{X}$ to deal with this situation. An object is a 4-tuple (X, I, U, u). X is a topological space; I is a set; U is a function $I \longrightarrow O(X)$, the open set lattice of X; and u is a function that chooses, for each *small* singular simplex $\sigma: \Delta \longrightarrow X$, that is one whose image is included in some $U(i)$, an element $u(\sigma) \in I$ such that the image of σ is included in $U\big(u(\sigma)\big)$. We assume that $\{U(i) \parallel i \in I\}$ is a simple open cover of X. A morphism $(f, F): (X, I, U, u) \longrightarrow (Y, J, V, v)$ consists of a continuous map $f: X \longrightarrow Y$ and a function $F: I \longrightarrow J$ subject to two conditions. The first is that for each $i \in I$, there be a (necessarily unique) arrow $f(i): U(i) \longrightarrow V\big(F(i)\big)$ such that

$$\begin{array}{ccc} U(i) & \xrightarrow{\langle i \rangle} & X \\ {\scriptstyle f(i)}\downarrow & & \downarrow{\scriptstyle f} \\ V\big(F(i)\big) & \xrightarrow[\langle F(i) \rangle]{} & Y \end{array}$$

commutes. We have introduced the notation $\langle i \rangle: U(i) \longrightarrow X$ for the inclusion arrow. The second condition is that for each singular simplex σ, $F\big(u(\sigma)\big) = v(f \circ \sigma)$. This latter condition is so exigent that one might wonder if there are any non-identity arrows. As we will see, there are enough for our purposes.

It is no problem at all to see that the obvious definition of composition makes $\mathcal{X}$ into a category. We now define an endofunctor G on $\mathcal{X}$ by

$$G(X, I, U, u) = (\hat{X}, I \times I, U \cap U, \hat{u})$$

where we denote by $\hat{X}$ the disjoint union of the sets $U(i)$. This requires some explanation. For $i \in I$, let $\langle i, i \rangle: U(i) \longrightarrow \hat{X}$ denote the inclusion of that summand. (The reason for this odd notation will become clear quickly.) Let $e: \hat{X} \longrightarrow X$ denote the unique map such that $e \circ \langle i, i \rangle = \langle i \rangle$. For i, $i' \in I$, let $\langle i', i \rangle: U(i') \cap U(i) \longrightarrow \hat{X}$ denote the composite $U(i') \cap U(i) \longrightarrow U(i) \xrightarrow{\langle i,i \rangle} \hat{X}$, the first arrow being inclusion. Then $U \cap U$ denotes the map that takes $(i', i) \in I \times I$ to the inclusion $\langle i', i \rangle$.

Before defining $\hat{u}$, we introduce a bit more notation. Suppose the singular simplex σ factors through $U(i)$. Let us denote by $\sigma(i): \Delta \longrightarrow U(i)$ the unique singular simplex such that $\sigma = \langle i \rangle \circ \sigma(i)$. In that case, let $[\sigma; i]$ denote the singular simplex $\langle i, i \rangle \circ \sigma(i)$ in $\hat{X}$. We observe that every singular simplex in $\hat{X}$ has the form $[\sigma; i]$ for a unique σ and i. For if $\tau: \Delta \longrightarrow \hat{X}$ is a singular simplex, it factors through a unique connected component $U(i)$ and then $\sigma = e \circ \tau$ is a singular simplex of X whose image is included in $U(i)$ so that $\tau = [\sigma; i]$. Now we can define $\hat{u}[\sigma; i] = \langle u(\sigma), i \rangle$. This makes sense because if σ factors through $U(i)$ and $U\big(u(\sigma)\big)$, it factors through their intersection.

Next we have to define G on maps. If (f, F) is a map as above, then we define $G(f, F) = (\hat{f}, F \times F)$, where $\hat{f}: \hat{X} \longrightarrow \hat{Y}$ is the unique map such that the square

$$\begin{array}{ccc} U(i) & \xrightarrow{\langle i,i \rangle} & \hat{X} \\ {\scriptstyle f(i)}\big\downarrow & & \big\downarrow{\scriptstyle \hat{f}} \\ V\big(F(i)\big) & \xrightarrow[\langle F(i),F(i) \rangle]{} & \hat{Y} \end{array}$$

commutes for each $i \in I$. To see that this is a morphism in $\mathcal{X}$, we first observe that from $\sigma = \langle i \rangle \circ \sigma(i)$, we get $f \circ \sigma = f \circ \langle i \rangle \circ \sigma(i) = \langle F(i) \rangle \circ f(i) \circ \sigma(i)$ which means by definition that $(f \circ \sigma)\big(F(i)\big) = f(i) \circ \sigma(i)$. Then,

$$(F \times F)(\hat{u}[\sigma; i]) = (F \times F)\langle u(\sigma), i \rangle = \big\langle F\big(u(\sigma)\big), F(i) \big\rangle$$

while

$$\begin{aligned} v(\hat{f} \circ [\sigma; i]) &= \hat{v}\big(\hat{f} \circ \langle i, i \rangle \circ \sigma(i)\big) = \hat{v}\big(\langle F(i), F(i) \rangle \circ f(i) \circ \sigma(i)\big) \\ &= \hat{v}\Big(\langle F(i), F(i) \rangle \circ (f \circ \sigma)\big(F(i)\big)\Big) = \hat{v}[f \circ \sigma; F(i)] \\ &= \langle v(f \circ \sigma), F(i) \rangle = \big\langle F\big(u(\sigma)\big), F(i) \big\rangle \end{aligned}$$

Define a map $\epsilon = (e, E): G \longrightarrow \mathrm{Id}$ as follows. The map e is already defined by $e \circ \langle i, i \rangle = \langle i \rangle$ on $\langle X, I, U, u \rangle$ and $E: I \times I \longrightarrow I$ is first projection. We have to show that this definition satisfies the requirements for being a map. To this end, for i, $i' \in I$, let $e(\langle i', i \rangle): U(i) \cap U(i') \longrightarrow U(i')$ be the inclusion. The first thing have to show is that $e \circ \langle i', i \rangle = \langle i' \rangle \circ e(\langle i', i \rangle)$. Since the composite of inclusions is an inclusion,

$$e \circ \langle i', i \rangle = e \circ \langle i, i \rangle \circ e(\langle i', i \rangle) = \langle i \rangle \circ e(\langle i', i \rangle)$$

is the inclusion of $U(i) \cap U(i')$ into X, which is also what $\langle i' \rangle \circ \langle i', i \rangle$ is. Second, we must show that for all singular simplexes $[\sigma; i]$ of $\hat{X}$, $E(\hat{u}[\sigma; i]) = u(e \circ [\sigma; i])$. We have

$$E(\hat{u}[\sigma; i]) = E\big(u(\sigma), i\big) = u(\sigma) = u(e \circ [\sigma; i])$$

Now let C be the chain complex functor on $\mathcal{X}$ that has in degree n the free abelian group generated by the small n-simplexes. This depends only on the space and cover, not on the index set or the choice function. C_{-1} is defined as the free abelian group generated by the connected components. It is a standard theorem of singular homology that the inclusion of $C_n(X, I, U, u)$ into the full singular chain group of the space X is a homotopy equivalence. See, for example, [Dold, 1980], Proposition III.7.3.

Since the covers are simple, the complex $CG \longrightarrow C_{-1}G \longrightarrow 0$ is contractible. I claim that for each $n \geq 0$, there is a natural transformation $\theta_n: C_n \longrightarrow C_nG$ such that $C_n\epsilon \circ \theta_n = \mathrm{id}$. It will then follow from Theorem 5.3 that C is ϵ-contractible. For a small $\sigma: \Delta_n \longrightarrow X$, define $\theta_n(\sigma) = [\sigma; u(\sigma)]$. Naturality requires the commutation of

$$\begin{array}{ccc} C_n(X, I, U, u) & \xrightarrow{\theta_n} & C_nG(X, I, U, u) \\ {\scriptstyle C_n(f,F)}\big\downarrow & & \big\downarrow{\scriptstyle C_nG(f,F)} \\ C_n(Y, J, V, v) & \xrightarrow[\theta_n]{} & C_nG(Y, J, V, v) \end{array}$$

The clockwise path applied to $\sigma \in C_n(X, I, U, u)$ is

$$C_nG(f, F) \circ \theta_n(\sigma) = C_n(\hat{f}, F \times F)[\sigma; u(\sigma)] = \hat{f} \circ [\sigma; u(\sigma)]$$

and the other path is

$$\theta_n(f \circ \sigma) = [f \circ \sigma; v(f \circ \sigma)] = \big[f \circ \sigma; F\big(u(\sigma)\big)\big]$$

Thus we have to show that $\hat{f} \circ [\sigma; u(\sigma)] = \big[f \circ \sigma; F\big(u(\sigma)\big)\big]$. But $\big[f \circ \sigma; F\big(u(\sigma)\big)\big]$ is characterized by two properties; first that its composite with e is $f \circ \sigma$ and second that its component is $F(i)$. For the first, we calculate $e \circ \hat{f} \circ [\sigma; i] = f \circ e \circ [\sigma; i] = f \circ \sigma$ while it follows from the definition of $\hat{f}$ that component of $\hat{f} \circ [\sigma; i]$ is $F(i)$.

Thus we conclude:

THEOREM 7.1. *Let K be the chain complex functor that has in degree n the free abelian group generated by the small singular n-simplexes, augmented over the free abelian group on the connected components. Then K is ϵ-contractible and $K \longrightarrow K_{-1} \longrightarrow 0$ is G-contractible with respect to the class of contractions.*

One thing to note is that although the class of contractions is the smallest possible class and gives natural homotopy equivalences, they are natural only with respect to the arrows in the category and there relatively few arrows.

Now let $p > 0$ be an integer and let $\mathcal{X}^p$ denote the subcategory of $\mathcal{X}$ consisting of those (X, I, U, u) for which X is a manifold of class C^p and the open cover is simple with contracting homotopies of class C^p and those maps (f, F) for which f has class C^p. It is known that every differentiable manifold has such a cover (See [Lefschetz, 1942] Section 46 of Chapter VIII and further references cited there), so that the results we get are true for all class C^p manifolds for all $p > 0$.

Now let $q \leq p$ and define the chain complex functor K^q as the free abelian group generated by the small simplexes of class C^q. As usual, K^q_{-1} is the free abelian group generated by the connected components. Exactly the same considerations used in Theorem 7.1 show that K^q is ϵ contractible and that $K^q \longrightarrow K^q_{-1} \longrightarrow 0$ is G-contractible. This is true in particular for $p = q$. There is an inclusion $K^p \longrightarrow K^q$ and $K^p_{-1} = K^q_{-1}$ so that

$K^p \subseteq K^q$ induces a homotopy equivalence. At first, this is seen to be natural only with respect to the arrows in $\mathcal{X}^p$, but in fact the inclusion is natural in the whole category of manifolds of class C^p. Thus the homotopy equivalence is best viewed as being a quasi-homotopy equivalence. Finally, each of these chain complex functors is quasi-homotopy equivalent to the homology based on all (not just small) simplexes of class C^q, resp. C^p. Thus we conclude:

THEOREM 7.2. *Let $q \le p$ be non-negative integers. Then on the category of manifolds of class C^p, the inclusion of the $K^p \longrightarrow K^q$ is a quasi-homotopy equivalence.*

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