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Landau–Kelly representation of statistical thermodynamics of a quantum plasma and electron emission from metals

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We have investigated the influence of a strong magnetic field on various aspects of a quantum Fermi plasma. Due to the strong magnetic field, the distribution function becomes anisotropic. First, we consider non-degenerate quantum, Landau and Kelly distribution function. It was found that the adiabatic equation is similar to the adiabatic equation for a Maxwell distribution function, when we include the magnetic field in the energy expression. Using the Kelly distribution for a degenerate, quantum Fermi gas, parallel and perpendicular components of the pressure were derived. It was found that perpendicular component of pressure never becomes zero and three-dimensional system always stay three-dimensional. Lastly, we investigated electron emission from metals and have shown the influence of the magnetic field. We calculated thermionic emission, the so-called Richardson effect. In addition, we investigate the influence of external electromagnetic radiation on the electron current density (Hallwachs effect) from metals.

Key words: quantum plasma

1. Introduction

Quantum plasmas are a subject of increasing interest due to their potential applications in modern emerging technologies (Lindsay 2010), e.g. metallic and semiconductor nanostructures, which include metallic nano-particles, metal clusters, thin films, spintronics, nanotubes, quantum wells, quantum dots, nanoplasmonic devices, quantum X-ray free electron lasers, etc. In the case of the degenerate Fermi gas, the shape of the Fermi surface provides information about the physical properties of a plasma. Fermi surface is conveniently considered spherical by considering the isotropic momentum distribution attributed to the Fermi gas particles. A lot of literature is available that describes various aspects of linear and nonlinear propagation characteristics of different electrostatic or electromagnetic modes in the context of isotropic Fermi surfaces (Shukla & Eliasson 2010). However, it is well known that there do exist certain situations where the concept of spherical symmetry of a Fermi surface is no longer valid, even in a collisionless regime of a Fermi gas

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(Lifshitz & Peschanskii 1959). In the presence of a magnetic field the momentum in the parallel and perpendicular directions will be different. A precise study in such scenarios demands elongated or even cylindrical Fermi surfaces (Landau & Lifshitz 1980). Tsintsadze and Tsintsadze developed a new type of quantum kinetic equations for Fermi particles of various species and subsequently obtained a set of hydrodynamic equations describing a quantum plasma (Tsintsadze & Tsintsadze 2009*a*,*b*). Based on these studies, the investigation of linear and nonlinear ion acoustic waves in quantum plasmas as well as ion acoustic solitary structures has attracted substantial attention (Eliasson & Shukla 2010; Rasheed, Murtaza & Tsintsadze 2010; Tsintsadze & Tsintsadze 2010; Shah *et al.* 2011; Tsintsadze *et al.* 2011).

In the magnetic field ($\boldsymbol{H}$), a Lorentz force $e/c\,(\boldsymbol{v}\times\boldsymbol{H})$ acts on a particle, with the charge e, in the perpendicular direction to a velocity $\boldsymbol{v}$, so it cannot produce work on the particle. Here, c is the speed of light. Hence, its energy does not depend on the magnetic field. However, as was shown by Landau that the situation radically changes in the quantum mechanical theory of magnetism. The point is that in a constant magnetic field the electrons, under the action of it, rotate in circular orbits in a plane perpendicular to the field $\boldsymbol{H}_0(0, 0, H_0)$. Therefore, the motion of the electrons can be resolved into two parts: one along the field, in which the longitudinal component of energy is not quantized ($E_\parallel = p_\parallel^2/2m_e$), and the second, quantized (Landau & Lifshitz 1948) in a plane perpendicular to $\boldsymbol{H}_0$ (the transverse component). Thus, in the non-relativistic case, the net energy of an electron in a magnetic field without taking into account its spin is $E(p_\parallel, l) = p_\parallel^2/2m_e + \hbar w_{ce}(l + 1/2)$, where m_e is the electron rest mass and $w_{ce} = |e|H_0/(m_e c)$ is the cyclotron frequency of the electron. Effects of Landau quantization on the longitudinal electric wave characteristic in a quantum plasma are considered in Tsintsadze (2010). Novel branches of longitudinal waves are found, which have no analogies without Landau quantization. Using Tsintsadze (2010), an effect of trapping in a degenerate quantum plasma in the presence of Landau quantization was considered in Tsintsadze *et al.* (2015). Our understanding of the thermodynamics of a Fermi quantum plasma, which is of great interest due to its important application in astrophysics (Landstreet 1967; Bisnovatyi-Kogan 1971; Shapiro & Teukolsky 1983; Lipunov 1987; Haensel, Potekhin & Yakovlev 2007), has recently undergone some appreciable theoretical progress. The influence of a strong magnetic field on the thermodynamic properties of a medium is an important issue in supernovae and neutron stars, the convective zone of the Sun and the early pre-stellar period of evolution of the universe. A wide range of new phenomena arises from the magnetic field in the Fermi gas such as the change of shape of the Fermi sphere and thermodynamics (the De Haas & Van Alphen (1930) and Shubnikov & de Haas (1930) effects). Quite recently, an adiabatic magnetization process was proposed in Tsintsadze & Tsintsadze (2014) for cooling the Fermi electron gas to ultra-low temperatures. It should be noted that the diamagnetic effect has a purely quantum nature and in the classical electron gas it is absent.

If a particle has a spin, the intrinsic magnetic moment of the particle interacts directly with the magnetic field. The correct expression for the energy is obtained by adding an extra term $\boldsymbol{\mu}\boldsymbol{H}_0$, corresponding to the energy of the magnetic moment $\boldsymbol{\mu}$ in the magnetic field $\boldsymbol{H}_0$. Hence, the electron energy levels $\varepsilon_e^{l,\delta}$ are determined in the non-relativistic limit by the expression

$$\varepsilon_e^{l,\delta} = \frac{p_\parallel^2}{2m_e} + (2l + 1 + \delta)\mu_B H_0, \tag{1.1}$$

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where l is the orbital quantum number ($l = 0, 1, 2, 3, \ldots$), δ is the operator to the z direction and describes the spin orientation $\boldsymbol{s} = \boldsymbol{\delta}/2(\delta = \pm 1)$ and $\mu_B = |e|\hbar/(2m_e c)$ is the Bohr magneton.

From the expression (1.1) one sees that the energy spectrum of the electrons consists of the lowest Landau level $l = 0, \delta = -1$ and pairs of degenerate levels with opposite polarization $\delta = 1$. Thus each value with $l \neq 0$ occurs twice and that with $l = 0$ once. Therefore, in the non-relativistic limit $\varepsilon_e^{l,\delta}$ can be rewritten as

$$\varepsilon_e^{l,\delta} = \varepsilon_e^{l} = \frac{p_\parallel^2}{2m_e} + \hbar w_{ce} l, \tag{1.2}$$

where $\hbar$ is the Plank constant divided by 2π.

2. Thermodynamics of magnetized plasmas

We investigated thermodynamic quantities of the quantum plasma using two types of distribution function: one is the non-degenerate quantum, Landau and Kelly distribution function and the second one is the Kelly distribution function for a degenerate Fermi gas. First, we consider non-degenerate electron gas in the strong magnetic field.

2.1. *Thermodynamics of Landau–Kelly distribution function*

As was shown by Landau & Lifshitz (1948, p. 90) and Kelly (1964): for particles executing small oscillations about some equilibrium positions (as we say, to an oscillator) that the distribution function of Landau–Kelly statistics has the form

$$f_0^{lk} = \exp\left(-\frac{p_z^2}{2mT} - \frac{p_\perp^2}{2m\varepsilon_\perp}\right), \tag{2.1}$$

where $\varepsilon_\perp = \hbar w_{ce}/2 \ \coth(\hbar w_{ce}/2T)$, T is the temperature in energy units, and H_0 is external magnetic field.

We note that in the magnetic field, the condition for non-degeneracy is

$$\varepsilon_{Fe} \ll T^{1/3}\varepsilon_\perp^{2/3}. \tag{2.2}$$

On the left side of the inequality the Fermi energy of electrons is derived from the Fermi distribution function in the absence of the magnetic field. On the right side is the energy of the electrons in the magnetic field derived from the Landau–Kelly distribution function. Therefore, the inequality gives the condition for non-degeneracy.

The equilibrium density of electrons is defined as

$$n_e = \frac{2}{(2\pi\hbar)^3}\int \mathrm{d}\boldsymbol{p} f_0^{lk}. \tag{2.3}$$

Here, the factor 2 is on account of the electron spin, $\mathrm{d}\boldsymbol{p} = 2\pi p_\perp \,\mathrm{d}p_\perp \,\mathrm{d}p_z$.

Substituting the Landau–Kelly distribution function (2.1) into (2.3), we obtain such expression for the density of electrons

$$n_e = 2\left(\frac{m}{2\pi\hbar^2}\right)^{3/2} T^{1/2}\varepsilon_\perp. \tag{2.4}$$

Let us present the asymptotic expression of (2.4).

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For small $x=\hbar w_c/2T$

$$n_e=2\left(\frac{m}{2\pi\hbar^2}\right)^{3/2}T^{3/2}=n_0, \tag{2.5}$$

and for large x,

$$n_e=\left(\frac{m}{2\pi\hbar^2}\right)^{3/2}T^{1/2}\hbar w_{ce}=n_0\frac{\hbar w_{ce}}{2T}\gg n_0. \tag{2.6}$$

We are using a strong magnetic field approximation, where $\hbar w_{ce}\gg 2K_BT$. Numerically, it corresponds to the inequality $H_0\gg 2\times 10^4T$. From here we can see that at temperatures 1–10°, the magnetic field should be greater than 10^4–10^5 G.

The mean kinetic energy for one electron is defined in the form

$$\langle\varepsilon\rangle=\frac{2}{(2\pi\hbar)^3n_e}\int_{-\infty}^{\infty}\mathrm{d}p_z\int_0^{\infty}2\pi p_\perp\mathrm{d}p_\perp * \left(\frac{p_z^2}{2m_e}+\frac{p_\perp^2}{2m_e}\right)f_0=\varepsilon_\perp+\frac{k_BT}{2}. \tag{2.7}$$

The specific heat for the electron gas is

$$C_V=\frac{\partial\langle\varepsilon\rangle}{\partial T}=k_B\left(\frac{1}{2}+\frac{x^2}{\sinh x^2}\right), \tag{2.8}$$

where $\sinh x$ is the sine hyperbolic function. It's expression in power series is

$$\sinh x=\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}. \tag{2.9}$$

For $x=\hbar w_{ce}/(2k_BT)\ll 1$, the magnetic field is very small and the specific heat is slightly less than $3/2k_B$, i.e.

$$C_V=\tfrac{3}{2}k_B(1-\tfrac{2}{9}x^2). \tag{2.10}$$

For the strong magnetic field $\hbar w_{ce}>k_BT$, we obtain the specific heat in a form

$$C_V=k_B\left(\frac{1}{2}+\left(\frac{\hbar w_{ce}}{k_BT}\right)^2\mathrm{e}^{-\hbar w_{ce}/k_BT}\right). \tag{2.11}$$

From the expression (2.8) we can conclude that the specific heat at the given temperature is a function of the magnetic field: it has a maximum value $C_V=3/2k_B$ when the magnetic field is zero, it has a tendency to decrease with an increase in magnetic field and goes to the constant limit $C_V=1/2k_B$ when the thermal energy is negligible due to the magnetic field energy.

For the calculation of the entropy per particle by the Landau–Kelly distribution function, we use the well-known expression $S=-k_B/n_e\int\mathrm{d}\boldsymbol{p}f\ln f$, which leads to the result

$$S=k_B\ln\frac{(2\pi m_e)^{3/2}T^{1/2}\varepsilon_\perp}{n_e}. \tag{2.12}$$

Since for an adiabatic process $\mathrm{d}S=0$, from (2.12) we obtain the adiabatic equation

$$\frac{T^{1/2}\varepsilon_\perp}{n_e}=\text{const}. \tag{2.13}$$

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First, in the case of a weak magnetic field, i.e. $k_BT > \hbar w_{ce}$, we have the following adiabatic equation:

$$\frac{T^{3/2}}{n_e}\left(1+\frac{1}{12}\left(\frac{\hbar w_{ce}}{k_BT}\right)^2\right) = \text{const.} \tag{2.14}$$

Next, in the case of a strong magnetic field, i.e. $\hbar w_c \gg k_BT$, which is a more interesting case, the adiabatic equation reads

$$\frac{T^{1/2}H}{n} = \text{const.} \tag{2.15}$$

We want to emphasize that the same adiabatic equation (2.15) was found in our paper (Tsintsadze & Tsintsadze 2014) in the same approximation. There we used Maxwell's distribution function with energy $\varepsilon = p_z^2/2m + \hbar w_{ce} l$ (where l is the orbital quantum number $l = 0, 1, 2, 3, \ldots$).

Thus the Maxwell distribution function, which is quite different from the Landau–Kelly distribution function, gives a similar expression for the adiabatic equation.

2.2. *Thermodynamics of the Kelly distribution function*

Next, we consider a degenerate Fermi electron gas. To describe the state of Fermi particles Kelly (1964) derived the distribution function

$$f_\alpha^k = \frac{2}{(2\pi\hbar)^3}\mathrm{e}^{-(p_x^2+p_y^2)/(m_\alpha \hbar w_{c\alpha})}\sum_{l=0}\frac{(-1)^l L_l\left(2\dfrac{p_x^2+p_y^2}{m_\alpha \hbar w_{c\alpha}}\right)}{\mathrm{e}^{(\varepsilon_l-\mu_\alpha)/T}+1}. \tag{2.16}$$

Magnetic field is along the z axis. Here suffix α stands for the particle species, $L_l(x)$ is the Laguerre polynomial of order l (Gradshteyn & Ryzhik 2010), for which the following condition exists $2(-1)^l \int \mathrm{e}^{-w^2} L_l(2w^2) w\,\mathrm{d}w = 1$, $\varepsilon_l = p_\parallel^2/2m_e + \hbar w_c l$, and μ_α is the chemical potential defined by the normalization condition.

$$n_\alpha = 2\int \mathrm{d}\boldsymbol{p} f_\alpha^k(\boldsymbol{p}_\perp, p_\parallel). \tag{2.17}$$

Here, the factor 2 is on account of the particle spin. For simplicity we use the notation $w_\alpha^2 = p_\perp^2/(m_\alpha \hbar w_{c\alpha}) = (p_x^2+p_y^2)/(m_\alpha \hbar w_{c\alpha})$. We note that such a distribution function was independently derived by Zilberman (1970). Kelly's distribution function represents a hybrid distribution function. In a plane perpendicular to the magnetic field $\boldsymbol{H}$, Kelly's distribution function is Boltzmannian, but along the magnetic field distribution function is a Fermi distribution function.

We investigate the Kelly distribution function in a strong magnetic field, when the magnetic energy is greater than or equal to the Fermi energy. As the Kelly distribution describes degenerate systems, we shall consider gas at the temperature limit $|l\hbar w_c - \mu| \gg T$, where the thermal energy is neglectable. In this case the Fermi distribution function is, to a good approximation, described by the Heaviside step function $H\left(\mu - \varepsilon^l\right)$, from which follows $\mu = \varepsilon_{Fe} = \varepsilon^l = p_\parallel^2/2m_e + l\hbar w_{ce}$. For the parallel component of momentum we get the expression $p_\parallel = \pm\sqrt{2m_e}\,(\varepsilon_{Fe} - l\hbar w_{ce})^{1/2}$. The last expression reads that the summation along l is limited by the condition $\varepsilon_{Fe} > l\hbar w_{ce}$, so that $l_{max} = \varepsilon_{Fe}/\hbar w_{ce}$. As we are interested in strong magnetic fields, when $\hbar w_{ce} >= \varepsilon_{Fe}$ the orbital number can have only two possible values $l = 0, 1$.

First, we consider the case when the magnetic energy is greater than the Fermi energy. Orbital number restriction yields $l = 0$. For the lowest Landau level $\delta = -1$ (see (1.1)). In this case Kelly's distribution function is

$$f_\alpha^k\left(\boldsymbol{p}_\perp, p_\parallel\right) = \frac{2\mathrm{e}^{-w_\alpha}}{(2\pi\hbar)^3} \frac{1}{\exp\left(\dfrac{p_\parallel^2/2m_\alpha - \mu_\alpha}{T_\alpha}\right) + 1}. \tag{2.18}$$

At $T = 0$, Kelly's distribution function (2.18) reads

$$f_\alpha^k\left(\boldsymbol{p}_\perp, p_\parallel\right) = \frac{2\mathrm{e}^{-w_\alpha}}{(2\pi\hbar)^3} H\left(\mu_\alpha - p_\parallel^2/2m_\alpha\right), \tag{2.19}$$

where $H(x)$ is the Heaviside step function and $\mu_\alpha = p_F^2/2m_\alpha$. Substituting the distribution function (2.19) into (2.17) we obtain the expression for the density

$$n_\alpha = \frac{m_\alpha \hbar w_{c\alpha} p_F}{\pi^2 \hbar^3}, \tag{2.20}$$

which is true for the lowest Landau level ($l = 0$), i.e. this expression is associated with the Pauli paramagnetism and the self-energy of particles. If we suppose that the density of electrons is constant, then from (2.20) follows an important statement, namely, that the Fermi momentum decreases along with the increase of the magnetic field, so that a pancake configuration of the Fermi energy thins.

Now, we calculate the mean energy of the particles at the lowest Landau level by Kelly's distribution function (2.18).

$$\langle\varepsilon\rangle = \langle\varepsilon_\perp\rangle + \langle\varepsilon_\parallel\rangle = \frac{2}{n_e}\int \mathrm{d}\boldsymbol{p}_\perp \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{p}_\parallel \left(\frac{p_\parallel^2}{2m} + \frac{p_\perp^2}{2m}\right) f_{0e}^k. \tag{2.21}$$

The result of the calculation is

$$\langle\varepsilon\rangle = \frac{\hbar w_c}{2} + \frac{\varepsilon_F}{3}\left(1 + \frac{\pi^2}{6}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right). \tag{2.22}$$

We know, that for the specific heat of a Fermi gas $C_V = (\partial\langle\varepsilon\rangle/\partial T)_V$.

We know that in all temperature regions a metal consists of two subsystems: a crystalline lattice of ions and a free electron gas. Therefore, the specific heat of a metal can be presented as a sum of two items

$$C_V = C_V^{\mathrm{lat}} + C_V^e, \tag{2.23}$$

where C_V^{lat} is the specific heat of the lattice and

$$\theta_D \ll T, \quad C_V^{\mathrm{lat}} = 3k_B N. \tag{2.24a,b}$$

$$\theta_D \gg T, \quad C_V^{\mathrm{lat}} = \frac{12\pi^4}{5} k_B N \left(\frac{T}{\theta_D}\right)^3, \tag{2.25a,b}$$

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where θ_D is the Debye characteristic temperature, N is the total number of particles, C_V^e is the specific heat for a free electron isotropic gas. For $T \gg T_F$

$$C_V^e = \tfrac{3}{2} k_B N, \tag{2.26}$$

and for $T \ll T_F$

$$C_V^e = \frac{\pi^2}{2} K_B N \left(\frac{T}{T_F} \right), \tag{2.27}$$

where $T_F = (3\pi^2)^{2/3} \hbar^2 n_e^{2/3} / (2 m_e k_B)$.

Comparison between $C_V^{\rm lat}$ and C_V^e shows us that for temperatures $T \geqslant 1^\circ$, $C_V^{\rm lat}$ is always more than C_V^e. As was shown by Tsintsadze (2010), a strong magnetic field leads to a reduction of the Fermi energy

$$\varepsilon_F = k_B T_F = \gamma \left(\frac{n}{H} \right)^2, \tag{2.28}$$

where $\gamma = \pi^4 \hbar^4 c^2 / (2 m_e e^2)$. In our case the specific heat follows from the expression of (2.22)

$$C_V^e = \frac{\pi^2}{9} k_B N \left(\frac{k_B T}{\varepsilon_F} \right), \tag{2.29}$$

Here, ε_F is defined by (2.28). Therefore, when the magnetic field increases, i.e. in this case the Fermi energy ε_F decreases, this leads to an increase of the specific heat.

We obtained the above expression equation (2.22)–(2.29) in the limit: $\hbar w_c > \varepsilon_F = \mu$. Now we suppose that $\hbar w_c = \varepsilon_F$. So, in this case, the orbital quantum number can be only $l = 0, 1$. In this limit, Kelly's distribution function is

$$f_0^k = \frac{2 \mathrm{e}^{-w^2}}{(2\pi\hbar)^3} \left(\frac{1}{\exp \dfrac{p_\parallel^2/2m_e - \mu}{k_B T} + 1} - \frac{L_1 \left(\dfrac{2 p_\perp^2}{m \hbar w_c} \right)}{\exp \dfrac{p_\parallel^2}{k_B T} + 1} \right), \tag{2.30}$$

where $L_1 (2 p_\perp^2 / m \hbar w_c) = 1 - 2 p_\perp^2 / m \hbar w_c$.

In such a case, from the last term of equation (2.30) it follows that $T \neq 0$. Using the anisotropic distribution function (2.30) we obtain the expression for the electron density

$$n_e = \frac{m w_c p_f}{\pi^2 \hbar^2} \left(1 + 0.5 \sqrt{\frac{k_B T}{\varepsilon_F}} \right). \tag{2.31}$$

For the mean kinetic energy of per particle

$$\langle \varepsilon \rangle = \langle \varepsilon_\perp \rangle + \langle \varepsilon_\parallel \rangle = \frac{5}{6} \hbar w_c + \frac{1}{3} \hbar w_c \sqrt{\frac{k_B T}{\hbar w_c}}. \tag{2.32}$$

Following (2.32), for the specific heat we obtain

$$C_V = \frac{\partial \langle \varepsilon \rangle}{\partial T} = \frac{k_B}{6} \sqrt{\frac{\hbar w_c}{k_B T}}. \tag{2.33}$$

To get the expression for the specific heat (2.33) we supposed, that $\hbar w_c \propto \varepsilon_F \gg k_B T$, but the temperature here cannot be zero. We can rewrite the relation as $\hbar w_c \propto \varepsilon_F = \gamma\, (n/H)^2$, where γ is defined under (2.27). Therefore the specific heat (2.33) can be called anomalous.

3. Parallel and perpendicular components of pressure and Fermi gas compressibility for the Kelly distribution function

We now derive the perpendicular component of the pressure using the Kelly distribution function (2.19) for electrons in the lowest Landau level ($l=0$, $\delta=-1$) for the temperature $T=0$.

$$P_{\perp e}=\frac{1}{3}\int \mathrm{d}\boldsymbol{p}\frac{(p_x^2+p_y^2)}{m_e}f_e^k(\boldsymbol{p}_\perp, p_\parallel). \tag{3.1}$$

After simple integration of (3.1) we obtain

$$P_{\perp e}=\tfrac{1}{3}\hbar w_{ce}n_e, \tag{3.2}$$

where n_e is the density defined by (2.20).

At temperatures lower than the degeneracy temperature, $T_F=\beta(n/H_0)^2$ (where $\beta=\pi^4\hbar^4c^2/(2m_ee^2)$; Tsintsadze (2010)) from (2.17) and (2.18) the density of the electrons follows the expression

$$n_e=\frac{m_e\hbar w_{ce}p_F}{\pi^2\hbar^3}\left(1-\frac{\pi^2}{24}\left(\frac{T}{T_F}\right)^2\right). \tag{3.3}$$

In this case n_e in (3.2) is governed by (3.3). It is obvious from (3.2) that at $l=0$, $P_\perp$ is not zero.

Next, for the parallel component of the pressure, in the same case, i.e. $l=0$ and $T=0$, we obtain

$$P_{\parallel e}=\frac{1}{3}\times 2\int \mathrm{d}\boldsymbol{p}\frac{p_\parallel^2}{m_e}f_e^k. \tag{3.4}$$

Use of (2.19) in (3.4) yields

$$p_{\parallel e}=\gamma\left(\frac{n_e}{H}\right)^2 n_e, \tag{3.5}$$

where $\gamma=\pi^4\hbar^4c^2/(9m_ee^2)$.

Having expressions (3.2) and (3.5), we can calculate the compressibilities to both directions.

$$u_\perp^2=\frac{1}{m_e}\frac{\partial P_\perp}{\partial n_e}=\frac{\hbar w_{ce}}{3m_e}. \tag{3.6}$$

As we can see, the perpendicular compressibility does not contain the density of particles as well as the temperature. It is just a function of the magnetic field. Thus this velocity is new.

The compressibility along the magnetic field reads

$$u_\parallel^2=\frac{1}{m_e}\frac{\partial P_\parallel}{\partial n_e}=\frac{3\gamma}{m_e}\left(\frac{n_e}{H}\right)^2. \tag{3.7}$$

We want to emphasize that, when the magnetic field increase, the transverse part of compressibility increases and the parallel part decreases.

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4. Richardson effect

Now we consider the Richardson effect (Greiner, Neise & Stöcker 1995, pp. 350–353) in two limit cases. According to the model of an ideal Fermi gas at finite temperature, for electrons in the conductive band of metals there will be a number of electrons with enough energy to leave the metal. This is the so-called Richardson effect, or thermionic emission, where an electron current evaporates from the heated metal. The work function φ is defined as the amount of energy necessary to leave the metal.

We shall assume that the conductive electrons in the metal are free and independent particles in a constant potential well of depth W, produced by the interaction of electrons and metallic ions. We want to suggest that all electrons which hit a surface area element $\mathrm{d}x\,\mathrm{d}y = \mathrm{d}S_z$ with moment p_z and fulfil the requirement $\varepsilon_z = p_z^2/2m_e \geqslant W$ can leave the metal, independent of their momentum component perpendicular to the surface normal.

The current density of electrons that leave the metal is given by the expression

$$J_z = \frac{2e}{(2\pi\hbar)^3}\int_{\sqrt{2m_eW}}^{\infty}\mathrm{d}p_z\frac{p_z}{m_e}\int_{-\infty}^{\infty}\mathrm{d}p_x\int_{-\infty}^{\infty}\mathrm{d}p_y f. \tag{4.1}$$

Where the z axis is along the normal to the metal surface. The factor two is due to the spin.

4.1. *Richardson effect for Landau–Kelly distribution*

First, we investigate the Richardson effect for the Landau–Kelly distribution function. Magnetic field is perpendicular to the metal surface. After simple integration of the current density equation (4.1) using the Landau–Kelly distribution (2.1) we get

$$J_z^{\perp} = \frac{em\varepsilon_{\perp}T}{2\pi^2\hbar^3}\mathrm{e}^{-W/T}. \tag{4.2}$$

If the magnetic field is weak $T \gg \hbar w_{ce}$, we obtain the current density for the classical Richardson effect

$$J_z^{\perp} = en\left(\frac{T}{2\pi m}\right)^{1/2}\mathrm{e}^{-W/T} = \frac{emT^2}{2\pi^2\hbar^3}\mathrm{e}^{-W/T} \equiv J_z^{RCM} \tag{4.3}$$

and the work function φ agrees with the bottom of the well in the classical case.

In the limit of a strong magnetic field, $T \ll \hbar w_{ce}$,

$$J_z(H) = \frac{em\hbar w_{ce}T}{4\pi^2\hbar^3}\mathrm{e}^{-W/T} = \left(\frac{\hbar w_{ce}}{2T}\right)J_z^{RCM}, \tag{4.4}$$

where J_z^R is the current density for the classical Richardson effect.

This expression shows us the strong increase of the Richardson effect in the strong, external magnetic field: $J_z(H) \gg J_z^R$.

Now we investigate the case where the magnetic field is across the surface and the surface normal is perpendicular to the field ($\boldsymbol{B} \perp \boldsymbol{n}$). The magnetic field is along the x axis and the surface normal is in the z direction: $\boldsymbol{B} = (B, 0, 0)$, $\boldsymbol{n} = (0, 0, n)$. Original indices in the Landau–Kelly distribution function defined by (2.1) are changed to reflect magnetic field orientation. We can rewrite the distribution function as

$$f=\exp\left(-\frac{p_x^2}{2mT}-\frac{p_z^2}{2m\varepsilon_\perp}-\frac{p_y^2}{2m\varepsilon_\perp}\right). \tag{4.5}$$

Integrating (4.1) will give the current density expression

$$J_z^{\parallel}=\frac{em\varepsilon_\perp\sqrt{\varepsilon_\perp T}}{2\pi^2\hbar^3}\mathrm{e}^{-W/\varepsilon_\perp}, \tag{4.6}$$

and therefore, if comparing current density expressions for parallel (4.6) and perpendicular (4.2) case,

$$J_z^{\parallel}/J_z^{\perp}=\sqrt{\varepsilon_\perp/T}\exp[W/T-W/\varepsilon_\perp]. \tag{4.7}$$

Since the perpendicular energy is increased by the magnetic field, it is clear that $\varepsilon_\perp > T$, and therefore $J_z^{\parallel}/J_z^{\perp} > 1$. This means that the magnetic field more efficiently increases the current density when it is along the surface.

4.2. *Richardson effect for the Kelly distribution*

Next, we shall consider the Richardson effect in the degenerate Fermi electron gas in the metal, with the presence of a strong external magnetic field. We use the Kelly distribution function.

The current density of electrons for the Kelly model can be calculated by using (4.1) with the Kelly distribution function f^K defined in (2.16).

$$J_z=\frac{2e}{(2\pi\hbar)^3}\int_{\sqrt{2m_eW}}^{\infty}\mathrm{d}p_z\frac{p_z}{m_e}\int_{-\infty}^{\infty}\mathrm{d}p_x\int_{-\infty}^{\infty}\mathrm{d}p_y f^k. \tag{4.8}$$

Integration of (4.1) by p_x and p_y leads us to the integral

$$J_z=\frac{e\hbar\omega_{ce}}{2\pi^2\hbar^3}\sum_{l=0}\int_{\sqrt{2m_eW}}^{\infty}\mathrm{d}p_z\frac{p_z}{\exp\left[\dfrac{p_z^2/(2m)+\hbar\omega_{ce}l-\mu}{T}\right]+1}. \tag{4.9}$$

We can rewrite (4.9) in the form

$$J_z=\frac{e\hbar\omega_{ce}m_eT}{2\pi^2\hbar^3}\sum_{l=0}^{\infty}\int_{W/T}^{\infty}\frac{\mathrm{d}x}{\mathrm{e}^{x+\delta_l}+1}, \tag{4.10}$$

where $x=p_z^2/(2mT)$ and $\delta=(\hbar\omega_{ce}l-\mu)/T$. The result of the integration over x is

$$J_z=\frac{e\hbar\omega_{ce}m_eT}{2\pi^2\hbar^3}\sum_{l=0}^{\infty}\ln\left(1+\exp\left[\frac{-(W-\mu)-\hbar\omega_{ce}l}{T}\right]\right). \tag{4.11}$$

The exponential term in (4.11) is very small, even at $l=0$ and $T=2000$ K. So we can expand the logarithm into power series.

$$J_z=\frac{e\hbar\omega_{ce}m_eT}{2\pi^2\hbar^3}\mathrm{e}^{(W-\mu)/T}\sum_{l=0}\mathrm{e}^{-(\hbar\omega_{ce}l)/T}, \tag{4.12}$$

and summation of the geometric progression gives

$$\sum_{l=0}^{\infty} \mathrm{e}^{-(\hbar\omega_{ce}l)/T} = \frac{1}{1-\mathrm{e}^{-(\hbar\omega_{ce})/T}}. \tag{4.13}$$

Finally, the current density which leaves the metal reads as

$$J_z = \frac{e\hbar\omega_{ce} m_e T}{2\pi^2\hbar^3} \frac{\mathrm{e}^{(W-\mu)/T}}{1-\mathrm{e}^{-(\hbar\omega_{ce})/T}}. \tag{4.14}$$

We can express work using the depth of the potential well W and the Fermi energy ε_F. Since the difference between the bottom of the well and the Fermi energy ($\varepsilon_F = p_F^2/2m$) at $T=0$ is the work function φ, where p_F is the momentum of the quasi-particles on the Fermi surface,

$$\varphi = W - \varepsilon_F > 0. \tag{4.15}$$

For the current density we have

$$J_z = \frac{e\hbar\omega_{ce} m_e T}{2\pi^2\hbar^3} \frac{\mathrm{e}^{-\varphi/T}}{1-\mathrm{e}^{-(\hbar\omega_{ce})/T}}. \tag{4.16}$$

Let suppose that the external magnetic field is zero ($\hbar\omega_{ce}=0$). Then, from (5.5) follows the Richardson current density for the quantum case J_z^{RQM}.

$$J_z = \frac{e m_e T^2}{2\pi^2\hbar^3}\mathrm{e}^{-\varphi/T} \equiv J_z^{RQM}. \tag{4.17}$$

In the case of a strong magnetic field $\hbar\omega_{ce} \gg T$

$$J_z = \frac{e\hbar\omega_{ce} m_e T}{2\pi^2\hbar^3}\mathrm{e}^{-\varphi/T}, \tag{4.18}$$

or in another form

$$J_z(H) = \frac{e m_e T^2}{2\pi^2\hbar^3}\mathrm{e}^{-\varphi/T}\frac{\hbar\omega_{ce}}{T} = J_z^{RQM}\frac{\hbar\omega_{ce}}{T}. \tag{4.19}$$

Comparing expressions (4.17) and (4.19), we clearly see that at the same temperature, the magnetic field helps electrons leave the metal and increases current density.

5. Hallwachs effect

We already considered thermionic emission, where electrons in the conductive band leave the metal using thermal energy. Now we calculate the current density of electrons emerging from the metal when it is illuminated with short wavelength photons, the so-called Hallwachs effect (Greiner *et al.* 1995, pp. 353–354). We use the same model of the metal as in the Richardson effect and assume that electrons in the conductive band, which scatter with photons of energy $\hbar\omega$, obtain the same amount of energy as additional kinetic energy. Electrons in the metal can absorb the whole energy of a photon because in the metal there are enough other particles (for example atoms) to satisfy momentum balance. We can say that such electrons can leave the metal in the z direction if the condition $p_z^2/2m + \hbar\omega > W$ is fulfilled.

For the current density we can write

$$J_z = \frac{2e}{(2\pi\hbar)^3} \int_{\sqrt{2m_e(W-\hbar\omega)}}^{\infty} \mathrm{d}p_z \frac{p_z}{m_e} \int_{-\infty}^{\infty} \mathrm{d}p_x \int_{-\infty}^{\infty} \mathrm{d}p_y f. \tag{5.1}$$

Here, we should note that J_z is the current density of only scattered electrons which leave the metal. The number of scattering processes increases proportionally to the intensity of the incoming radiation. In the expression for the total current density we will have constant of proportionality, which will depend on the reflecting power of the metal, the intensity of the radiation and the cross-section of the scattering process.

5.1. *Hallwachs effect for the Landau–Kelly distribution function*

First, we investigate the Hallwachs effect for the Landau–Kelly distribution function. After simple integration of the current density equation (5.1) using the Landau–Kelly distribution (2.1) we get

$$J_z = \frac{em\varepsilon_\perp T}{2\pi^2\hbar^3} \mathrm{e}^{(\hbar\omega-W)/T}. \tag{5.2}$$

This correspond to pure thermal emission with the work function decreased by $\hbar\omega$. Increasing the photon energy (frequency) exponentially increases the current density. In the special case where the radiation just provides the work function, i.e. $\hbar\omega = W$, the current density is $J_z = em\varepsilon_\perp T/(2\pi^2\hbar^3)$ and therefore is significantly different from zero because of thermal excitation.

5.2. *Hallwachs effect for the Kelly distribution*

Next, we shall consider the Hallwachs effect in the degenerate Fermi electron gas in the metal. For this purpose we use the Kelly distribution function. The current density of electrons for the Kelly model can be calculated by using equation (5.1) with the Kelly distribution function f^K defined in (2.16). Using the same method as for the Richardson effect for the Kelly distribution, we arrive at the current density expression

$$J_z = \frac{e\hbar\omega_{ce} m_e T}{2\pi^2\hbar^3} \sum_{l=0}^{\infty} \ln\left(1 + \exp\left[\frac{\mu + \hbar\omega - W - \hbar\omega_{ce} l}{T}\right]\right). \tag{5.3}$$

Here, we cannot say that the exponential term in (5.3) is very small, since the photon energy can compensate for the difference between the depth of the potential well and the magnetic energy.

First, we investigate (4.12) in the limit $\exp[(\mu + \hbar\omega - W)/T] \ll 1$. We expand the logarithm into power series and then summation of the geometric progression will give

$$J_z = \frac{e\hbar\omega_{ce} m_e T}{2\pi^2\hbar^3} \frac{\mathrm{e}^{-(\varphi-\hbar\omega)/T}}{1 - \mathrm{e}^{-(\hbar\omega_{ce})/T}}. \tag{5.4}$$

Current density is therefore increased due to the radiation because radiation decreases the work function and more electrons, which scatter on the photons, can leave the metal.

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Secondly, we investigate a metal in strong radiation with energy exceeding the work function: $\hbar\omega - W - \mu \gg T$, the magnetic field is strong and satisfies the condition $\hbar\omega_{ce} \gg \hbar\omega - W - \mu$. From (5.3) we obtain the current density expression in the form

$$J_z = \frac{e\hbar\omega_{ce} m_e}{2\pi^2\hbar^3} \left(\hbar\omega + \mu - W\right). \tag{5.5}$$

Current density is linearly proportional to the single photon energy and increases with an increase of it.

6. Summary

To summarize, we studied various thermodynamic quantities of the quantum plasma in a strong, external, uniform and constant magnetic field, using two anisotropic distribution functions for non-degenerate and degenerate quantum Fermi plasmas. The Landau–Kelly distribution function gives us the specific heat that has maximum and minimum values which depends on the external magnetic field: specific heat decreases with an increasing magnetic field. For the Kelly distribution function we investigate the specific heat for two cases: (i) When the magnetic energy exceeds the Fermi energy, an orbital number can have only a $l = 0$ value and the specific heat increases with an increase of the magnetic field. (ii) When the magnetic energy is equal to the Fermi energy ($l = 0,\ 1$), specific heat still increases with the increase of magnetic energy, also it increases with a decrease of temperature, so it can be called anomalous. We derived the parallel and perpendicular components of pressure and we calculated compressibility of a Fermi gas for the Kelly distribution function. We have shown that the perpendicular compressibility increases and the parallel compressibility decreases with an increase of the magnetic field. We also note that perpendicular compressibility depends only on the magnetic field and its expression is new.

We compare the Richardson effect current density expression with and without a magnetic field. The Richardson effect expressions for classical and quantum systems without the magnetic field are well known. In degenerate systems, the work function is smaller than in non-degenerate systems, that is why the electron current for a Fermi distribution is greater than for a Maxwell distribution. Adding an external magnetic field perpendicular to the metal surface significantly increases the current for both the Landau–Kelly and Kelly distribution functions. In addition, for strong magnetic fields, with an increase of the external magnetic field the current increases two times faster for the Kelly distribution than for the Landau–Kelly distribution function.

Moreover, when we illuminate the metal surface with short wavelength radiation, photons give additional kinetic energy to the metal electrons and the current density increases and exceeds the thermionic current density for both distribution functions. In the special case where the photon energy compensates the work function, the electron current is significantly different from zero due to the thermal excitation and external magnetic field.

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