Parametric Envelopes

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1. *Introduction*

This note is about the parametric description of curves. It is triggered by a description of curve stitching in a book by Millington [1], who uses what he calls, for ellipses, a 'principle of inverses'. My wife uses the book, as one source among others, in her primary school teaching. I have also taught, and organised, mathematics masterclasses for 25 years, to 13-year-olds and to 10-year-olds [2, 3]. I describe a different proof from that indicated by Millington, using as an example an envelope which is a closed curve. Then I explore to what extent the same type of proof can be applied to the cardioid, which can be viewed as consisting of two opposed open envelopes.

2. *Principle of inverses*

To describe this 'principle' in an x, y plane using cartesian coordinates, we draw two parallel lines $x = \pm a$ with any fixed $a > 0$. We mark points on them at $y = b$ on $x = a$, and at $y = 1/b$ on $x = -a$, for a sequence of values $b = \ldots, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, \ldots$. The points are joined in pairs by straight lines, as illustrated for those five central values.

We wish to find the envelope of the family of lines as b is varied continuously. The name 'principle of inverses' was evidently used by Millington because the construction consists of joining pairs of points having coordinates $x = a$, $y = b$ and $x = -a$, $y = 1/b$ for an eventually continuous sequence of values of b, with any fixed $a \neq 0$. The associated pairs of values of y, namely b and $1/b$, are the inverses of each other.

3. *Analysis*

The equation of the line joining $x = a$, $y = b$ to $x = -a$, $y = 1/b$ is $\dfrac{y - b}{x - a} = \dfrac{\frac{1}{b} - b}{-a - a}$, which simplifies to $\dfrac{y - b}{x - a} = \dfrac{b^2 - 1}{2ab}$. We have chosen the distance $2a$ between the vertical starting lines to be fixed. Then we want to calculate the *envelope* of all the lines $y - b = \dfrac{(b^2 - 1)(x - a)}{2ab}$ as b varies, with fixed a. This is an equation of the form $F(x, y, b) = 0$, in which the function $F = y - b - \dfrac{(b^2 - 1)(x - a)}{2ab}$.

Siddons, Snell and Morgan [4] tell us that to find such an envelope we have to find the equation $\frac{\partial F}{\partial b} = 0$, and then eliminate the variable parameter b between this equation and $F = 0$. We see that

$$\frac{\partial F}{\partial b} = -1 - \frac{(x - a)}{2a}\left(1 + \frac{1}{b^2}\right). \qquad (1)$$

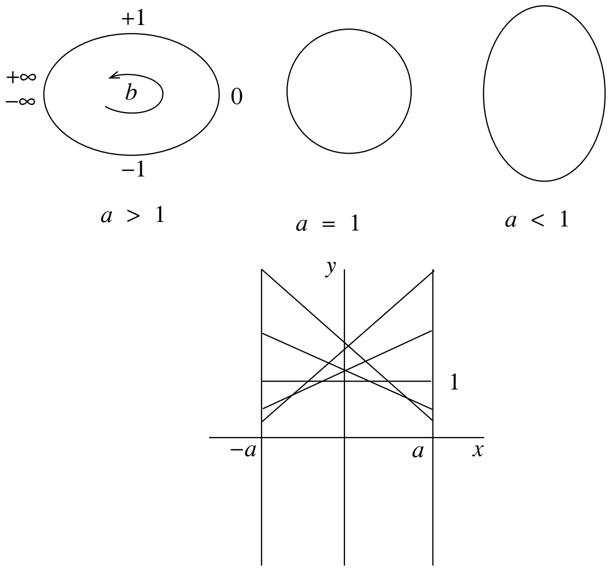
https://doi.org/10.1017/mag.2015.30 Published online by Cambridge University Press

This is zero when $b^2 = \dfrac{a - x}{a + x}$. Combining this property of $\frac{\partial F}{\partial b} = 0$ with the above $F = 0$ leads to $b = \dfrac{a - x}{ay}$ and thence, by eliminating b, to

$$\frac{x^2}{a^2} + y^2 = 1. \tag{2}$$

This is a circle if $a = 1$, an ellipse whose major axis is along the y-axis if $a < 1$, and an ellipse whose major axis is along the x-axis if $a > 1$. (Millington's analysis uses different ideas from those outlined here, and it appears to consider only the case $a > 1$).

The progress of the parameter b round the ellipse monotonically increases anticlockwise from $-\infty$ at the left-hand end of its x-axis, continuing through increasing but negative values to -1 at the bottom of its y-axis, then on through increasing but still negative values to 0 at the right-hand end of its x-axis; and increasing to $+1$ at the top end of its y-axis, and finally reaching $+\infty$ at the left-hand end of its x-axis. This progress is indicated in Figure 1.



ellipse

FIGURE 1

4. *The Cardioid*

The construction of the cardioid provides another example of how to devise an envelope, when approached by the same method as that just illustrated for ellipses; but it turns out to have some very different analytical properties. The basic heart shape is well known. We illustrate it by starting

https://doi.org/10.1017/mag.2015.30 Published online by Cambridge University Press

with the unit circle $x^2 + y^2 = 1$, using cartesian coordinates, with the origin at the centre, x positive to the right and y positive upwards. We mark a discrete number (n) of points on the circumference at equal angles of every t degrees, so that $nt = 360$. We choose $t = 10$ in the diagram so that there are $n = 36$ points there.

We use p to label the points. The top one is $p = 0$ (or $p = 36$) at $x = 0, y = 1$. Moving clockwise from the top, the coordinates of the pth point are $x = \sin b$, $y = \cos b$ where we have written $pt = b$ to be the clockwise angle from the vertical of this particular radius. This b will eventually have the role of a continuously varying parameter, as it did for the ellipse in the previous section, but over a different range. These ranges are $0 \leqslant b \leqslant \pi$ for the right-hand semicircle, and $\pi \leqslant b \leqslant 2\pi$ for the left-hand semicircle. The equation of the chord joining p to its double $2p$ at $x = \sin 2b$, $y = \cos 2b$ is

$$\frac{y - \cos b}{x - \sin b} = \frac{\cos 2b - \cos b}{\sin 2b - \sin b}. \tag{3}$$

Elementary trigonometric formulae show the right-hand side to be $-\tan \frac{3b}{2}$. Therefore this chord can be written as $F(x, y, b) = 0$, where the function F of the coordinates x, y on the chord and the parameter b is

$$F(x, y, b) = y - \cos b + (x - \sin b)\tan\frac{3b}{2}, \tag{4}$$

which has the property

$$\frac{\partial F}{\partial b} = -\frac{1}{2}\sin b - \cos b \tan\frac{3b}{2} + \frac{3}{2}x + \frac{3}{2}(x - \sin b)\tan^2\frac{3b}{2}. \tag{5}$$

We now have the ingredients which are required to determine implicitly the equation of the cardioid, as an envelope of the stated continuous sequence of chords, by eliminating the parameter b from the pair of equations $F = 0$ and $\frac{\partial F}{\partial b} = 0$. In this calculation b is now to be regarded as a continuous variable, having a similar role to that which generated the ellipse in the previous section, but only having a finite range here. It goes from 0 to π in generating the right-hand half of the cardioid, and from π to 2π to generate the left-hand half which is a mirror image of the right-hand half. We can write $F = 0$ as

$$\tan\frac{3b}{2} = \frac{y - \cos b}{\sin b - x} \tag{6}$$

and then $\frac{\partial F}{\partial b} = 0$ as

$$3(x^2 + y^2) - 4x\sin b - 4y\cos b + 1 = 0. \tag{7}$$

So these two equations together provide a parametric description, via b varying from 0 to 2π, of the cardioid shown in the x, y plane of the diagram, where the x, y origin is at the centre of the circle. The particular value $b = \pi$ of the parameter will identify the cusp of the cardioid shown in the diagram, where the last two equations reduce to

$$\pm\infty = \frac{y + 1}{-x}, \qquad 3(x^2 + y^2) + 4y + 1 = 0. \tag{8}$$

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These require $x = 0$ and $y = -\frac{1}{3}$. The conclusion is that the cusp of this cardioid is at

$$x = 0, y = -\tfrac{1}{3}, \tag{9}$$

i.e. one third of the distance down from the centre of the circle to the circumference. Other versions of heart-shaped curves have been known for three hundred years, as Google (for example) makes clear, but I have not seen this parametric version before. When I demonstrated this shape to my ten-year-old Masterclass pupils as the reflection of an oblique beam of light off the internal side of a coffee cup onto the liquid surface, they immediately announced another name: Looks like a bum!

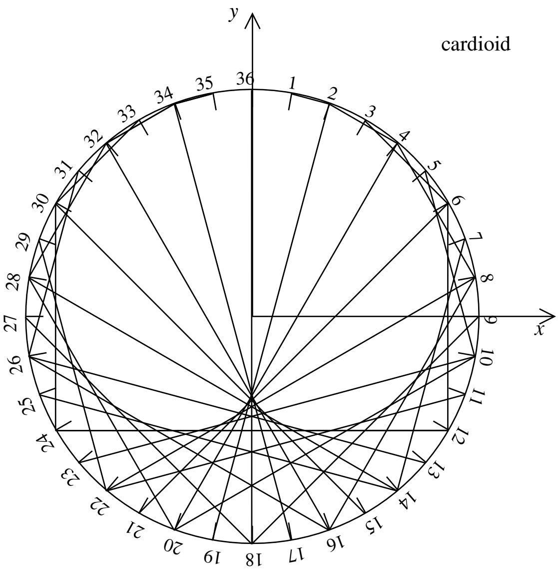

FIGURE 2

References

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2. M. J. Sewell, *Mathematics masterclasses – stretching the imagination*, Oxford University Press (1997).
3. M. J. Sewell, *Mathematics masterclasses for ten-year-olds*, University of Reading Mathematics Department (2014).
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doi:10.1017/mag.2015.30

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https://doi.org/10.1017/mag.2015.30 Published online by Cambridge University Press