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QUASICONFORMAL HARMONIC MAPPINGS BETWEEN DOMAINS CONTAINING INFINITY

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Abstract

Assume that Ω and D are two domains with compact smooth boundaries in the extended complex plane $\overline{\mathbf{C}}$. We prove that every quasiconformal mapping between Ω and D mapping ∞ onto itself is bi-Lipschitz continuous with respect to both the Euclidean and Riemannian metrics.

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1. Introduction

Let U and V be two open domains in the complex plane $\mathbf{C}$. We say that a twice-differentiable mapping $f = u + iv : U \to V$ is harmonic if $\Delta f := \Delta u + i\Delta v = 0$ in U. By the Lewy theorem, any harmonic homeomorphism is a diffeomorphism. If its Jacobian J_f is positive, then it is sense-preserving. In that case $J_f = |f_z|^2 - |f_{\bar z}|^2 > 0$.

We say that a harmonic mapping f is quasiconformal (abbreviated q.c.) if there is a constant k with $0 \le k < 1$ so that $|f_{\bar z}(z)| \le k|f_z(z)|$ for $z \in U$. The family of quasiconformal harmonic mappings was first considered by Martio in [17]. The class of q.c. harmonic mappings contains the conformal mappings and this explains its importance in geometric function theory.

Pavlović [19] showed that a harmonic quasiconformal mapping of the unit disk $\mathbf{U}$ onto itself is bi-Lipschitz continuous. To see the importance of his result, consider the following two separate results. If we assume that the mapping $f : \mathbf{U} \to \mathbf{U}$ is merely quasiconformal, then it is only Hölder continuous with the Hölder coefficient $\alpha = (1 - k)/(1 + k)$. This is a celebrated theorem of Mori. On the other hand, if $f : \mathbf{U} \to \mathbf{U}$ is merely a harmonic diffeomorphism, then by a result of Hengartner and Schober it has a continuous extension up to the boundary (see [6, Theorem 4.3] or [3, Section 3.3]). However, in view of the well-known Radó–Kneser–Choquet theorem, this is the best regularity that such a mapping can have at the boundary.

We can formulate the result of Pavlović precisely and give some extensions of it in terms of the Poisson integral. Define the Poisson kernel by

$$P(z,\theta) = \frac{1}{2\pi}\frac{1-|z|^2}{|z-e^{i\theta}|^2}, \quad |z|<1, \theta\in[0,2\pi).$$

The function $z \mapsto P(z,\theta)$ is harmonic. For a mapping $f \in L^1(\mathbf{T})$, where $\mathbf{T}$ is the unit circle, we define the Poisson integral by

$$w(z) = \mathcal{P}[f](z) = \int_0^{2\pi} P(z,\theta) f(e^{i\theta})\, d\theta.$$

The Radó–Kneser–Choquet theorem states that, if f is a homeomorphism of the unit circle onto a convex Jordan curve γ, then its Poisson integral is a harmonic diffeomorphism of the unit disk $\mathbf{U}$ onto the Jordan domain Ω bounded by γ. If $f = u + iv$ is a harmonic function defined in a smooth Jordan domain D, then its harmonic conjugate is the harmonic function $\tilde{f} = \tilde{u} + i\tilde{v}$ if $u + i\tilde{u}$ and $v + i\tilde{v}$ are analytic functions. Notice that $\tilde{f}$ is uniquely determined up to an additive constant.

Let χ be the boundary value of f and assume that $\tilde{\chi}$ is the boundary value of $\tilde{f}$. Then $\tilde{\chi}$ is called the Hilbert transform of χ and we denote it by $\tilde{\chi} = H(\chi)$. We assume that $\tilde{\chi} \in L^1(\partial D)$. In particular, the Hilbert transform of a function $\chi \in L^1(\mathbf{T})$ is defined by

$$\tilde{\chi}(\tau) = H(\chi)(\tau) = -\frac{1}{\pi}\int_{0^+}^{\pi} \frac{\chi(\tau+t)-\chi(\tau-t)}{2\tan(t/2)}\, dt.$$

Here $\int_{0^+}^{\pi} \Phi(t)\, dt := \lim_{\epsilon\to 0^+} \int_{\epsilon}^{\pi} \Phi(t)\, dt$. This integral is improper and converges for almost all $\tau \in [0,2\pi]$; this and other facts concerning the operator H used in this paper can be found in Zygmund [21, Ch. VII]. Assume that $\chi, \tilde{\chi} = H(\chi) \in L^1(\mathbf{T})$. Then

$$\mathcal{P}[\tilde{\chi}] = \widetilde{\mathcal{P}[\chi]},$$

where $\tilde{k}(z)$ denotes the harmonic conjugate of $k(z)$ (see [20, Theorem 6.1.3]).

A special situation is $\gamma = \mathbf{T}$. Heinz [5] proved that, if f is a harmonic diffeomorphism of the unit disk onto itself, then the Hilbert–Schmidt norm of its derivative is given by

$$\|Df\|^2 = |f_x|^2 + |f_y|^2 \geq c, \tag{1.1}$$

where $c > 0$ depends only on $f(0)$. It follows from (1.1) that the inverse of a quasiconformal harmonic mapping of the unit disk onto itself is Lipschitz continuous. So, the main achievement of Pavlović in [19] was to prove that a harmonic quasiconformal mapping of the unit disk onto itself is Lipschitz continuous on the closure of the domain. Another form of his result can be formulated as the following proposition.

Proposition 1.1. *The harmonic diffeomorphism $f = \mathcal{P}[e^{i\varphi(t)}](z)$ is quasiconformal if and only if the function φ is bi-Lipschitz and the Hilbert transform of φ' is essentially bounded on* $\mathbf{R}$.

In [8], the author proved that every quasiconformal harmonic mapping between Jordan domains with $C^{1,\alpha}$ boundaries is Lipschitz continuous on the closure of the domain. Later this result was extended to Jordan domains with only Dini-smooth boundaries [11] (see the rest of this section for the precise statement of the result). A Dini-smooth boundary is the weakest assumption that we have to impose to get the Lipschitz continuity of such mappings. In fact, Lesley and Warschawski [15] gave an example of a conformal mapping f of the unit disk onto a domain with merely C^1 Jordan boundary, so that f is not Lipschitz continuous. Let Ω be a Jordan domain with rectifiable boundary and let γ be an arc-length parametrisation of $\partial\Omega$. We say that $\partial\Omega$ is C^1 if $\gamma \in C^1$. Then $\arg\gamma'$ is continuous and we let ω be its modulus of continuity. If ω satisfies

$$\int_0^\delta \frac{\omega(t)}{t}\,dt < \infty \quad (\delta > 0),$$

we say that $\partial\Omega$ is Dini-smooth. Denote by $C^{1,\varpi}$ the class of all Dini-smooth Jordan curves. This leads to the following proposition.

Proposition 1.2 [11]. *Let Ω be Jordan domain such that $\partial\Omega \in C^{1,\varpi}$ and let $f : \mathbf{U} \to \Omega$ be a harmonic homeomorphism.*

(a) *If f is quasiconformal, then f is Lipschitz.*
(b) *If Ω is convex and f is q.c., then f is bi-Lipschitz.*
(c) *If Ω is convex, then f is q.c. if and only* $\log|F'|, H(F') \in L^\infty(\partial D)$.

Remark 1.3. If Ω is the unit disk, then Proposition 1.2 coincides with the main result of Pavlović [19].

A bi-Lipschitz characterisation for harmonic quasiconformal mappings of the half-plane onto itself has been established by the author and Pavlović in [12]. Further, it has been shown in [10] that a quasiconformal harmonic mapping between $C^{1,1}$ (not necessarily convex) Jordan domains is bi-Lipschitz continuous. The same conclusion was obtained in [2] by Božin and Mateljević for merely $C^{1,\alpha}$ domains. Further results in the two-dimensional case can be found in [13] and for several dimensions in [1] and [14]. For a different setting for the class of quasiconformal harmonic mappings, we refer to the papers [16, 18]. For example, [16] deals with the following problem for the class of quasiconformal harmonic mappings. The quasihyperbolic metric k_D in a domain D of the complex plane is defined as follows. For points $z_1, z_2 \in D$, set $d_{k_D}(z_1, z_2) = \inf \int_\gamma d(z, \partial D)^{-1}|dz|$, where the infimum is taken over all rectifiable arcs γ joining x_1 and x_2 in D. Manojlović in [16] proved that, if $f : D \to D'$ is a quasiconformal and harmonic mapping, then it is bi-Lipschitz with respect to quasihyperbolic metrics on D and D'.

In this note we prove that a harmonic quasiconformal mapping between two domains with $C^{1,\alpha}$ compact boundary in the extended complex plane containing infinity is bi-Lipschitz continuous, provided that it maps infinity to infinity. On the other hand, we prove that every harmonic quasiconformal mapping between two

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domains with $C^{1,\alpha}$ compact boundary in the extended complex plane containing infinity is a quasi-isometry with respect to the Riemannian metric. The proofs are given in the next section.

To conclude this introduction, we mention the existence problem. In view of the Radó–Kneser–Choquet theorem for convex Jordan domains, a counterpart for domains in the extended complex plane could be formulated as the following conjecture.

Conjecture 1.4. *Let* $\tilde{\mathbf{U}} = \{z : |z| > 1\} \subset \tilde{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. *Assume that* F *is a homeomorphism of the unit circle* $\mathbf{T}$ *onto itself. Then there exists a harmonic diffeomorphism* $f = Ex[F] : \tilde{\mathbf{U}} \to \tilde{\mathbf{U}}$, *possibly having one singularity* a *such that* $f(a) = \infty$, *which has a continuous extension equal to* F *on* $\mathbf{T}$.

2. The main results and their proofs

2.1. Quasiconformal and harmonic mappings and Euclidean quasi-isometry.

Theorem 2.1. *Assume that* Ω *and* D *are domains in* $\mathbf{C} \cup \{\infty\}$ *with* $C^{1,\alpha}$ *boundary so that* $\infty \in \Omega \cap D$. *If* f *is a quasiconformal harmonic mapping between* Ω *and* D *so that* $f(\infty) = \infty$, *then* f *is bi-Lipschitz continuous.*

The theorem gives the following immediate consequence.

Corollary 2.2. *Assume that* f *is a quasiconformal harmonic mapping of the complement of the unit disk onto the complement of a Jordan domain with* $C^{1,\alpha}$ *boundary so that* $f(\infty) = \infty$. *Then* f *is bi-Lipschitz continuous.*

For the sake of completeness, we give here a formal definition of quasiconformal mappings. The map $w : \mathbb{D} \to \mathbb{C}$ of the unit disk to the complex plane is quasiconformal if it is a sense-preserving homeomorphism that has locally L^2-integrable weak partial derivatives and, for almost every $z \in \mathbb{D}$, it satisfies the distortion inequality $|w_{\bar{z}}| \le k|w_z|$, where $k < 1$. In this situation we say that w is K-quasiconformal, where the constant K is defined by $K := (1 + k)/(1 - k)$.

To prove our results, we need the following propositions.

Proposition 2.3 [7]. *Let* $\tilde{\mathbf{U}} = \{z : |z| > 1\}$ *and assume that* f *is a univalent harmonic mapping of* $\tilde{\mathbf{U}}$ *so that* $f(\infty) = \infty$. *Then*

$$f(z) = a_0 + \alpha z + \beta\bar{z} + \sum_{k=1}^{\infty} a_k z^{-k} + \sum_{k=1}^{\infty} b_k \bar{z}^{-k} + A \log|z|. \tag{2.1}$$

Moreover, $0 \le |\beta| < |\alpha|$ *and* $a(z) = \overline{f_{\bar{z}}}/f_z$ *is an analytic function of* $\tilde{\mathbf{U}}$ *onto* $\mathbf{U}$.

Proposition 2.4 [2]. *Let* $\alpha \in (0, 1]$ *and assume that* $w = f(z)$ *is a* K*-quasiconformal harmonic mapping between planar Jordan domains* Ω_1 *and* Ω *with* $C^{1,\alpha}$ *boundaries. Suppose in addition that* $a_0 \in \Omega_1$ *and* $b_0 = f(a_0)$. *Then* w *is bi-Lipschitz. Moreover, there exists a positive constant* $c = c(K, \Omega, \Omega_1, a_0, b_0) \ge 1$ *such that*

$$|Df(z)| \le c \quad \text{and} \quad |Df^{-1}(f(z))| \le c \quad \text{for } z \in \Omega_1.$$

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Here and in the sequel $|Df(z)| = |f_z(z)| + |f_{\bar z}(z)|$. The case $\alpha = 1$ of Proposition 2.4 has been previously proved in [10].

Proposition 2.5 (Kellogg, see [4]). *Let $0 < \alpha < 1$. If D and Ω are Jordan domains having $C^{1,\alpha}$ boundaries and ω is a conformal mapping of D onto Ω, then $\omega' \in C^\alpha(\overline{D})$ and $(\omega^{-1})' \in C^\alpha(\overline{\Omega})$.*

Corollary 2.6. *Let $w = f(z)$ be a K-quasiconformal harmonic mapping between planar domains Ω_1 and Ω with $C^{1,\alpha}$ compact boundaries. Suppose in addition that $a_0 \in \Omega_1$ and $b_0 = f(a_0)$. Then w is bi-Lipschitz. Moreover, there exists a positive constant $c = c(K, \Omega, \Omega_1, a_0, b_0) \geq 1$ such that*

$$\frac{1}{c}|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq c|z_1 - z_2| \quad \text{for } z_1, z_2 \in \Omega_1.$$

Proof of Corollary 2.6. Let $b = f(a) \in \partial\Omega$. As $\partial\Omega \in C^{1,\alpha}$, it follows that there exists a $C^{1,\alpha}$ Jordan curve $\gamma_b \subset \partial\Omega$ whose interior D_b lies in Ω and such that $\partial\Omega \cap \gamma_b$ is a neighbourhood of b. (See [8, Theorem 2.1] for an explicit construction of such a Jordan curve.) Let $D_a = f^{-1}(D_b)$ and take a conformal mapping g_a of the unit disk onto D_a. Then $f_a = f \circ g_a$ is a q.c. harmonic mapping of the unit disk onto the $C^{1,\alpha}$ domain D_b. According to Proposition 2.4, f_a is bi-Lipschitz. According to Kellogg's theorem, $f = f_a \circ g_a^{-1}$ and its inverse f^{-1} are Lipschitz in some small neighbourhoods of a and $b = f(a)$, respectively. This means that Df is bounded in some neighbourhood of a. Since $\partial\Omega_1$ is compact, it follows that ∇f is bounded in $\partial\Omega_1$. The same holds for Df^{-1} with respect to $\partial\Omega$. This implies that f is bi-Lipschitz. □

Proof of Theorem 2.1. Since $\infty \in \Omega$, there is a real number $R > 0$ such that the set $A_R = \{z : |z| > R\} \subset \Omega$. Then $B_R = f(A_R)$ is a domain with $C^{1,\alpha}$ boundary containing ∞. Moreover, $P_R := \Omega \setminus A_{R+1}$ is a planar domain whose boundary consists of finitely many $C^{1,\alpha}$ Jordan curves. Then $Q_R = f(A_{R+1})$ is also a planar domain whose boundary consists of finitely many $C^{1,\alpha}$ Jordan curves.

From (2.1), $f_z = \alpha + O(1/|z|)$ and $f_{\bar z} = \beta + O(1/|z|)$. Therefore, f is Lipschitz in a neighbourhood of $z = \infty$, that is, in a domain A_R for a big enough R. Moreover, $|\alpha| > |\beta|$. Therefore,

$$J_f(z) = |\alpha|^2 - |\beta^2| + O(1/|z|) \quad \text{as } z \to \infty.$$

From this it follows that f is locally bi-Lipschitz continuous in $|z| > R$. Thus, there is a constant C_1 so that

$$|Df(z)| \leq C_1 \quad \text{and} \quad |Df^{-1}(f(z)| \leq C_1 \quad \text{for } z \in A_R.$$

By Proposition 2.4, the mapping $f : P_R \to Q_R$ is bi-Lipschitz continuous. Thus, there is a constant C_2 so that

$$|Df(z)| \leq C_2 \quad \text{and} \quad |Df^{-1}(f(z)| \leq C_2 \quad \text{for } z \in P_R.$$

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Since $\Omega = P_R \cup A_R$, we now need to prove that f is bi-Lipschitz in Ω with the bi-Lipschitz constant $C = C(C_1, C_2, \Omega, D)$, that is, we prove the double inequality

$$\frac{1}{C}|z-w| \le |f(z) - f(w)| \le C|z-w| \quad \text{for } z, w \in \Omega.$$

First we prove that Ω is a chord-arc domain. We show that there is a positive constant M so that, if $z, w \in \Omega$, then there is a rectifiable curve α joining z and w with length $l(\alpha) \le M|z-w|$. In order to prove this, let $\delta_i = [z, w] \cap \Omega_i$, where Ω_i is the Jordan domain bounded by γ_i. If one of the sets δ_i is not empty, then there are two points z' and w' such that $[z, z') \subset \Omega$, $z' \in \delta_i$, $(w', w] \subset \Omega$ and $w' \in \delta_j$ for some i and j from the set $\{1, \ldots, n\}$, allowing the possibility that i and j are equal.

Now we construct a curve α joining z and w. If $\delta_i = \sum_{k=1}^{m_i}[z_k^i, z_{k+1}^i]$, then we denote by $\gamma_i([z_k^i, z_{k+1}^i])$ the shorter of the two Jordan arcs of γ_i that join z_k^i and z_{k+1}^i. Then we define the portion $\alpha_i = \sum_{k=1}^{m_i} \gamma_i([z_k^i, z_{k+1}^i])$. Since γ_i is smooth, there is a constant B_i so that $l(\alpha_i) \le B_i \sum_{k=1}^{m_i} |[z_k^i, z_{k+1}^i]|$. Now we define $\alpha = [z, z'] + \sum_{k=1}^{n} \alpha_i + [w', w]$, which gives $l(\alpha) \le \max\{1, \max\{B_i : i \in [0, n]\}\}|z-w|$. So, we can choose M to be a little bigger than the constant $\max\{1, \max\{B_i : i \in [0, n]\}\}$ but close enough to it in order to allow α to be entirely in Ω and to consist of linear segments $[z_k, z_{k+1}]$, for $k = 0, \ldots, n$, on which f is locally $C = \max\{C_1, C_2\}$ bi-Lipschitz. Therefore,

$$\begin{aligned}|f(z) - f(w)| &= \Big|\sum_{k=0}^{m} f(z_{k+1}) - f(z_k)\Big| \\ &\le \sum_{k=0}^{m} |f(z_{k+1}) - f(z_k)| \le C \sum_{k=0}^{m} |z_{k+1} - z_k| \le CM|z-w|.\end{aligned}$$

Similarly, we prove that f^{-1} is Lipschitz. □

2.2. Quasiconformal and harmonic mappings and Riemannian quasi-isometry. If, in the notation of Theorem 2.1, $f(a) = \infty$ instead of $f(\infty) = \infty$, then the Möbius transformation $m(z) = (z + 1/\bar{a})/(1 + z/a)$ gives a harmonic mapping of $\tilde{\mathbf{U}}$ onto itself defined by $F(z) = f(m(z))$ so that $F(\infty) = f(a) = \infty$ and such that

$$|DF| = |Df|\frac{|a|^2 - 1}{|a+z|^2} \le L \tag{2.2}$$

and

$$l(DF) = l(Df)\frac{|a|^2 - 1}{|a+z|^2} \ge 1/L \tag{2.3}$$

for $z \in \tilde{\mathbf{U}}$. Here $L > 1$ is a constant and $l(A) = \inf_{|z|=1} |Az|$.

Let dR denote the spherical (Riemannian) metric $dR = |dz|/(1 + |z|^2)$. The Riemannian distance between given points $z, w \in \overline{\mathbf{C}}$ is given by

$$d_R(z, w) = \tan^{-1}\frac{|z-w|}{|1 + z\overline{w}|}. \tag{2.4}$$

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The relation (2.4) may not be new, but since we could not find a reference we give its proof here. The conformal Riemannian isometries of the unit sphere are given by

$$m(\zeta) = \frac{a\zeta + b}{\bar{a} - \bar{b}\zeta}$$

and it is easy to verify that they satisfy the relation

$$|m'(z)| = \frac{1 + |m(z)|^2}{1 + |z|^2}.$$

So, for given points z and w, choosing the isometry $m(\zeta) = e^{-it}(\zeta - z)/(1 + \zeta\bar{z})$ gives $m(z) = 0$ and $p = m(w) = e^{-it}(w - z)/1 + w\bar{z}) > 0$ for a certain real constant t. Therefore,

$$d_R(z, w) = d_R(0, p) = \inf_\gamma \int_\gamma \frac{|dz|}{1 + |z|^2},$$

where the infinum is taken over the whole class of rectifiable arcs γ, so that $0, p \in \gamma$.

Now we use the following proposition to deduce (2.4).

Proposition 2.7 [9]. *Let $l \in \mathbf{R}$. If the metric $\rho = \rho_\Sigma$ in the geodesic disk $D_\rho(0, r)$ of a Riemann surface Σ is given by $\rho_\Sigma(z) = h(|z|^2)$, then the intrinsic distance between $lz, z \in D_\rho(0, r)$, with $[lz, z] \subset D_\rho(0, r)$, is given by*

$$d_\Sigma(lz, z) = \left| \int_{l|z|}^{|z|} h(t^2)\, dt \right|.$$

Next we prove that a q.c. harmonic mapping between $\tilde{\mathbf{U}}$ and Ω is a quasi-isometry with respect to this metric. Let γ be a Jordan curve joining two different points z and w. Let $\delta = f(\gamma)$. We need to show that there exists a constant $M > 0$ independent of z and w so that

$$\int_\delta \frac{|d\zeta|}{1 + |\zeta|^2} \le M \int_\gamma \frac{|dz|}{1 + |z|^2}. \tag{2.5}$$

From (2.2), for $\zeta = f(z)$,

$$|d\zeta| \le L\frac{|a + z|^2}{|a|^2 - 1}|dz|. \tag{2.6}$$

From (2.3),

$$|\zeta - f(a)| \ge \frac{1}{L} \int_{[a,z]} l(Df)(\zeta)|d\zeta| \ge \frac{1}{L(|a|^2 - 1)} \int_{[z,a]} |a + \zeta|^2 |d\zeta|.$$

Thus,

$$|\zeta - f(a)| \ge \frac{1}{L(|a|^2 - 1)} \frac{||z| - |a||^3}{3}. \tag{2.7}$$

From (2.6) and (2.7), it is clear that we can find a constant $M = M(a, L)$ so that (2.5) holds. Moreover, a converse inequality can be proved in a similar way by using the fact that f is bi-Lipschitz continuous. In other words, if d_R is the Riemannian distance on the extended complex plane, then we have the following statement.

Theorem 2.8. *If $f : \tilde{\mathbf{U}} \to \Omega$ is a quasiconformal harmonic mapping, where Ω is a domain with $C^{1,\alpha}$ boundary containing ∞, then there is a constant $M > 0$ so that for every $z, w \in \tilde{\mathbf{U}}$,*

$$\frac{1}{M} d_R(z, w) \le d_R(f(z), f(w)) \le M d_R(z, w).$$

References

[1] K. Astala and V. Manojlović, 'On Pavlović theorem in space', *Potential Anal.* **43**(3) (2015), 361–370.

[2] V. Božin and M. Mateljević, 'Quasiconformal and HQC mappings between Lyapunov Jordan domains', *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5),* to appear, 23 pages, doi: 10.2422/20362145.201708_013.

[3] P. Duren, *Harmonic Mappings in the Plane*, Cambridge Tracts in Mathematics, 156 (Cambridge University Press, Cambridge, 2004).

[4] G. M. Goluzin, *Geometric Function Theory of a Complex Variable*, Translations of Mathematical Monographs, 26 (American Mathematical Society, Providence, RI, 1969).

[5] E. Heinz, 'On one-to-one harmonic mappings', *Pacific J. Math.* **9** (1959), 101–105.

[6] W. Hengartner and G. Schober, 'Harmonic mappings with given dilatation', *J. Lond. Math. Soc. Ser. II* **33** (1986), 473–483.

[7] W. Hengartner and G. Schober, 'Univalent harmonic functions', *Trans. Amer. Math. Soc.* **299** (1987), 1–31.

[8] D. Kalaj, 'Quasiconformal harmonic mapping between Jordan domains', *Math. Z.* **260**(2) (2008), 237–252.

[9] D. Kalaj, 'Harmonic maps between annuli on Riemann surfaces', *Israel J. Math.* **182** (2011), 123–147.

[10] D. Kalaj, 'Harmonic mappings and distance function', *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **10**(3) (2011), 669–681.

[11] D. Kalaj, 'Quasiconformal harmonic mappings between Dini's smooth Jordan domains', *Pacific J. Math.* **276** (2015), 213–228.

[12] D. Kalaj and M. Pavlović, 'Boundary correspondence under quasiconformal harmonic diffeomorphisms of a half-plane', *Ann. Acad. Sci. Fenn. Math.* **30**(1) (2005), 159–165.

[13] D. Kalaj and E. Saksman, 'Quasiconformal maps with controlled Laplacian', *J. Anal. Math.* **137**(1) (2019), 251–268.

[14] D. Kalaj and A. Zlatičanin, 'Quasiconformal mappings with controlled Laplacian and Hölder continuity', *Ann. Acad. Sci. Fenn. Math.* **44**(2) (2019), 797–803.

[15] F. D. Lesley and S. E. Warschawski, 'On conformal mappings with derivative in VMOA', *Math. Z.* **158** (1978), 275–283.

[16] V. Manojlović, 'Bi-Lipschicity of quasiconformal harmonic mappings in the plane', *Filomat* **23**(1) (2009), 85–89.

[17] O. Martio, 'On harmonic quasiconformal mappings', *Ann. Acad. Sci. Fenn. Ser. A* I **425** (1986), 10 pages.

[18] D. Partyka, K.-I. Sakan and J.-F. Zhu, 'Quasiconformal harmonic mappings with the convex holomorphic part', *Ann. Acad. Sci. Fenn. Math.* **43**(1) (2018), 401–418; erratum, *Ann. Acad. Sci. Fenn. Math.* **43**(2) (2018), 1085–1086.

[19] M. Pavlović, 'Boundary correspondence under harmonic quasiconformal homeomorphisms of the unit disc', *Ann. Acad. Sci. Fenn. Math.* **27** (2002), 365–372.

[20] M. Pavlović, *Introduction to Function Spaces on the Disk* (Matematički Institut SANU, Belgrade, 2004).

[21] A. Zygmund, *Trigonometric Series I* (Cambrige University Press, Cambridge, 1958).

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