Integrals evaluated in terms of Catalan's constant

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Catalan's constant, named after E. C. Catalan (1814-1894) and usually denoted by G, is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \dots .$$

It is, of course, a close relative of

$$\sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2} = \tfrac{3}{4}\zeta(2) = \frac{\pi^2}{8}.$$

The numerical value is $G \approx 0.9159656$. It is not known whether G is irrational: this remains a stubbornly unsolved problem. The best hope for a solution might appear to be the method of Beukers [1] to prove the irrationality of $\zeta(2)$ directly from the series, but it is not clear how to adapt this method to G.

A remarkable assortment of seemingly very different definite integrals equate to G, or are evaluated in terms of G. A compilation of no fewer than eighty integral and series representations for G, with proofs, is given by Bradley [2]. Another compilation, based on *Mathematica*, is [3]. Here we will present a selection of some of the simpler integrals and double integrals that are evaluated in terms of G, including a few that are actually not to be found in [2] or [3].

The most basic one is

$$\int_0^1 \frac{\tan^{-1} x}{x}\, dx = G, \tag{1}$$

obtained at once by termwise integration of the series

$$\frac{\tan^{-1} x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n + 1}.$$

Termwise integration (for those who care) is easily justified. Write

$$s_{2n}(x) = \sum_{r=0}^{n} (-1)^r \frac{x^{2r}}{2r + 1}.$$

Since the series is alternating, with terms decreasing in magnitude, we have $\frac{\tan^{-1} x}{x} = s_{2n}(x) + r_{2n}(x)$, where $|r_{2n}(x)| \leqslant \frac{x^{2n+2}}{2n + 3}$, so that $\int_0^1 r_{2n}(x)\, dx \to 0$ as $n \to \infty$.

We now embark on a round trip of integrals derived directly from (1). Most of them can be seen in [4], but we repeat them here for completeness. First, the substitution $x = \tan\theta$ gives

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$$G = \int_0^{\pi/4} \frac{\theta}{\tan\theta} \sec^2\theta \, d\theta = \int_0^{\pi/4} \frac{\theta}{\sin\theta \cos\theta} \, d\theta. \tag{2}$$

Since $\tan\theta + \cot\theta = \dfrac{1}{\sin\theta \cos\theta}$, (2) can be rewritten

$$\int_0^{\pi/4} \theta \left(\tan\theta + \cot\theta\right) d\theta = G. \tag{3}$$

The substitution $\theta = 2\phi$ gives one of the most important equivalent forms:

$$\int_0^{\pi/2} \frac{\theta}{\sin\theta} \, d\theta = \int_0^{\pi/4} \frac{4\phi}{\sin 2\phi} \, d\phi = \int_0^{\pi/4} \frac{2\phi}{\sin\phi \cos\phi} \, d\phi = 2G. \tag{4}$$

We mention in passing that (4) can be rewritten in terms of the gamma function. Recall that by Euler's reflection formula,

$$\Gamma(1 + x)\Gamma(1 - x) = \frac{\pi x}{\sin \pi x}.$$

Substituting $\theta = \pi x$ in (4), we deduce

$$\int_0^{1/2} \Gamma(1 + x)\Gamma(1 - x) \, dx = \frac{2G}{\pi}.$$

The substitution $x = e^{-t}$ in (1) gives at once

$$G = \int_0^\infty \tan^{-1}\left(e^{-t}\right) dt.$$

Next, we integrate by parts in (1), obtaining

$$G = \left[\tan^{-1} x \log x\right]_0^1 - \int_0^1 \frac{\log x}{1 + x^2} \, dx.$$

The first term is zero, since $\tan^{-1} x \log x \sim x \log x \to 0$ as $x \to 0^+$, hence

$$\int_0^1 \frac{\log x}{1 + x^2} \, dx = -G. \tag{5}$$

Now substituting $x = \frac{1}{y}$ in (5), we deduce

$$\int_1^\infty \frac{\log x}{1 + x^2} \, dx = G. \tag{6}$$

Substituting $x = e^y$ in (6), we obtain

$$G = \int_0^\infty \frac{y}{1 + e^{2y}} e^y \, dy = \int_0^\infty \frac{y}{2 \cosh y} \, dy,$$

hence

$$\int_0^\infty \frac{x}{\cosh x} \, dx = 2G.$$

Now substituting $x = \tan\theta$ in (6), we obtain

$$-G = \int_0^{\pi/4} \frac{\log \tan\theta}{\sec^2\theta} \sec^2\theta \, d\theta = \int_0^{\pi/4} \log \tan\theta \, d\theta. \tag{7}$$

Since $\dfrac{1 + \cos\theta}{1 - \cos\theta} = \cot^2\dfrac{\theta}{2}$, we can deduce

$$\int_0^{\pi/2} \log\frac{1+\cos\theta}{1-\cos\theta}\,d\theta = -2\int_0^{\pi/2} \log\tan\frac{\theta}{2}\,d\theta = -4\int_0^{\pi/4} \log\tan\phi\,d\phi = 4G.$$

This, in turn, leads to an interesting series representation. Given the series

$$\frac{1}{2}\log\frac{1+x}{1-x} = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$$

and the well-known identity

$$\int_0^{\pi/2} \cos^{2n+1}\theta\,d\theta = \int_0^{\pi/2} \sin^{2n+1}\theta\,d\theta = \frac{2.4.\,\ldots\,.(2n)}{1.3.\,\ldots\,.(2n+1)}, \qquad (8)$$

we deduce

$$2G = \sum_{n=0}^{\infty}\frac{2.4.\,\ldots\,.(2n)}{1.3.\,\ldots\,.(2n-1)(2n+1)^2} = \sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2}{(2n)!\,(2n+1)^2} = \sum_{n=0}^{\infty}\frac{2^{2n}}{\binom{2n}{n}(2n+1)^2}.$$

Using (8) again (but none of the earlier integrals), we now establish

$$\int_0^{\pi/2} \sinh^{-1}(\sin\theta)\,d\theta = G.$$

Write

$$a_n = \frac{1.3.\,\ldots\,.(2n-1)}{2.4.\,\ldots\,.(2n)}.$$

By the binomial series,

$$\frac{1}{(1+x^2)^{1/2}} = \sum_{n=0}^{\infty}(-1)^n a_n x^{2n}.$$

Hence

$$\sinh^{-1}x = \int_0^x \frac{1}{(1+t^2)^{1/2}}\,dt = \sum_{n=0}^{\infty}(-1)^n\frac{a_n}{2n+1}x^{2n+1}.$$

As above,

$$\int_0^{\pi/2} \sin^{2n+1}\theta\,d\theta = \frac{1}{a_n(2n+1)},$$

so, by a neat cancellation of a_n,

$$\int_0^{\pi/2} \sinh^{-1}(\sin\theta)\,d\theta = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = G.$$

We remark that the same method, applied to $\sin^{-1}(\sin\theta)\,(=\theta)$ delivers the series

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \int_0^{\pi/2}\theta\,d\theta = \frac{\pi^2}{8}.$$

Integrals of logsine type and applications

A further application of (7) is to integrals of the logsine type, of which there are a rich variety. We start by stating the most basic such integral, which does not involve G:

$$\int_0^{\pi/2} \log \sin\theta \, d\theta = \int_0^{\pi/2} \log \cos\theta \, d\theta = -\frac{\pi}{2} \log 2. \tag{9}$$

Our proof follows [5, p. 246]. The substitution $\theta = \frac{\pi}{2} - \phi$ shows that the two integrals are equal: denote them both by I. Also, the substitution $\theta = \pi - \phi$ gives $\int_{\pi/2}^{\pi} \log \sin\theta \, d\theta = I$. Hence $2I = \int_0^{\pi} \log \sin\theta \, d\theta$. Now substituting $\theta = 2\phi$ and using $\sin 2\phi = 2 \sin\phi \cos\phi$, we have

$$\begin{aligned} 2I &= 2\int_0^{\pi/2} \log \sin 2\phi \, d\phi \\ &= 2\int_0^{\pi/2} \left(\log \sin\phi + \log \cos\phi + \log 2\right) d\phi \\ &= 4I + \pi \log 2, \end{aligned}$$

hence (9).

Note that the equality of the two integrals in (9) (without knowing their value) implies that $\int_0^{\pi/2} \log \tan\theta \, d\theta = 0$. Also, (9) can be restated neatly as follows:

$$\int_0^{\pi/2} \log\left(2 \sin\theta\right) d\theta = \int_0^{\pi/2} \log\left(2 \cos\theta\right) d\theta = 0.$$

The integral (9) has numerous equivalent forms. For example, first substituting $x = \sin\theta$ and then integrating by parts, we find

$$\int_0^1 \frac{\sin^{-1} x}{x} \, dx = \int_0^{\pi/2} \theta \cot\theta \, d\theta = -\int_0^{\pi/2} \log \sin\theta \, d\theta = \frac{\pi}{2} \log 2.$$

Further equivalents, and a survey of Euler's work in this area, are given in [6].

Catalan's constant enters the scene when we integrate from 0 to $\frac{\pi}{4}$ instead of $\frac{\pi}{2}$. Let

$$I_S = \int_0^{\pi/4} \log \sin\theta \, d\theta, \qquad I_C = \int_0^{\pi/4} \log \cos\theta \, d\theta.$$

Substituting $\theta = \frac{\pi}{2} - \phi$, we have $I_C = \int_{\pi/4}^{\pi/2} \log \sin\theta \, d\theta$. So by (9), $I_S + I_C = \int_0^{\pi/2} \log \sin\theta \, d\theta = -\frac{\pi}{2} \log 2$. Meanwhile by (7),

$$I_S - I_C = \int_0^{\pi/4} \log \tan\theta \, d\theta = -G.$$

So we conclude

$$I_S = -\frac{1}{2}G - \frac{\pi}{4} \log 2, \qquad I_C = \frac{1}{2}G - \frac{\pi}{4} \log 2. \tag{10}$$

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Again there is a neat restatement:

$$\int_0^{\pi/4} \log(2\sin\theta)\,d\theta = -\tfrac{1}{2}G, \qquad \int_0^{\pi/4} \log(2\cos\theta)\,d\theta = \tfrac{1}{2}G.$$

However, (10) will be more useful in the ensuing applications.

Alternatively, (9) and (10) can be derived from the series

$$\log(2\sin\theta) = -\sum_{n=1}^{\infty}\frac{1}{n}\cos 2n\theta, \qquad \log(2\cos\theta) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos 2n\theta$$

(for example see [7]); however, justification of termwise convergence is more delicate in this case.

The substitution $x = \tan\theta$ delivers the following reformulation of (10) in terms of the log function:

$$\int_0^1 \frac{\log(1+x^2)}{1+x^2}\,dx = \int_0^{\pi/4} \log\sec^2\theta\,d\theta = -2\int_0^{\pi/4}\log\cos\theta\,d\theta = \frac{\pi}{2}\log 2 - G.$$

One can easily check that by substituting $x = \frac{1}{y}$ and combining with (6) one obtains

$$\int_1^{\infty} \frac{\log(1+x^2)}{1+x^2}\,dx = G + \frac{\pi}{2}\log 2.$$

Now integrate by parts in (10): we find

$$\int_0^{\pi/4} \log\cos\theta\,d\theta = [\theta\log\cos\theta]_0^{\pi/4} + \int_0^{\pi/4}\theta\frac{\sin\theta}{\cos\theta}\,d\theta$$

$$= -\frac{\pi}{8}\log 2 + \int_0^{\pi/4}\theta\tan\theta\,d\theta,$$

so that

$$\int_0^{\pi/4}\theta\tan\theta\,d\theta = \frac{1}{2}G - \frac{\pi}{8}\log 2, \qquad (11)$$

an identity not given in [2] or [3]. Similarly, or by (11) combined with (3), we have $\int_0^{\pi/4}\theta\cot\theta\,d\theta = \frac{1}{2}G + \frac{\pi}{8}\log 2$.

Next, write

$$I = \int_0^{\pi/2}\log(1+\sin\theta)\,d\theta = \int_0^{\pi/2}\log(1+\cos\theta)\,d\theta.$$

By (10) and the identity $1 + \cos\theta = 2\cos^2\frac{1}{2}\theta$, we obtain

$$I = \int_0^{\pi/2}\left(\log 2 + 2\log\cos\tfrac{1}{2}\theta\right)d\theta$$

$$= \frac{\pi}{2}\log 2 + 4\int_0^{\pi/4}\log\cos\phi\,d\phi$$

$$= \frac{\pi}{2}\log 2 + 2G - \pi\log 2$$

$$= 2G - \frac{\pi}{2}\log 2, \qquad (12)$$

another integral not given in [2] or [3]. Combining (12) and (9), we have the pleasingly simple result

$$\int_0^{\pi/2} \log(1 + \operatorname{cosec}\theta)\,d\theta = \int_0^{\pi/2} \left(\log(1 + \sin\theta) - \log\sin\theta\right)d\theta = 2G, \quad (13)$$

and of course the same applies with cosec θ replaced by sec θ.

Writing $(1 + \sin\theta)(1 - \sin\theta) = \cos^2\theta$, we deduce further

$$\int_0^{\pi/2} \log(1 - \sin\theta)\,d\theta = 2\int_0^{\pi/2} \log\cos\theta\,d\theta - \int_0^{\pi/2} \log(1 + \sin\theta)\,d\theta$$

$$= -2G - \frac{\pi}{2}\log 2.$$

Integrating by parts in (12), we have

$$\int_0^{\pi/2} \log(1 + \sin\theta)\,d\theta = [\theta\log(1 + \sin\theta)]_0^{\pi/2} - J = \frac{\pi}{2}\log 2 - J,$$

where

$$J = \int_0^{\pi/2} \frac{\theta\cos\theta}{1 + \sin\theta}\,d\theta,$$

hence

$$J = \pi\log 2 - 2G.$$

Furthermore, the substitution $x = \sin\theta$ gives

$$\int_0^1 \frac{\sin^{-1}x}{1 + x}\,dx = J.$$

A further deduction from (10) is derived using the identity $\cos\theta + \sin\theta = \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$:

$$\int_0^{\pi/2} \log(\cos\theta + \sin\theta)\,d\theta = \int_0^{\pi/2} \left(\frac{1}{2}\log 2 + \log\cos\left(\theta - \frac{\pi}{4}\right)\right)d\theta$$

$$= \frac{\pi}{4}\log 2 + \int_{-\pi/4}^{\pi/4} \log\cos\phi\,d\phi$$

$$= \frac{\pi}{4}\log 2 + G - \frac{\pi}{2}\log 2$$

$$= G - \frac{\pi}{4}\log 2. \quad (14)$$

Alternatively, (14) can be deduced from (12) and the identity $(\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta$. By (14) and (9), we have

$$\int_0^{\pi/2} \log(1 + \tan\theta)\,d\theta = \int_0^{\pi/2} \left(\log(\cos\theta + \sin\theta) - \log\cos\theta\right)d\theta$$

$$= G - \frac{\pi}{4}\log 2 + \frac{\pi}{2}\log 2$$

$$= G + \frac{\pi}{4}\log 2, \quad (15)$$

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and hence also, with the substitution $x = \tan\theta$,

$$\int_0^\infty \frac{\log(1 + x)}{1 + x^2}\, dx = G + \frac{\pi}{4}\log 2,$$

which is proved by a rather longer method in [2].

Of course, $\tan\theta$ can be replaced by $\cot\theta$ in (15). With (12) and (13), this means that we have found the values of $\int_0^{\pi/2} \log\left[1 + f(\theta)\right] d\theta$, where $f(\theta)$ can be any one of the six trigonometric functions.

Yet further integrals can be derived by integrating by parts in (14) and (15) (rather better with $\cot\theta$): we leave it to the reader to explore this.

Double integrals

An obvious double-integral representation of G, not involving any trigonometric or logarithmic functions, follows at once from the geometric series. For $x, y \in [0, 1)$, we have $\frac{1}{1 + x^2y^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} y^{2n}$. Since

$$\int_0^1 \int_0^1 x^{2n} y^{2n} dx\, dy = \frac{1}{(2n + 1)^2},$$

termwise integration gives

$$\int_0^1 \int_0^1 \frac{1}{1 + x^2y^2}\, dx\, dy = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^2} = G. \tag{16}$$

Termwise integration is justified as for (1). (Alternatively, one stage of integration immediately equates the double integral to (1).)

Substituting $x = e^{-u}$ and $y = e^{-v}$, we deduce

$$G = \int_0^\infty \int_0^\infty \frac{e^{-u}e^{-v}}{1 + e^{-2u}e^{-2v}}\, du\, dv = \int_0^\infty \int_0^\infty \frac{1}{2\cosh(u + v)}\, du\, dv.$$

(This is really a pair of successive single-variable substitutions, not a full-blooded two-variable one.)

Next we establish a much less transparent double-integral representation:

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{1 + \cos\theta\cos\phi}\, d\theta\, d\phi = 2G. \tag{17}$$

This, again, is not in [2] or [3]. It was given in [4], possibly its first appearance. Here we present a proof based on (2). We actually show

$$\int_0^{\pi/4} \int_0^{\pi/4} \frac{1}{1 + \cos 2\theta\cos 2\phi}\, d\theta\, d\phi = \frac{1}{2}G, \tag{18}$$

from which (17) follows by substituting $\theta = 2\theta'$ and $\phi = 2\phi'$. Write

$$J(\theta) = \int_0^{\pi/4} \frac{1}{1 + \cos 2\theta\cos 2\phi}\, d\phi.$$

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Now

$$1 + \cos 2\theta \cos 2\phi$$
$$= (\cos^2\theta + \sin^2\theta)(\cos^2\phi + \sin^2\phi) + (\cos^2\theta - \sin^2\theta)(\cos^2\phi - \sin^2\phi)$$
$$= 2(\cos^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi). \tag{19}$$

The substitution $t = \tan\phi$ gives

$$\int_0^{\pi/4} \frac{1}{a^2 \cos^2\phi + b^2 \sin^2\phi}\, d\phi = \frac{1}{ab} \tan^{-1} \frac{b}{a}.$$

Applied with $a = \cos\theta$ and $b = \sin\theta$, this gives

$$J(\theta) = \frac{\theta}{2 \sin \cos\theta}.$$

By (2), it follows that $\int_0^{\pi/4} J(\theta)\, d\theta = \frac{1}{2}G$, which is (18).

Alternatively, substitute $x = \tan\theta$ and $y = \tan\phi$ in (16). Again, (18) is obtained after applying (19). (This is the method of [4].)

We now establish two further double integrals:

$$\int_0^1 \int_0^1 \frac{1}{2 - x^2 - y^2}\, dx\, dy = G, \tag{20}$$

$$\int_0^1 \int_0^1 \frac{1}{(x + y)(1 - x)^{1/2}(1 - y)^{1/2}}\, dx\, dy = 4G. \tag{21}$$

Result (21) is identity (44) in [2], achieved with some effort; we present a somewhat simpler proof. The substitution $x = 1 - u^2$, $y = 1 - v^2$ reduces (21) to (20). Denote the integral in (20) by I. Since the integrand satisfies $f(y, x) = f(x, y)$, the contributions with $x \leqslant y$ and $y \leqslant x$ are the same, hence

$$I = 2 \int_0^1 \int_0^x \frac{1}{2 - x^2 - y^2}\, dy\, dx.$$

Transform to polar coordinates:

$$I = \int_0^{\pi/4} \int_0^{\sec\theta} \frac{2r}{2 - r^2}\, dr\, d\theta.$$

Now

$$\int_0^{\sec\theta} \frac{2r}{2 - r^2}\, dr = \left[-\log(2 - r^2)\right]_0^{\sec\theta} = \log \frac{2}{2 - \sec^2\theta}$$

and

$$\frac{2}{2 - \sec^2\theta} = \frac{2}{1 - \tan^2\theta} = \frac{\tan 2\theta}{\tan\theta},$$

so

$$I = \int_0^{\pi/4} (\log \tan 2\theta - \log \tan\theta)\, d\theta.$$

As seen in (9), $\int_0^{\pi/4} \log \tan 2\theta = \frac{1}{2}\int_0^{\pi/2} \log \tan \phi \, d\phi = 0$. By (7), we deduce that $I = G$.

The discussion site [8] gives the following variant of (20):

$$\int_0^1 \int_0^1 \frac{1}{4 - x^2 - y^2}\, dx\, dy = \frac{1}{3} G.$$

The method is similar, but the writer resorts to *Mathematica* for the final stage, the integral $\int_0^{\pi/4} \log\left(\frac{4}{4 - \sec^2\theta}\right) d\theta$; with a bit of effort, this can be deduced from our results above using the identity $\frac{4}{4 - \sec^2\theta} = \frac{2\sin 2\theta \cos\theta}{\sin 3\theta}$.

Applications to elliptic integrals

The 'complete elliptic integral of the first kind' is defined, for $0 \leqslant t < 1$, by

$$K(t) = \int_0^{\pi/2} \frac{1}{(1 - t^2 \sin^2\theta)^{1/2}}\, d\theta.$$

The value of $K(t)$ is given explicitly by a theorem of Gauss:

$$K(t) = \frac{\pi}{2M(1, t^*)},$$

where $M(a, b)$ is the arithmetic-geometric mean of a and b, and $t^* = (1 - t^2)^{1/2}$. Two quite different proofs of this theorem can be seen in [9] and [10]. However, without any reference to Gauss's theorem, we can apply (4) to evaluate $\int_0^1 K(t)\, dt$. Indeed, reversing the implied double integral, we have $\int_0^1 K(t)\, dt = \int_0^{\pi/2} F(\theta)$, where

$$F(\theta) = \int_0^1 \frac{1}{(1 - t^2 \sin^2\theta)^{1/2}}\, dt.$$

Substituting $t \sin\theta = \sin\phi$, we have

$$F(\theta) = \int_0^\theta \frac{1}{\cos\phi} \frac{\cos\phi}{\sin\theta}\, d\phi = \frac{\theta}{\sin\theta}.$$

Hence, by (4),

$$\int_0^1 K(t)\, dt = 2G.$$

Of course, this really amounts to another double-integral representation of G.

The 'complete elliptic integral of the second kind' is

$$E(t) = \int_0^{\pi/2} (1 - t^2 \sin^2\theta)^{1/2}\, d\theta.$$

In the same way, we have $\int_0^1 E(t) = \int_0^{\pi/2} G(\theta)\,d\theta$, where

$$\begin{aligned} G(\theta) &= \int_0^1 \left(1 - t^2 \sin^2\theta\right)^{1/2} dt \\ &= \int_0^\theta \cos\phi \frac{\cos\phi}{\sin\theta}\, d\phi \\ &= \frac{1}{2\sin\theta} \int_0^\theta \left(1 + \cos 2\phi\right) d\phi \\ &= \frac{\theta}{2\sin\theta} + \frac{1}{2}\cos\theta, \end{aligned}$$

hence

$$\int_0^1 E(t)\,dt = G + \frac{1}{2}.$$

A generalisation: the function $\mathrm{Ti}_2(x)$

Going back to the original integral in (1), we can define a function of x by considering the integral on $[0, x]$. The more or less standard notation is

$$\mathrm{Ti}_2(x) = \int_0^x \frac{\tan^{-1} t}{t}\, dt.$$

Clearly, $G = \mathrm{Ti}_2(1)$. Integrating termwise as in (1), we have for $|x| \leqslant 1$,

$$\mathrm{Ti}_2(x) = x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}. \tag{22}$$

This can be compared with 'dilogarithm' function $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. We remark that G appears in the evaluation $\mathrm{Li}_2(i) = -\frac{1}{8}\zeta(2) + iG$.

We can use (22) to calculate values for $|x| \leqslant 1$, for example $\mathrm{Ti}_2\left(\frac{1}{2}\right) \approx 0.48722$.

Since $\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan\theta}$ for $0 < \theta < \frac{\pi}{2}$, we have

$$\tan^{-1} x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$$

for $x > 0$. This translates into a corresponding functional equation for $\mathrm{Ti}_2(x)$:

$$\mathrm{Ti}_2(x) - \mathrm{Ti}_2\left(\frac{1}{x}\right) = \frac{\pi}{2}\log x.$$

It is sufficient to prove this for $x \geqslant 1$. Write $\mathrm{Ti}_2(x) - \mathrm{Ti}_2\left(\frac{1}{x}\right) = F(x)$. Then

$$F(x) = \int_{1/x}^{x} \frac{\tan^{-1} t}{t}\, dt = \int_{1/x}^{x} \frac{1}{u} \tan^{-1} \frac{1}{u}\, du$$
$$= \int_{1/x}^{x} \frac{1}{u} \left(\frac{\pi}{2} - \tan^{-1} u\right) du$$
$$= \pi \log x - F(x).$$

So for $x > 1$, we have the following series expansion in powers of $\frac{1}{x}$:

$$\mathrm{Ti}_2(x) = \frac{\pi}{2} \log x + \frac{1}{x} - \frac{1}{3^2 x^3} + \frac{1}{5^2 x^5} - \dots,$$

showing very clearly the nature of $\mathrm{Ti}_2(x)$ for large x.

Of course, the procedures that gave equivalent expressions for G can be also applied to $\mathrm{Ti}_2(x)$. For example, the substitution $t = \tan\theta$ leads to

$$\mathrm{Ti}_2(x) = \frac{1}{2} \int_0^{2\tan^{-1} x} \frac{\theta}{\sin\theta}\, d\theta.$$

Integration by parts and then again the substitution $t = \tan\theta$ gives

$$\mathrm{Ti}_2(x) = \tan^{-1} x \log x - \int_0^x \frac{\log t}{1 + t^2}\, dt$$
$$= \tan^{-1} x \log x - \int_0^{\tan^{-1} x} \log \tan\theta\, d\theta.$$

As with the function $\mathrm{Li}_2(x)$, not many other explicit values of $\mathrm{Ti}_2(x)$ are known. One, proved by ingenious methods in [2] (formula (33)), is:

$$\mathrm{Ti}_2(2 - \sqrt{3}) = \frac{2}{3} G - \frac{1}{12} \pi \log(2 + \sqrt{3}).$$

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The answers to the *Nemo* page from November on measuring were:

1.	Lawrence Binyon	Song	
2.	Henry James	The Portrait of a Lady	Part II, Chapter 44
3.	Emily Dickinson	Life, CXVI	
4.	William Wordsworth	The Cave at Staffa	
5.	Oliver Goldsmith	The Good-Natured Man	Act 2
6.	WB Yeats	The Dawn	

Congratulations to Ian Anderson, Martin Lukarevski and Henry Ricardo on tracking all of these down. This month we take another look at proof. The quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 31st May 2017.

1. What is now proved was once only imagined.

2. He proves by algebra that Hamlet's grandson is Shakespeare's grandfather and that he himself is the ghost of his own father.

3. Life set me larger – problems –
Some I shall keep – to solve
Till Algebra is easier –
Or simpler proved – above –

Continued on page 68.