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COMPARE THE RATIO OF SYMMETRIC POLYNOMIALS OF ODDS TO ONE AND STOP

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Abstract

In this paper we deal with an optimal stopping problem whose objective is to maximize the probability of selecting k out of the last ℓ successes, given a sequence of independent Bernoulli trials of length N, where k and ℓ are predetermined integers satisfying $1 \le k \le \ell < N$. This problem includes some odds problems as special cases, e.g. Bruss' odds problem, Bruss and Paindaveine's problem of selecting the last ℓ successes, and Tamaki's multiplicative odds problem for stopping at any of the last m successes. We show that an optimal stopping rule is obtained by a threshold strategy. We also present the tight lower bound and an asymptotic lower bound for the probability of a win. Interestingly, our asymptotic lower bound is attained by using a variation of the well-known secretary problem, which is a special case of the odds problem. Our approach is based on the application of Newton's inequalities and optimization technique, which gives a unified view to the previous works.

Keywords: Optimal stopping; odds problem; lower bound; secretary problem; Newton's inequality

2010 Mathematics Subject Classification: Primary 60G40
Secondary 60L15

1. Introduction

In this paper we discuss a variation of the odds problem, which is an extension of Bruss' odds problem first discussed in [3]. Let $X_1, X_2, \dots, X_N$ denote a sequence of independent Bernoulli random variables. If $X_j = 1$, we say that the outcome of random variable X_j is a *success*. Otherwise ($X_j = 0$), we say that the outcome of X_j is a *failure*. These random variables can be regarded as results of an underlying discrete stochastic process. For example, we can assume that they constitute the record process. This paper deals with an optimal stopping problem of maximizing the probability of selecting k out of the last ℓ successes, where $1 \le k \le \ell < N$. More precisely, the problem may be stated as follows.

We consider a game in which a player is given the digits (realization of random variables) one by one and allowed to select the index of the variable when he/she observes a success. The number of selected indices of variables must be less than or equal to k. The player *wins* if he/she selected exactly k indices of variables contained in the set of the last ℓ successes. For example, consider the case with $N = 8$, $k = 3$, and $\ell = 4$. When $(X_1, X_2, \dots, X_8)$ has a vector of realized values $(0, 1, 1, 0, 0, 1, 1, 1)$, the player wins if he/she selected exactly three indices of

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variables in the set $\{X_3, X_6, X_7, X_8\}$. We deal with a problem of maximizing a probability of a win. It is easy to see that the player wins if and only if the first selected variable is in $\{X_3, X_6\}$ by simply enumerating following $k = 3$ successes. Under this strategy, the player wins if the set of selected indices corresponds to either $\{X_3, X_6, X_7\}$ or $\{X_6, X_7, X_8\}$. Thus, the player need only observe the sequence with an objective to correctly predict the occurrence of the mth last success satisfying $k \le m \le \ell$. From the above, the problem becomes a 'single' stopping problem of maximizing the probability of stopping on a random variable X_m satisfying $X_m = 1$ and $\ell \ge X_m + X_{m+1} + \cdots + X_N \ge k$. We present an optimal stopping rule and an asymptotic lower bound for the probability of a '*win*' (i.e. obtaining the mth last success with $k \le m \le \ell$).

When $\mathbb{P}[X_i = 1] = 1/i$, our problem becomes a variation of the secretary problem [9]. In particular, in the case that $\ell = k = 1$, the problem is equivalent to the classical secretary problem. One of the reasons why the odds problems are popular in the optimal stopping theory is that it includes the secretary problem as a special case.

Although our problem setting looks artificial, it includes some odds problems as special cases (see Table 1). When $\ell = k = 1$, the problem is equivalent to the well-known Bruss' odds problem [3], which has an elegant and simple optimal stopping rule known as the *odds theorem* or *sum-the-odds theorem*. A typical lower bound for an asymptotic optimal value (the probability of a win), when N approaches ∞, has been shown to be e^{-1} by Bruss [4], which is equal to that for the classical secretary problem. If $\ell = k \ge 1$, Bruss and Paindaveine [5] showed that an optimal stopping rule is obtained by a threshold strategy. When $\ell \ge k = 1$, Tamaki [13] demonstrated the *sum-the-multiplicative-odds theorem*, which gives an optimal stopping rule obtained using a threshold strategy. Recently, we discussed his model and gave a lower bound for the probability of a win [10]. Bruss and Paindaveine [5] and Tamaki [13] also discussed the corresponding secretary problem and derived asymptotic optimal values. The related problem of the distribution of the rank of the accepted candidate has been studied by Bartoszyński [1] and of the last record rank before the last acceptance by Bruss [2].

TABLE 1: Previous results and our results (•). (⋆) An optimal stopping rule is attained by the threshold strategy defined by the minimum index i satisfying the key inequality in the last column (see (2) for details), where $\boldsymbol{r} = (r_i, r_{i+1}, \ldots, r_N)$ and other notations are defined by (1).

Model	Condition	Lower bound	Key inequality (⋆)
[3]	$\ell = k = 1$	e^{-1} [4]	$\dfrac{r_i + r_{i+1} + \cdots + r_N}{1} = \dfrac{\boldsymbol{e}_1(\boldsymbol{r})}{\boldsymbol{e}_0(\boldsymbol{r})} < 1$ [3]
[5]	$\ell = k \ge 1$	$\dfrac{\ell^\ell}{(\ell!)\mathrm{e}^\ell}$ (•)	$\dfrac{\boldsymbol{e}_\ell(\boldsymbol{r})}{\boldsymbol{e}_{\ell-1}(\boldsymbol{r})} < 1$ [5]
[13]	$\ell \ge k = 1$	$\exp\Big(-(\ell!)^{1/\ell}\Big)\sum_{m=1}^{\ell}\dfrac{(\ell!)^{m/\ell}}{m!}$ [10]	$\dfrac{\boldsymbol{e}_\ell(\boldsymbol{r})}{\boldsymbol{e}_0(\boldsymbol{r})} < 1$ [13]
This paper	$\ell \ge k \ge 1$	(‡) See below (•)	$\dfrac{\boldsymbol{e}_\ell(\boldsymbol{r})}{\boldsymbol{e}_{k-1}(\boldsymbol{r})} < 1$ (•)

‡$\exp\bigg(-\bigg(\dfrac{\ell!}{(k-1)!}\bigg)^{1/(\ell-k+1)}\bigg)\displaystyle\sum_{m=k}^{\ell}\bigg(\dfrac{1}{m!}\bigg(\dfrac{\ell!}{(k-1)!}\bigg)^{m/(\ell-k+1)}\bigg).$

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In this paper we describe an optimal stopping rule and derive the greatest lower bound for the probability of a win for the problem of selecting k out of the last ℓ successes. The asymptotic value of our lower bound is equivalent to the asymptotic optimal value for the corresponding secretary problem appearing in [4], [5], and [13]. A special feature of our proof is the application of Newton's inequalities [11] and optimization technique to obtain our bound.

2. Elementary symmetric polynomials

For any pair of positive integers m and N satisfying $1 \le m \le N$ and a vector $\boldsymbol{r} \in \mathbb{R}^N$, $\boldsymbol{e}_m(\boldsymbol{r})$ denotes the mth *elementary symmetric polynomial* (function) of $\boldsymbol{r} = (r_1, r_2, \dots, r_N)$ defined by

$$\boldsymbol{e}_m(\boldsymbol{r}) = \sum_{1 \le i_1 < i_2 < \cdots < i_m \le N} r_{i_1} r_{i_2} \cdots r_{i_m} = \sum_{B \subseteq \{1,2,\dots,N\} \text{ and } |B|=m} \prod_{i \in B} r_i, \tag{1}$$

which is the sum of the $\binom{N}{m}$ terms. We also define $\boldsymbol{e}_0(\boldsymbol{r}) = 1$. The *mth elementary symmetric mean* of $\boldsymbol{r}$ is defined by

$$S_m(\boldsymbol{r}) = \boldsymbol{e}_m(\boldsymbol{r}) \Big/ \binom{N}{m} \quad \text{for all } m \in \{1, 2, \dots, N\} \quad \text{and} \quad S_0(\boldsymbol{r}) = 1.$$

We abbreviate $S_m(\boldsymbol{r})$ to S_m when there is no ambiguity. The elementary symmetric polynomials satisfy the following inequalities shown by Newton.

Theorem 1. (Newton's inequalities [11].) *For every nonnegative vector $\boldsymbol{r} \in \mathbb{R}_+^N$ and a positive integer $1 \le m < N$,*

$$S_m(\boldsymbol{r})^2 \ge S_{m-1}(\boldsymbol{r}) S_{m+1}(\boldsymbol{r}),$$

with equality exactly when all the r_i are equal.

Newton's inequalities directly imply the following.

Lemma 1. *For any positive vector $\widetilde{\boldsymbol{r}} = (\widetilde{r}_1, \widetilde{r}_2, \dots, \widetilde{r}_N) > \boldsymbol{0}$ and integers (m, ℓ) satisfying $1 \le m \le \ell \le N$, the inequality $\boldsymbol{e}_{\ell-1}(\widetilde{\boldsymbol{r}}) / \boldsymbol{e}_{m-1}(\widetilde{\boldsymbol{r}}) \ge \boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}}) / \boldsymbol{e}_m(\widetilde{\boldsymbol{r}})$ holds.*

Proof. The positivity of $\widetilde{\boldsymbol{r}}$ implies that $S_{m'}(\widetilde{\boldsymbol{r}}) > 0$ (for all m', $0 \le m' \le N$). Newton's inequalities are equivalent to the midpoint log-concavity $\log(S_{m'}) \ge \frac{1}{2}(\log(S_{m'-1}) + \log(S_{m'+1}))$, which directly yields the concavity of the sequence $(\log(S_0), \log(S_1), \log(S_2), \dots, \log(S_N))$ and the following inequalities:

$$\begin{gathered}
\tfrac{1}{2}(\log(S_m) + \log(S_{\ell-1})) \ge \tfrac{1}{2}(\log(S_{m-1}) + \log(S_\ell)), \\
S_m S_{\ell-1} \ge S_{m-1} S_\ell, \\
\binom{N}{m-1} \boldsymbol{e}_{\ell-1}(\widetilde{\boldsymbol{r}}) \Big/ \binom{N}{\ell-1} \boldsymbol{e}_{m-1}(\widetilde{\boldsymbol{r}}) = \frac{S_{\ell-1}}{S_{m-1}} \ge \frac{S_\ell}{S_m} = \binom{N}{m} \boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}}) \Big/ \binom{N}{\ell} \boldsymbol{e}_m(\widetilde{\boldsymbol{r}}), \\
\frac{\boldsymbol{e}_{\ell-1}(\widetilde{\boldsymbol{r}})}{\boldsymbol{e}_{m-1}(\widetilde{\boldsymbol{r}})} \ge \left(\frac{N-m+1}{N-\ell+1}\right)\left(\frac{\ell}{m}\right) \frac{\boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}})}{\boldsymbol{e}_m(\widetilde{\boldsymbol{r}})} \ge \frac{\boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}})}{\boldsymbol{e}_m(\widetilde{\boldsymbol{r}})}.
\end{gathered}$$

□

Lemma 2. *For any positive vector $\widetilde{\boldsymbol{r}} = (\widetilde{r}_1, \widetilde{r}_2, \dots, \widetilde{r}_N) > \boldsymbol{0}$ and integers (m, ℓ, N) satisfying $0 \le m \le \ell < N$ and $N \ge 2$, the inequality $\boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}}) / \boldsymbol{e}_m(\widetilde{\boldsymbol{r}}) \ge \boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}}_{-1}) / \boldsymbol{e}_m(\widetilde{\boldsymbol{r}}_{-1})$ holds, where $\widetilde{\boldsymbol{r}}_{-1} = (\widetilde{r}_2, \dots, \widetilde{r}_N)$.*

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Proof. When $m = 0$, the result is obvious from the positivity of $\widetilde{\boldsymbol{r}}$. Let us consider the cases when $m \geq 1$. If we apply Lemma 1 to the positive vector $\widetilde{\boldsymbol{r}}_{-1}$ then we obtain the inequality $\boldsymbol{e}_{\ell-1}(\widetilde{\boldsymbol{r}}_{-1})/\boldsymbol{e}_{m-1}(\widetilde{\boldsymbol{r}}_{-1}) \geq \boldsymbol{e}_{\ell}(\widetilde{\boldsymbol{r}}_{-1})/\boldsymbol{e}_{m}(\widetilde{\boldsymbol{r}}_{-1})$, which directly implies that

$$\frac{\boldsymbol{e}_{\ell}(\widetilde{\boldsymbol{r}})}{\boldsymbol{e}_{m}(\widetilde{\boldsymbol{r}})} = \frac{\widetilde{r}_1 \boldsymbol{e}_{\ell-1}(\widetilde{\boldsymbol{r}}_{-1}) + \boldsymbol{e}_{\ell}(\widetilde{\boldsymbol{r}}_{-1})}{\widetilde{r}_1 \boldsymbol{e}_{m-1}(\widetilde{\boldsymbol{r}}_{-1}) + \boldsymbol{e}_{m}(\widetilde{\boldsymbol{r}}_{-1})} \geq \frac{\boldsymbol{e}_{\ell}(\widetilde{\boldsymbol{r}}_{-1})}{\boldsymbol{e}_{m}(\widetilde{\boldsymbol{r}}_{-1})}. \qquad \square$$

3. Threshold strategy

We deal with a sequence of independent $0/1$ random variables $X_1, X_2, \dots, X_N$, where N is a given positive integer and the distribution is $\mathbb{P}[X_i = 1] = p_i, \mathbb{P}[X_i = 0] = 1 - p_i = q_i, 0 < p_i < 1$ for each i. We define $r_i = p_i/q_i$ for each i. The r_i are called *odds*. Given a pair of integers (k, ℓ) satisfying $1 \leq k \leq \ell < N$, we discuss a problem to predict the mth last success satisfying $k \leq m \leq \ell$, if any, with maximum probability at the time of its occurrence.

In the rest of this section we denote the subvector $(r_i, r_{i+1}, \dots, r_N)$ by $\boldsymbol{r}^{[i]}$ and introduce the notation

$$W_i := \mathbb{P}[k \leq X_{i+1} + \cdots + X_N \leq \ell] = \frac{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r}^{[i+1]})}{\prod_{j=i+1}^{N}(1 + r_j)},$$

$$V_i := \mathbb{P}[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] = \frac{\sum_{m=k-1}^{\ell-1} \boldsymbol{e}_m(\boldsymbol{r}^{[i+1]})}{\prod_{j=i+1}^{N}(1 + r_j)}.$$

We define an index i_* by

$$i_* := \min\left\{ i \;\middle|\; 1 \leq i \leq N - \ell \text{ and } \frac{\boldsymbol{e}_{\ell}(\boldsymbol{r}^{[i+1]})}{\boldsymbol{e}_{k-1}(\boldsymbol{r}^{[i+1]})} < 1 \right\}. \tag{2}$$

When the minimum in (2) is taken on the empty set, we set $i_* := N - \ell + 1$.

Now we give an optimal rule.

Theorem 2. *Let us consider the problem of stopping at the mth last success with $k \leq m \leq \ell$ defined on $X_1, X_2, \dots, X_N$ satisfying $r_i > 0$ for all i and $1 \leq k \leq \ell < N$. An optimal stopping rule is obtained by stopping at the first success $X_i = 1$ with $i \geq i_*$, and the corresponding probability of a win is equal to W_{i_*-1}.*

Proof. First, we show a property of the ratio W_i/V_i. The definition of i_* and Lemma 2 directly induce the following:

$$\frac{\boldsymbol{e}_{\ell}(\boldsymbol{r}^{[i+1]})}{\boldsymbol{e}_{k-1}(\boldsymbol{r}^{[i+1]})} \begin{cases} \geq 1 & (\text{for all } i, 0 \leq i \leq i_* - 1), \\ < 1 & (\text{for all } i, i_* \leq i \leq N - \ell). \end{cases} \tag{3}$$

The definitions of W_i and V_i imply that

$$\frac{W_i}{V_i} = \frac{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r}^{[i+1]})}{\prod_{j=i+1}^{N}(1 + r_j)} \frac{\prod_{j=i+1}^{N}(1 + r_j)}{\sum_{m=k-1}^{\ell-1} \boldsymbol{e}_m(\boldsymbol{r}^{[i+1]})} = \frac{\sum_{m=k}^{\ell-1} \boldsymbol{e}_m(\boldsymbol{r}^{[i+1]}) + \boldsymbol{e}_{\ell}(\boldsymbol{r}^{[i+1]})}{\sum_{m=k}^{\ell-1} \boldsymbol{e}_m(\boldsymbol{r}^{[i+1]}) + \boldsymbol{e}_{k-1}(\boldsymbol{r}^{[i+1]})} \tag{4}$$

and, thus, we have

$$\frac{W_i}{V_i} \begin{cases} \geq 1 & (\text{for all } i,\ 0 \leq i \leq i_* - 1), \\ < 1 & (\text{for all } i,\ i_* \leq i \leq N - \ell). \end{cases} \tag{5}$$

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From property (5), our problem becomes a monotone stopping problem and the one-stage look-ahead strategy gives an optimal stopping rule (see, for example, [6]–[8], [12]). Thus, an optimal stopping rule is attained by the threshold strategy with the threshold value

$$\tau := \min\left\{ i \;\middle|\; 1 \le i \le N - \ell \text{ and } \frac{W_i}{V_i} < 1 \right\}.$$

When the minimum in the above definition is taken on the empty set, we set $\tau := N - \ell + 1$. If we employ the optimal threshold strategy defined by τ, it is also known that the corresponding probability of a win is equal to $W_{\tau-1}$.

From (4), we have

$$\frac{W_i}{V_i} < 1 \quad \text{if and only if} \quad \frac{\boldsymbol{e}_\ell(\boldsymbol{r}^{[i+1]})}{\boldsymbol{e}_{k-1}(\boldsymbol{r}^{[i+1]})} < 1$$

and, thus, $i_* = \tau$. As a result, an optimal stopping rule is obtained by the threshold strategy with the threshold value i_*, which does not select any index less than i_* and selects the first variable $X_i = 1$ satisfying $i_* \le i$. The corresponding probability of a win is equal to $W_{\tau-1} = W_{i_*-1}$. This completes the proof. □

4. Lower bound

In this section we discuss a lower bound for the probability of a win under the optimal stopping rule. First, we discuss a lemma which plays an important role in this section.

Lemma 3. *Every positive vector* $\widetilde{\boldsymbol{r}} = (\widetilde{r}_1, \widetilde{r}_2, \dots, \widetilde{r}_N) > 0$ *satisfies*

$$(S_{k-1}(\widetilde{\boldsymbol{r}}))^{\ell-m} \ge (S_m(\widetilde{\boldsymbol{r}}))^{\ell-k+1}(S_\ell(\widetilde{\boldsymbol{r}}))^{k-1-m} \quad \textit{for all } m \in \{0, 1, 2, \dots, k-1\}, \tag{6}$$

$$(S_m(\widetilde{\boldsymbol{r}}))^{\ell-k+1} \ge (S_{k-1}(\widetilde{\boldsymbol{r}}))^{\ell-m}(S_\ell(\widetilde{\boldsymbol{r}}))^{m-k+1} \quad \textit{for all } m \in \{k, k+1, \dots, \ell\}, \tag{7}$$

$$(S_\ell(\widetilde{\boldsymbol{r}}))^{m-k+1} \ge (S_{k-1}(\widetilde{\boldsymbol{r}}))^{m-\ell}(S_m(\widetilde{\boldsymbol{r}}))^{\ell-k+1} \quad \textit{for all } m \in \{\ell+1, \dots, N\}. \tag{8}$$

Proof. In the following we abbreviate $S_m(\widetilde{\boldsymbol{r}})$ to S_m for simplicity. Newton's inequalities directly imply the concavity of the sequence $(\log(S_0), \dots, \log(S_N))$ and, thus, we have the following inequalities:

$$\log(S_{k-1}) \ge \frac{(\ell-k+1)\log(S_m) + (k-1-m)\log(S_\ell)}{\ell-m} \quad \text{for all } m \in \{0, 1, \dots, k-1\},$$

$$\log(S_m) \ge \frac{(\ell-m)\log(S_{k-1}) + (m-k+1)\log(S_\ell)}{\ell-k+1} \quad \text{for all } m \in \{k, k+1, \dots, \ell\},$$

$$\log(S_\ell) \ge \frac{(m-\ell)\log(S_{k-1}) + (\ell-k+1)\log(S_m)}{m-k+1} \quad \text{for all } m \in \{\ell+1, \ell+2, \dots, N\}.$$

Consequently, (6)–(8) follow, completing the proof. □

Theorem 3. *Let us consider the problem of stopping at the mth last success with* $k \le m \le \ell$ *defined on* $X_1, X_2, \dots, X_N$ *satisfying*

(i) $r_i > 0$ *for all* i,

(ii) $1 \le k \le \ell < N$,

(iii) $1 > \boldsymbol{e}_\ell(\bar{\boldsymbol{r}})/\boldsymbol{e}_{k-1}(\bar{\boldsymbol{r}})$, *where* $\bar{\boldsymbol{r}} = (r_{N-\ell+1}, r_{N-\ell+2}, \dots, r_N) \in \mathbb{R}^\ell$, *and*

(iv) $\boldsymbol{e}_\ell(\boldsymbol{r})/\boldsymbol{e}_{k-1}(\boldsymbol{r}) \ge 1$.

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Under the optimal stopping rule, the greatest lower bound for the probability of a win is equal to

$$\frac{\sum_{m=k}^{\ell} \binom{N}{m} \theta^m}{(1+\theta)^N}, \quad \textit{where } \theta = \left(\binom{N}{k-1} \Big/ \binom{N}{\ell} \right)^{1/(\ell-k+1)}.$$

Proof. Since the optimal stopping rule defined by (2) is a threshold strategy, the truncation of the subsequence $X_1, X_2, \dots, X_{i_*-1}$ does not affect the probability of a win. Thus, we need only consider a case where

$$\boldsymbol{e}_{k-1}(r_2, r_3, \dots, r_N) - \boldsymbol{e}_\ell(r_2, r_3, \dots, r_N) > 0, \tag{9}$$

$$\boldsymbol{e}_{k-1}(r_1, r_2, r_3 \dots, r_N) - \boldsymbol{e}_\ell(r_1, r_2, r_3 \dots, r_N) \le 0. \tag{10}$$

Under assumptions (9) and (10), the optimal stopping rule is obtained by setting $i_* = 1$, and the probability of a win is equal to

$$W_0 = \frac{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r})}{(1+r_1)(1+r_2)\cdots(1+r_N)}.$$

Thus, the greatest lower bound for the probability of a win under the optimal stopping rule is equal to the optimal value of an optimization problem

P1: minimize

$$\mathbb{P}_{\text{win}}(N) := \frac{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r})}{(1+r_1)(1+r_2)\cdots(1+r_N)}$$

subject to $0 < r_i$ for all $i \in \{1, 2, \dots, N\}$,

$$\boldsymbol{e}_{k-1}(\boldsymbol{r}_{-1}) - \boldsymbol{e}_\ell(\boldsymbol{r}_{-1}) > 0,$$

$$\boldsymbol{e}_{k-1}(\boldsymbol{r}) - \boldsymbol{e}_\ell(\boldsymbol{r}) \le 0, \tag{11}$$

where $\boldsymbol{r}_{-1} = (r_2, r_3, \dots, r_N)$.

We show that we need only consider feasible solutions satisfying constraint (11) by equality. Let $\boldsymbol{r}'$ be a feasible solution of (P1) satisfying $\boldsymbol{e}_{k-1}(\boldsymbol{r}') - \boldsymbol{e}_\ell(\boldsymbol{r}') < 0$. We introduce a function $f(r)\colon [0, r'_1] \to \mathbb{R}$ defined by

$$f(r) = \boldsymbol{e}_{k-1}(r, r'_2, r'_3, \dots, r'_N) - \boldsymbol{e}_\ell(r, r'_2, r'_3, \dots, r'_N),$$

which is obtained by fixing $N-1$ variables $\{r'_2, r'_3, \dots, r'_N\}$. The assumption on $\boldsymbol{r}'$ directly implies that

$$f(r'_1) = \boldsymbol{e}_{k-1}(\boldsymbol{r}') - \boldsymbol{e}_\ell(\boldsymbol{r}') < 0 < \boldsymbol{e}_{k-1}(\boldsymbol{r}'_{-1}) - \boldsymbol{e}_\ell(\boldsymbol{r}'_{-1}) = f(0).$$

From the continuity of $f(r)$, the mean-value theorem implies the existence of a value $r'' \in (0, r'_1)$ satisfying $f(r'') = 0$. Obviously, $(r'', r'_2, r'_3, \dots, r'_N)$ is feasible to (P1). The objective function value $\mathbb{P}'_{\text{win}}(N)$ corresponding to $\boldsymbol{r}'$ becomes

$$\begin{aligned}\mathbb{P}'_{\text{win}}(N) &= \frac{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r}')}{(1+r'_1)(1+r'_2)\cdots(1+r'_N)} \\ &= \frac{\sum_{m=k}^{\ell} (\boldsymbol{e}_m(\boldsymbol{r}'_{-1}) + r'_1 \boldsymbol{e}_{m-1}(\boldsymbol{r}'_{-1}))}{(1+r'_1)(1+r'_2)\cdots(1+r'_N)}\end{aligned}$$

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$$= \frac{\sum_{m=k}^{\ell}(\boldsymbol{e}_m(\boldsymbol{r}'_{-1}) - \boldsymbol{e}_{m-1}(\boldsymbol{r}'_{-1}) + (1+r'_1)\boldsymbol{e}_{m-1}(\boldsymbol{r}'_{-1}))}{(1+r'_1)(1+r'_2)\cdots(1+r'_N)}$$
$$= \left[(-1)\frac{\boldsymbol{e}_{k-1}(\boldsymbol{r}'_{-1}) - \boldsymbol{e}_{\ell}(\boldsymbol{r}'_{-1})}{1+r'_1} + \sum_{m=k}^{\ell}\boldsymbol{e}_{m-1}(\boldsymbol{r}'_{-1})\right][(1+r'_2)\cdots(1+r'_N)]^{-1}.$$

Since $\boldsymbol{e}_{k-1}(\boldsymbol{r}'_{-1}) - \boldsymbol{e}_{\ell}(\boldsymbol{r}'_{-1}) > 0$ and $r'' \in (0, r'_1)$, the objective function value of $(r'', r'_2, r'_3, \dots, r'_N)$ is strictly less than that of $\boldsymbol{r}'$. As a result, we have shown that if a solution $\boldsymbol{r}'$ feasible to (P1) satisfies $\boldsymbol{e}_{k-1}(\boldsymbol{r}') - \boldsymbol{e}_{\ell}(\boldsymbol{r}') < 0$, then there exists a feasible solution $\boldsymbol{r}''$ satisfying $\boldsymbol{e}_{k-1}(\boldsymbol{r}'') - \boldsymbol{e}_{\ell}(\boldsymbol{r}'') = 0$ with a strictly smaller objective value. Thus, we need only consider a set of feasible solutions of (P1) satisfying $\boldsymbol{e}_{k-1}(\boldsymbol{r}) - \boldsymbol{e}_{\ell}(\boldsymbol{r}) = 0$.

Let $\boldsymbol{r}^*$ be a feasible solution of (P1) satisfying $\boldsymbol{e}_{k-1}(\boldsymbol{r}^*) - \boldsymbol{e}_{\ell}(\boldsymbol{r}^*) = 0$. Next, we derive an upper bound and/or a lower bound for $\boldsymbol{e}_m(\boldsymbol{r}^*)$. For simplicity, we introduce the notation

$$\alpha := \left(\frac{(S_{k-1})^{\ell}}{(S_{\ell})^{k-1}}\right)^{1/(\ell-k+1)} \quad \text{and} \quad \theta := \left(\frac{S_{\ell}}{S_{k-1}}\right)^{1/(\ell-k+1)}.$$

The equality $\boldsymbol{e}_{k-1}(\boldsymbol{r}^*) - \boldsymbol{e}_{\ell}(\boldsymbol{r}^*) = 0$ directly implies that

$$\theta = \left(\frac{S_{\ell}}{S_{k-1}}\right)^{1/(\ell-k+1)}$$
$$= \left(\binom{N}{k-1}\boldsymbol{e}_{\ell}(\boldsymbol{r}^*) \Big/ \binom{N}{\ell}\boldsymbol{e}_{k-1}(\boldsymbol{r}^*)\right)^{1/(\ell-k+1)}$$
$$= \left(\binom{N}{k-1} \Big/ \binom{N}{\ell}\right)^{1/(\ell-k+1)}.$$

(i) Inequality (6) implies that, for any $m \in \{0, 1, 2, \dots, k-1\}$,

$$\boldsymbol{e}_m(\boldsymbol{r}^*) = \binom{N}{m}S_m$$
$$\le \binom{N}{m}\left(\frac{(S_{k-1})^{\ell-m}}{(S_{\ell})^{k-1-m}}\right)^{1/(\ell-k+1)}$$
$$= \binom{N}{m}\left(\frac{(S_{k-1})^{\ell}}{(S_{\ell})^{k-1}}\frac{(S_{\ell})^m}{(S_{k-1})^m}\right)^{1/(\ell-k+1)}$$
$$= \binom{N}{m}\alpha\theta^m.$$

(ii) For each $m \in \{k, k+1, \dots, \ell\}$, inequality (7) gives a lower bound (not upper bound)

$$\boldsymbol{e}_m(\boldsymbol{r}^*) = \binom{N}{m}S_m$$
$$\ge \binom{N}{m}((S_{k-1})^{\ell-m}(S_{\ell})^{m-k+1})^{1/(\ell-k+1)}$$
$$= \binom{N}{m}\left(\frac{(S_{k-1})^{\ell}}{(S_{\ell})^{k-1}}\frac{(S_{\ell})^m}{(S_{k-1})^m}\right)^{1/(\ell-k+1)}$$
$$= \binom{N}{m}\alpha\theta^m.$$

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(iii) Inequality (8) implies that, for any $m \in \{\ell+1, \ell+2, \dots, N\}$,

$$\begin{aligned}
\boldsymbol{e}_m(\boldsymbol{r}^*) &= \binom{N}{m} S_m \\
&\le \binom{N}{m} \bigg(\frac{(S_\ell)^{m-k+1}}{(S_{k-1})^{m-\ell}} \bigg)^{1/(\ell-k+1)} \\
&= \binom{N}{m} \bigg(\frac{(S_{k-1})^{\ell}}{(S_\ell)^{k-1}} \frac{(S_\ell)^{m}}{(S_{k-1})^{m}} \bigg)^{1/(\ell-k+1)} \\
&= \binom{N}{m} \alpha\theta^m.
\end{aligned}$$

Then the objective function value $\mathbb{P}^*_{\text{win}}(N)$ corresponding to $\boldsymbol{r}^*$ satisfies

$$\begin{aligned}
\frac{1}{\mathbb{P}^*_{\text{win}}(N)} &= \frac{(1+r_1^*)(1+r_2^*)\cdots(1+r_N^*)}{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r}^*)} \\
&= \frac{\sum_{m=0}^{N} \boldsymbol{e}_m(\boldsymbol{r}^*)}{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r}^*)} \\
&= \frac{\sum_{m=0}^{k-1} \boldsymbol{e}_m(\boldsymbol{r}^*)}{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r}^*)} + \frac{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r}^*)}{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r}^*)} + \frac{\sum_{m=\ell+1}^{N} \boldsymbol{e}_m(\boldsymbol{r}^*)}{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r}^*)} \\
&\le \frac{\sum_{m=0}^{k-1} \binom{N}{m}\alpha\theta^m}{\sum_{m=k}^{\ell} \binom{N}{m}\alpha\theta^m} + 1 + \frac{\sum_{m=\ell+1}^{N} \binom{N}{m}\alpha\theta^m}{\sum_{m=k}^{\ell} \binom{N}{m}\alpha\theta^m} \\
&= \frac{\alpha \sum_{m=0}^{N} \binom{N}{m}\theta^m}{\alpha \sum_{m=k}^{\ell} \binom{N}{m}\theta^m} \\
&= \frac{(1+\theta)^N}{\sum_{m=k}^{\ell} \binom{N}{m}\theta^m}
\end{aligned}$$

and, thus,

$$\mathbb{P}^*_{\text{win}}(N) \ge \frac{\sum_{m=k}^{\ell} \binom{N}{m}\theta^m}{(1+\theta)^N}. \tag{12}$$

Now we discuss the tightness of the above lower bound. If we consider the case where $\widehat{r}_1 = \widehat{r}_2 = \cdots = \widehat{r}_N = \theta$, then we have

$$\begin{aligned}
&\boldsymbol{e}_{k-1}(\widehat{\boldsymbol{r}}) - \boldsymbol{e}_\ell(\widehat{\boldsymbol{r}}) \\
&\quad = \binom{N}{k-1}\theta^{k-1} - \binom{N}{\ell}\theta^\ell \\
&\quad = \binom{N}{k-1}\bigg(\binom{N}{k-1}\bigg/\binom{N}{\ell}\bigg)^{(k-1)/(\ell-k+1)} - \binom{N}{\ell}\bigg(\binom{N}{k-1}\bigg/\binom{N}{\ell}\bigg)^{\ell/(\ell-k+1)} \\
&\quad = \binom{N}{k-1}^{1+(k-1)/(\ell-k+1)} \bigg/ \binom{N}{\ell}^{(k-1)/(\ell-k+1)} \\
&\qquad - \binom{N}{k-1}^{\ell/(\ell-k+1)} \bigg/ \binom{N}{\ell}^{-1+\ell/(\ell-k+1)} \\
&\quad = 0
\end{aligned}$$

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and

$$\begin{aligned}
\boldsymbol{e}_{k-1}(\widehat{\boldsymbol{r}}_{-1}) - \boldsymbol{e}_{\ell}(\widehat{\boldsymbol{r}}_{-1}) &= \binom{N-1}{k-1}\theta^{k-1} - \binom{N-1}{\ell}\theta^{\ell} \\
&= \left(\frac{N-k+1}{N}\right)\binom{N}{k-1}\theta^{k-1} - \left(\frac{N-\ell}{N}\right)\binom{N}{\ell}\theta^{\ell} \\
&= \left(\frac{N-k+1}{N}\right)\boldsymbol{e}_{k-1}(\widehat{\boldsymbol{r}}) - \left(\frac{N-\ell}{N}\right)\boldsymbol{e}_{\ell}(\widehat{\boldsymbol{r}}) \\
&= \left(\frac{N-k+1}{N}\right)\boldsymbol{e}_{k-1}(\widehat{\boldsymbol{r}}) - \left(\frac{N-\ell}{N}\right)\boldsymbol{e}_{k-1}(\widehat{\boldsymbol{r}}) \\
&= \frac{\ell-k+1}{N}\boldsymbol{e}_{k-1}(\widehat{\boldsymbol{r}}) \\
&> 0.
\end{aligned}$$

Thus, $\widehat{\boldsymbol{r}}$ is feasible for (P1) and the corresponding probability of a win (under the optimal stopping rule) attains the lower bound appearing in the right-hand side of (12). From the above, $\widehat{\boldsymbol{r}}$ is optimal for (P1), which induces the tightness of our lower bound. □

Finally, we consider an asymptotic lower bound that is independent of N. The greatest lower bound for the probability of a win (under the optimal stopping rule) is nonincreasing with respect to N. Thus, we discuss the case that $N \to \infty$ and present a general lower bound.

Corollary 1. *Under the assumptions in Theorem 3, the probability of a win is greater than*

$$\exp\biggl(-\left(\frac{\ell!}{(k-1)!}\right)^{1/(\ell-k+1)}\biggr)\sum_{m=k}^{\ell}\biggl(\frac{1}{m!}\left(\frac{\ell!}{(k-1)!}\right)^{m/(\ell-k+1)}\biggr).$$

Proof. It is easy to see that

$$\frac{\sum_{m=k}^{\ell}\binom{N}{m}\theta^m}{(1+\theta)^N} \geq \exp(-N\theta)\sum_{m=k}^{\ell}\binom{N}{m}\theta^m = \exp\left(-\binom{N}{1}\theta\right)\sum_{m=k}^{\ell}\binom{N}{m}\theta^m.$$

For each $m \in \{0, 1, \dots, N\}$, we can find an asymptotic value for $\binom{N}{m}\theta^m$, i.e.

$$\begin{aligned}
\binom{N}{m}\theta^m &= \binom{N}{m}\left(\binom{N}{k-1}\bigg/\binom{N}{\ell}\right)^{m/(\ell-k+1)} \\
&= \frac{N!}{(N-m)!\,m!}\left(\frac{\ell!\,(N-\ell)!}{(k-1)!\,(N-k+1)!}\right)^{m/(\ell-k+1)} \\
&= \frac{1}{m!}\left(\frac{\ell!}{(k-1)!}\right)^{m/(\ell-k+1)}\frac{N!}{(N-m)!\,N^m}\left(\frac{(N-\ell)!\,N^{\ell-k+1}}{(N-k+1)!}\right)^{m/(\ell-k+1)} \\
&= \frac{1}{m!}\left(\frac{\ell!}{(k-1)!}\right)^{m/(\ell-k+1)} \\
&\quad\times\frac{(1-0/N)(1-1/N)\cdots(1-(m-1)/N)}{((1-(k-1)/N)(1-k/N)\cdots(1-(\ell-1)/N))^{m/(\ell-k+1)}} \\
&\to \frac{1}{m!}\left(\frac{\ell!}{(k-1)!}\right)^{m/(\ell-k+1)} \qquad \text{as } N \to \infty.
\end{aligned}$$

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From the above discussion, we obtain the asymptotic lower bound

$$\lim_{N\to\infty} \frac{\sum_{m=k}^{\ell} \binom{N}{m}\theta^m}{(1+\theta)^N} \geq \lim_{N\to\infty} \exp\left(-\binom{N}{1}\theta\right) \sum_{m=k}^{\ell} \binom{N}{m}\theta^m$$
$$= \exp\left(-\left(\frac{\ell!}{(k-1)!}\right)^{1/(\ell-k+1)}\right) \sum_{m=k}^{\ell} \left(\frac{1}{m!}\left(\frac{\ell!}{(k-1)!}\right)^{m/(\ell-k+1)}\right),$$

completing the proof. □

5. Conclusion

In this paper we considered an optimal stopping problem of maximizing the probability of selecting k out of the last ℓ successes, where $1 \leq k \leq \ell < N$. Our results thus cover quite a general class of odds problems which include the original Bruss' odds problem [3], as well as the results of Bruss and Paindaveine [5] and Tamaki [13]. We showed that an optimal stopping rule is given by a threshold strategy. We also gave a lower bound for the probability of a win. Our proofs are based on Newton's inequalities and optimization technique.

Our general lower bound for the probability of a win is attained by corresponding odds problems and/or secretary problems:

- e^{-1} (if $\ell = k = 1$), which is a well-known bound for the classical secretary problem and a lower bound for Bruss' odds problem shown by Bruss [4];
- $\ell^\ell/(\ell!)e^\ell$ (if $\ell = k \geq 1$) shown by Bruss and Paindaveine [5] for the secretary problem;
- $\exp(-(\ell!)^{1/\ell})\sum_{m=1}^{\ell}(\ell!)^{m/\ell}/m!$ (if $\ell \geq k = 1$) shown by Tamaki [13] for the secretary problem, and by Matsui and Ano [10] for a variation of the odds problem proposed by Tamaki.

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