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Rigidity dimension of algebras

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Abstract

A new homological dimension, called rigidity dimension, is introduced to measure the quality of resolutions of finite dimensional algebras (especially of infinite global dimension) by algebras of finite global dimension and big dominant dimension. Upper bounds of the dimension are established in terms of extensions and of Hochschild cohomology, and finiteness in general is derived from homological conjectures. In particular, the rigidity dimension of a non-semisimple group algebra is finite and bounded by the order of the group. Then invariance under stable equivalences is shown to hold, with some exceptions when there are nodes in case of additive equivalences, and without exceptions in case of triangulated equivalences. Stable equivalences of Morita type and derived equivalences, both between self-injective algebras, are shown to preserve rigidity dimension as well.

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1. *Introduction*

A resolution of a mathematical object relates this object as closely as possible with another object that has better properties. In this sense, an algebra E of finite global (cohomological) dimension may be a resolution of an arbitrary algebra A. The *rigidity dimension* of A, to be introduced in this article, is intended to measure in homological terms the quality of a best possible resolution of A.

A classical correspondence, often named after Morita and Tachikawa and independently proved by Müller (Section 2·2), suggests to proceed in resolving a given finite dimensional algebra A by first choosing an A-module M that is a finitely generated generator-cogenerator, and then choose for E the endomorphism ring $\mathrm{End}_A(M)$. When E has finite global dimension, we call E a *resolution algebra* of A, and M a *resolving module*. Then A and E are closely related by a double centraliser property on the balanced A-E-bimodule M: there are equalities $E = \mathrm{End}_A(M)$ and $A = \mathrm{End}_E(M)$. A construction due to Iyama ensures the existence of at least one resolution algebra of A.

How to measure the quality of a resolution and what means 'best possible'? Increasing multiplicities of summands of M and thus replacing E by a Morita equivalent algebra if necessary, we may assume that there is an idempotent $e = e^2 \in E$ such that $A = eEe$. Multiplication $e \cdot -$ by the idempotent e defines an exact functor $e \cdot - : E\text{-mod} \to A\text{-mod}$. The right E-module M is faithful and projective-injective and as E-module determined by these properties, up to multiplicities of direct summands. Morita-Tachikawa correspondence implies that E has dominant dimension at least two. Informally, the dominant dimension of E controls how strongly the exact functor $e \cdot -$ relates the module categories and in particular the cohomology over E and over A. Larger dominant dimension of E signifies a better resolution.

The importance of dominant dimension in this context is suggested not only by Morita–Tachikawa correspondence, but also by the role of dominant dimension in Auslander's representation dimension and Iyama's higher representation dimension [24], by Rouquier's theory of quasi-hereditary covers [33] and, closest to the current context, by results [12, 13, 14, 16] on Schur algebras and more generally on endomorphism algebras of generators over symmetric algebras, making precise how the dominant dimension of E controls which cohomology groups over E and over A are identified by the functor $e \cdot - : E\text{-mod} \to A\text{-mod}$.

Equivalently, the dominant dimension of E measures the vanishing of self-extensions of the resolving A-module M. This explains the term rigidity in the main definition of this article and provides another reason why large dominant dimension of E indicates that the resolving module M over A has been chosen particularly well. This moreover relates M to Iyama's cluster tilting modules [25], which are also known as maximal orthogonal modules. The rigidity dimension is defined to be the supremum of dominant dimensions taken over all resolution algebras E. It always will be at least two, but it is open if it always takes a finite value.

A representation theoretic example to keep in mind is Schur–Weyl duality between Schur algebras of general linear groups, or their quantised versions, and group algebras of symmetric groups, or their Hecke algebras. As Schur algebras always do have finite global dimension, they are non-commutative resolution algebras of the group algebras or Hecke algebras. Their dominant dimensions have been computed, which gives a lower bound for the rigidity dimension of group algebras of symmetric groups. In a subsequent article dealing

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with methods and examples, we will show that in some cases at least this bound is optimal and thus some Schur algebras are best possible resolution algebras of group algebras of symmetric groups.

The main concept introduced in this article is a new homological dimension, which provides information on the 'best possible' resolution algebra of a fixed algebra A:

$$\mathrm{rigdim}(A) := \sup\left\{ \mathrm{domdim}(E) \;\middle|\; E \text{ is a resolution algebra of } A \right\}.$$

As we know already, $\mathrm{rigdim}(A) \geq 2$ for any A. By Theorem 2·7 (Müller), when M is a generator-cogenerator in A-mod, the dominant dimension of its endomorphism ring is determined by the rigidity degree of M (Definition 2·5), which measures vanishing of self-extensions of M. Therefore, the new dimension is called **rigidity dimension** of A. Note that non-semisimple algebras of finite global dimension always have finite dominant dimension, so the supremum is taken over a set of natural numbers, unless A is semisimple and thus has infinite rigidity dimension. When A has infinite global dimension, $\mathrm{rigdim}(A)$ measures how close A can come to a resolution algebra E.

A combination of global and dominant dimension also occurs in the definitions of Auslander's representation dimension, which is always finite, and Iyama's higher representation dimension, which is often infinite. It turns out that rigidity dimension controls finiteness of higher representation dimension: $\mathrm{repdim}_n(A)$ is finite if and only if $\mathrm{rigdim}(A) \geq n+1$. Thus, rigidity dimension can be viewed as a companion of higher representation dimension.

We will address two basic questions about this new dimension: *finiteness* and *invariance under equivalences*.

Three approaches are developed to establish *finiteness*. The first approach provides an upper bound for $\mathrm{rigdim}(A)$ in terms of the smallest degree $n \geq 1$, if existent, for which $\mathrm{Ext}^n_A(\mathrm{D}(A), A)$ does not vanish (Theorem 3·1). Here D is the usual k-duality over the ground field k. This can be applied for instance to Schur algebras of algebraic groups and to blocks of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ of semisimple complex Lie algebras.

The second approach provides an upper bound in terms of the smallest positive degree of a non-nilpotent homogenous generator of Hochschild cohomology (Theorem 3·5). This implies finiteness of rigidity dimension for all non-semisimple group algebras (Theorem 3·6). Symonds' proof of Benson's regularity conjecture then implies that the order of the group is an explicit, but weak, upper bound for $\mathrm{rigdim}(kG)$.

The third approach derives finiteness of rigidity dimension for all non-semisimple algebras from homological conjectures (Theorem 3·7): assuming Tachikawa's first conjecture (Section 3·3 (TC1)) yields finiteness for algebras that are not self-injective, as an application of the first approach. Finiteness in general is shown to follow from Yamagata's conjecture (Section 3·3 (YC)). These conjectures are in general open; for finite-dimensional algebras, no counterexamples are known. Tachikawa's first conjecture is part of a reformulation of Nakayama's conjecture, which states that an algebra is self-injective if and only if it has infinite dominant dimension.

Which equivalences of categories do preserve rigidity dimension? Morita equivalences are easily seen to preserve rigidity dimension in general. For stable or derived equivalences the problem is much more subtle. In fact, in the presence of nodes (which at least for self-injective algebras does not happen frequently), rigidity dimension can change under stable equivalence, as we show by examples. However, algebras without nodes always have minimal rigidity dimension in their stable equivalence class (Theorem 4·4 and Corollary 4·5).

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This implies that rigidity dimension is invariant under stable equivalences between algebras without nodes. In general, invariance is guaranteed when the equivalence preserves the triangulated structure which stable categories of self-injective algebras are known to have (Corollary 4·9).

Stable equivalences of Morita type, and thus also derived equivalences, between self-injective algebras do leave rigidity dimension invariant. More precisely, stable equivalences of adjoint type always preserve rigidity dimension for all finite dimensional algebras (Theorem 5·2).

In the subsequent paper [7], rigidity dimensions of classes of examples will be determined.

Notation. Throughout this paper, k is an arbitrary but fixed field. Unless stated otherwise, all algebras are finite-dimensional associative k-algebras with units, and all modules are finite-dimensional left modules. The set of positive integers is denoted by $\mathbb{N}$; the set of non-negative integers is denoted by $\mathbb{N}_0$.

Let A be an algebra. A-mod denotes the category of all left A-modules. The syzygy and cosyzygy operators of A-mod are denoted by Ω_A and Ω_A^-, respectively. Let A^{op} be the opposite algebra of A. Then $\mathrm{D} := \mathrm{Hom}_k(-, k)$ is a duality between A-mod and A^{op}-mod.

Let $\mathcal{X}$ be a class of A-modules. By $\mathrm{add}(\mathcal{X})$, we denote the smallest full subcategory of A-mod which contains $\mathcal{X}$ and is closed under finite direct sums and direct summands. When $\mathcal{X}$ consists of only one object X, we write $\mathrm{add}(X)$ for $\mathrm{add}(\mathcal{X})$. In particular, $\mathrm{add}({}_AA)$ is exactly the category of projective A-modules and also denoted by A-proj. Let $\mathscr{P}_A$ and $\mathscr{I}_A$ stand for the set of isomorphism classes of indecomposable projective and injective A-modules, respectively.

An A-module M is called *basic* if it is a direct sum of pairwise non-isomorphic indecomposable submodules. Moreover, the head, radical and socle of M are denoted by $\mathrm{hd}(M)$, $\mathrm{rad}(M)$ and $\mathrm{soc}(M)$, respectively. The composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ in A-mod is denoted by $fg : X \to Z$. In this sense, $\mathrm{Hom}_A(X, Y)$ is an $\mathrm{End}_A(X)$-$\mathrm{End}_A(Y)$-bimodule. Particularly, M is a right $\mathrm{End}_A(M)$-module.

2. *Rigidity dimension*

The main object of study in this paper, rigidity dimension, will be introduced in the second subsection. Before, we recall the definitions of global and dominant dimension and define the rigidity degree of a module. The third subsection then provides basic properties and examples. This includes the connection with Iyama's higher representation dimension, the relation with cluster tilting modules, which can be used to provide lower bounds for rigidity dimension, and finally a result on rigidity dimension of weakly Calabi-Yau self-injective algebras, which implies that preprojective algebras of Dynkin type have rigidity dimension exactly three.

2·1. *Global and dominant dimension, and rigidity degree*

An A-module M is called a *generator* if $A \in \mathrm{add}(M)$; a *cogenerator* if $\mathrm{D}(A_A) \in \mathrm{add}(M)$. M is called a *generator-cogenerator* if it is both a generator and a cogenerator.

The *global dimension* of an algebra A, denoted by gldim A, is defined to be the maximal number t or ∞ such that $\mathrm{Ext}_A^t(M, N) \neq 0$ for some $M, N \in A$-mod. To compute the global dimension of endomorphism algebras, the following result due to Auslander [1, Chapter III, Section 3] is very useful.

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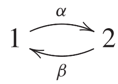
PROPOSITION 2·1. *Let ${}_AM$ be a generator-cogenerator and let $n \geq 2$. Then* $\operatorname{gldim} \operatorname{End}_A(M) \leq n$ *if and only if for each indecomposable A-module X, there exists an exact sequence of A-modules*

$$\xi: \quad 0 \longrightarrow M_{n-2} \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

such that $M_i \in \operatorname{add}(M)$ for all i and the sequence $\operatorname{Hom}_A(M, \xi)$ is exact.

The following example is an illustration. It will be continued in Example 4·6.

Example 2·2. Let B be the k-algebra given by the quiver



and relations $\{\alpha\beta, \beta\alpha\}$. Our composition of arrows in the quiver is taken from right to left. In total, there are up to isomorphism four indecomposable B-modules, the projective B-modules P_1 and P_2 and their simple heads S_1 and S_2 respectively. Let $M = B \oplus S_1$. Since B is self-injective, M is a generator-cogenerator. Moreover, the following sequences are exact and remain exact after applying $\operatorname{Hom}_B(M, -)$:

$$0 \longrightarrow S_1 \longrightarrow S_1 \longrightarrow 0, \quad 0 \longrightarrow P_1 \longrightarrow P_1 \longrightarrow 0, \quad 0 \longrightarrow P_2 \longrightarrow P_2 \longrightarrow 0,$$
$$0 \longrightarrow S_1 \longrightarrow P_2 \longrightarrow S_2 \longrightarrow 0.$$

By Proposition 2·1, $\operatorname{gldim} \operatorname{End}_B(M) \leq 3$.

The following is the key ingredient in our main definition (Definition 2·8) in this article.

Definition 2·3. An A-module M is called a *resolving module* if:

(1) M is a generator-cogenerator;
(2) $\operatorname{gldim} \operatorname{End}_A(M) < \infty$.

Dominant dimension was introduced by Nakayama and later systematically studied by Tachikawa, Morita, Müller, Yamagata and many others, see [**31**, **36**, **38**]. See also [**9**, **12**, **14**, **15**, **16**, **26**, **39**] for some recent developments on dominant dimension partly motivating this paper.

Definition 2·4. The *dominant dimension* of A, denoted by $\operatorname{domdim} A$, is the largest $t \in \mathbb{N}_0$, or ∞, such that in a minimal injective resolution

$$0 \longrightarrow {}_AA \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^{t-1} \longrightarrow I^t \longrightarrow \cdots$$

all I^i are projective for $0 \leq i < t$.

Note that $\operatorname{domdim} A = \operatorname{domdim} A^{\mathrm{op}}$ (see [**31**, theorem 4]). If $\operatorname{domdim} A \geq 1$, then the injective envelope of ${}_AA$ is faithful and projective. If $\operatorname{domdim} A \geq 2$, then any faithful projective-injective A-module P has the *double centralizer property*, that is, with $\Lambda = \operatorname{End}_A(P)$ there is an isomorphism $A \cong \operatorname{End}_{\Lambda^{\mathrm{op}}}(P)^{\mathrm{op}}$. In this case, also $A \cong \operatorname{End}_\Lambda(\mathrm{D}(P))$, and $\mathrm{D}(P)$ is a generator-cogenerator in Λ -mod. In general, for calculating dominant dimensions of endomorphism algebras, the following definition is useful.

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Definition 2·5. Let M be an A-module. The *rigidity degree* of M, denoted by $\mathrm{rd}(_AM)$, is the maximal $n \in \mathbb{N}_0$, or ∞, such that $\mathrm{Ext}^i_A(M, M)$ vanishes for all $1 \leq i \leq n$. In other words, $\mathrm{rd}(_AM) \geq n$ if and only if $\mathrm{Ext}^i_A(M, M) = 0$ for all $1 \leq i \leq n$.

LEMMA 2·6. *Let $M = M_1 \oplus \cdots \oplus M_s$ be a decomposition of the A-module M into indecomposable direct summands. Then*

$$\mathrm{rd}(M) = \min\{\mathrm{rd}(M_i \oplus M_j) \mid 1 \leq i, j \leq s\}.$$

Proof. For $1 \leq i, j \leq s$, set $M_{i,j} = M_i \oplus M_j$. Since $M_{i,j} \in \mathrm{add}(M)$, we have $\mathrm{rd}(M) \leq \mathrm{rd}(M_{i,j})$. If $\mathrm{rd}(M)$ is infinite, then there is nothing to prove. Suppose $\mathrm{rd}(M) = t < \infty$. Then $\mathrm{Ext}^p_A(M, M) = 0$ for $1 \leq p \leq t$ and $\mathrm{Ext}^{t+1}_A(M, M) \neq 0$. In particular, there is a pair (u, v) of integers with $1 \leq u, v \leq s$ such that $\mathrm{Ext}^{t+1}_A(M_u, M_v) \neq 0$. This implies $\mathrm{rd}(M_{u,v}) \leq t$, and thus $\mathrm{rd}(M_{u,v}) = t$.

In this paper, rigidity degrees of modules, measuring the vanishing of self-extensions, are of particular interest, due to their connections with both dominant dimensions and Hochschild cohomology rings. The connection with dominant dimension is provided by a result due to Müller:

THEOREM 2·7 (Müller [31, lemma 3]). *Let A be an algebra and M a generator-cogenerator in A* -mod. *Then* $\mathrm{domdim}\,\mathrm{End}_A(M) = \mathrm{rd}(_AM) + 2$.

Indeed, applying $\mathrm{Hom}_A(M, -)$ to the minimal injective resolution of $0 \to M \to I_0 \to I_1 \to \cdots \to I_n \to \cdots$ of M, yields a complex in $\mathrm{End}_A(M)$ -mod (by our convention on composition of maps):

$$0 \to \mathrm{End}_A(M) \to \mathrm{Hom}_A(M, I_0) \to \cdots \to \mathrm{Hom}_A(M, I_n) \to \cdots$$

Now $\mathrm{Hom}_A(M, I_i) \in \mathrm{add}(\mathrm{D}(M))$, and $\mathrm{D}(M)$ is a projective-injective $\mathrm{End}_A(M)$-module since M is a generator-cogenerator. Theorem 2·7 then says that the exactness of the above complex characterises the dominant dimension of $\mathrm{End}_A(M)$.

2·2. *Definition of rigidity dimension*

Both Rouquier's theory of quasi-hereditary covers [33] and the theory of non-commutative crepant resolutions due to van den Bergh, Iyama and Wemyss, see for instance [24], 'resolve' algebras of infinite global dimension by algebras of finite global dimension. The quality of such a resolution can be measured by the dominant dimension of the resolution algebra or by the rigidity degree of the generator-cogenerator over the algebra being resolved. The homological dimension to be defined now, aims to measure the quality of such resolutions.

Definition 2·8. The rigidity dimension of an algebra A is defined to be

$$\mathrm{rigdim}(A) = \sup\left\{\mathrm{domdim}\,\mathrm{End}_A(M) \mid M \textit{ is a resolving } A\textit{-module}\right\}.$$

The construction in the proof of Iyama's finiteness theorem on representation dimension, [23, lemma 2·2], ensures the existence of at least one resolving module (Definition 2·3) for any finite dimensional algebra. Thus, the supremum is taken over a non-empty set.

The rigidity dimension of a semisimple algebra is always ∞, as the dominant dimension of semisimple algebras is ∞. In Section 3, we will give criteria to check finiteness of rigidity dimension for many non-semisimple algebras. In particular, we will prove finiteness for all non-selfinjective algebras after assuming Tachikawa's first conjecture, and for all non-semisimple algebras after assuming Yamagata's conjecture.

It was shown by Morita and Tachikawa in an unpublished but circulating paper, and independently by Müller [**31**], that finite dimensional algebras E with dominant dimension at least 2 are exactly the endomorphism rings $\mathrm{End}_A(M)$, where A is an algebra and M is a generator-cogenerator in A-mod. This result is frequently cited as *Morita-Tachikawa correspondence*, see also [**38**, theorem 3·3·1] for more details. A consequence is a lower bound for rigidity dimension.

COROLLARY 2·9. *For any algebra A,* $\mathrm{rigdim}\, A \geq 2$.

This also follows from a reformulation of the definition, using Müller's Theorem 2·7:

$$\mathrm{rigdim}(A) = \sup\left\{ \mathrm{rd}({}_A M) \mid M \text{ is a resolving } A\text{-module}\right\} + 2.$$

As a consequence of this reformulation, we have the following result:

Let M be a generator-cogenerator in A-mod. If ${}_A M$ is not projective and $\mathrm{rd}(M) \geq \mathrm{rigdim}(A) - 1$, then $\mathrm{gldim}\,\mathrm{End}_A(M) = \infty$.

2·3. *Basic properties, examples and connections*

PROPOSITION 2·10. *Let A and B be algebras. Then*:

(1) $\mathrm{rigdim}(A) = \mathrm{rigdim}(A^{\mathrm{op}})$ *and* $\mathrm{rigdim}(A \times B) = \min\{\mathrm{rigdim}(A), \mathrm{rigdim}(B)\}$;
(2) *If A and B are Morita equivalent, then* $\mathrm{rigdim}(A) = \mathrm{rigdim}(B)$;
(3) *If k is perfect, then* $\mathrm{rigdim}(A \otimes_k B) \geq \min\{\mathrm{rigdim}(A), \mathrm{rigdim}(B)\}$.

Proof. (1) and (2) are consequences of well-known facts: both global dimension and dominant dimension are invariant under taking opposite algebras or passing to Morita equivalent algebras; the dominant dimension (resp. global dimension) of the product of two algebras is the minimum (resp. the maximum) of their dominant dimensions (resp. global dimensions).

(3) Let X and Y be generator-cogenerators in A-mod and B-mod, respectively. Then $X \otimes_k Y$ is a generator-cogenerator in $(A \otimes_k B)$-mod. Now $\mathrm{End}_{A\otimes_k B}(X \otimes_k Y) \cong \mathrm{End}_A(X) \otimes_k \mathrm{End}_B(Y)$ as k-algebras, and $\mathrm{domdim}(\mathrm{End}_A(X) \otimes_k \mathrm{End}_B(Y)) = \min\{\mathrm{domdim}\,\mathrm{End}_A(X), \mathrm{domdim}\,\mathrm{End}_B(Y)\}$ by [**31**, lemma 6], and $\mathrm{gldim}(\mathrm{End}_A(X) \otimes_k \mathrm{End}_B(Y)) = \mathrm{gldim}\,\mathrm{End}_A(X) + \mathrm{gldim}\,\mathrm{End}_B(Y)$ whenever k is a perfect field. Therefore, $\mathrm{rigdim}(A \otimes_k B) \geq \min\{\mathrm{rigdim}(A), \mathrm{rigdim}(B)\}$.

2·3·1. *Relation with higher representation dimension*

The following result exhibits rigidity dimension as a counterpart of the higher representation dimension introduced by Iyama [**24**, definition 5·4]. Recall that for a natural number n, the nth representation dimension $\mathrm{repdim}_n(A)$ of an algebra A is defined to be

$$\mathrm{repdim}_n(A) = \inf\left\{ \mathrm{gldim}\,\mathrm{End}_A(M) \,\middle|\, \begin{array}{l} M \text{ is a generator-cogenerator in } A\text{-mod} \\ \text{and } \mathrm{domdim}\,\mathrm{End}_A(M) \geq n+1. \end{array}\right\}$$

Auslander's classical representation dimension is repdim_1. As Iyama has shown [**23**], repdim_1 is always finite. For $n \geq 2$, infinite values do occur.

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PROPOSITION 2·11. *Let A be an algebra and n a positive integer. Then* $\operatorname{repdim}_n(A) < \infty$ *if and only if* $\operatorname{rigdim}(A) \geq n+1$.

Proof. If $\operatorname{repdim}_n(A) < \infty$, then $\operatorname{rigdim}(A) \geq n+1$ by definition. Conversely, if $\operatorname{rigdim}(A) \geq n+1$, then there exists a resolving A-module M such that $\operatorname{domdim} \operatorname{End}_A(M) \geq n+1$. Hence $\operatorname{repdim}_n(A) < \infty$.

When $n = 1$, the statement $\operatorname{rigdim}(A) \geq 2$ for all A in Corollary 2·9 is a reformulation of Iyama's finiteness result, which has been used in the definition of rigidity dimension.

Example 2·12. Let A be a finite dimensional non-simple self-injective local k-algebra with $\operatorname{rad}^3(A) = 0$. Then $\operatorname{rigdim}(A) = 2$, since by [**20**, theorem 3·4] every non-projective A-module has non-trivial self-extensions.

2·3·2. *Relation with cluster tilting modules, and a lower bound for rigidity dimension*

Recall that a module M is said to be $(n+1)$-*cluster tilting* (also known as maximal n-orthogonal) for some $n \geq 1$ if it satisfies $M^{\perp_n} = {}^{\perp_n}M = \operatorname{add}(M)$ [**25**, definition 1·1], where

$$M^{\perp_n} = \{X \in A\text{-mod} \mid \operatorname{Ext}^i_A(M, X) = 0,\ 1 \leq i \leq n\},$$
$${}^{\perp_n}M = \{X \in A\text{-mod} \mid \operatorname{Ext}^i_A(X, M) = 0,\ 1 \leq i \leq n\}.$$

Note that an $(n+1)$-cluster tilting module M automatically is a generator-cogenerator and its endomorphism algebra always has finite global dimension (see [**24**, **25**]).

PROPOSITION 2·13. *Let A be a non-semisimple algebra and n a natural number. If there exists an $(n+1)$-cluster tilting A-module, then* $\operatorname{rigdim}(A) \geq n+2$.

Proof. Let ${}_AM$ be an $(n+1)$-cluster tilting A-module. By [**24**, theorem 0·2], M is a resolving module with $\operatorname{rd}(M) = n$. Therefore, $\operatorname{rigdim}(A) \geq n+2$.

Remark. In Proposition 2·13, if furthermore either $1 \leq \operatorname{injdim}({}_AA) \leq n+1$ or $1 \leq \operatorname{injdim}(A_A) \leq n+1$, then $\operatorname{rigdim}(A) = n+2$. Indeed, under this assumption $\operatorname{rigdim}(A) \leq n+2$ by Theorem 3·1(2) below, and therefore $\operatorname{rigdim}(A) = n+2$.

2·3·3. *Weakly Calabi–Yau self-injective algebras*

Let A be a self-injective algebra. The stable module category $A\text{-}\underline{\mathrm{mod}}$ of A is a k-linear Hom-finite triangulated category, and its shift functor Σ is the cosyzygy functor Ω_A^{-1} [**19**, section 2·6]. Recall that $A\text{-}\underline{\mathrm{mod}}$ is said to be *weakly n-Calabi–Yau* for a natural number n if there are natural k-linear isomorphisms

$$\underline{\mathrm{Hom}}_A(Y, \Sigma^n(X)) \cong \mathrm{D}\,\underline{\mathrm{Hom}}_A(X, Y)$$

for any $X, Y \in A\text{-}\underline{\mathrm{mod}}$. Since $A\text{-}\underline{\mathrm{mod}}$ has a Serre duality $\Omega_A\nu_A$ [**11**, proposition 1·2], A is weakly n-Calabi-Yau if and only if Ω_A^{-n} and $\Omega_A\nu_A$ are naturally isomorphic as auto-equivalences of $A\text{-}\underline{\mathrm{mod}}$. Equivalently, $\Omega_A^{n+1}\nu_A$ is naturally isomorphic to the identity functor of $A\text{-}\underline{\mathrm{mod}}$. If A is symmetric, then it is weakly n-Calabi-Yau if and only if Ω_A^{n+1} is naturally isomorphic to the identity functor of $A\text{-}\underline{\mathrm{mod}}$.

PROPOSITION 2·14. *Let A be a non-semisimple self-injective algebra. If A-$\underline{\text{mod}}$ is weakly n-Calabi–Yau with $n \geq 1$, then* $\text{rigdim}(A) \leq n+1$.

Proof. Since A is self-injective and $n \geq 1$, we have $\text{Ext}^n_A(X, X) \cong \underline{\text{Hom}}_A(X, \Omega_A^{-n}(X))$ for any A-module X. Recall that the shift functor Σ of A-$\underline{\text{mod}}$ is given by the cosyzygy functor Ω_A^{-1}. Since A-$\underline{\text{mod}}$ is weakly n-Calabi-Yau, it follows that $\underline{\text{Hom}}_A(X, \Omega_A^{-n}(X)) \cong \underline{\text{Hom}}_A(X, \Sigma^n(X)) \cong \text{D}\,\underline{\text{Hom}}_A(X, X)$. Thus $\text{Ext}^n_A(X, X) \cong \text{D}\,\underline{\text{Hom}}_A(X, X)$. This implies that if X is non-projective, then $\text{Ext}^n_A(X, X)$ does not vanish. In this case, $\text{rd}({}_AX) \leq n-1$. Now, it is clear that $\text{rigdim}(A) \leq n+1$.

COROLLARY 2·15. *Let A be a preprojective algebra of Dynkin type over an algebraically closed field. Then* $\text{rigdim}(A) = 3$.

Proof. By [**10**, lemma 1], A-$\underline{\text{mod}}$ is weakly 2-Calabi–Yau. Proposition 2·14 implies $\text{rigdim}(A) \leq 3$. Further, by [**17**, theorem 2·2 and corollary 2·3] there exists a 2-cluster tilting A-module. It follows from Proposition 2·13 that $\text{rigdim}(A) \geq 3$. Thus $\text{rigdim}(A) = 3$.

3. *Finiteness*

Defining a homological dimension, a basic question is: On which algebras does it take finite values? Semisimple algebras have infinite rigidity dimension, for trivial reasons, which distinguish them from all other algebras. In the first two subsections we provide two methods to prove finiteness. The first one is using extension groups between injective and projective modules; this works for algebras of finite global dimension and for gendo-symmetric algebras. The second one is using Hochschild cohomology; this works for group algebras of finite groups. The third subsection then derives finiteness in general from (still unproven) homological conjectures due to Tachikawa and Yamagata.

3·1. *Finiteness I: relation with the extension groups* $\text{Ext}^*_A(\text{D}(A), A)$

Since $\text{D}(A) \oplus A$ appears as a direct summand of every generator-cogenerator M (up to multiplicities), the groups $\text{Ext}^*_A(\text{D}(A), A)$ naturally occur in the computation of the Yoneda algebra $\text{Ext}^*_A(M, M)$, and therefore in the computation of the rigidity dimension of A.

THEOREM 3·1. *Let A be a non-selfinjective k-algebra. Then*:

(1) $\text{rigdim}(A) \leq \sup\{n \in \mathbb{N}_0 \mid \text{Ext}^j_A(\text{D}(A), A) = 0 \text{ for } 1 \leq j \leq n\} + 2$. *Equality holds if the endomorphism algebra of $A \oplus \text{D}(A)$ has finite global dimension;*

(2) $\text{rigdim}(A) \leq \text{injdim}({}_AA) + 1 \leq \text{gldim}(A) + 1$.

Proof. (1) Let $d = \sup\{n \in \mathbb{N}_0 \mid \text{Ext}^j_A(\text{D}(A), A) = 0 \text{ for } 1 \leq j \leq n\}$. Then $d = \text{rd}({}_AA \oplus \text{D}(A))$ since $\text{Ext}^i_A(\text{D}(A), A) \cong \text{Ext}^i_A(A \oplus \text{D}(A), A \oplus \text{D}(A))$ for any $i \geq 1$. Now let M be a generator-cogenerator in A-mod. Since $A \oplus \text{D}(A) \in \text{add}({}_AM)$, it follows that $\text{rd}({}_AM) \leq \text{rd}({}_AA \oplus \text{D}(A))$ and therefore $\text{rigdim}(A) \leq d+2$. If furthermore $\text{gldim}\,\text{End}_A(A \oplus \text{D}(A)) < \infty$, then $\text{rigdim}(A) = \text{rd}({}_AA \oplus \text{D}(A)) + 2 = d + 2$.

(2) Let $m = \text{injdim}({}_AA)$. Then $m \geq 1$ since otherwise, A is self-injective. When m is infinite, there is nothing to show. Suppose m is finite. Then $\text{Ext}^m_A(\text{D}(A), A) \neq 0$ and (1) implies $\text{rigdim}(A) \leq m - 1 + 2 = m + 1$.

Example 3·2. Let A be a non-semisimple hereditary algebra. Then Corollary 2·9 and Theorem 3·1 imply that $\mathrm{rigdim}(A) = 2$.

Recall that an algebra is called *gendo-symmetric* if it is the endomorphism algebra of a generator over a symmetric algebra, see [15, 16] and also [39]. Such algebras arise extensively in algebraic Lie theory. Hecke algebras, quantized Schur algebras, and blocks of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of semisimple complex Lie algebras are typical examples.

COROLLARY 3·3. *If A is a gendo-symmetric algebra, then* $\mathrm{rigdim}(A) \leq \mathrm{domdim}(A)$.

Proof. By [15, proposition 3·3], the dominant dimension of A is at least two, and equals

$$\sup\{n \in \mathbb{N}_0 \mid \mathrm{Ext}^i_A(\mathrm{D}(A), A) = 0,\ 1 \leq i \leq n\} + 2.$$

If A is self-injective, then $\mathrm{domdim}(A) = \infty$. Otherwise, Theorem 3·1(1) implies $\mathrm{rigdim}(A) \leq \mathrm{domdim}(A)$.

An application of this corollary is that any gendo-symmetric algebra A with $\mathrm{domdim}(A) = 2$ has rigidity dimension 2. Examples of such algebras are the non-simple blocks of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of semisimple complex Lie algebras. The global dimension of these algebras always is an even number, which can be arbitrarily large.

3·2. *Finiteness II: relation with Hochschild cohomology*

Let A be a k-algebra and A^{op} its opposite algebra. The Hochschild cohomology ring $\mathrm{HH}^*(A)$ is the Yoneda extension algebra $\mathrm{Ext}^*_{A^{\mathrm{ev}}}(A, A)$ where $A^{\mathrm{ev}} = A \otimes_k A^{\mathrm{op}}$, the enveloping algebra of A. So, $\mathrm{HH}^*(A)$ is the direct sum of $\mathrm{HH}^i(A) := \mathrm{Ext}^i_{A^{\mathrm{ev}}}(A, A)$ for $i \in \mathbb{N}_0$. This is an $\mathbb{N}_0$-graded k-algebra. In general, it is not commutative, but graded commutative. Moreover, it may be infinite-dimensional as a vector space over k. However, if A^{ev} has finite global dimension, then $\mathrm{HH}^*(A)$ is finite-dimensional. The *reduced Hochschild cohomology ring* $\overline{\mathrm{HH}}^*(A)$ is the quotient $\mathrm{HH}^*(A)/\mathcal{N}$ where $\mathcal{N}$ is the ideal of $\mathrm{HH}^*(A)$ generated by homogeneous nilpotent elements. Although the problem whether the (reduced) Hochschild cohomology ring is finitely generated has been widely studied, very little seems to be known about the degrees of the homogeneous generators. We will show that rigidity dimension of A is closely related to the minimal degree of non-nilpotent homogeneous generators of positive degree.

We will use the following result, which is essentially combining [5, theorems 2·13 and 5·9] in our situation. Note that a main tool in [5] is the *grade* defined by Auslander and Bridger, which is closely related to dominant dimension.

THEOREM 3·4 (**Buchweitz**). *Let M be a generator-cogenerator in A*-mod *and let* $E = \mathrm{End}_A(M)$. *Then there is an $\mathbb{N}_0$-graded algebra homomorphism $\varphi : \mathrm{HH}^*(E) \to \mathrm{HH}^*(A)$ such that $\mathrm{HH}^i(E) \to \mathrm{HH}^i(A)$ is an isomorphism for each $0 \leq i \leq \mathrm{rd}({}_A M)$.*

The connection with rigidity dimension is as follows:

THEOREM 3·5. *Let A be an algebra over a perfect field k. Suppose that $\overline{\mathrm{HH}}^*(A)$ is not concentrated in degree zero. Then* $\mathrm{rigdim}(A)$ *is finite.*

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More precisely, let $\delta(A) = \inf\{i \geq 1 \mid \overline{\mathrm{HH}}^i(A) \neq 0\}$. *Then* $\mathrm{rigdim}(A) \leq \delta(A) + 1$. *If k has characteristic different from* 2 *and* $\mathrm{rigdim}(A)$ *is even, then* $\mathrm{rigdim}(A) \leq \delta(A)$.

Proof. Let M be a generator-cogenerator in A -mod and let $E = \mathrm{End}_A(M)$. By Theorem 3·4, there is a graded algebra homomorphism $\varphi : \mathrm{HH}^*(E) \to \mathrm{HH}^*(A)$ such that $\varphi_i : \mathrm{HH}^i(E) \to \mathrm{HH}^i(A)$ is an isomorphism for $0 \leq i \leq \mathrm{rd}({}_A M) = \mathrm{domdim}\, E - 2$, where the final equality follows from Müller's Theorem 2·7.

If $\mathrm{gldim}\, E < \infty$, then k being a perfect field yields $\mathrm{gldim}\, E^{\mathrm{ev}} < \infty$, and thus $\mathrm{HH}^*(E)$ is a finite dimensional k-algebra, and therefore all homogeneous elements in $\mathrm{HH}^*(E)$ of positive degrees are nilpotent. Using φ and φ_m, all homogenous elements in $\mathrm{HH}^*(A)$ of degree m with $1 \leq m \leq \mathrm{domdim}\, E - 2$ are seen to be nilpotent. Hence, $\mathrm{domdim}\, E - 1 \leq \delta(A)$ since $\delta(A)$ detects the minimal degree of homogeneous generators of positive degrees in $\overline{\mathrm{HH}}^*(A)$. Thus $\mathrm{rigdim}(A) \leq \delta(A) + 1$.

If k has characteristic different from 2, then homogeneous elements in $\mathrm{HH}^*(A)$ of odd degrees are nilpotent, and therefore $\delta(A)$ must be an even number or ∞. When $\mathrm{rigdim}(A)$ is an even number, then $\mathrm{rigdim}(A) \leq \delta(A)$ as claimed.

Remark. (1) Theorem 3·5 provides an upper bound for the rigidity dimension of A as long as some information on its Hochschild cohomology ring is known. Conversely, a lower bound for rigidity dimension is a lower bound for the degree of homogeneous generators in the reduced Hochschild cohomology ring. Examples in [**7**] will show that the bound in Theorem 3·5 is optimal.

(2) The reduced Hochschild cohomology ring $\overline{\mathrm{HH}}^*(A)$ is not finitely generated in general. The first example has been a seven dimensional k-algebra A found by Xu [**37**]. If k has characteristic two, then $\delta(A) = 1$ [**37**], see also [**34**, theorem 4·5], and therefore $\mathrm{rigdim}(A) = 2$.

(3) Even when A has infinite global dimension, the reduced Hochschild cohomology ring $\overline{\mathrm{HH}}^*(A)$ can be concentrated in degree zero, see [**6**] for examples.

Theorem 3·5 can be applied to group algebras, for which we obtain finiteness in all non-semisimple cases.

THEOREM 3·6. *Let G be a finite group and k a perfect field of characteristic* $p \geq 0$. *Then* $\mathrm{rigdim}(kG)$ *is finite if and only if p divides the order* $|G|$ *of G (if and only if kG is not semisimple). In this case,* $\mathrm{rigdim}(kG) \leq |G|$.

Proof. By [**27**, proposition 4·5], there is an injective k-algebra homomorphism which is graded by construction

$$\theta : \mathrm{H}^*(G, k) \longrightarrow \mathrm{HH}^*(kG),$$

where $\mathrm{H}^*(G, k) = \mathrm{Ext}^*_{kG}(k, k)$ is the cohomology ring of the group algebra kG. Let $\overline{\mathrm{H}}^*(G, k)$ be the cohomology ring $\mathrm{H}^*(G, k)$ modulo nilpotent elements. Then θ induces an injective morphism $\overline{\mathrm{H}}^*(G, k) \to \overline{\mathrm{HH}}^*(kG)$.

If p does not divide $|G|$, then kG is semisimple and hence $\mathrm{rigdim}(kG) = \infty$. If p divides $|G|$, then k is not a projective kG-module, and by Chouinard's theorem in group cohomology, see [**4**, lemma 5·2·3 and theorem 5·2·4],

$$\gamma(kG) := \inf\{i \geq 1 \mid \overline{\mathrm{H}}^i(G, k) \neq 0\} < \infty.$$

The embedding $\overline{\mathrm{H}}^*(G, k) \to \overline{\mathrm{HH}}^*(kG)$ gives $\gamma(kG) \geq \delta(kG)$, the minimal degree of homogeneous generators of $\overline{\mathrm{HH}}^*(kG)$ of positive degrees. Using Theorem 3·5, it follows that

$$\mathrm{rigdim}(kG) \leq \delta(kG) + 1 \leq \gamma(kG) + 1 < \infty.$$

The explicit upper bound follows from a major result by Symonds, who proved Benson's regularity conjecture. According to [**35**, proposition 0·3], which is a consequence of the main result in [**35**], for a finite group G with more than one element, group cohomology $\mathrm{H}^*(G, k)$ has a set of homogeneous generators of degree at most $|G| - 1$. As a consequence, the reduced group cohomology ring has a set of generators in degrees smaller than $|G|$, and thus $\gamma(kG) \leq |G| - 1$.

Remark. (1) Symonds' result [**35**] provides an upper bound for the degrees of homogeneous generators of $\mathrm{H}^*(G, k)$. The proof of Theorem 3·6 shows that the rigidity dimension of the group algebra kG may be used to provide a lower bound for the degrees of homogeneous generators of the Hochschild cohomology ring modulo nilpotents.

(2) In [**7**], the rigidity dimensions of defect one blocks of group algebras will be determined.

3·3. *Finiteness III: using homological conjectures*

Can it be expected that rigidity dimension is always finite, except for semisimple algebras? Some evidence is provided here by deriving this statement from some homological conjectures.

- (TC1) **Tachikawa's first conjecture** [**36**, p. 115] Let A be a finite dimensional k-algebra. Suppose $\mathrm{Ext}^i_A(\mathrm{D}(A), A) = 0$ for all $i \geq 1$. Then A is self-injective.
- (TC2) **Tachikawa's second conjecture** [**36**, p. 116] Let A be a finite dimensional self-injective k-algebra and M a finitely generated A-module. Suppose $\mathrm{Ext}^i_A(M, M) = 0$ for all $i \geq 1$. Then M is a projective A-module.
- (YC) **Yamagata's conjecture** [**38**, p. 876] There exists a function $\varphi : \mathbb{N} \to \mathbb{N}$ such that for any finite dimensional k-algebra A with finite dominant dimension, $\mathrm{domdim}\, A \leq \varphi(n)$ where n is the number of isomorphism classes of simple A-modules.

(TC1) and (TC2) together are equivalent to Nakayama's conjecture.

THEOREM 3·7. *Let A be a finite dimensional non-semisimple k-algebra.*

(1) *If A is not self-injective, then (TC1) implies* $\mathrm{rigdim}(A) < \infty$.
(2) *If* (YC) holds, then $\mathrm{rigdim}(A) < \infty$.

Proof. (1) If A is not self-injective, then by (TC1), $\mathrm{Ext}^n_A(\mathrm{D}(A), A) \neq 0$ for some natural number $n \geq 1$. Therefore, by Theorem 3·1, $\mathrm{rigdim}(A) \leq n + 2$.

(2) Let M be a generator-cogenerator in A -mod. Up to multiplicities of direct summands, suppose ${}_AM = A \oplus \mathrm{D}(A) \oplus \bigoplus_{i=1}^m M_i$, where $m \geq 1$, and M_i is either zero or indecomposable, non-projective and non-injective. Then by Lemma 2·6, $\mathrm{rd}(M) = \mathrm{rd}(A \oplus \mathrm{D}(A) \oplus M_u \oplus M_v)$ for some $1 \leq u, v \leq m$. Now, we set $N_{u,v} = A \oplus \mathrm{D}(A) \oplus M_u \oplus M_v$ and assume that $E := \mathrm{End}_A(M)$ has finite global dimension.

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By assumption, A is not semisimple, hence E is not semisimple either. It follows that $\operatorname{domdim} E \leq \operatorname{gldim} E < \infty$. Moreover, by Theorem 2·7, $\operatorname{domdim} E = \operatorname{rd}(M) + 2$ and $\operatorname{domdim} \operatorname{End}_A(N_{u,v}) = \operatorname{rd}(N_{u,v}) + 2$. Then the equality $\operatorname{rd}(N_{u,v}) = \operatorname{rd}(M)$ implies that $\operatorname{domdim} \operatorname{End}_A(N_{u,v}) = \operatorname{domdim} E < \infty$. Denote by d the number of isomorphism classes of indecomposable objects in $\operatorname{add}(A \oplus \mathrm{D}(A))$. Then $N_{u,v}$ has at least d and at most $d+2$ non-isomorphic indecomposable direct summands. Applying (YC) to $\operatorname{End}_A(N_{u,v})$ yields the inequality $\operatorname{domdim} \operatorname{End}_A(N_{u,v}) \leq \max\{\varphi(d), \varphi(d+1), \varphi(d+2)\}$. Consequently, $\operatorname{domdim} E$ has a uniform upper bound, which only depends on the function φ and d. Thus $\operatorname{rigdim}(A) < \infty$.

4. *Stable equivalences and invariance I*

In this and the next section, the invariance of rigidity dimension under stable or derived equivalences is discussed. As examples show, this fails in general. However, the first main result (Theorem 4·4) in this section implies that algebras without nodes have minimal rigidity dimension in their stable equivalence class. In particular, two stably equivalent algebras without nodes have the same rigidity dimension. The result shows that more generally only certain nodes matter, depending on the functor providing the equivalence. The second main result (Theorem 4·8) provides stronger information for stable equivalences between self-injective algebras. In particular, stable equivalences preserving the triangulated structure are shown to preserve rigidity dimension.

4·1. *Notation and definitions*

Let A be an algebra. By $A\text{-}\underline{\mathrm{mod}}$ we denote the stable category of A modulo projectives. It has the same objects as $A\text{-mod}$, but the morphism set between two A-modules X and Y is given by $\underline{\mathrm{Hom}}(X, Y) := \mathrm{Hom}_A(X, Y)/\mathscr{P}(X, Y)$ where $\mathscr{P}(X, Y)$ consists of homomorphisms factoring through projective A-modules. This category is usually called the *stable module category* of A. Dually, one can define the stable category $A\text{-}\overline{\mathrm{mod}}$ of A modulo injectives, which is the quotient category of $A\text{-mod}$ modulo injective modules. As usual, we denote by $\tau_A := \mathrm{D}\,\mathrm{Tr}_A$ the Auslander-Reiten translation of A. Then $\tau_A : A\text{-}\underline{\mathrm{mod}} \to A\text{-}\overline{\mathrm{mod}}$ is an additive equivalence, see [3, IV].

Definition 4·1. Two algebras A and B are *stably equivalent* if $A\text{-}\underline{\mathrm{mod}}$ and $B\text{-}\underline{\mathrm{mod}}$ are equivalent as additive categories.

The main complication when studying invariance of homological dimensions under stable equivalences, comes from a particular class of modules called nodes. These are rather exceptional, and do not occur much in nature. In particular, by [3, proposition X·1·8], an indecomposable self-injective algebra has a node if and only if it is a Nakayama algebra with radical square zero. In such a case, the only non-projective indecomposable modules are simple modules which are all nodes.

Definition 4·2. An indecomposable A-module S is called a *node* if it is neither projective nor injective, and there is an almost split sequence $0 \to S \to P \to T \to 0$ with P a projective A-module.

A node S must be simple, see, for example, [3, theorem V·3·3]. A node does not occur as a composition factor of $\operatorname{rad}(A)/\operatorname{soc}(A)$.

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Two stably equivalent algebras without nodes and without semi-simple summands share many homological invariants, such as stable Grothendieck groups, global dimensions and dominant dimensions (see [30]). Such stable equivalences do preserve projective dimensions. However, if one algebra has a node, simple examples of stable equivalences already show that projective and global dimensions are not preserved.

We will use various correspondences between objects in two equivalent stable categories and in related categories.

Let $F : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence. Then the following functor

$$F' := \tau_B \circ F \circ \tau_A^{-1} : A\text{-}\overline{\mathrm{mod}} \longrightarrow B\text{-}\overline{\mathrm{mod}}$$

is also an additive equivalence. Denote by $A\text{-mod}_{\mathscr{P}}$ (resp. $A\text{-mod}_{\mathscr{I}}$) the full subcategory of A-mod consisting of all modules without projective (resp. injective) direct summands. Then F (resp. F') induces a bijection $A\text{-mod}_{\mathscr{P}} \to B\text{-mod}_{\mathscr{P}}$ (resp. $A\text{-mod}_{\mathscr{I}} \to B\text{-mod}_{\mathscr{I}}$) on isomorphism classes of objects. Formally, we set $F(P) = 0$ and $F'(I) = 0$ when ${}_AP$ is projective and ${}_AI$ is injective.

Throughout this section, we also regard F and F' as correspondences (not functors) from A-mod to B-mod. If X is indecomposable, not projective, not injective and not a node, then $F(X) \cong F'(X)$ (see [2, lemma 3·4] or [3, corollary X·1·7]).

A particular set of nodes will turn out to play a crucial role (compare [18, section 3]):

Definition 4·3. A node S in A-mod is called an *F-exceptional node* if $F(S) \not\cong F'(S)$. Let $\mathfrak{n}_F(A)$ be the set of isomorphism classes of F-exceptional nodes of A.

In general, $\mathfrak{n}_F(A)$ is a proper subset of the set of isomorphism classes of nodes. It can be empty even though there are nodes (see Example 4·6).

Let $F^{-1} : B\text{-}\underline{\mathrm{mod}} \to A\text{-}\underline{\mathrm{mod}}$ be a quasi-inverse of F. Then $\mathfrak{n}_{F^{-1}}(B)$ denotes the set of isomorphism classes of F^{-1}-exceptional nodes of B.

4·2. *Invariance under general stable equivalences and under triangulated stable equivalences*

The main results are stated and a brief outline is given of the proofs, which will occupy the rest of this section. Moreover, examples are provided to show that the assumptions are necessary and algebras with nodes really behave differently.

THEOREM 4·4. *Let $F : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence between algebras A and B. Suppose A has no F-exceptional nodes. Then* $\mathrm{rigdim}(A) \le \mathrm{rigdim}(B)$. *If additionally B has no F^{-1}-exceptional nodes, then* $\mathrm{rigdim}(A) = \mathrm{rigdim}(B)$.

The above theorem implies the following result, which may be easier to apply, since it uses a property of the algebra, not of the given equivalence.

COROLLARY 4·5. *Let A and B be stably equivalent algebras.*
(a) *If A has no nodes, then* $\mathrm{rigdim}(A) \le \mathrm{rigdim}(B)$.
(b) *If neither A nor B has nodes, then* $\mathrm{rigdim}(A) = \mathrm{rigdim}(B)$.

Remark. In Corollary 4·5, the assertion (a) implies that the rigidity dimension of an algebra without nodes is minimal in its stable equivalence class; (b) follows from (a), which

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may also be obtained by combining classical results of Auslander and Reiten and of Martinez-Villa with the work of Guo, see [**2**, theorem 3·6], [**30**, proposition 2·2] and [**18**, lemma 3·5].

In general, in the presence of nodes, stable equivalences do not preserve rigidity dimensions, as the following example illustrates.

Example 4·6. Let A_1 be the path algebra of the quiver $1 \to 2$ over a field k, and let B be the k-algebra in Example 2·2. Then $A := A_1 \times A_1$ is stably equivalent to B, since both stable categories have just two indecomposable objects (up to isomorphism) - the two simple modules - and no nonzero morphisms between them. Moreover, A has no nodes, but B has two nodes, and $\mathrm{rigdim}(A) = 2$, $\mathrm{rigdim}(B) = 3$.

Since A_1 is hereditary, Proposition 2·10 (1) and Example 3·2 imply $\mathrm{rigdim}(A) = \mathrm{rigdim}(A_1) = 2$. To calculate $\mathrm{rigdim}(B)$, let S_1 and S_2 denote the simple B-modules corresponding to the vertices 1 and 2, respectively. Note that B is self-injective with radical square zero and has only four basic generators (up to isomorphism): B, $B \oplus S_1$, $B \oplus S_2$ and $B \oplus S_1 \oplus S_2$. Their endomorphism algebras, except for B itself, have finite global dimension (see Example 2·2 for an illustration). In other words, the last three modules are all resolving modules. Moreover, $\mathrm{domdim}\,\mathrm{End}_B(B \oplus S_1) = 3 = \mathrm{domdim}\,\mathrm{End}_B(B \oplus S_2)$, but $\mathrm{domdim}\,\mathrm{End}_B(B \oplus S_1 \oplus S_2) = 2$. Thus $\mathrm{rigdim}(B) = 3 > \mathrm{rigdim}(A)$. This illustrates that the inequality in Corollary 4·5 (a) (and thus also Theorem 4·4) cannot be improved. In this example, no matter what the equivalence F is, the set $\mathfrak{n}_F(A)$ is empty and $\mathfrak{n}_{F^{-1}}(B) = \{S_1, S_2\}$.

Let C be the k-algebra given by the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ with the relation $\beta\alpha$. Denote by T_i the simple C-module corresponding to the vertex i for $i = 1, 2$. Then there is a stable equivalence $G : B\text{-}\underline{\mathrm{mod}} \to C\text{-}\underline{\mathrm{mod}}$ which sends S_i to T_i. In this situation, $\mathfrak{n}_G(B) = \{S_1\} \subsetneqq \{S_1, S_2\}$ and $\mathfrak{n}_{G^{-1}}(C)$ is empty, though C has a unique node T_2. Since $\mathrm{gldim}(C) = 2$, we have $\mathrm{rigdim}(C) \leq 3$ by Theorem 3·1. Moreover, both the global dimension and the dominant dimension of $\mathrm{End}_C(C \oplus \mathrm{D}(C))$ are equal to 3. Thus $\mathrm{rigdim}(C) = 3$. So, two stably equivalent algebras, one of them having no exceptional nodes and the other having exceptional nodes, may have the same rigidity dimension.

The stable module category of a self-injective algebra carries an additional structure; it is a triangulated category (see [**3**, **19**]). A stable equivalence between self-injective algebras may be a triangulated equivalence or just an additive equivalence. However, the following example shows that stably equivalent self-injective algebras may have different rigidity dimensions.

Example 4·7. Let B be as in Example 4·6 and let $D := k[x]/(x^2) \times k[x]/(x^2)$. Then D is self-injective, with the stable category just the same as that for B according to Example 4·6. Moreover, D has nodes and $\mathrm{rigdim}(D) = 2 \neq \mathrm{rigdim}(B)$. Note that $B\text{-}\underline{\mathrm{mod}}$ and $D\text{-}\underline{\mathrm{mod}}$ are triangulated categories, the shift functor of $B\text{-}\underline{\mathrm{mod}}$ permutes simple modules and the shift functor of $D\text{-}\underline{\mathrm{mod}}$ is the identity. This means that $B\text{-}\underline{\mathrm{mod}}$ and $D\text{-}\underline{\mathrm{mod}}$ are equivalent as additive categories, but not as triangulated categories.

Self-injective algebras with nodes are quite well-known. The non-projective indecomposable A-modules are all simple, and each simple A-module S is a node satisfying

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$\tau_A(S) \cong \Omega_A(S)$ and $\Omega_A^m(S) \cong S$, where m equals the number of isomorphism classes of simple A-modules (see [3, Chapter X]).

For a self-injective algebra A with nodes, we denote by $\rho(A)$ the smallest positive integer d such that there is a node ${}_AS$ and an isomorphism $S \cong \Omega_A^d(S)$. Clearly, $\rho(A)$ equals the minimum of the numbers of isomorphism classes of simple modules over all blocks of A with nodes.

The second main result of this section provides sufficient conditions for stably equivalent self-injective algebras to have the same rigidity dimension.

THEOREM 4·8. *Let A and B be stably equivalent self-injective algebras.*

(1) *If A has no nodes, then* $\mathrm{rigdim}(A) = \mathrm{rigdim}(B)$.
(2) *If A and B have nodes, then both* $\mathrm{rigdim}(A)$ *and* $\mathrm{rigdim}(B)$ *are finite and* $|\,\mathrm{rigdim}(A) - \mathrm{rigdim}(B)| \leq |\rho(A) - \rho(B)|$.

If additionally $\rho(A) = \rho(B)$*, then* $\mathrm{rigdim}(A) = \mathrm{rigdim}(B)$.

A consequence of Theorem 4·8 is the following result.

COROLLARY 4·9. *Let A and B be stably equivalent self-injective algebras. Then* $\mathrm{rigdim}(A) = \mathrm{rigdim}(B)$ *in the following cases*:

(i) *A and B are symmetric*;
(ii) $A\text{-}\underline{\mathrm{mod}}$ *and* $B\text{-}\underline{\mathrm{mod}}$ *are equivalent as triangulated categories.*

In the proof of the two theorems, it will be important to control what happens to generator-cogenerators and their endomorphism algebras, under stable equivalences. In every step, nodes will cause problems. Therefore, in the first part of the proof, we will have to use various correspondences between objects in the stable categories. These correspondences are compatible with Guo's results [18], which thus can be used to compare global dimensions of endomorphism algebras. The core of the proof then is to compare also dominant dimensions, which needs another technically involved subsection. Finally, everything can be put together to derive Theorem 4·4. To prove Theorem 4·8 we will in addition use the description of self-injective algebras with nodes in [3].

4·3. *Further preparations for the proofs*

Throughout this subsection, A and B are stably equivalent algebras, possibly with nodes. We continue setting up correspondences between objects in the two stable categories and in related categories.

Let $F : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a fixed equivalence of additive categories throughout this section. To analyse the behavior of the functors F and F' on indecomposable A-modules, and the interaction with syzygy operators, consider the following subsets of indecomposable A-modules (see also [18]):

$$\triangle_A := \mathfrak{n}_F(A) \dot{\cup} (\mathscr{P}_A \setminus \mathscr{I}_A) \quad \text{and} \quad \bigtriangledown_A := \mathfrak{n}_F(A) \dot{\cup} (\mathscr{I}_A \setminus \mathscr{P}_A).$$

Let $\triangle_A^c$ be the class of indecomposable, non-injective A-modules which do not belong to $\triangle_A$. Then each module $Y \in A\text{-mod}_{\mathscr{I}}$ admits a unique decomposition (up to isomorphism)

$$Y \cong Y_\triangle \oplus Y'$$

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with $Y_\triangle \in \mathrm{add}(\triangle_A)$ and $Y' \in \mathrm{add}(\triangle_A^c)$. The module $Y_\triangle$ is called the $\triangle_A$-*component* of Y.

In the following, we denote by $\mathcal{GCN}_F(A)$ the class of basic A-modules X which are generator-cogenerators with $\mathfrak{n}_F(A) \subseteq \mathrm{add}(X)$. In particular, if A has no nodes, then $\mathcal{GCN}_F(A)$ is exactly the class of basic generator-cogenerators for A-mod. In the same situation, we similarly use the notation $\mathcal{GCN}_{F^{-1}}(B)$.

When working with generator-cogenerators, the following correspondences will be used:

$$\Phi: \quad A\text{-mod} \longrightarrow B\text{-mod} \qquad U \mapsto F(U) \oplus \bigoplus_{Q \in \mathscr{P}_B} Q,$$

$$\Psi: \quad B\text{-mod} \longrightarrow A\text{-mod} \qquad V \mapsto F^{-1}(V) \oplus \bigoplus_{P \in \mathscr{P}_A} P.$$

Here are some results from [18] for later use.

LEMMA 4·10 (compare [18, lemmas 3·1 and 3·2]).

(1) *There are one-to-one correspondences*

$$F: \nabla_A \longrightarrow \nabla_B, \quad F': \triangle_A \longrightarrow \triangle_B, \quad F': \triangle_A^c \longrightarrow \triangle_B^c.$$

(2) *The correspondences* Φ *and* Ψ *restrict to one-to-one correspondences between* $\mathcal{GCN}_F(A)$ *and* $\mathcal{GCN}_{F^{-1}}(B)$. *Moreover, if* $X \in \mathcal{GCN}_F(A)$, *then* $\Phi(X) \cong F'(X) \oplus \bigoplus_{I \in \mathscr{I}_B} I$.

Proof. (1) By [18, lemma 3·1], we only need to show that $X \in \mathfrak{n}_F(A)$ implies $F'(X) \in \triangle_B$.

Let $X \in \mathfrak{n}_F(A)$. Then X is is not injective. Consequently, $F'(X)$ is not injective. Recall that $F^{-1}(V) \cong (F^{-1})'(V)$ for any $V \in \triangle_B^c$. Suppose $F'(X) \in \triangle_B^c$. Then $F'(X)$ is not projective and $F^{-1}(F'(X)) \cong (F^{-1})'(F'(X)) \cong X$. It follows that $F'(X) \cong FF^{-1}(F'(X)) \cong F(X)$. This is a contradiction since $X \in \mathfrak{n}_F(A)$. Thus $F'(X) \in \triangle_B$.

Note that $F(U) \cong F'(U)$ for any $U \in \triangle_A^c$. Now, (2) follows from (1).

Generator-cogenerators in $\mathcal{GCN}_F(A)$ are better to control under stable equivalences. In particular, global dimensions of their endomorphism algebras are preserved under stable equivalences:

LEMMA 4·11 ([18, lemma 3·5]). *If* $X \in \mathcal{GCN}_F(A)$, *then* $\mathrm{gldim}\,\mathrm{End}_A(X) = \mathrm{gldim}\,\mathrm{End}_B(\Phi(X))$.

4·4. *Stable equivalences and dominant dimension of endomorphism algebras*

To compare rigidity dimension under stable equivalences, we need an analogue of Lemma 4·11 for dominant dimensions; this is the main part of the proof of Theorem 4·4.

PROPOSITION 4·12. *If* $X \in \mathcal{GNC}_F(A)$, *then* $\mathrm{domdim}\,\mathrm{End}_A(X) = \mathrm{domdim}\,\mathrm{End}_B(\Phi(X))$.

The proof of Proposition 4·12 will need three lemmas. The first one extends the first part of [18, lemma 3·3].

LEMMA 4·13. *Assume that* ${}_AZ$ *is indecomposable and non-projective. Let*

$$0 \longrightarrow X \oplus X' \longrightarrow Y \oplus P \xrightarrow{g} Z \longrightarrow 0$$

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be an exact sequence of A-modules without a split exact sequence as a direct summand, such that $X \in \mathrm{add}(\triangle_A^c)$, $X' \in \mathrm{add}(\triangle_A)$, $Y \in A\text{-mod}_{\mathscr{P}}$ *and* $P \in \mathrm{add}({}_AA)$. *Then there is an exact sequence of B-modules*

$$0 \longrightarrow F(X) \oplus F'(X') \longrightarrow F(Y) \oplus Q \xrightarrow{g'} F(Z) \longrightarrow 0$$

without a split direct summand such that $g' = F(g)$ *in* $B\text{-}\underline{\mathrm{mod}}$ *with* $Q \in \mathrm{add}(B)$.

Proof. By the proof of the first part of [**18**, lemma 3·3], the following two statements hold.

(1) There is an exact sequence of B-modules

$$0 \longrightarrow N \oplus N' \longrightarrow F(Y) \oplus Q \xrightarrow{g'} F(Z) \longrightarrow 0$$

without a split direct summand such that $g' = F(g)$ in $B\text{-}\underline{\mathrm{mod}}$, and that $N \in \mathrm{add}(\triangle_B^c)$, $N' \in \mathrm{add}(\triangle_B)$ and $Q \in \mathrm{add}(B)$.

(2) There is an isomorphism $F'(X \oplus X') \cong N \oplus N'$ in $(B\text{-}\underline{\mathrm{mod}})\text{-mod}$.

Note that $F'(X) \cong F(X)$ since $X \in \mathrm{add}(\triangle_A^c)$. By (2), we have $F(X) \oplus F'(X') \cong N \oplus N'$. Since $F(X) \in \mathrm{add}(\triangle_B^c)$ and $F'(X') \in \triangle_B$ by Lemma 4·10(1), there are isomorphisms $F(X) \cong N$ and $F'(X') \cong N'$. By (1), Lemma 4·13 follows.

The second lemma establishes a connection between different syzygy modules under stable equivalences.

LEMMA 4·14. *Let* $X \in A\text{-mod}$ *and n a positive integer. Then*

$$F\big(\Omega_A^n(X)\big) \oplus \bigoplus_{j=1}^{n} \Omega_B^{n-j}\big(F'(\Omega_A^j(X)_\triangle)\big) \cong \Omega_B^n(F(X)) \oplus \bigoplus_{j=1}^{n} \Omega_B^{n-j}\big(F(\Omega_A^j(X)_\triangle)\big),$$

where $\Omega_A^j(X)_\triangle$ *stands for the* $\triangle_A$*-component of the A-module* $\Omega_A^j(X)$.

Proof. We prove Lemma 4·14 by induction on n. If $n = 1$, then it suffices to check

$$F\big(\Omega_A(X)\big) \oplus F'(\Omega_A(X)_\triangle) \cong \Omega_B(F(X)) \oplus F(\Omega_A(X)_\triangle). \qquad (*)$$

If X is projective, then both sides are zero and there is nothing to prove. Now, assume that X is indecomposable and non-projective. Let $0 \to \Omega_A(X) \to P \to X \to 0$ be an exact sequence such that P is a projective cover of X. Then this sequence contains no split direct summand and $Y := \Omega_A(X) \in A\text{-mod}_{\mathscr{I}}$. So Y has a decomposition as $Y \cong Y_\triangle \oplus Z$, where $Y_\triangle \in \mathrm{add}(\triangle_A)$ and $Z \in \mathrm{add}(\triangle_A^c)$. Applying Lemma 4·13 to the sequence $0 \to Z \oplus Y_\triangle \to P \to X \to 0$ yields the exact sequence of B-modules

$$0 \longrightarrow F(Z) \oplus F'(Y_\triangle) \longrightarrow Q \longrightarrow F(X) \longrightarrow 0$$

without split direct summands, such that $Q \in \mathrm{add}(B)$. Thus Q is a projective cover of $F(X)$ and further, $\Omega_B(F(X)) \cong F(Z) \oplus F'(Y_\triangle)$. Hence

$$F(Y) \oplus F'(Y_\triangle) \cong F(Y_\triangle) \oplus F(Z) \oplus F'(Y_\triangle) \cong \Omega_B(F(X)) \oplus F(Y_\triangle).$$

This shows the isomorphism $(*)$.

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Let $n \geq 2$. Suppose that for any A-module U, there is an isomorphism of B-modules

$$F\big(\Omega_A^{n-1}(U)\big) \oplus \bigoplus_{j=1}^{n-1} \Omega_B^{n-1-j}\big(F'(\Omega_A^j(U)_\triangle)\big) \cong \Omega_B^{n-1}(F(U)) \oplus \bigoplus_{j=1}^{n-1} \Omega_B^{n-1-j}\big(F(\Omega_A^j(U)_\triangle)\big).$$

Choosing $U = Y$ gives rise to

$$F\big(\Omega_A^n(X)\big) \oplus \bigoplus_{j=1}^{n-1} \Omega_B^{n-1-j}\big(F'(\Omega_A^{j+1}(X)_\triangle)\big) \cong$$
$$\Omega_B^{n-1}(F(\Omega_A(X))) \oplus \bigoplus_{j=1}^{n-1} \Omega_B^{n-1-j}\big(F(\Omega_A^{j+1}(X)_\triangle)\big). \qquad (**)$$

Thus

$$F\big(\Omega_A^n(X)\big) \oplus \bigoplus_{j=1}^{n} \Omega_B^{n-j}\big(F'(\Omega_A^j(X)_\triangle)\big)$$
$$\cong F\big(\Omega_A^n(X)\big) \oplus \bigoplus_{j=1}^{n-1} \Omega_B^{n-1-j}\big(F'(\Omega_A^{j+1}(X)_\triangle)\big) \oplus \Omega_B^{n-1}(F'(\Omega_A(X)_\triangle))$$
$$\cong \Omega_B^{n-1}(F(\Omega_A(X))) \oplus \bigoplus_{j=1}^{n-1} \Omega_B^{n-1-j}\big(F(\Omega_A^{j+1}(X)_\triangle)\big) \oplus \Omega_B^{n-1}(F'(\Omega_A(X)_\triangle)) \qquad \big(\text{by } (**)\big)$$
$$\cong \Omega_B^{n-1}\big(F(\Omega_A(X)) \oplus F'(\Omega_A(X)_\triangle)\big) \oplus \bigoplus_{j=1}^{n-1} \Omega_B^{n-1-j}\big(F(\Omega_A^{j+1}(X)_\triangle)\big)$$
$$\cong \Omega_B^n(F(X)) \oplus \Omega_B^{n-1}\big(F(\Omega_A(X)_\triangle)\big) \oplus \bigoplus_{j=1}^{n-1} \Omega_B^{n-(j+1)}\big(F(\Omega_A^{j+1}(X)_\triangle)\big) \qquad \big(\text{by } (*)\big)$$
$$\cong \Omega_B^n(F(X)) \oplus \bigoplus_{j=1}^{n} \Omega_B^{n-j}\big(F(\Omega_A^j(X)_\triangle)\big).$$

This shows the isomorphism in Lemma 4·14.

The third lemma identifies extension groups of modules under stable equivalences.

LEMMA 4·15. *Let $X \in A$-mod, $Y \in \mathcal{GCN}_F(A)$ and n a positive integer. Then the following holds:*

(1) $\mathrm{Ext}_A^1(X, Y) \cong \mathrm{Ext}_B^1(\Phi(X), \Phi(Y))$*;*
(2) *If* $\mathrm{Ext}_B^i(N, \Phi(Y)) = 0$ *for each* $N \in \nabla_B$ *and* $1 \leq i \leq n$*, then*

$$\mathrm{Ext}_A^{n+1}(X, Y) \cong \mathrm{Ext}_B^{n+1}(\Phi(X), \Phi(Y)).$$

Proof. By Auslander–Reiten formula (see [3]), we have the following isomorphisms:

$$\mathrm{Ext}_A^1(X, Y) \cong \mathrm{D}\,\underline{\mathrm{Hom}}_A(\tau_A^{-1}Y, X) \cong \mathrm{D}\,\underline{\mathrm{Hom}}_B(F\tau_A^{-1}(Y), F(X))$$
$$\cong \mathrm{D}\,\underline{\mathrm{Hom}}_B(\tau_B^{-1}F'(Y), F(X)) \cong \mathrm{Ext}_B^1(F(X), F'(Y)).$$

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By Lemma 4·10(2), $\Phi(Y) = F'(Y) \oplus \bigoplus_{I \in \mathscr{I}_B} I$, due to $Y \in \mathcal{GCN}_F(A)$. Also, $\Phi(X) = F(X) \oplus \bigoplus_{Q \in \mathscr{P}_B} Q$ by definition. Consequently, $\mathrm{Ext}^1_A(X, Y) \cong \mathrm{Ext}^1_B(\Phi(X), \Phi(Y))$. Thus (1) holds

The proof of (1) also implies

$$\mathrm{Ext}^{n+1}_A(X, Y) \cong \mathrm{Ext}^1_A(\Omega^n_A(X), Y) \cong \mathrm{Ext}^1_B(F(\Omega^n_A(X)), F'(Y)) \cong \mathrm{Ext}^1_B(F(\Omega^n_A(X)), \Phi(Y)).$$

Define

$$L = \bigoplus_{j=1}^{n} \Omega^{n-j}_B\big(F'(\Omega^j_A(X)_\triangle)\big) \quad \text{and} \quad R = \bigoplus_{j=1}^{n} \Omega^{n-j}_B\big(F(\Omega^j_A(X)_\triangle)\big).$$

Then $F\big(\Omega^n_A(X)\big) \oplus L \cong \Omega^n_B(F(X)) \oplus R$ in B -mod by Lemma 4·14. Since

$$\Omega^j_A(X)_\triangle \in \mathrm{add}(\triangle_A) = \mathrm{add}\,\big(\mathfrak{n}_F(A)\dot{\cup}(\mathscr{P}_A \setminus \mathscr{I}_A)\big),$$

it follows from Lemma 4·10(1) that

$$F'(\Omega^j_A(X)_\triangle) \in \mathrm{add}(\triangle_B) \quad \text{and} \quad F(\Omega^j_A(X)_\triangle) \in \mathrm{add}\,\big(F(\mathfrak{n}_F(A))\big) \subseteq \mathrm{add}(\nabla_B),$$

where $\nabla_B = \mathfrak{n}_{F^{-1}}(B)\dot{\cup}(\mathscr{I}_B \setminus \mathscr{P}_B)$. So, if $\mathrm{Ext}^i_B(N, \Phi(Y)) = 0$ for each $N \in \nabla_B$ and $1 \le i \le n$, then $\mathrm{Ext}^1_B(L, \Phi(Y)) = 0 = \mathrm{Ext}^1_B(R, \Phi(Y))$ and thus

$$\begin{aligned}\mathrm{Ext}^1_B(F(\Omega^n_A(X)), \Phi(Y)) &\cong \mathrm{Ext}^1_B(\Omega^n_B(F(X)), \Phi(Y)) \\ &\cong \mathrm{Ext}^{n+1}_B(F(X), \Phi(Y)) \cong \mathrm{Ext}^{n+1}_B(\Phi(X), \Phi(Y)).\end{aligned}$$

This shows (2).

Proof of Proposition 4·12. If $\mathrm{domdim}\,\mathrm{End}_B(\Phi(X)) = n+2$ for some $n \ge 0$, then $\mathrm{Ext}^i_B(\Phi(X), \Phi(X)) = 0$ for $1 \le i \le n$ and $\mathrm{Ext}^{n+1}_B(\Phi(X), \Phi(X)) \ne 0$ due to Theorem 2·7. By Lemma 4·10(2), $\Phi(X) \in \mathcal{GCN}_{F^{-1}}(B)$. Thus, $\nabla_B \subseteq \mathrm{add}(\Phi(X))$. Therefore, Lemma 4·15 implies $\mathrm{Ext}^j_A(X, X) \cong \mathrm{Ext}^j_B(\Phi(X), \Phi(X))$ for $1 \le j \le n+1$. Then $\mathrm{domdim}\,\mathrm{End}_A(X) = n+2$, again by Theorem 2·7. Similarly, if $\mathrm{domdim}\,\mathrm{End}_B(\Phi(X)) = \infty$, then $\mathrm{domdim}\,\mathrm{End}_A(X) = \infty$.

Remark. When neither A nor B has nodes, both Lemma 4·14 and Lemma 4·15 can be simplified. In Lemma 4·14, the isomorphism becomes $F\big(\Omega^n_A(X)\big) \oplus Q \cong \Omega^n_B(F(X))$, where Q is a projective B-module without injective direct summands. This implies a stronger form of Lemma 4·15: if $X \in A$ -mod and $Y \in \mathcal{GCN}_F(A)$, then $\mathrm{Ext}^n_A(X, Y) \cong \mathrm{Ext}^n_B(\Phi(X), \Phi(Y))$ for any $n \ge 1$. These isomorphisms also can be obtained from [**2**, theorem 3·6] and [**30**, section 1, corollary; proposition 2·2].

4·5. *Completion of proofs and an application to representation dimension*

Proof of Theorem 4·4. Let $X \in \mathcal{GCN}_F(A)$. Then $\Phi(X) \in \mathcal{GCN}_{F^{-1}}(B)$. By Lemma 4·11, $\mathrm{End}_A(X)$ and $\mathrm{End}_B(\Phi(X))$ have the same global dimension. Moreover, by Proposition 4·12, they have the same dominant dimension. Since A has no F-exceptional nodes, $\mathfrak{n}_F(A)$ is the empty set and $\mathcal{GCN}_F(A)$ is the class of basic generator-cogenerators in A -mod. It follows that $\mathrm{rigdim}(A) \le \mathrm{rigdim}(B)$. If B has no F^{-1}-exceptional nodes, then $\mathrm{rigdim}(B) \le \mathrm{rigdim}(A)$, and thus $\mathrm{rigdim}(A) = \mathrm{rigdim}(B)$.

To prepare for the proof of Theorem 4·8, we describe the rigidity dimensions of Nakayama self-injective algebras with radical square zero.

LEMMA 4·16. *Let A be a non-simple Nakayama self-injective algebra with radical square zero. Then* $\mathrm{rigdim}(A) = \rho(A) + 1$.

Proof. In view of Proposition 2·10 (1), we may assume that A is indecomposable. Let e be the number of isomorphism classes of simple A-modules. It suffices to show $\mathrm{rigdim}(A) = e + 1$.

Since the radical square of A is zero, every indecomposable, non-projective A-module is simple. So, for any generator M in A-mod, if $\mathrm{gldim}\,\mathrm{End}_A(M) < \infty$, then M contains at least one simple module, say S, as a direct summand. By Theorem 2·7, $\mathrm{domdim}\,\mathrm{End}_A(M) \leq \mathrm{domdim}\,\mathrm{End}_A(A \oplus S)$. Note that $\Omega_A^e(S) \cong S$ and $\{\Omega_A^i(S) \mid 1 \leq i \leq e\}$ is the complete set of isomorphism classes of indecomposable, non-projective A-modules. The equalities $S^{\perp_{(e-1)}} = \mathrm{add}(A \oplus S) = {}^{\perp_{(e-1)}}S$ can be verified by writing down projective and injective resolutions of S. This implies that $A \oplus S$ is an e-cluster tilting module. By [**24**, theorem 0·2], $\mathrm{domdim}\,\mathrm{End}_A(A \oplus S) = e + 1 = \mathrm{gldim}\,\mathrm{End}_A(A \oplus S)$. Thus $\mathrm{rigdim}(A) = \mathrm{domdim}\,\mathrm{End}_A(A \oplus S) = e + 1$.

Proof of Theorem 4·8. (1) Let $F : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence between algebras A and B. Since A and B are self-injective, it follows from [**3**, proposition X·1·6] that F and F^{-1} restrict to one-to-one correspondences between the sets of isomorphism classes of nodes of A and of B. If A has no nodes, then so does B. Thus (1) holds by Corollary 4·5.

(2) Suppose that A and B have nodes. Let $A = A_1 \times A_2$ and $B = B_1 \times B_2$ be decompositions of algebras, such that A_2 and B_2 are the products of all blocks of A and B without nodes, respectively. In other words, all nodes of A and B only belong to A_1-mod and B_1-mod, respectively. By [**3**, proposition X·1·8], A_1 and B_1 are products of indecomposable Nakayama algebras with radical square zero. Then all indecomposable non-projective A_1-modules (and similarly, B_1-modules) are nodes. Consequently, F restricts to a stable equivalence between A_1 and B_1 and also a stable equivalence between A_2 and B_2. Note that $\mathrm{rigdim}(A_1) = \rho(A_1) + 1$ and $\mathrm{rigdim}(B_1) = \rho(B_1) + 1$ by Lemma 4·16. Combining this with Proposition 2·10 (1) yields

$$\mathrm{rigdim}(A) = \min\{\rho(A_1) + 1, \mathrm{rigdim}(A_2)\} \leq \rho(A_1) + 1 < \infty,$$
$$\mathrm{rigdim}(B) = \min\{\rho(B_1) + 1, \mathrm{rigdim}(B_2)\} \leq \rho(B_1) + 1 < \infty.$$

Clearly, $\rho(A_1) = \rho(A)$ and $\rho(B_1) = \rho(B)$ since A_2 and B_2 have no nodes. Moreover, $\mathrm{rigdim}(A_2) = \mathrm{rigdim}(B_2)$ by (1). Thus $|\,\mathrm{rigdim}(A) - \mathrm{rigdim}(B)| \leq |\rho(A) - \rho(B)|$.

Proof of Corollary 4·9. By Theorem 4·8 (2), it is enough to show $\rho(A) = \rho(B)$ in the two cases of Corollary 4·9. A sufficient condition to guarantee $\rho(A) = \rho(B)$ is that $F(\Omega_A(S)) \cong \Omega_B(F(S))$ in $B\text{-mod}_{\mathscr{P}}$ for any node S of A. In this situation, both S and $F(S)$ are Ω-periodic of the same period.

When A and B are symmetric algebras, it follows from [**3**, proposition X·1·12] that the correspondence F between objects in $A\text{-mod}_{\mathscr{P}}$ and $B\text{-mod}_{\mathscr{P}}$ commutes with the syzygy functor Ω. This shows the case (i). Recall that the shift functor of the triangulated category $A\text{-}\underline{\mathrm{mod}}$ is the cosyzygy functor Ω_A^{-1} [**19**, theorem 2·6]. So, in the case (ii), F commutes with Ω^{-1}, and thus also with Ω.

Finally, we explain how our results can be used to compare higher representation dimensions of stably equivalent algebras. Recall that classical representation dimension repdim_1 is preserved under arbitrary stable equivalences of algebras (see [**18**]).

COROLLARY 4·17. *Let $F : A\text{-}\underline{\mathrm{mod}} \rightarrow B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence between algebras A and B, and let n be a positive integer. Suppose that A has no F-exceptional nodes and $n+1 \leq \mathrm{rigdim}(A)$. Then $\mathrm{repdim}_n(B) \leq \mathrm{repdim}_n(A) < \infty$. If additionally B has no F^{-1}-exceptional nodes, then $\mathrm{repdim}_n(A) = \mathrm{repdim}_n(B)$.*

Proof. Since $n+1 \leq \mathrm{rigdim}(A)$, Proposition 2·11 implies $\mathrm{repdim}_n(A) < \infty$. By Theorem 4·4, $\mathrm{rigdim}(A) \leq \mathrm{rigdim}(B)$. This forces $n+1 \leq \mathrm{rigdim}(B)$, and thus $\mathrm{repdim}_n(B) < \infty$. Since A has no F-exceptional nodes, $\mathfrak{n}_F(A)$ is the empty set and $\mathcal{GCN}_F(A)$ is exactly the class of basic generator-cogenerators in A-mod. Now, our desired result follows from Proposition 4·12 and Lemma 4·11.

5. *Stable equivalences and invariance II*

In this section, invariance of rigidity dimension under stable equivalences of adjoint type is established. This implies invariance under stable equivalences of Morita type under very mild assumptions, and thus also invariance under certain derived equivalences.

5·1. *Definitions and main result*

Definition 5·1. Two algebras A and B are *stably equivalent of Morita type* if there exist an (A, B)-bimodule M and a (B, A)-bimodule N such that:

(i) M and N are both projective as one sided modules;
(ii) $M \otimes_B N \cong A \oplus P$ as A-A-bimodules for some projective A-A–bimodule P;
(iii) $N \otimes_A M \cong B \oplus Q$ as B-B-bimodules for some projective B-B-bimodule Q.

Further, if $(M \otimes_B -, N \otimes_A -)$ and $(N \otimes_A -, M \otimes_B -)$ are adjoint pairs of functors, then A and B are *stably equivalent of adjoint type*.

If A and B are stably equivalent of Morita type, then $(M \otimes_B -, N \otimes_A -)$ induces a stable equivalence between A and B.

THEOREM 5·2. (*a*) *Let A and B be stably equivalent of adjoint type. Then* $\mathrm{rigdim}(A) = \mathrm{rigdim}(B)$.

(*b*) *Let A and B be stably equivalent of Morita type. Then* $\mathrm{rigdim}(A) = \mathrm{rigdim}(B)$ *in each of the following three cases*:

(1) *A and B have no simple blocks*;
(2) *A and B are algebras over a perfect field k*;
(3) *A and B are self-injective algebras.*

Remark. It seems to be still unknown whether rigidity dimension is invariant under stable equivalence of Morita type in general, though Theorem 5·2 gives a positive answer in many cases. Our proof of Theorem 5·2 (b) depends on the relevant functors forming an adjoint pair, and thus part (a) being applicable. For an arbitrary stable equivalence of Morita type, it is not clear if tensor functors induced by two bimodules preserve injective modules.

5·2. *Proof of the main result*

In the proof of Theorem 5·2, the following result will be used, which is likely to be known to experts. We thank Yuming Liu for pointing out the present proof.

LEMMA 5·3. *Let $A = A_1 \times A_2$ and $B = B_1 \times B_2$, where A_1 and B_1 are separable, and A_2 and B_2 have no separable blocks. If A and B are stably equivalent of Morita type, then so are A_2 and B_2.*

Proof. It suffices to verify the following claim:

If Λ is a non-zero algebra and S is a separable algebra, then Λ and $\Lambda \times S$ are stably equivalent of Morita type.

For checking this, let $\Gamma = \Lambda \times S$, $M = \Lambda \times (\Lambda \otimes_k S)$ and $N = \Lambda \times (S \otimes_k \Lambda)$. Then M can be endowed with a Λ-Γ-bimodule structure: For $\lambda, \lambda_1, \lambda_2 \in \Lambda$ and $s, s' \in S$,

$$\lambda(\lambda_1, \lambda_2 \otimes s) = (\lambda\lambda_1,\ \lambda\lambda_2 \otimes s) \text{ and } (\lambda_1, \lambda_2 \otimes s)(\lambda', s') = (\lambda_1\lambda', \lambda_2 \otimes ss').$$

Similarly, N can be endowed with a Γ-Λ-bimodule structure. Moreover, $M \otimes_\Gamma N \cong \Lambda \oplus (\Lambda \otimes_k S \otimes_k \Lambda)$ as Λ-Λ-bimodules and $N \otimes_\Lambda M \cong \Lambda \oplus \Lambda \otimes_k S \oplus S \otimes_k \Lambda \oplus S \otimes_k \Lambda \otimes_k S$ as Γ-Γ bimodules. Then $\Lambda \otimes_k S \otimes_k \Lambda$ is a projective Λ-Λ-bimodule. As

$$\Gamma \otimes_k \Gamma^{\mathrm{op}} \cong (\Lambda \otimes_k \Lambda^{\mathrm{op}}) \times (\Lambda \otimes_k S^{\mathrm{op}}) \times (S \otimes_k \Lambda^{\mathrm{op}}) \times (S \otimes_k S^{\mathrm{op}}),$$

$S \otimes_k S$, $\Lambda \otimes_k S$ and $S \otimes_k \Lambda$ are projective Γ-Γ-bimodules. The S-S-bimodule S is projective, since S is a separable algebra. Furthermore, the multiplication map $S \otimes_k S \to S$ is a homomorphism of S-S-bimodules, and S is a direct summand of $S \otimes_k S$ as bimodules. Since $S \otimes_k \Lambda \otimes_k S \cong (S \otimes_k S)^n$ with $n := \dim({}_k\Lambda) \geq 1$, it follows that S is a direct summand of the projective bimodule $S \otimes_k \Lambda \otimes_k S$. Consequently, there is a projective Γ-Γ-bimodule Q such that $N \otimes_\Lambda M \cong (\Lambda \oplus S) \oplus Q \cong \Gamma \oplus Q$ as Γ-Γ-bimodules. So Λ and Γ are stably equivalent of Morita type.

Proof of Theorem 5·2. Let ${}_AM_B$ and ${}_BN_A$ be bimodules defining a stable equivalence of Morita type (not necessarily of adjoint type) between A and B. Let ${}_AX$ be a generator in A-mod. We claim that $N \otimes_A X$ is a generator in B-mod and $\operatorname{gldim}\operatorname{End}_A(X) = \operatorname{gldim}\operatorname{End}_B(N \otimes_A X)$. Indeed, since $N \otimes_A M \cong B \oplus Q$ as B-bimodules for some projective B-bimodule Q, it follows that ${}_BB \in \operatorname{add}(N \otimes_A M)$. Then ${}_AM$ being projective implies $B \in \operatorname{add}({}_BN)$. In other words, ${}_BN$ is a projective generator, and thus ${}_BN \otimes_A X$ is a generator. By [29, theorem 1·1], $\operatorname{End}_A(X)$ and $\operatorname{End}_B(N \otimes_A X)$ are stably equivalent of Morita type. Since global dimensions are preserved by stable equivalences of Morita type, we have $\operatorname{gldim}\operatorname{End}_A(X) = \operatorname{gldim}\operatorname{End}_B(N \otimes_A X)$.

(a) Now, suppose that the pair (M, N) defines a stable equivalence of adjoint type. In other words, the pairs $(M \otimes_B -, N \otimes_A -)$ and $(N \otimes_A -, M \otimes_B -)$ are adjoint pairs of functors. Further, assume X to be a cogenerator in A-mod. We claim that $N \otimes_A X$ is a cogenerator in B-mod and $\operatorname{domdim}\operatorname{End}_A(X) = \operatorname{domdim}\operatorname{End}_B(N \otimes_A X)$. Since ${}_AM \otimes_B -$ is exact with a right adjoint $N \otimes_A -$, injective A-modules are sent to injective B-modules by $N \otimes_A -$. Similarly, $M \otimes_B -$ sends injective B-modules to injective A-modules. Moreover, $N \otimes_A M \otimes_B \mathrm{D}(B) \cong \mathrm{D}(B) \oplus Q \otimes_B \mathrm{D}(B)$. In particular, $N \otimes_A \mathrm{D}(A_A)$ is a cogenerator in B-mod, and $N \otimes_A (A \oplus \mathrm{D}(A_A))$ is a generator-cogenerator in B-mod. This implies that $N \otimes_A X$ is a generator-cogenerator in B-mod.

Since A and B are stably equivalent of adjoint type, it follows from [29, theorem 1·3] that $\mathrm{End}_A(X)$ and $\mathrm{End}_B(N \otimes_A X)$ are stably equivalent of adjoint type, too. As such stable equivalences preserve dominant dimension by [29, lemma 4·2(2)], we get $\mathrm{domdim}\,\mathrm{End}_A(X) = \mathrm{domdim}\,\mathrm{End}_B(N \otimes_A X)$. By the definition of rigidity dimension, $\mathrm{rigdim}(A) \leq \mathrm{rigdim}(B)$. Swapping the roles of A and B yields $\mathrm{rigdim}(B) \leq \mathrm{rigdim}(A)$. Thus $\mathrm{rigdim}(A) = \mathrm{rigdim}(B)$.

(b) Under some mild assumptions, stable equivalences of Morita type are of adjoint type. Using this, the claims in (b) can be derived from (a) as follows:

(1) If neither A nor B has a simple block, then each stable equivalence of Morita type between A and B is of adjoint type due to [8, lemma 4·1] and [28, lemma 4·8(1)]. Thus $\mathrm{rigdim}(A) = \mathrm{rigdim}(B)$ by (a).

(2) Let $A = A_1 \times A_2$ and $B = B_1 \times B_2$ such that A_1 and B_1 are semi-simple and that A_2 and B_2 have no semi-simple blocks. Since k is perfect, the class of finite-dimensional semisimple k-algebras coincides with that of finite-dimensional separable k-algebras. So both A_1 and B_1 are separable. By Lemma 5·3, both A_2 and B_2 are stably equivalent of Morita type. It follows from (1) that $\mathrm{rigdim}(A_2) = \mathrm{rigdim}(B_2)$. Since $\mathrm{rigdim}(A_1) = \mathrm{rigdim}(B_1) = \infty$, Proposition 2·10 (1) implies $\mathrm{rigdim}(A) = \mathrm{rigdim}(B)$.

(3) By [28, lemma 4·8(3)], if two self-injective algebras without separable blocks are stably equivalent of Morita type, then there is a stable equivalence of adjoint type between them. Separable algebras are semi-simple, hence have infinite rigidity dimension. Now, (3) follows from Lemma 5·3 and (a) together with Proposition 2·10 (1).

5·3. *Applications to derived equivalences*

Any derived equivalence between self-injective algebras induces a stable equivalence of Morita type, see [32, corollary 2·2]. The following result is a consequence of Theorem 5·2(b)(3). Alternatively, it can be derived from Corollary 4·9 (ii), since any derived equivalence between self-injective algebras induces a triangle equivalence between their stable module categories.

COROLLARY 5·4. *Let A and B be self-injective algebras. Suppose A and B are derived equivalent. Then* $\mathrm{rigdim}(A) = \mathrm{rigdim}(B)$.

This can be extended to algebras that are not necessarily self-injective, by restricting the class of derived equivalences to certain derived equivalences which induce stable equivalences of Morita type. These are the almost ν-stable derived equivalences introduced in [21]. Every derived equivalence between two self-injective algebras induces an almost ν-stable derived equivalence (see [21, proposition 3·8]). In general, there are still many examples of almost ν-stable derived equivalences, for example, between algebras constructed in some way from self-injective algebras.

LEMMA 5·5 ([22, corollary 1·2]). *Let A be a self-injective algebra and X an A-module. Then* $\mathrm{End}_A(A \oplus X)$ *and* $\mathrm{End}_A(A \oplus \Omega_A(X))$ *are almost ν-stable derived equivalent.*

Almost ν-stable derived equivalences in many respects have better properties than general derived equivalences, for instance in the following way.

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LEMMA 5·6 ([21, theorem 1·1]). *If A and B are almost ν-stable derived equivalent, then they are stably equivalent of Morita type. In this case, A and B have the same global dimension and dominant dimension.*

The following result extends Corollary 5·4 to non-selfinjective algebras.

PROPOSITION 5·7. *If A and B are almost ν-stable derived equivalent, then* $\operatorname{rigdim}(A) = \operatorname{rigdim}(B)$.

Proof. Let $A = A_1 \times A_2$ and $B = B_1 \times B_2$ such that A_1 and B_1 are semi-simple and that A_2 and B_2 have no semi-simple blocks. Since derived equivalences preserve semi-simplicity of blocks, both A_i and B_i are derived equivalent for $i = 1, 2$. Moreover, A_2 and B_2 are almost ν-stable derived equivalent. By Lemma 5·6, they are stably equivalent of Morita type. It follows from Theorem 5·2(b)(1) that $\operatorname{rigdim}(A_2) = \operatorname{rigdim}(B_2)$. Since a semi-simple algebra has infinite rigidity dimension, $\operatorname{rigdim}(A) = \operatorname{rigdim}(B)$ by Proposition 2·10 (1).

Proposition 5·7 and Lemma 5·5 imply the following result.

COROLLARY 5·8. *Let A be a self-injective algebra and let X be an A-module. Then* $\operatorname{End}_A(A \oplus X)$ *and* $\operatorname{End}_A(A \oplus \Omega_A(X))$ *have the same rigidity dimension.*

Finally, we point that, in general, rigidity dimensions are not preserved under derived equivalences.

Example 5·9. Let H be the path algebra over k given by the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, and let C be the quotient algebra of H modulo the ideal generated by $\beta\alpha$. Then H and C are derived equivalent via the tilting H-module $T = S_1 \oplus S_3 \oplus He_1$ of projective dimension one, where S_1 and S_3 denote the simple H-modules corresponding to the vertices 1 and 3, respectively, while $e_1 = e_1^2 \in H$ corresponds to the vertex 1. Since H is hereditary, $\operatorname{rigdim}(H) = 2$ by Example 3·2. Note that $\operatorname{rigdim}(C) = 3$ by Example 4·6. Thus $\operatorname{rigdim}(H) \neq \operatorname{rigdim}(C)$.

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