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THE GROUP OF AUTOMORPHISMS OF THE ALGEBRA OF ONE-SIDED INVERSES OF A POLYNOMIAL ALGEBRA. II

V. V. BAVULA

Department of Pure Mathematics, University of Sheffield, Hicks Building, Sheffield S3 7RH, UK (v.bavula@sheffield.ac.uk)

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Abstract The algebra $\mathbb{S}_n$ of one-sided inverses of a polynomial algebra P_n in n variables is obtained from P_n by adding commuting *left* (but not two-sided) inverses of the canonical generators of the algebra P_n. The algebra $\mathbb{S}_n$ is isomorphic to the algebra

$$K\left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \int_1, \dots, \int_n \right\rangle$$

of scalar integro-differential operators provided that $\mathrm{char}(K) = 0$. Ignoring the non-Noetherian property, the algebra $\mathbb{S}_n$ belongs to a family of algebras like the nth Weyl algebra A_n and the polynomial algebra P_{2n}. Explicit generators are found for the group G_n of automorphisms of the algebra $\mathbb{S}_n$ and for the group $\mathbb{S}_n^*$ of units of $\mathbb{S}_n$ (both groups are huge). An analogue of the *Jacobian* homomorphism $\mathrm{Aut}_{K-\mathrm{alg}}(P_n) \to K^*$ is introduced for the group G_n (notice that the algebra $\mathbb{S}_n$ is non-commutative and neither left nor right Noetherian). The polynomial Jacobian homomorphism is unique. Its analogue is also unique for $n > 2$ but for $n = 1, 2$ there are exactly two of them. The proof is based on the following theorem that is proved in the paper:

$$G_n/[G_n, G_n] \simeq \begin{cases} K^* \times K^* & \text{if } n = 1, \\ \mathbb{Z}/2\mathbb{Z} \times K^* \times \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z}/2\mathbb{Z} \times K^* & \text{if } n > 2. \end{cases}$$

Keywords: group of automorphisms; group generators; inner automorphisms; Fredholm operators; index of an operator; semi-direct and exact products of groups

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1. Introduction

Throughout, ring means an associative ring with 1; module means a left module; $\mathbb{N} := \{0, 1, \dots\}$ is the set of natural numbers; K is a field and K^* is its group of units; $P_n := K[x_1, \dots, x_n]$ is a polynomial algebra over K;

$$\partial_1 := \frac{\partial}{\partial x_1}, \quad \dots, \quad \partial_n := \frac{\partial}{\partial x_n}$$

are the partial derivatives (K-linear derivations) of P_n; $\mathrm{End}_K(P_n)$ is the algebra of all K-linear maps from P_n to P_n and $\mathrm{Aut}_K(P_n)$ is its group of units (i.e. the group of all the invertible linear maps from P_n to P_n); the subalgebra $A_n := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ of $\mathrm{End}_K(P_n)$ is called the nth *Weyl* algebra provided that $\mathrm{char}(K) = 0$.

Definition 1.1 (Bavula [4]). The *algebra* $\mathbb{S}_n$ of *one-sided inverses* of P_n is an algebra generated over a field K of characteristic zero by $2n$ elements $x_1, \ldots, x_n, y_n, \ldots, y_n$ that satisfy the defining relations

$$y_1 x_1 = 1, \ldots, y_n x_n = 1, \qquad [x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 0 \quad \text{for all } i \neq j,$$

where $[a, b] := ab - ba$ is the algebra commutator of elements a and b.

By the very definition, the algebra $\mathbb{S}_n$ is obtained from the polynomial algebra P_n by adding commuting, left (but not two-sided) inverses of its canonical generators. The algebra $\mathbb{S}_1$ is a well-known primitive algebra [**10**, Example 2, p. 35]. Over the field $\mathbb{C}$ of complex numbers, the completion of the algebra $\mathbb{S}_1$ is the *Toeplitz algebra*, which is the C^*-algebra generated by a unilateral shift on the Hilbert space $l^2(\mathbb{N})$ (note that $y_1 = x_1^*$). The Toeplitz algebra is the universal C^*-algebra generated by a proper isometry.

Example 1.2 (Bavula [4]). Consider a vector space $V = \bigoplus_{i \in \mathbb{N}} K e_i$ and two shift operators on V, $X \colon e_i \mapsto e_{i+1}$ and $Y \colon e_i \mapsto e_{i-1}$ for all $i \geqslant 0$, where $e_{-1} := 0$. The subalgebra of $\mathrm{End}_K(V)$ generated by the operators X and Y is isomorphic to the algebra $\mathbb{S}_1(X \mapsto x, Y \mapsto y)$. By taking the nth tensor power $V^{\otimes n} = \bigoplus_{\alpha \in \mathbb{N}^n} K e_\alpha$ of V we see that the algebra $\mathbb{S}_n \simeq \mathbb{S}_1^{\otimes n}$ is isomorphic to the subalgebra of $\mathrm{End}_K(V^{\otimes n})$ generated by the $2n$ shifts $X_1, Y_1, \ldots, X_n, Y_n$ that act in different directions. In particular, when the field K has characteristic zero, $V = P_1 = \bigoplus_{i \in \mathbb{N}} K e_i$, $e_i := x^i/i!$, $Y = \mathrm{d}/\mathrm{d}x$ and $X = \int \colon P_1 \to P_1$, $x^i \mapsto x^{i+1}/i+1$ (the integration), the algebra $\mathbb{S}_1$ is isomorphic to the algebra $K\langle \mathrm{d}/\mathrm{d}x, \int \rangle$ of scalar integro-differential operators. By taking the nth tensor power $P_1^{\otimes n} = P_n = \bigoplus_{\alpha \in \mathbb{N}^n} K e_\alpha$, $e_\alpha = \prod_{i=1}^n x_i^{\alpha_i}/\alpha_i!$, and setting $Y_i = \partial/\partial x_i$ and $\int_i \colon P_n \to P_n$, $x^\alpha \mapsto x_i x^\alpha/(\alpha_i + 1)$, we see that the algebra $\mathbb{S}_n$ is isomorphic to the algebra $K\langle \partial/\partial x_1, \ldots, \partial/\partial x_n, \int_1, \ldots, \int_n \rangle$ of scalar integro-differential operators.

The algebra $\mathbb{S}_n$ is a non-commutative non-Noetherian algebra that is not a domain either. Moreover, it contains the algebra of infinite-dimensional matrices. The Gelfand–Kirillov dimension and the classical Krull dimension of the algebra $\mathbb{S}_n$ is $2n$, but the global dimension and the weak homological dimension of the algebra $\mathbb{S}_n$ is n [**4**].

1.1. Explicit generators for the group G_n

Let $G_n := \mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{S}_n)$ be the group of automorphisms of the algebra $\mathbb{S}_n$ and let $\mathbb{S}_n^*$ be the group of units of the algebra $\mathbb{S}_n$. The groups G_n and $\mathbb{S}_n^*$ are huge, e.g. both of them contain the group

$$\underbrace{\mathrm{GL}_\infty(K) \ltimes \cdots \ltimes \mathrm{GL}_\infty(K)}_{2^n - 1 \text{ times}},$$

which is a small part of them. A semi-direct product ${}^{\mathrm{semi}}\prod_{i=1}^m H_i = H_1 \ltimes H_2 \ltimes \cdots \ltimes H_m$ of several groups means that $H_1 \ltimes (H_2 \ltimes (\cdots \ltimes (H_{m-1} \ltimes H_m) \cdots))$.

Theorem 1.3 (Bavula [7]).

(1) $G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$.

(2) $G_1 \simeq \mathbb{T}^1 \ltimes \mathrm{GL}_\infty(K)$.

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In the theorem above, $S_n = \{s \in S_n \mid s(x_i) = x_{s(i)},\ s(y_i) = y_{s(i)}\}$ is the symmetric group, $\mathbb{T}^n := \{t_\lambda \mid t_\lambda(x_i) = \lambda_i x_i,\ t_\lambda(y_i) = \lambda_i^{-1} y_i,\ \lambda = (\lambda_i) \in K^{*n}\}$ is the n-dimensional algebraic torus, $\mathrm{Inn}(\mathbb{S}_n)$ is the group of inner automorphisms of the algebra $\mathbb{S}_n$ (which is huge) and $\mathrm{GL}_\infty(K)$ is the group of all the invertible infinite-dimensional matrices of the type $1 + M_\infty(K)$ where the algebra (without 1) of infinite-dimensional matrices $M_\infty(K) := \varinjlim M_d(K) = \bigcup_{d \geqslant 1} M_d(K)$ is the injective limit of matrix algebras. Theorem 1.3 is a difficult one (see the introduction of [**7**], where the structure and the main ideas of the proof are explained).

The results of the papers [**2**, **4**–**7**] and of the present paper show that (when ignoring non-Noetherian property) the algebra $\mathbb{S}_n$ belongs to a family of algebras like the nth Weyl algebra A_n, the polynomial algebra P_{2n} and the Jacobian algebra $\mathbb{A}_n$ (see [**2**, **6**]). Moreover, the algebras $\mathbb{S}_n$, $\mathbb{A}_n$ and A_n are *generalized Weyl algebras*. The structure of the group $G_1 \simeq \mathbb{T}^1 \ltimes \mathrm{GL}_\infty(K)$ is another confirmation of the 'similarity' of the algebras P_2, A_1 and $\mathbb{S}_1$. The groups of automorphisms of the polynomial algebra P_2 and the Weyl algebra A_1 were found by Jung [**11**], Van der Kulk [**15**]; and Dixmier [**8**], respectively. These two groups have almost identical structure in that they are 'infinite GL-groups' in the sense that they are generated by the algebraic torus $\mathbb{T}^1$ and by the obvious automorphisms: $x \mapsto x + \lambda y^i$, $y \mapsto y$; $x \mapsto x$, $y \mapsto y + \lambda x^i$, where $i \in \mathbb{N}$ and $\lambda \in K$, which are sort of 'elementary infinite-dimensional matrices' (i.e. 'infinite-dimensional transvections'). The same picture holds as for the group G_1. In prime characteristic, the group of automorphism of the Weyl algebra A_1 was found by Makar-Limanov [**12**] (see also [**3**] for a different approach and for further developments).

A next step in explicitly finding the group G_n and its generators is done in [**5**], where explicit generators are found for the group G_2 and the following theorem is proved.

Theorem 1.4 (Bavula [5, Theorem 2.12]). *$G_2 \simeq S_2 \ltimes \mathbb{T}^2 \ltimes \mathbb{Z} \ltimes ((K^* \ltimes E_\infty(\mathbb{S}_1)) \boxtimes_{\mathrm{GL}_\infty(K)} (K^* \ltimes E_\infty(\mathbb{S}_1)))$, where $E_\infty(\mathbb{S}_1)$ is the subgroup of $\mathrm{GL}_\infty(\mathbb{S}_1)$ generated by the elementary matrices.*

The aim of the present paper is to find explicitly the group G_n (see Theorem 4.2) and its generators for $n \geqslant 2$. We show that these are given explicitly by the following theorem (Theorem 4.6).

Theorem. *Let $J_s := \{1, \dots, s\}$, where $s = 1, \dots, n$. The group $G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$ is generated by the transpositions (ij) where $i < j$; the elements $t_{(\lambda,1,\dots,1)} \colon x_1 \mapsto \lambda x_1$, $y_1 \mapsto \lambda^{-1} y_1$, $x_k \mapsto x_k$, $y_k \mapsto y_k$, $k = 2, \dots, n$; and the inner automorphisms $\omega_u \colon a \mapsto uau^{-1}$, where u belongs to the following sets:*

(1) $\theta_{s,1}(J_s) := (1 + (y_s - 1)\prod_{i=1}^{s-1}(1 - x_i y_i)) \cdot (1 + (x_1 - 1)\prod_{j=2}^{s}(1 - x_j y_j))$, $s = 2, \dots, n$;

(2) $1 + x_n^t E_{0\alpha}(J_s)$, $1 + x_n^t E_{\alpha 0}(J_s)$, $1 + y_n^t E_{0\alpha}(J_s)$ *and* $1 + y_n^t E_{\alpha 0}(J_s)$, *where* $t \in \mathbb{N} \setminus \{0\}$, $s = 1, \dots, n-1$ *and* $\alpha \in \mathbb{N}^s \setminus \{0\}$;

(3) $1 + (\lambda - 1)E_{00}(J_s)$, $1 + E_{0\alpha}(J_s)$ *and* $1 + E_{\alpha 0}(J_s)$, *where* $\lambda \in K^*$, $s = 1, \dots, n$ *and* $\alpha \in \mathbb{N}^s \setminus \{0\}$,

where $E_{00}(J_s) := \prod_{i=1}^{s}(1 - x_i y_i)$, $E_{0\alpha}(J_s) := \prod_{i=1}^{s}(y_i^{\alpha_i} - x_i y_i^{\alpha_i + 1})$ *and* $E_{\alpha 0}(J_s) := \prod_{i=1}^{s}(x_i^{\alpha_i} - x_i^{\alpha_i + 1} y_i)$.

1.2. The structure and main ideas of finding the generators for the groups G_n and $\mathbb{S}_n^*$

A first step is the following theorem.

Theorem 1.5 (Bavula [5, 6]).

(1) $\mathbb{S}_n^* = K^* \times (1+\mathfrak{a}_n)^*$, *where the ideal* $\mathfrak{a}_n$ *of the algebra* $\mathbb{S}_n$ *is the sum of all the height one prime ideals of the algebra* $\mathbb{S}_n$.

(2) *The centre of the group* $\mathbb{S}_n^*$ *is* K^* *and the centre of the group* $(1+\mathfrak{a}_n)^*$ *is* $\{1\}$.

(3) *The map* $(1+\mathfrak{a}_n)^* \to \mathrm{Inn}(\mathbb{S}_n)$, $u \mapsto \omega_u$ *is a group isomorphism.*

Theorems 1.3 and 1.5 reduce the problem of finding the group G_n to the problem of finding the group of units $(1+\mathfrak{a}_n)^*$. To save on notation, we often identify the groups $(1+\mathfrak{a}_n)^*$ and $\mathrm{Inn}(\mathbb{S}_n)$ via $u \mapsto \omega_u$.

The polynomial algebra P_n is a faithful $\mathbb{S}_n$-module (see the example above), hence $\mathbb{S}_n \subset \mathrm{End}_K(P_n)$. The ideals of the algebra $\mathbb{S}_n$ commute ($IJ = JI$) [4]. There are precisely n height 1 prime ideals of the algebra $\mathbb{S}_n$, say $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. They are found explicitly in [4] and they form a single G_n-orbit. In particular, the ideals $\mathfrak{a}_{n,s} := \sum_{i_1<\cdots<i_s} \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_s}$, $s = 1, \dots, n$, are G_n-invariant ideals of the algebra $\mathbb{S}_n$. The group $(1+\mathfrak{a}_n)^*$ has the strictly descending chain of G_n-invariant (hence, normal) subgroups

$$(1+\mathfrak{a}_n)^* = (1+\mathfrak{a}_{n,1})^* \supset \cdots \supset (1+\mathfrak{a}_{n,s})^* \supset \cdots \supset (1+\mathfrak{a}_{n,n-1})^* \supset (1+\mathfrak{a}_{n,n})^*.$$

Briefly, to prove results for the group $(1+\mathfrak{a}_n)^*$ we first prove similar results for the subgroups $(1+\mathfrak{a}_{n,s})^*$, $s = 1, \dots, n-1$, using a double induction on (n, s) starting with $(n, n-1)$ in the second part of the induction (the induction on s is a downward induction, the group $(1+\mathfrak{a}_{n,n})^*$ is isomorphic to $\mathrm{GL}_\infty(K)$ and contains no essential information about the overgroups, that is why we have to start with $(n, n-1)$). The initial case $(n, n-1)$ is the most difficult one. We devote the entire of § 3 to treating it.

The difficulty in finding the group $(1+\mathfrak{a}_n)^*$ stems from two facts: (i) $\mathbb{S}_n^* \subsetneqq \mathbb{S}_n \cap \mathrm{Aut}_K(P_n)$, i.e. there are non-units of the algebra $\mathbb{S}_n$ that are invertible linear maps in P_n; and (ii) some units of the algebra $\mathbb{S}_n$ are products of *non-units*. To tackle the second problem the so-called *current groups* $\Theta_{n,s}$, $s = 1, \dots, n-1$, are introduced. These are finitely generated subgroups of $(1+\mathfrak{a}_n)^*$ generated by explicit generators and each of the generators is a product of two non-units of $\mathbb{S}_n^*$ (they are even non-units of $\mathrm{End}_K(P_n)$). The current groups turn out to be the most important subgroups of $(1+\mathfrak{a}_n)^*$ in that they control the most difficult parts of the structure of the group $(1+\mathfrak{a}_n)^*$.

In dealing with the case $(n, n-1)$, we use the Fredholm linear maps/operators and their indices. This technique is not available in other cases, i.e. when $(n, s) \neq (n, n-1)$, but the point is that other cases can be reduced to the initial one but over a larger coefficient *ring* (not a field). The indices of operators are used to construct several group homomorphisms. The most difficult part of § 3 is to prove that the homomorphisms are well-defined maps (as their constructions are based on highly non-unique decompositions). As a result, the group $(1+\mathfrak{a}_n)^*$ is found explicitly to be given by (see Theorem 4.2)

$$(1+\mathfrak{a}_n)^* = \Theta'_{n,1}\mathbb{E}_{n,1}\Theta'_{n,2}\mathbb{E}_{n,2}\cdots\Theta'_{n,n-1}\mathbb{E}_{n,n-1},$$

where the sets $\Theta'_{n,s} \subseteq \Theta_{n,s}$ and the groups $\mathbb{E}_{n,s}$ are given explicitly (see (2.22) and (4.3)). As a consequence, we have explicit generators for the group $(1+\mathfrak{a}_n)^*$ (see Theorem 4.5).

Theorem. *The group $(1+\mathfrak{a}_n)^*$ is generated by the following elements:*

(1)

$$\theta_{\max(J),j}(J) := \Bigg(1+(y_{\max(J)}-1)\prod_{i\in J\setminus\max(J)}(1-x_iy_i)\Bigg)\cdot\Bigg(1+(x_j-1)\prod_{k\in J\setminus j}(1-x_ky_k)\Bigg),$$

where J runs through all the subsets of $\{1,\dots,n\}$ that contain at least two elements, $j\in J\setminus\max(J)$ and $\max(J)$ is the maximal number in J;

(2) $1+x_i^tE_{0\alpha}(I)$, $1+x_i^tE_{\alpha 0}(I)$, $1+y_i^tE_{0\alpha}(I)$ *and* $1+y_i^tE_{\alpha 0}(I)$, *where I runs through all the subsets of $\{1,\dots,n\}$ such that $|I|=1,\dots,n-1$, $t\in\mathbb{N}\setminus\{0\}$, $i\notin I$, $\alpha\in\mathbb{N}^I\setminus\{0\}$;*

(3) $1+(\lambda-1)E_{00}(I)$, $1+E_{0\alpha}(I)$ *and* $1+E_{\alpha 0}(I)$, *where $\lambda\in K^*$, $I\neq\emptyset$ and $\alpha\in\mathbb{N}^I\setminus\{0\}$.*

It is then easy to obtain explicit generators for the group G_n (see Theorem 4.6).

1.3. An analogue of the polynomial Jacobian homomorphism for the group G_n

For the polynomial algebra P_n there is an important group homomorphism

$$\mathcal{J}_n\colon \mathcal{P}_n := \mathrm{Aut}_{K-\mathrm{alg}}(P_n)\to K^*,\quad \sigma\mapsto \det\left(\frac{\partial\sigma(x_i)}{\partial x_j}\right), \tag{1.1}$$

the so-called *Jacobian* homomorphism. Note that the Jacobian homomorphism is a determinant. Each automorphism $\sigma\in\mathcal{P}_n$ is a unique product $\xi\sigma_{\mathrm{aff}}$ of an affine automorphism $\sigma_{\mathrm{aff}}\in\mathrm{Aff}_n$ and an element ξ of the *Jacobian* group Σ_n (see § 5 for details), and the Jacobian of σ is uniquely determined by its affine part, i.e. $\mathcal{J}(\sigma)=\mathcal{J}(\sigma_{\mathrm{aff}})$. This property *uniquely* characterizes the Jacobian homomorphism. There are two different ways of defining the Jacobian homomorphism: by the explicit formula (1.1) or as a group homomorphism from $\mathcal{P}_n/[\mathcal{P}_n,\mathcal{P}_n]$ to K^* that is defined naturally (i.e. as the determinant) on the affine subgroup Aff_n of $\mathcal{P}_n$.

The group $G_n = S_n\ltimes\mathbb{T}^n\ltimes\mathrm{Inn}(\mathbb{S}_n)$ has a similar structure to the group $\mathcal{P}_n$, where $\mathrm{aff}_n := S_n\ltimes\mathbb{T}^n$ is an affine part of G_n and the group $\mathrm{Inn}(\mathbb{S}_n)$ of inner automorphisms plays the role of the Jacobian group. In § 5, we introduce an analogue $\mathbb{J}_n\colon G_n\to K^*$ of the Jacobian homomorphism using the second definition of the polynomial Jacobian map as a guiding principle: the map $\mathbb{J}_n$ is a homomorphism $\mathbb{J}_n\colon G_n/[G_n,G_n]\to K^*$ such that on the affine group aff_n it is defined in exactly the same way as in the polynomial case. For $n > 2$, the homomorphism $\mathbb{J}_n$ is *unique* (see Theorem 5.8) since (Corollary 5.5 (3))

$$G_n/[G_n, G_n] \simeq \mathrm{aff}_n/[\mathrm{aff}_n, \mathrm{aff}_n] \simeq \mathbb{Z}/2\mathbb{Z} \times K^*.$$

But for $n = 1, 2$, the homomorphism $\mathbb{J}_n$ is not unique. There are exactly two of them since (see Corollary 5.5 (3))

$$G_n/[G_n, G_n] \simeq \mathrm{aff}_n/[\mathrm{aff}_n, \mathrm{aff}_n] \times \begin{cases} K^* & \text{if } n = 1, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2. \end{cases}$$

More informally, for $n = 1$ the appearance of the second homomorphism, the so-called exotic homomorphism $\mathbb{J}_1^{\mathrm{ex}}$, has a connection with existence of the determinant (homomorphism) on the group $\mathrm{GL}_\infty(K)$, but for $n = 2$ the exotic $\mathbb{J}_2^{\mathrm{ex}}$ is explained by the fact that the current group Θ_2 does not belong to the commutant $[G_2, G_2]$. For $n = 1$, $\mathbb{J}_1$ and $\mathbb{J}_1^{\mathrm{ex}}$ are algebraically independent characters of the group G_1, but for $n = 2$, $\mathbb{J}_2^2 = (\mathbb{J}_2^{\mathrm{ex}})^2$ (see Theorem 5.8).

The proofs are based on finding explicitly the commutant $[G_n, G_n]$ of the group G_n (see Theorem 5.4 (1)) and proving that (see Theorem 5.4 (2))

$$G_n/[G_n, G_n] \simeq \begin{cases} K^* \times K^* & \text{if } n = 1, \\ \mathbb{Z}/2\mathbb{Z} \times K^* \times \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z}/2\mathbb{Z} \times K^* & \text{if } n > 2. \end{cases}$$

The most surprising thing is that despite the fact that the algebra $\mathbb{S}_n$ is non-commutative, non-Noetherian, of Gelfand–Kirillov dimension $2n$ (not n) and not a domain, the unique homomorphism $\mathbb{J}_n$ 'coincides' with the polynomial Jacobian homomorphism $\mathcal{J}_n$ for the polynomial algebra P_n (not P_{2n}): for $\sigma \in G_n$,

$$\mathbb{J}_n(\sigma) = \det\left(\frac{\partial \bar{\sigma}(x_i)}{\partial x_j}\right), \tag{1.2}$$

i.e. the homomorphism $\mathbb{J}_n$ is the composition of the homomorphism

$$G_n \to \mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{S}_n/\mathfrak{a}_n \simeq L_n), \quad \sigma \mapsto \bar{\sigma}\colon a + \mathfrak{a}_n \mapsto \sigma(a) + \mathfrak{a}_n,$$

where $L_n := K[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ is the Laurent polynomial algebra (see (2.7)), and the Jacobian map

$$\mathcal{J}_n\colon \mathrm{Aut}_{K-\mathrm{alg}}(L_n) \to L_n^*, \quad \tau \mapsto \det\left(\frac{\partial \tau(x_i)}{\partial x_j}\right).$$

Proof of (1.2). Since $G_n = \mathrm{aff}_n \ltimes \mathrm{Inn}(\mathbb{S}_n)$ and the factor algebra $\mathbb{S}_n/\mathfrak{a}_n \simeq L_n$ is *commutative*, the homomorphism (1.2) acts trivially on $\mathrm{Inn}(\mathbb{S}_n)$ (i.e. $\mathbb{J}_n(\mathrm{Inn}(\mathbb{S}_n)) = 1$), but on aff_n the map (1.2) acts exactly as in the polynomial case: for each element $s \cdot t_\lambda \in \mathrm{aff}_n$, where $s \in S_n$ and $t_\lambda \in \mathbb{T}^n$, $\mathbb{J}_n(s \cdot t_\lambda) = \mathrm{sgn}(s) \cdot \prod_{i=1}^n \lambda_i$, where $\mathrm{sgn}(s) \in \{\pm 1\}$ is the sign/parity of the permutation s. $\square$

So, for each element $\sigma = s \cdot t_\lambda \cdot \omega_u \in G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$, where $s \in S_n$, $t_\lambda \in \mathbb{T}^n$ and $\omega_u \in \mathrm{Inn}(\mathbb{S}_n)$,

$$\mathbb{J}_n(\sigma) = \mathrm{sgn}(s) \cdot \prod_{i=1}^{n} \lambda_i. \tag{1.3}$$

One may have noticed that $\mathbb{J}_n(\sigma)$ depends only on $\sigma(x_1), \dots, \sigma(x_n)$, and the set $\{x_1, \dots, x_n\}$ is not a generating set for the algebra $\mathbb{S}_n$. It is a trivial observation that an algebra endomorphism is uniquely determined by its action on a set of algebra generators but for the algebra $\mathbb{S}_n$, an algebra endomorphism is uniquely determined by its action on either of the sets $\{x_1, \dots, x_n\}$ or $\{y_1, \dots, y_n\}$ (which are not algebra generating sets).

Theorem 1.6 (rigidity of the group G_n [7, Theorem 3.7]). *Let $\sigma, \tau \in G_n$. Then the following statements are equivalent.*

(1) $\sigma = \tau$.

(2) $\sigma(x_1) = \tau(x_1), \dots, \sigma(x_n) = \tau(x_n)$.

(3) $\sigma(y_1) = \tau(y_1), \dots, \sigma(y_n) = \tau(y_n)$.

2. The group $(1 + \mathfrak{a}_n)^*$ and its subgroups

In this section, we collect some results without proofs on the algebras $\mathbb{S}_n$ that will be used in this paper; their proofs can be found in [**4**]. Several important subgroups of the group $(1 + \mathfrak{a}_n)^*$ are introduced. The most interesting of these are the current subgroups $\Theta_{n,s}$, $s = 1, \dots, n-1$. They encapsulate the most difficult parts of the groups $\mathbb{S}_n^*$ and G_n. This section sets the scene for proving the main results of the paper.

2.1. The algebra of one-sided inverses of a polynomial algebra

Clearly, $\mathbb{S}_n = \mathbb{S}_1(1) \otimes \cdots \otimes \mathbb{S}_1(n) \simeq \mathbb{S}_1^{\otimes n}$, where $\mathbb{S}_1(i) := K\langle x_i, y_i \mid y_i x_i = 1\rangle \simeq \mathbb{S}_1$ and

$$\mathbb{S}_n = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} K x^\alpha y^\beta,$$

where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $y^\beta := y_1^{\beta_1} \cdots y_n^{\beta_n}$ and $\beta = (\beta_1, \dots, \beta_n)$. In particular, the algebra $\mathbb{S}_n$ contains two polynomial subalgebras P_n and $Q_n := K[y_1, \dots, y_n]$ and is equal, as a vector space, to their tensor product $P_n \otimes Q_n$. Note also that the Weyl algebra A_n is a tensor product (as a vector space) $P_n \otimes K[\partial_1, \dots, \partial_n]$ of two polynomial subalgebras.

When $n = 1$, we usually drop the subscript '1' if this does not lead to confusion. So, $\mathbb{S}_1 = K\langle x, y \mid yx = 1\rangle = \bigoplus_{i,j \geqslant 0} K x^i y^j$. For each natural number $d \geqslant 1$, let $M_d(K) := \bigoplus_{i,j=0}^{d-1} K E_{ij}$ be the algebra of d-dimensional matrices, where $\{E_{ij}\}$ are the matrix units, and let

$$M_\infty(K) := \varinjlim M_d(K) = \bigoplus_{i,j \in \mathbb{N}} K E_{ij}$$

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be the algebra (without 1) of infinite-dimensional matrices. The algebra $\mathbb{S}_1$ contains the ideal $F := \bigoplus_{i,j\in\mathbb{N}} KE_{ij}$, where

$$E_{ij} := x^i y^j - x^{i+1}y^{j+1}, \quad i,j \geqslant 0. \tag{2.1}$$

For all natural numbers i, j, k and l, $E_{ij}E_{kl} = \delta_{jk}E_{il}$, where δ_{jk} is the Kronecker delta function. The ideal F is an algebra (without 1) isomorphic to the algebra $M_\infty(K)$ via $E_{ij} \mapsto E_{ij}$. For all $i, j \geqslant 0$

$$xE_{ij} = E_{i+1,j}, \quad yE_{ij} = E_{i-1,j} \quad (E_{-1,j} := 0), \tag{2.2}$$

$$E_{ij}x = E_{i,j-1}, \quad E_{ij}y = E_{i,j+1} \quad (E_{i,-1} := 0). \tag{2.3}$$

The algebra

$$\mathbb{S}_1 = K \oplus xK[x] \oplus yK[y] \oplus F \tag{2.4}$$

is the direct sum of vector spaces. It follows that

$$\mathbb{S}_1/F \simeq K[x, x^{-1}] =: L_1, \quad x \mapsto x, \quad y \mapsto x^{-1}, \tag{2.5}$$

since $yx = 1$, $xy = 1 - E_{00}$ and $E_{00} \in F$.

The algebra $\mathbb{S}_n = \bigotimes_{i=1}^n \mathbb{S}_1(i)$ contains the ideal

$$F_n := F^{\otimes n} = \bigoplus_{\alpha,\beta\in\mathbb{N}^n} KE_{\alpha\beta},$$

where

$$E_{\alpha\beta} := \prod_{i=1}^n E_{\alpha_i\beta_i}(i), \quad E_{\alpha_i\beta_i}(i) := x_i^{\alpha_i} y_i^{\beta_i} - x_i^{\alpha_i+1} y_i^{\beta_i+1}.$$

Note that $E_{\alpha\beta}E_{\gamma\rho} = \delta_{\beta\gamma}E_{\alpha\rho}$ for all elements $\alpha, \beta, \gamma, \rho \in \mathbb{N}^n$, where $\delta_{\beta\gamma}$ is the Kronecker delta function; $F_n = \bigotimes_{i=1}^n F(i)$ and $F(i) := \bigoplus_{s,t\in\mathbb{N}} KE_{st}(i)$.

2.2. The involution η on $\mathbb{S}_n$

The algebra $\mathbb{S}_n$ admits the *involution*

$$\eta\colon \mathbb{S}_n \to \mathbb{S}_n, \quad x_i \mapsto y_i, \quad y_i \mapsto x_i, \quad i = 1, \dots, n.$$

It is a K-algebra anti-isomorphism ($\eta(ab) = \eta(b)\eta(a)$ for all $a, b \in \mathbb{S}_n$) such that $\eta^2 = \mathrm{id}_{\mathbb{S}_n}$, the identity map on $\mathbb{S}_n$. So, the algebra $\mathbb{S}_n$ is *self-dual* (i.e. it is isomorphic to its opposite algebra, $\eta\colon \mathbb{S}_n \simeq \mathbb{S}_n^{op}$). The involution η acts on the 'matrix' ring F_n as the transposition

$$\eta(E_{\alpha\beta}) = E_{\beta\alpha}. \tag{2.6}$$

The canonical generators x_i, y_j ($1 \leqslant i, j \leqslant n$) determine the ascending filtration $\{\mathbb{S}_{n,\leqslant i}\}_{i\in\mathbb{N}}$ on the algebra $\mathbb{S}_n$ in the obvious way (i.e. by the total degree of the generators): $\mathbb{S}_{n,\leqslant i} := \bigoplus_{|\alpha|+|\beta|\leqslant i} Kx^\alpha y^\beta$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ ($\mathbb{S}_{n,\leqslant i}\mathbb{S}_{n,\leqslant j} \subseteq \mathbb{S}_{n,\leqslant i+j}$ for all $i, j \geqslant 0$). Then $\dim(\mathbb{S}_{n,\leqslant i}) = \binom{i+2n}{2n}$ for $i \geqslant 0$ and so the Gelfand–Kirillov dimension $\mathrm{GK}(\mathbb{S}_n)$ of the algebra $\mathbb{S}_n$ is equal to $2n$. It is not difficult to show that the algebra $\mathbb{S}_n$ is neither left nor right Noetherian. Moreover, it contains infinite direct sums of left and right ideals (see [**4**]). The proof of the following statements can be found in [**4**].

- The algebra $\mathbb{S}_n$ is central, prime and catenary. Every non-zero ideal of $\mathbb{S}_n$ is an essential left and right submodule of $\mathbb{S}_n$.
- The ideals of $\mathbb{S}_n$ commute ($IJ = JI$); and the set of ideals of $\mathbb{S}_n$ satisfy the ascending chain condition (a.c.c.).
- The classical Krull dimension $\text{cl.Kdim}(\mathbb{S}_n)$ of $\mathbb{S}_n$ is $2n$.
- Let I be an ideal of $\mathbb{S}_n$. Then the factor algebra $\mathbb{S}_n/I$ is left (or right) Noetherian if and only if the ideal I contains all the height 1 primes of $\mathbb{S}_n$.

2.3. The set of height 1 primes of $\mathbb{S}_n$

Consider the ideals of the algebra $\mathbb{S}_n$:

$$\mathfrak{p}_1 := F \otimes \mathbb{S}_{n-1}, \quad \mathfrak{p}_2 := \mathbb{S}_1 \otimes F \otimes \mathbb{S}_{n-2}, \dots, \mathfrak{p}_n := \mathbb{S}_{n-1} \otimes F.$$

Then $\mathbb{S}_n/\mathfrak{p}_i \simeq \mathbb{S}_{n-1} \otimes (\mathbb{S}_1/F) \simeq \mathbb{S}_{n-1} \otimes K[x_i, x_i^{-1}]$ and $\bigcap_{i=1}^n \mathfrak{p}_i = \prod_{i=1}^n \mathfrak{p}_i = F^{\otimes n} = F_n$. Clearly, $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for all $i \neq j$.

- The set $\mathcal{H}_1$ of height 1 prime ideals of the algebra $\mathbb{S}_n$ is $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.

Let $\mathfrak{a}_n := \mathfrak{p}_1 + \dots + \mathfrak{p}_n$. Then the factor algebra

$$\mathbb{S}_n/\mathfrak{a}_n \simeq (\mathbb{S}_1/F)^{\otimes n} \simeq \bigotimes_{i=1}^n K[x_i, x_i^{-1}] = K[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}] =: L_n \tag{2.7}$$

is a Laurent polynomial algebra in n variables and so $\mathfrak{a}_n$ is a prime ideal of height and co-height n of the algebra $\mathbb{S}_n$.

Proposition 2.1 (Bavula [4]). *The polynomial algebra P_n is the only (up to isomorphism) faithful simple $\mathbb{S}_n$-module.*

In more detail, ${}_{\mathbb{S}_n}P_n \simeq \mathbb{S}_n/(\sum_{i=0}^n \mathbb{S}_n y_i) = \bigoplus_{\alpha \in \mathbb{N}^n} K x^\alpha \bar{1}$, $\bar{1} := 1 + \sum_{i=1}^n \mathbb{S}_n y_i$; and the action of the canonical generators of the algebra $\mathbb{S}_n$ on the polynomial algebra P_n is given by the rule

$$x_i * x^\alpha = x^{\alpha + e_i}, \quad y_i * x^\alpha = \begin{cases} x^{\alpha - e_i} & \text{if } \alpha_i > 0, \\ 0 & \text{if } \alpha_i = 0 \end{cases} \quad \text{and} \quad E_{\beta\gamma} * x^\alpha = \delta_{\gamma\alpha} x^\beta,$$

where the set $e_1 := (1, 0, \dots, 0), \dots, e_n := (0, \dots, 0, 1)$ is the canonical basis for the free $\mathbb{Z}$-module $\mathbb{Z}^n$. We identify the algebra $\mathbb{S}_n$ with its image in the algebra $\text{End}_K(P_n)$ of all the K-linear maps from the vector space P_n to itself, i.e. $\mathbb{S}_n \subset \text{End}_K(P_n)$.

For each non-empty subset I of the set $\{1, \dots, n\}$, let $\mathbb{S}_I := \bigotimes_{i \in I} \mathbb{S}_1(i) \simeq \mathbb{S}_{|I|}$, where $|I|$ is the number of elements in the set I, $F_I := \bigotimes_{i \in I} F(i) \simeq M_\infty(K)$, let $\mathfrak{a}_I$ be the ideal of the algebra $\mathbb{S}_I$ generated by the vector space $\bigoplus_{i \in I} F(i)$, i.e. $\mathfrak{a}_I := \sum_{i \in I} F(i) \otimes \mathbb{S}_{I \backslash i}$. The factor algebra $L_I := \mathbb{S}_I/\mathfrak{a}_I \simeq K[x_i, x_i^{-1} : i \in I]$ is a Laurent polynomial algebra. For elements $\alpha = (\alpha_i)_{i \in I}, \beta = (\beta_i)_{i \in I} \in \mathbb{N}^I$, let $E_{\alpha\beta}(I) := \prod_{i \in I} E_{\alpha_i \beta_i}(i)$. Then $E_{\alpha\beta}(I) E_{\xi\rho}(I) = \delta_{\beta\xi} E_{\alpha\rho}(I)$ for all $\alpha, \beta, \xi, \rho \in \mathbb{N}^I$.

2.4. The G_n-invariant normal subgroups $(1+\mathfrak{a}_{n,s})^*$ of $(1+\mathfrak{a}_n)^*$

We will often use the following two obvious lemmas.

Lemma 2.2 (Bavula [7]). *Let R be a ring and $I_1,\dots,I_n$ be ideals of the ring R such that $I_iI_j=0$ for all $i\neq j$. Let $a=1+a_1+\cdots+a_n\in R$, where $a_1\in I_1,\dots,a_n\in I_n$. The element a is a unit of the ring R if and only if all the elements $1+a_i$ are units and, in this case, $a^{-1}=(1+a_1)^{-1}(1+a_2)^{-1}\cdots(1+a_n)^{-1}$.*

Let R be a ring, R^* be its group of units, I be an ideal of R such that $I\neq R$ and let $(1+I)^*$ be the group of units of the multiplicative monoid $1+I$.

Lemma 2.3 (Bavula [7]). *Let R and I be as above. Then*

(1) $R^*\cap(1+I)=(1+I)^*$.

(2) $(1+I)^*$ *is a normal subgroup of* R^*.

For each subset I of the set $\{1,\dots,n\}$, let $\mathfrak{p}_I:=\bigcap_{i\in I}\mathfrak{p}_i$ and $\mathfrak{p}_\emptyset:=\mathbb{S}_n$. Each $\mathfrak{p}_I$ is an ideal of the algebra $\mathbb{S}_n$ and $\mathfrak{p}_I=\prod_{i\in I}\mathfrak{p}_i$. The complement to the subset I is denoted by CI. For a one-element subset $\{i\}$, we write Ci rather than $C\{i\}$. In particular, $\mathfrak{p}_{Ci}:=\mathfrak{p}_{C\{i\}}=\bigcap_{j\neq i}\mathfrak{p}_j$.

For each number $s=1,\dots,n$, let $\mathfrak{a}_{n,s}:=\sum_{|I|=s}\mathfrak{p}_I$. By the very definition, the ideals $\mathfrak{a}_{n,s}$ are G_n-invariant ideals (since the set $\mathcal{H}_1$ of all the height 1 prime ideals of the algebra $\mathbb{S}_n$ is $\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$ and $\mathcal{H}_1$ is a G_n-orbit). We have the strictly descending chain of G_n-invariant ideals of the algebra $\mathbb{S}_n$:

$$\mathfrak{a}_n=\mathfrak{a}_{n,1}\supset\mathfrak{a}_{n,2}\supset\cdots\supset\mathfrak{a}_{n,s}\supset\cdots\supset\mathfrak{a}_{n,n}=F_n\supset\mathfrak{a}_{n,n+1}:=0.$$

These are also ideals of the subalgebra $K+\mathfrak{a}_n$ of $\mathbb{S}_n$. Each set $\mathfrak{a}_{n,s}$ is an ideal of the algebra $K+\mathfrak{a}_{n,t}$ for all $t\leqslant s$, and the group of units of the algebra $K+\mathfrak{a}_{n,s}$ is the direct product of its two subgroups (see Lemma 2.3 (1))

$$(K+\mathfrak{a}_{n,s})^*=K^*\times(1+\mathfrak{a}_{n,s})^*,\quad s=1,\dots,n.$$

The groups $(K+\mathfrak{a}_{n,s})^*$ and $(1+\mathfrak{a}_{n,s})^*$ are G_n-invariant. There is the descending chain of G_n-invariant (hence, normal) subgroups of $(1+\mathfrak{a}_n)^*$:

$$(1+\mathfrak{a}_n)^*=(1+\mathfrak{a}_{n,1})^*\supset\cdots\supset(1+\mathfrak{a}_{n,s})^*\supset\cdots\supset(1+\mathfrak{a}_{n,n})^*=(1+F_n)^*\supset(1+\mathfrak{a}_{n,n+1})^*=\{1\}.$$

For each number $s=1,\dots,n$, the factor algebra

$$(K+\mathfrak{a}_{n,s})/\mathfrak{a}_{n,s+1}=K\oplus\bigoplus_{|I|=s}\bar{\mathfrak{p}}_I$$

contains the idempotent ideals $\bar{\mathfrak{p}}_I:=(\mathfrak{p}_I+\mathfrak{a}_{n,s+1})/\mathfrak{a}_{n,s+1}$ such that $\bar{\mathfrak{p}}_I\bar{\mathfrak{p}}_J=0$ for all $I\neq J$ such that $|I|=|J|=s$.

Recall that for a Laurent polynomial algebra $L = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $\mathrm{K}_1(L) \simeq L^*$ (see [**1**, **13**, **14**]),

$$\mathrm{GL}_\infty(L) = U(L) \ltimes E_\infty(L), \tag{2.8}$$

where $E_\infty(L)$ is the subgroup of $\mathrm{GL}_\infty(L)$ generated by all the *elementary matrices* $\{1 + aE_{ij} \mid a \in L,\ i, j \in \mathbb{N},\ i \neq j\}$ and $U(L) := \{\mu(u) := uE_{00} + 1 - E_{00} \mid u \in L^*\} \simeq L^*$, $\mu(u) \leftrightarrow u$. The group $E_\infty(L)$ is a normal subgroup of $\mathrm{GL}_\infty(L)$. This is true for an arbitrary coefficient ring L.

By Lemma 2.2 and (2.8), the group of units of the algebra $(K + \mathfrak{a}_{n,s})/\mathfrak{a}_{n,s+1} =: K + \mathfrak{a}_{n,s}/\mathfrak{a}_{n,s+1}$ is the direct product of groups,

$$\begin{aligned}(K + \mathfrak{a}_{n,s}/\mathfrak{a}_{n,s+1})^* \\ &= K^* \times \prod_{|I|=s} (1 + \bar{\mathfrak{p}}_I)^* \simeq K^* \times \prod_{|I|=s} \mathrm{GL}_\infty(L_{CI}) \simeq K^* \times \prod_{|I|=s} U(L_{CI}) \ltimes E_\infty(L_{CI}),\end{aligned}$$

since $(1 + \bar{\mathfrak{p}}_I)^* \simeq (1 + M_\infty(L_{CI}))^* = \mathrm{GL}_\infty(L_{CI})$, where

$$L_{CI} = \mathbb{S}_{CI}/\mathfrak{a}_{CI} = \bigotimes_{i \in CI} K[x_i, x_i^{-1}]$$

is the Laurent polynomial algebra. In more detail, for each non-empty subset I of $\{1, \dots, n\}$, let $\mathbb{Z}^I := \bigoplus_{i \in I} \mathbb{Z}e_i$. It is a subgroup of $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i$. Similarly, $\mathbb{N}^I := \bigoplus_{i \in I} \mathbb{N}e_i$. By (2.8),

$$(1 + \bar{\mathfrak{p}}_I)^* = U(L_{CI}) \ltimes E_\infty(L_{CI}) = (U_I(K) \times \mathbb{X}_{CI}) \ltimes E_\infty(L_{CI}), \tag{2.9}$$

where

$$\begin{aligned}
U(L_{CI}) &:= \{\mu_I(u) := uE_{00}(I) + 1 - E_{00}(I) \mid u \in L_{CI}^*\} \simeq L_{CI}^*, && \mu_I(u) \leftrightarrow u,\\
L_{CI}^* &= \{\lambda x^\alpha \mid \lambda \in K^*,\ \alpha \in \mathbb{Z}^{CI}\},\\
U_I(K) &:= \{\mu_I(\lambda) := \lambda E_{00}(I) + 1 - E_{00}(I) \mid \lambda \in K^*\} \simeq K^*, && \mu_I(\lambda) \leftrightarrow \lambda,\\
\mathbb{X}_{CI} &:= \{\mu_I(x^\alpha) := x^\alpha E_{00}(I) + 1 - E_{00}(I) \mid \alpha \in \mathbb{Z}^{CI}\}\\
&\simeq \mathbb{Z}^{CI} \simeq \mathbb{Z}^{n-s}, && \mu_I(x^\alpha) \leftrightarrow \alpha,\\
E_\infty(L_{CI}) &:= \langle 1 + aE_{\alpha\beta}(I) \mid a \in L_{CI},\ \alpha, \beta \in \mathbb{N}^I,\ \alpha \neq \beta\rangle.
\end{aligned}$$

The algebra epimorphism $\psi_{n,s} \colon K + \mathfrak{a}_{n,s} \to (K + \mathfrak{a}_{n,s})/\mathfrak{a}_{n,s+1}$, $a \mapsto a + \mathfrak{a}_{n,s+1}$ yields a group homomorphism of their groups of units $(K + \mathfrak{a}_{n,s})^* \to (K + \mathfrak{a}_{n,s}/\mathfrak{a}_{n,s+1})^*$. The kernel of this homomorphism is $(1 + \mathfrak{a}_{n,s+1})^*$. As a result we have the exact sequence of group homomorphisms:

$$\begin{array}{ccccccc}
1 \longrightarrow & (1 + \mathfrak{a}_{n,s+1})^* & \longrightarrow & (K + \mathfrak{a}_{n,s})^* & \longrightarrow & (K + \mathfrak{a}_{n,s}/\mathfrak{a}_{n,s+1})^* \\
& \big\downarrow = & & \big\downarrow = & & \big\downarrow = \\
& (1 + \mathfrak{a}_{n,s+1})^* & & K^* \times (1 + \mathfrak{a}_{n,s})^* & & K^* \times \prod_{|I|=s} (1 + \bar{\mathfrak{p}}_I)^*
\end{array}$$

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which yields the exact sequence of group homomorphisms in which $\mathcal{Z}_{n,s} := \operatorname{coker}(\psi_{n,s})$:

$$1 \to (1+\mathfrak{a}_{n,s+1})^* \to (1+\mathfrak{a}_{n,s})^* \xrightarrow{\psi_{n,s}} \prod_{|I|=s} (1+\bar{\mathfrak{p}}_I)^* \simeq \prod_{|I|=s} \mathrm{GL}_\infty(L_{CI}) \to \mathcal{Z}_{n,s} \to 1. \tag{2.10}$$

For $s = n$ the map $\psi_{n,n}$ is the identity map and so $\mathcal{Z}_{n,n} = \{1\}$. Intuitively, the group $\mathcal{Z}_{n,s}$ represents 'relations' that determine the image $\operatorname{im}(\psi_{n,s})$ as the subgroup of $\prod_{|I|=s}(1+\bar{\mathfrak{p}}_I)^*$. We will see later that the group $\mathcal{Z}_{n,s}$ is a free abelian group of rank $\binom{n}{s+1}$ (see Corollary 4.3). So, the image of the map $\psi_{n,s}$ is large. Note that $\mathfrak{a}_{n,s+1}$ and $\mathfrak{p}_I$ (where $|I| = s$) are ideals of the algebra $K + \mathfrak{a}_{n,s}$. By Lemma 2.3, the groups $(1+\mathfrak{a}_{n,s+1})^*$ and $(1+\mathfrak{p}_I)^*$ (where $|I| = s$) are normal subgroups of $(1+\mathfrak{a}_{n,s})^*$. Then the subgroup $\Upsilon_{n,s}$ of $(1+\mathfrak{a}_{n,s})^*$ generated by these normal subgroups is a normal subgroup of $(1+\mathfrak{a}_{n,s})^*$. As a subset of $(1+\mathfrak{a}_{n,s})^*$, the group $\Upsilon_{n,s}$ is equal to the product of the groups $(1+\mathfrak{a}_{n,s+1})^*$ and $(1+\mathfrak{p}_I)^*$, $|I| = s$, in *arbitrary* order (by their normality), i.e.

$$\Upsilon_{n,s} = \prod_{|I|=s} (1+\mathfrak{p}_I)^* \cdot (1+\mathfrak{a}_{n,s+1})^*. \tag{2.11}$$

By Theorem 1.3 and Theorem 1.5, the group $\Upsilon_{n,s}$ is a G_n-invariant (hence, normal) subgroup of $\mathbb{S}_n^*$. We will see that the factor group $(1+\mathfrak{a}_{n,s})^*/\Upsilon_{n,s}$ is a free abelian group of rank $\binom{n}{s+1}s$ (see (4.5)).

By (2.9), the direct product of groups $\prod_{|I|=s}(1+\bar{\mathfrak{p}}_I)^* = \mathbb{X}_{n,s} \ltimes \bar{\Gamma}_{n,s}$ is the semi-direct product of its two subgroups

$$\mathbb{X}_{n,s} := \prod_{|I|=s} \mathbb{X}_{CI} \simeq \mathbb{Z}^{\binom{n}{s}(n-s)} \quad \text{and} \quad \bar{\Gamma}_{n,s} := \prod_{|I|=s} U_I(K) \ltimes E_\infty(L_{CI}). \tag{2.12}$$

For each subset I of $\{1,\dots,n\}$ such that $|I| = s$, $U_I(K) \ltimes E_\infty(\mathbb{S}_{CI})$ is a subgroup of $(1+\mathfrak{p}_I)^*$, where

$$U_I(K) = \{\mu_I(\lambda) \mid \lambda \in K^*\} \simeq K^*, \qquad E_\infty(\mathbb{S}_{CI}) := \langle 1 + aE_{\alpha\beta}(I) \mid a \in \mathbb{S}_{CI},\ \alpha \neq \beta \in \mathbb{N}^I \rangle, \tag{2.13}$$

where $\mu_I(\lambda) := \lambda E_{00}(I) + 1 - E_{00}(I)$. Clearly,

$$\psi_{n,s}|_{U_I(K)} \colon U_I(K) \simeq U_I(K), \quad \mu_I(\lambda) \mapsto \mu_I(\lambda),$$

and $\psi_{n,s}(U_I(K) \ltimes E_\infty(\mathbb{S}_{CI})) = U_I(K) \ltimes E_\infty(L_{CI})$ for all subsets I with $|I| = s$. The subgroup of $(1+\mathfrak{a}_{n,s})^*$,

$$\Gamma_{n,s} := \psi_{n,s}^{-1}(\bar{\Gamma}_{n,s}) =^{\text{set}} \prod_{|I|=s} (U_I(K) \ltimes E_\infty(\mathbb{S}_{CI})) \cdot (1+\mathfrak{a}_{n,s+1})^*, \tag{2.14}$$

is a normal subgroup as the pre-image of a normal subgroup. The upper script 'set' was added to indicate that this is a product of subgroups but in general not the direct product. It is obvious that $\psi_{n,s}(\Gamma_{n,s}) = \bar{\Gamma}_{n,s}$ and $\Gamma_{n,s} \subseteq \Upsilon_{n,s}$. We will see that, in fact,

$\Gamma_{n,s} = \Upsilon_{n,s}$ (see Theorem 4.4). Let $\Delta_{n,s} := (1+\mathfrak{a}_{n,s})^*/\Gamma_{n,s}$. The group homomorphism $\psi_{n,s}$ (see (2.10)) induces a group monomorphism

$$\bar{\psi}_{n,s}\colon \Delta_{n,s} \to \prod_{|I|=s}(1+\bar{\mathfrak{p}}_I)^*)/\bar{\Gamma}_{n,s} \simeq \mathbb{X}_{n,s} \simeq \mathbb{Z}^{\binom{n}{s}(n-s)}.$$

This means that the group $\Delta_{n,s}$ is a free abelian group of rank less than or equal to $\binom{n}{s}(n-s)$. In fact, the rank is equal to $\binom{n}{s+1}s$ (see (4.4)).

For each subset I with $|I| = s$, consider the free abelian group $\mathbb{X}'_{CI} := \bigoplus_{j\in CI}\mathbb{Z}(j,I) \simeq \mathbb{Z}^{n-s}$, where $\{(j,I) \mid j \in CI\}$ is its free basis. Let

$$\mathbb{X}'_{n,s} := \bigoplus_{|I|=s}\mathbb{X}'_{CI} = \bigoplus_{|I|=s}\bigoplus_{j\in CI}\mathbb{Z}(j,I) \simeq \mathbb{Z}^{\binom{n}{s}(n-s)}.$$

For each subset I, consider the isomorphism of abelian groups

$$\mathbb{X}_{CI} \to \mathbb{X}'_{CI}, \quad \mu_I(x_j) := x_jE_{00}(I) + 1 - E_{00}(I) \mapsto (j,I).$$

These isomorphisms yield the group isomorphism

$$\mathbb{X}_{n,s} \to \mathbb{X}'_{n,s}, \quad \mu_I(x_j) \mapsto (j,I). \tag{2.15}$$

Each element a of $\mathbb{X}_{n,s}$ is a unique product $a = \prod_{|I|=s}\prod_{j\in CI}\mu_I(x_j)^{n(j,I)}$, where $n(j,I) \in \mathbb{Z}$. Each element a' of the group $\mathbb{X}'_{n,s}$ is a unique sum $a' = \sum_{|I|=s}\sum_{j\in CI}n(j,I)\cdot(j,I)$, where $n(j,I) \in \mathbb{Z}$. The map (2.15) sends a to a'. To make computations more readable we set $e_I := E_{00}(I)$. Then $e_Ie_J = e_{I\cup J}$.

2.5. The current groups $\Theta_{n,s}$, $s = 1,\dots,n-1$

The current groups $\Theta_{n,s}$ are the most important subgroups of the group $(1+\mathfrak{a}_n)^*$. They are finitely generated groups and their generators are given explicitly. The adjective 'current' comes from the action of the generators on the monomial basis for the polynomial algebra P_n. If we visualize the algebra P_n as a liquid and the monomials $\{x^\alpha\}$ as its atoms, then the action of the generators of the group $\Theta_{n,s}$ on the monomials resembles a current (see (2.16)). The generators shift the liquid only on the faces of the positive cone $\mathbb{N}^n \approx P_n$. The generators of the groups $\Theta_{n,s}$ are units of the algebra $\mathbb{S}_n$ but they are defined as a product of two *non-units*. As a result the groups $\Theta_{n,s}$ capture the most delicate phenomena about the structure and the properties of the groups $\mathbb{S}_n^*$ and G_n.

For each non-empty subset I of $\{1,\dots,n\}$ with $s := |I| < n$ and an element $i \in CI$, let

$$X(i,I) := \mu_I(x_i) = x_iE_{00}(I)+1-E_{00}(I) \quad \text{and} \quad Y(i,I) := \mu_I(y_i) = y_iE_{00}(I)+1-E_{00}(I).$$

Then $Y(i,I)X(i,I) = 1$, $\ker Y(i,I) = P_{C(I\cup i)}$ and $P_n = \operatorname{im} X(i,I) \oplus P_{C(I\cup i)}$, where $P_{C(I\cup i)} := K[x_j]_{j\in C(I\cup i)}$. As an element of the algebra $\operatorname{End}_K(P_n)$, the map $X(i,I)$ is injective (but not bijective) and the map $Y(i,I)$ is surjective (but not bijective).

Definition 2.4. For each subset J of $\{1,\dots,n\}$ with $|J| = s+1 \geqslant 2$ and for two distinct elements i and j of the set J,

$$\theta_{ij}(J) := Y(i, J\setminus i)X(j, J\setminus j) \in (1+\mathfrak{p}_{J\setminus i}+\mathfrak{p}_{J\setminus j})^* \subseteq (1+\mathfrak{a}_{n,s})^*.$$

The *current group* $\Theta_{n,s}$ is the subgroup of $(1+\mathfrak{a}_{n,s})^*$ generated by all the elements $\theta_{ij}(J)$ (for all the possible choices of J, i and j).

In more detail, the element $\theta_{ij}(J)$ belongs to the set $1+\mathfrak{p}_{J\setminus i}+\mathfrak{p}_{J\setminus j}$ and $\theta_{ij}(J)^{-1} = \theta_{ji}(J) \in 1+\mathfrak{p}_{J\setminus i}+\mathfrak{p}_{J\setminus j}$. This follows from the action of the element $\theta_{ij}(J)$ on the monomial basis of the polynomial algebra P_n, where $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$,

$$\theta_{ij}(J) * x^\alpha = \begin{cases} x^\alpha & \text{if } \exists k \in J\setminus\{i,j\}\colon \alpha_k \neq 0, \\ x^\alpha & \text{if } \forall k \in J\setminus\{i,j\}\colon \alpha_k = 0,\ \alpha_i > 0,\ \alpha_j > 0, \\ x^{\alpha-e_i} & \text{if } \forall k \in J\setminus\{i,j\}\colon \alpha_k = 0,\ \alpha_i > 0,\ \alpha_j = 0, \\ x^{\alpha+e_j} & \text{if } \forall k \in J\setminus\{i,j\}\colon \alpha_k = 0,\ \alpha_i = 0,\ \alpha_j \geqslant 0. \end{cases} \tag{2.16}$$

Alternatively, note that $\mu_{J\setminus j}(x_jy_j) = \mu_{J\setminus j}(1-e_{\{j\}}) = 1-e_{\{j\}}e_{J\setminus j} = 1-e_J$ and (using (2.2)) $\mu_{J\setminus i}(y_i)e_J = (1+(y_i-1)e_{J\setminus i})e_J = e_J+(y_i-1)e_J = e_J-e_J = 0$. Then

$$\begin{aligned} \theta_{ij}(J)\theta_{ji}(J) &= \mu_{J\setminus i}(y_i)\mu_{J\setminus j}(x_j)\cdot\mu_{J\setminus j}(y_j)\mu_{J\setminus i}(x_i) \\ &= \mu_{J\setminus i}(y_i)\cdot\mu_{J\setminus j}(x_jy_j)\cdot\mu_{J\setminus i}(x_i) \\ &= \mu_{J\setminus i}(y_i)\cdot(1-e_J)\cdot\mu_{J\setminus i}(x_i) \\ &= \mu_{J\setminus i}(y_ix_i) \\ &= \mu_{J\setminus i}(1) \\ &= 1. \end{aligned}$$

By symmetry, $\theta_{ji}(J)\theta_{ij}(J) = 1$, i.e.

$$\theta_{ij}(J) = \theta_{ji}(J)^{-1}. \tag{2.17}$$

Therefore, the unit $\theta_{ij}(I)$ is the product of an injective map and a surjective map, neither of which is a bijection.

Suppose that i, j and k are distinct elements of the set J (hence $|J| \geqslant 3$). Then,

$$\theta_{ij}(J)\theta_{jk}(J) = \theta_{ik}(J). \tag{2.18}$$

Indeed,

$$\begin{aligned} \theta_{ij}(J)\theta_{jk}(J) &= \mu_{J\setminus i}(y_i)\cdot\mu_{J\setminus j}(x_j)\mu_{J\setminus j}(y_j)\cdot\mu_{J\setminus k}(x_k) \\ &= \mu_{J\setminus i}(y_i)\cdot\mu_{J\setminus j}(x_jy_j)\cdot\mu_{J\setminus k}(x_k) \\ &= \mu_{J\setminus i}(y_i)\cdot(1-e_J)\cdot\mu_{J\setminus k}(x_k) \\ &= \mu_{J\setminus i}(y_i)\mu_{J\setminus k}(x_k) \\ &= \theta_{ik}(J). \end{aligned}$$

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For each number $s = 1, \dots, n-1$ the free abelian group $\mathbb{X}'_{n,s}$ admits the decomposition $\mathbb{X}'_{n,s} = \bigoplus_{|J|=s+1} \bigoplus_{j \cup I = J} \mathbb{Z}(j, I)$ and using it, for each subset J with $|J| = s+1$, we define a character (a homomorphism) χ'_J:

$$\chi'_J \colon \mathbb{X}'_{n,s} \to \mathbb{Z}, \qquad \sum_{|J'|=s+1} \sum_{j \cup I = J'} n_{j,I}(j, I) \mapsto \sum_{j \cup I = J} n_{j,I}.$$

Let $\max(J)$ be the maximal number of the set J. The group $\mathbb{X}'_{n,s}$ is the direct sum

$$\mathbb{X}'_{n,s} = \mathbb{K}'_{n,s} \oplus \mathbb{Y}'_{n,s} \tag{2.19}$$

of its free abelian subgroups

$$\begin{aligned} \mathbb{K}'_{n,s} &= \bigcap_{|J|=s+1} \ker(\chi'_J) \\ &= \bigoplus_{|J|=s+1} \bigoplus_{j \in J \setminus \max(J)} \mathbb{Z}(-(\max(J), J \setminus \max(J)) + (j, J \setminus j)) \\ &\simeq \mathbb{Z}^{\binom{n}{s+1}s}, \\ \mathbb{Y}'_{n,s} &= \bigoplus_{|J|=s+1} \mathbb{Z}(\max(J), J \setminus \max(J)) \simeq \mathbb{Z}^{\binom{n}{s+1}}. \end{aligned}$$

Consider the group homomorphism $\psi'_{n,s} \colon (1 + \mathfrak{a}_{n,s})^* \to \mathbb{X}'_{n,s}$, defined as the composition of the following group homomorphisms:

$$\psi'_{n,s} \colon (1 + \mathfrak{a}_{n,s})^* \to (1 + \mathfrak{a}_{n,s})^* / \Gamma_{n,s} \xrightarrow{\bar{\psi}_{n,s}} \prod_{|I|=s} (1 + \bar{\mathfrak{p}}_I)^* / \bar{\Gamma}_{n,s} \simeq \mathbb{X}_{n,s} \simeq \mathbb{X}'_{n,s}.$$

Then,

$$\psi'_{n,s}(\theta_{ij}(J)) = -(i, J \setminus i) + (j, J \setminus j). \tag{2.20}$$

It follows that

$$\psi'_{n,s}(\Theta_{n,s}) = \mathbb{K}'_{n,s} \tag{2.21}$$

since, by (2.20), $\psi'_{n,s}(\Theta_{n,s}) \supseteq \mathbb{K}'_{n,s}$ (as the free basis for $\mathbb{K}'_{n,s}$, introduced above, belongs to the set $\psi'_{n,s}(\Theta_{n,s})$); again, by (2.20), $\psi'_{n,s}(\Theta_{n,s}) \subseteq \bigcap_{|J|=s+1} \ker(\chi'_J) = \mathbb{K}'_{n,s}$.

Let $H, H_1, \dots, H_m$ be subsets (usually subgroups) of a group H. We say that H is the *product* of $H_1, \dots, H_m$ and write $H = {}^{\mathrm{set}}\prod_{i=1}^m H_i = H_1 \cdots H_m$ if each element h of H is a product $h = h_1 \cdots h_m$, where $h_i \in H_i$. We add the subscript 'set' (sometimes) in order to distinguish it from the direct product of groups. We say that H is the *exact product* of $H_1, \dots, H_m$ and write $H = {}^{\mathrm{exact}}\prod_{i=1}^m H_i = H_1 \times_{\mathrm{ex}} \cdots \times_{\mathrm{ex}} H_m$, if each element h of H is a *unique* product $h = h_1 \cdots h_m$ where $h_i \in H_i$. The order in the definition of the exact product is important.

The subgroup of $(1 + \mathfrak{a}_{n,s})^*$ generated by the groups $\Theta_{n,s}$ and $\Gamma_{n,s}$ is equal to their product $\Theta_{n,s}\Gamma_{n,s}$, by the normality of $\Gamma_{n,s}$. The subgroup $\Gamma_{n,s}$ of the group $\Theta_{n,s}\Gamma_{n,s}$ is a normal subgroup. Hence, the intersection $\Theta_{n,s} \cap \Gamma_{n,s}$ is a normal subgroup of $\Theta_{n,s}$.

Lemma 2.5. *For each number $s = 1, \dots, n-1$ the group $\Theta_{n,s}\Gamma_{n,s}$ is the exact product*

$$\Theta_{n,s}\Gamma_{n,s} =^{\text{exact}} \prod_{|J|=s+1} \prod_{j \in J \setminus \max(J)} \langle \theta_{\max(J),j}(J) \rangle \cdot \Gamma_{n,s},$$

i.e. each element $a \in \Theta_{n,s}\Gamma_{n,s}$ is a unique product

$$a = \prod_{|J|=s+1} \prod_{j \in J \setminus \max(J)} \theta_{\max(J),j}(J)^{n(j,J)} \cdot \gamma,$$

where $n(j, J) \in \mathbb{Z}$ and $\gamma \in \Gamma_{n,s}$. Moreover, the group $\Theta_{n,s}\Gamma_{n,s}$ is the semi-direct product

$$\Theta_{n,s}\Gamma_{n,s} =^{\text{semi}} \prod_{|J|=s+1} \prod_{j \in J \setminus \max(J)} \langle \theta_{\max(J),j}(J) \rangle \ltimes \Gamma_{n,s},$$

where the order in the double product is arbitrary and 'semi' indicates that this product is semi-direct.

Proof. The lemma follows at once from (2.21) and the fact that the elements $\psi'_{n,s}(\theta_{\max(J),j}(J)) = -(\max(J), J \setminus \max(J)) + (j, J \setminus j)$ form a basis for the free abelian group $\mathbb{K}'_{n,s}$. □

For each number $s = 1, \dots, n-1$ consider the subset of $(1 + \mathfrak{a}_{n,s})^*$,

$$\Theta'_{n,s} :=^{\text{exact}} \prod_{|J|=s+1} \prod_{j \in J \setminus \max(J)} \langle \theta_{\max(J),j}(J) \rangle, \tag{2.22}$$

which is the exact product of cyclic groups (each of them is isomorphic to $\mathbb{Z}$) since each element u of $\Theta'_{n,s}$ is a *unique* product

$$u = \prod_{|J|=s+1} \prod_{j \in J \setminus \max(J)} \theta_{\max(J),j}(J)^{n(j,J)},$$

where $n(j, J) \in \mathbb{Z}$ (see Lemma 2.5). The order in the product is arbitrary but fixed.

By Lemma 2.5, $\Theta_{n,s}/\Theta_{n,s} \cap \Gamma_{n,s} \simeq \Theta_{n,s}\Gamma_{n,s}/\Gamma_{n,s} \simeq \mathbb{K}'_{n,s} \simeq \mathbb{Z}^{\binom{n}{s+1}s}$ and so

$$[\Theta_{n,s}, \Theta_{n,s}] \subseteq \Gamma_{n,s}. \tag{2.23}$$

The next theorem is the pinnacle of finding the explicit generators for the groups $\mathbb{S}_n^*$ and G_n.

Theorem 2.6. $\psi'_{n,s}((1 + \mathfrak{a}_{n,s})^*) = \psi'_{n,s}(\Theta_{n,s})$ *for* $s = 1, \dots, n-1$.

Rough sketch of the proof. The proof is rather long and is given in § 4. We use an induction on n (the case $n = 2$ was considered in [**5**]) and then, for a fixed n, we use a second downward induction on $s = 1, \dots, n-1$ starting with $s = n-1$. The initial step $(n, s) = (n, n-1)$ is the most difficult one. We spend the entire of § 3 giving its proof. The remaining cases, using double induction, can be deduced from the initial one (for different n', i.e. when n' runs from 1 till n). The key idea in the proof of the case $(n, n-1)$ is to use the Fredholm operators and their indices. Then, using well-known results on indices, some (new) results on the Fredholm operators and their indices from [**5**], and their generalizations obtained in § 3, we construct several index maps (using various indices of the Fredholm operators). The most difficult part is to prove that these maps are well defined (as their constructions are based on highly non-unique decompositions). Then the proof follows from the properties of these index maps. □

3. The groups $(1 + \mathfrak{a}_{n,n-1})^*$ and $\Theta_{n,n-1}$

In this section, the group $(1 + \mathfrak{a}_{n,n-1})^*$ is found (see Corollary 3.11). We mentioned already in the introduction that the key idea in finding the group G_n is to use indices of operators. That is why we start this section by collecting known results on indices and proving new ones. These results are used in many proofs that follow.

3.1. The index ind of linear maps and its properties

Let $\mathcal{C}$ be the class of all K-linear maps with finite-dimensional kernel and cokernel (such maps are called the *Fredholm linear maps/operators*). So, $\mathcal{C}$ is the family of *Fredholm* linear maps/operators. For vector spaces V and U, let $\mathcal{C}(V, U)$ be the set of all Fredholm operators from V to U with finite-dimensional kernel and cokernel. So, we have the disjoint union $\mathcal{C} = \bigcup_{V,U} \mathcal{C}(V, U)$.

Definition 3.1. For a linear map $\varphi \in \mathcal{C}$, the integer $\operatorname{ind}(\varphi) := \dim \ker(\varphi) - \dim \operatorname{coker}(\varphi)$ is called the *index* of the map φ.

For vector spaces V and U, let $\mathcal{C}(V, U)_i := \{\varphi \in \mathcal{C}(V, U) \mid \operatorname{ind}(\varphi) = i\}$. Then $\mathcal{C}(V, U) = \bigcup_{i \in \mathbb{Z}} \mathcal{C}(V, U)_i$ is the disjoint union and the class $\mathcal{C}$ is the disjoint union $\bigcup_{i \in \mathbb{Z}} \mathcal{C}_i$, where $\mathcal{C}_i := \{\varphi \in \mathcal{C} \mid \operatorname{ind}(\varphi) = i\}$. When $V = U$, we write $\mathcal{C}(V) := \mathcal{C}(V, V)$ and $\mathcal{C}(V)_i := \mathcal{C}(V, V)_i$.

Example 3.2. Note that $\mathbb{S}_1 \subset \operatorname{End}_K(P_1)$. The map $x^i \in \operatorname{End}_K(P_1)$ is an injection with $P_1 = (\bigoplus_{j=0}^{i-1} K x^j) \oplus \operatorname{im}(x^i)$; the map $y^i \in \operatorname{End}_K(P_1)$ is a surjection with $\ker(y^i) = \bigoplus_{j=0}^{i-1} K x^j$. Hence,

$$\operatorname{ind}(x^i) = -i \quad \text{and} \quad \operatorname{ind}(y^i) = i, \quad i \geqslant 1. \tag{3.1}$$

Lemma 3.3 shows that $\mathcal{C}$ is a multiplicative semigroup with zero element (If the composition of two elements of C is undefined we set their product to be 0). The next two lemmas are well known (see [**9**, Lemmas A.2.4 and A.2.5]).

Lemma 3.3. *Let $\psi \colon M \to N$ and $\varphi \colon N \to L$ be K-linear maps. If two of the three maps ψ, φ and $\varphi\psi$ belong to the set $\mathcal{C}$, then so does the third and, in this case,*

$$\operatorname{ind}(\varphi\psi) = \operatorname{ind}(\varphi) + \operatorname{ind}(\psi).$$

By Lemma 3.3, $\mathcal{C}(N, L)_i \mathcal{C}(M, N)_j \subseteq \mathcal{C}(M, L)_{i+j}$ for all $i, j \in \mathbb{Z}$.

Lemma 3.4. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \\ 0 & \longrightarrow & U_1 & \longrightarrow & U_2 & \longrightarrow & U_3 & \longrightarrow & 0 \end{array}$$

be a commutative diagram of K-linear maps with exact rows. Suppose that $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{C}$. Then

$$\operatorname{ind}(\varphi_2) = \operatorname{ind}(\varphi_1) + \operatorname{ind}(\varphi_3).$$

Let V and U be vector spaces. Define $\mathcal{I}(V,U) := \{\varphi \in \operatorname{Hom}_K(V,U) \mid \dim \operatorname{im}(\varphi) < \infty\}$, and when $V = U$ we write $\mathcal{I}(V) := \mathcal{I}(V,V)$.

Theorem 3.5 (Bavula [5]). *Let V and U be vector spaces. Then $\mathcal{C}(V,U)_i + \mathcal{I}(V,U) = \mathcal{C}(V,U)_i$ for all $i \in \mathbb{Z}$.*

Lemma 3.6 (Bavula [5]). *Let V and V' be vector spaces and let $\varphi\colon V \to V'$ be a linear map such that the vector spaces $\ker(\varphi)$ and $\operatorname{coker}(\varphi)$ are isomorphic. Fix subspaces $U \subseteq V$ and $W \subseteq V'$ such that $V = \ker(\varphi) \oplus U$ and $V' = W \oplus \operatorname{im}(\varphi)$ and fix an isomorphism $f\colon \ker(\varphi) \to W$ (this is possible since $\ker(\varphi) \simeq \operatorname{coker}(\varphi) \simeq W$) and extend it to a linear map $f\colon V \to V'$ by setting $f(U) = 0$. Then the map $\varphi + f\colon V \to V'$ is an isomorphism.*

Corollary 3.7.

(1) $1 + F_n \subseteq \mathcal{C}(P_n)_0$.

(2) $\mathbb{S}_n^* + F_n \subseteq \mathcal{C}(P_n)_0$.

Proof. Both statements follow from Theorem 3.5 (since $\mathbb{S}_n^* \subseteq \mathcal{C}(P_n)_0$ and $F_n \subseteq \mathcal{I}(P_n)$), but we give short independent proofs (that do not use Theorem 3.5).

(1) Since $1 + F_n \simeq 1 + M_\infty(K)$, statement (1) is obvious.

(2) Let $u \in \mathbb{S}_n^*$ and $f \in F_n$. Then $u^{-1}f \in F_n$. By statement (1), the element $1 + u^{-1}f \in \mathcal{C}(P_n)_0$. Since $u \in \mathcal{C}(P_n)_0$, we have $u + f = u(1 + u^{-1}f) \in \mathcal{C}(P_n)_0$, by Lemma 3.3.

□

3.2. The subgroup $\Theta_{n,n-1}$ of $(1 + \mathfrak{a}_{n,n-1})^*$ for $n \geqslant 2$

For each pair of indices $i \neq j$, the element

$$\theta_{ij} := \theta_{ij}(\{1,\dots,n\}) := \Big(1 + (y_i - 1)\prod_{k \neq i} E_{00}(k)\Big) \cdot \Big(1 + (x_j - 1)\prod_{l \neq j} E_{00}(l)\Big) \in (1 + \mathfrak{a}_{n,n-1})^*$$

is a unit and

$$\theta_{ij}^{-1} = \Big(1 + (y_j - 1)\prod_{l \neq j} E_{00}(l)\Big) \cdot \Big(1 + (x_i - 1)\prod_{k \neq i} E_{00}(k)\Big) \in (1 + \mathfrak{a}_{n,n-1})^*, \tag{3.2}$$

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i.e. $\theta_{ij}^{-1} = \theta_{ji}$. This is obvious since

$$\theta_{ij} * x^{\alpha} = \begin{cases} x_i^{\alpha_i - 1} & \text{if } \alpha_i > 0 \quad \forall k \neq i \colon \alpha_k = 0, \\ x_j^{\alpha_j + 1} & \text{if } \alpha_j \geqslant 0 \quad \forall l \neq j \colon \alpha_l = 0, \\ x^{\alpha} & \text{otherwise}, \end{cases}$$

and

$$\theta_{ij}^{-1} * x^{\alpha} = \begin{cases} x_i^{\alpha_i + 1} & \text{if } \alpha_i \geqslant 0 \quad \forall k \neq i \colon \alpha_k = 0, \\ x_j^{\alpha_j - 1} & \text{if } \alpha_j > 0 \quad \forall l \neq j \colon \alpha_l = 0, \\ x^{\alpha} & \text{otherwise}. \end{cases}$$

Using the above action of the elements θ_{ij} on the monomial basis for the polynomial algebra P_n, it is easy to show that the elements θ_{ij} *commute* modulo $(1+F_n)^*$; $\theta_{jk}\theta_{ij} \equiv \theta_{ik} \bmod (1+F_n)^*$ for all distinct elements i, j and k; and $\theta_{ij}^m * 1 = x_j^m$ for all $m \geqslant 1$. Recall that $\Theta_{n,n-1}$ is the subgroup of $(1+\mathfrak{a}_{n,n-1})^*$ generated by the elements θ_{ij}. It follows from

$$\left(1 + (y_i - 1) \prod_{k \neq i} E_{00}(k)\right) * x^{\alpha} = \begin{cases} x_i^{\alpha_i - 1} & \text{if } \alpha_i > 0 \quad \forall k \neq i \colon \alpha_k = 0, \\ 0 & \text{if } \alpha = 0, \\ x^{\alpha} & \text{otherwise} \end{cases}$$

that the map $1 + (y_i - 1)\prod_{k \neq i} E_{00}(k) \in \mathrm{End}_K(P_n)$ is a surjection with kernel equal to K and so

$$\operatorname{ind}\left(1 + (y_i - 1) \prod_{k \neq i} E_{00}(k)\right) = 1. \tag{3.3}$$

Similarly, it follows from

$$\left(1 + (x_j - 1) \prod_{l \neq j} E_{00}(l)\right) * x^{\alpha} = \begin{cases} x_j^{\alpha_j + 1} & \text{if } \forall l \neq j \colon \alpha_l = 0, \\ x^{\alpha} & \text{otherwise} \end{cases}$$

that the map $1 + (x_j - 1)\prod_{l \neq j} E_{00}(l) \in \mathrm{End}_K(P_n)$ is an injection such that $P_n = K \oplus \mathrm{im}(1 + (x_j - 1)\prod_{l \neq j} E_{00}(l))$ and so

$$\operatorname{ind}\left(1 + (x_j - 1) \prod_{l \neq j} E_{00}(l)\right) = -1. \tag{3.4}$$

We see that the unit θ_{ij} of the algebra $\mathbb{S}_n$ is the product of two non-units having non-zero indices of opposite sign (note that $\operatorname{ind}(\theta_{ij}) = 0$ and so the sum of the two indices is equal to 0). Lemma 3.8 shows that this is a general phenomenon and so the group $(1+\mathfrak{a}_{n,n-1})^*$ is a sophisticated group in the sense that in producing units, non-units are involved.

Lemma 3.8. *Let $u = 1 + \sum_{i=1}^n a_i \in (1 + \mathfrak{a}_{n,n-1})^*$, where $a_i \in \mathfrak{p}_{Ci}$. Then the following hold:*

(1) $1 + a_i \in \mathcal{C}(P_n)$ *for all* $i = 1, \dots, n$ *and* $\sum_{i=1}^n \mathrm{ind}(1 + a_i) = 0$;

(2) *if* $u = 1 + \sum_{i=1}^n a'_i$, *where* $a'_i \in \mathfrak{p}_{Ci}$, *then* $\mathrm{ind}(1 + a_i) = \mathrm{ind}(1 + a'_i)$ *for all* $i = 1, \dots, n$.

Proof. (1) Since $a_i \in \mathfrak{p}_{Ci}$ for all i, we have $a_i a_j \in F_n$, provided that $i \neq j$. It follows that the elements $f := u - (1 + a_1)(1 + a_2) \cdots (1 + a_n)$ and $f' := u - (1 + a_2) \cdots (1 + a_n)(1 + a_1)$ belong to the ideal F_n. By Corollary 3.7 (2), $u - f, u - f' \in \mathcal{C}(P_n)_0$. It then follows from the equalities $u - f = (1 + a_1)(1 + a_2) \cdots (1 + a_n)$ and $u - f' = (1 + a_2) \cdots (1 + a_n)(1 + a_1)$ that

$$\mathrm{im}(1 + a_1) \supseteq \mathrm{im}(u - f) \quad \text{and} \quad \ker(1 + a_1) \subseteq \ker(u - f').$$

This means that $1 + a_1 \in \mathcal{C}(P_n)$. By symmetry, $1 + a_i \in \mathcal{C}(P_n)$ for all i. By Corollary 3.7 (2) and Lemma 3.3,

$$0 = \mathrm{ind}(u) = \mathrm{ind}(u - f) = \mathrm{ind}(1 + a_1) \cdots (1 + a_n) = \sum_{i=1}^n \mathrm{ind}(1 + a_i).$$

(2) For each number i, $f_i := a'_i - a_i = -\sum_{j \neq i}(a'_j - a_j) \in \mathfrak{p}_{Ci} \bigcap \mathfrak{p}_i = \bigcap_{j=1}^n \mathfrak{p}_j = F_n$. Since $F_n \subseteq \mathcal{I}(P_n)$, we see that $\mathrm{ind}(1 + a'_i) = \mathrm{ind}(1 + a_i + f_i) = \mathrm{ind}(1 + a_i)$, by Theorem 3.5. $\square$

By Lemma 3.8, for each number $i = 1, \dots, n$ there is a well-defined map

$$\mathrm{ind}_i \colon (1 + \mathfrak{a}_{n,n-1})^* \to \mathbb{Z}, \quad u = 1 + \sum_{j=1}^n a_j \mapsto \mathrm{ind}(1 + a_i) \tag{3.5}$$

(where $a_i \in \mathfrak{p}_{Ci}$ for $i = 1, \dots, n$) that is a group homomorphism

$$\begin{aligned}
\mathrm{ind}_i(uu') &= \mathrm{ind}_i\left(\left(1 + \sum_{j=1}^n a_j\right)\left(1 + \sum_{k=1}^n a'_k\right)\right) \\
&= \mathrm{ind}(1 + a_i + a'_i + a_i a'_i) \\
&= \mathrm{ind}((1 + a_i)(1 + a'_i)) \\
&= \mathrm{ind}(1 + a_i) + \mathrm{ind}(1 + a'_i) \\
&= \mathrm{ind}_i(u) + \mathrm{ind}_i(u')
\end{aligned}$$

since $a_j a'_j \in \mathfrak{p}_{Cj}$ for all j and $a_j a'_k \in F_n$ for all $j \neq k$. Let $\mathcal{K}_{n,n-1}$ be the kernel of the group epimorphism

$$\bigoplus_{i=1}^{n-1} \mathrm{ind}_i \colon (1 + \mathfrak{a}_{n,n-1})^* \to \mathbb{Z}^{n-1} = \bigoplus_{i=1}^{n-1} \mathbb{Z} e_i, 1 + \sum_{i=1}^n a_i \mapsto \sum_{i=1}^{n-1} \mathrm{ind}(1 + a_i) \cdot e_i,$$

where $a_i \in \mathfrak{p}_{Ci}$ for $i = 1, \dots, n$. The restriction of the epimorphism to the subset $\Theta'_{n,n-1} := {}^{\mathrm{exact}}\prod_{j=1}^{n-1} \langle \theta_{n,j} \rangle$ is a bijection since (by (3.3) and (3.4))

$$\bigoplus_{i=1}^{n-1} \mathrm{ind}_i(\theta_{j,j+1}) = \begin{cases} e_j - e_{j+1} & \text{if } j < n - 1, \\ e_{n-1} & \text{if } j = n - 1. \end{cases}$$

Therefore,

$$(1+\mathfrak{a}_{n,n-1})^* = {}^{\text{exact}}\Theta'_{n,n-1}\cdot \mathcal{K}_{n,n-1}, \quad \text{where } \mathcal{K}_{n,n-1} = \bigcap_{i=1}^{n} \ker(\text{ind}_i), \tag{3.6}$$

by Lemma 3.8 (1). So, $\mathcal{K}_{n,n-1}$ is a normal subgroup of the group $(1+\mathfrak{a}_{n,n-1})^*$, $\Theta'_{n,n-1}\cap \mathcal{K}_{n,n-1}=\{1\}$ and each element u of the group $(1+\mathfrak{a}_{n,n-1})^*$ is a unique product vw for some elements $v\in\Theta'_{n,n-1}$ and $w\in\mathcal{K}_{n,n-1}$. The subgroups $(1+\mathfrak{p}_{Ci})^*$, $i=1,\dots,n$, of the groups $(1+\mathfrak{a}_{n,n-1})^*$ and $(1+\mathfrak{a}_n)^*$ are normal and $(1+\mathfrak{p}_{Ci})^*\bigcap(1+\mathfrak{p}_{Cj})^*=(1+F_n)^*$ for all $i\neq j$. The product $\prod_{i=1}^n(1+\mathfrak{p}_{Ci})^* := \{u_1\cdots u_n \mid u_i\in(1+\mathfrak{p}_{Ci})^*,\ i=1,\dots,n\}$ is a normal subgroup of $(1+\mathfrak{a}_{n,n-1})^*$ and $(1+\mathfrak{a}_n)^*$. In fact, the order in the product can be arbitrary (by normality). Clearly, $\prod_{i=1}^n(1+\mathfrak{p}_{Ci})^*\subseteq\mathcal{K}_{n,n-1}$. In fact, equality holds, as the next proposition shows.

Proposition 3.9.

(1) $\mathcal{K}_{n,n-1}=\prod_{i=1}^n(1+\mathfrak{p}_{Ci})^*$.

(2)

$$\begin{aligned}(1+\mathfrak{a}_{n,n-1})^* &= {}^{\text{exact}}\Theta'_{n,n-1}\cdot\left(\prod_{i=1}^n(1+\mathfrak{p}_{Ci})^*\right)\\ &= \langle\theta_{n,1}\rangle\ltimes\cdots\ltimes\langle\theta_{n,n-1}\rangle\ltimes\left(\prod_{i=1}^n(1+\mathfrak{p}_{Ci})^*\right).\end{aligned}$$

Proof. (1) It suffices to show that each element $u=1+\sum_{i=1}^n a_i$ (where $a_i\in\mathfrak{p}_{Ci}$) of the group $\mathcal{K}_{n,n-1}$ is a product $u_1\cdots u_n$ of some elements $u_i\in(1+\mathfrak{p}_{Ci})^*$. By Lemma 3.8, $1+a_1\in\mathcal{C}(P_n)_0$ since $u\in\mathcal{K}_{n,n-1}$. Fix a subspace, say W, of P_n such that $P_n=\ker(1+a_1)\oplus W$ and $W=\bigoplus_{\alpha\in I}Kx^\alpha$, where I is a subset of $\mathbb{N}^n$. By Lemma 3.6, we can find an element $f_1\in F_n$ (since $\dim\ker(1+a_1)<\infty$, W has a monomial basis and $f_1(W)=0$) such that $u_1:=1+a_1+f_1\in\text{Aut}_K(P_n)$. We claim that $u_1\in(1+\mathfrak{p}_{C1})^*$. It is a subtle point since *not all* elements of the algebra $\mathbb{S}_n$ that are invertible linear maps in P_n are invertible in $\mathbb{S}_n$, i.e. $\mathbb{S}_n^*\subsetneqq\mathbb{S}_n\cap\text{Aut}_K(P_n)$, but (see [**7**])

$$(1+F_n)^*=(1+F_n)\cap\text{Aut}_K(P_n).$$

The main idea in the proof of the claim is to use this equality. Similarly, for each $i\geqslant 2$, we can find an element $f_i\in F_n$ such that $v_i:=1+a_i+f_i\in\text{Aut}_K(P_n)$. Then $v:=v_2\cdots v_n\in\text{Aut}_K(P_n)$, $u=u_1v+g_1$ and $u=vu_1+g_2$ for some elements $g_i\in F_n$. Hence,

$$u_1vu^{-1}=1-g_1u^{-1}\quad\text{and}\quad u^{-1}vu_1=1-u^{-1}g_2,$$

and so $1-g_1u^{-1}, 1-u^{-1}g_2\in(1+F_n)\cap\text{Aut}_K(P_n)=(1+F_n)^*$. It follows that

$$u_1^{-1}=vu^{-1}(1-g_1u^{-1})^{-1}\in(1+\mathfrak{p}_{C1})^*$$

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since

$$1 \equiv 1 - g_1 u^{-1} \equiv u_1 v u^{-1} \equiv v u^{-1} \mod \mathfrak{p}_{C1}.$$

This proves the claim. Clearly,

$$u_2' := v + u_1^{-1} g_1 \in 1 + \sum_{j=2}^{n} \mathfrak{p}_{Cj},$$

where $v \in 1 + \sum_{j=2}^{n} \mathfrak{p}_{\mathcal{C}_j}$ and $u_1^{-1} g_1 \in F_n$. It then follows from the equality $u = u_1 v + g_1 = u_1(v + u_1^{-1} g_1) = u_1 u_2'$ that $u_2' = u_1^{-1} u \in (1 + \sum_{j=2}^{n} \mathfrak{p}_{Cj})^*$. Repeating the same argument for the element u_2' we find an element $u_2 \in (1 + \mathfrak{p}_{C2})^*$ such that $u_3' := u_2^{-1} u_2' \in (1 + \sum_{j=3}^{n} \mathfrak{p}_{Cj})^*$. Repeating the same argument again and again (or using induction) we find elements $u_i \in (1 + \mathfrak{p}_{Ci})^*$ and elements $u_i' \in (1 + \sum_{j=i+1}^{n} \mathfrak{p}_{Cj})^*$ such that $u_i' = u_{i-1}^{-1} u_{i-1}'$, and hence $u = u_1 u_2' = u_1 u_2 u_3' = \cdots = u_1 u_2 \cdots u_n$, as required.

(2) Statement (2) follows from statement (1) and (3.6). □

For each number $i = 1, \dots, n$ the group of units of the monoid $1 + \mathfrak{p}_{Ci} = 1 + \mathbb{S}_1(i) \otimes \bigotimes_{j \neq i} F(j) \simeq 1 + M_\infty(\mathbb{S}_1(i))$ is equal to $(1 + \mathfrak{p}_{Ci})^* \simeq \mathrm{GL}_\infty(\mathbb{S}_1(i))$. This group contains the semi-direct product $U_{Ci}(K) \ltimes E_\infty(\mathbb{S}_1(i))$ of its two subgroups, where

$$U_{Ci}(K) := \left\{ \lambda \prod_{j \neq i} E_{00}(j) + 1 - \prod_{j \neq i} E_{00}(j) \;\middle|\; \lambda \in K^* \right\} \simeq K^*$$

and the group $E_\infty(\mathbb{S}_1(i))$ is generated by all the elementary matrices $1 + aE_{kl}(Ci)$, where $k, l \in \mathbb{N}^{n-1}$, $k \neq l$, $E_{kl}(Ci) := \prod_{j \neq i} E_{k_j l_j}(j)$ and $a \in \mathbb{S}_1(i)$. We will see in Proposition 3.10 that the group $(1 + \mathfrak{p}_{Ci})^*$ coincides with the semi-direct product.

The set F_n is an ideal of the algebra $K + \mathfrak{p}_{Ci} = K(1 + \mathfrak{p}_{Ci})$, which is a subalgebra of the algebra $\mathbb{S}_n$, and $(K + \mathfrak{p}_{Ci})/F_n = K(1 + \mathfrak{p}_{Ci}/F_n) \simeq K(1 + M_\infty(L_i))$, where $L_i := K[x_i, x_i^{-1}] \simeq \mathbb{S}_1(i)/F(i)$ is the Laurent polynomial algebra. The algebra L_i is a Euclidean domain, and hence $\mathrm{GL}_\infty(L_i) = U(L_i) \ltimes E_\infty(L_i)$, where

$$U(L_i) := \left\{ a \prod_{j \neq i} E_{00}(j) + 1 - \prod_{j \neq i} E_{00}(j) \;\middle|\; a \in L_i^* \right\} \simeq L_i^* = K^* \times \{x_i^m \mid m \in \mathbb{Z}\}$$

and $E_\infty(L_i)$ is the subgroup of $\mathrm{GL}_\infty(L_i)$ generated by all the elementary matrices. This statement follows from two facts: (i) every matrix over a Euclidean domain is conjugate to a diagonal matrix and (ii) every diagonal matrix in GL_∞ over an arbitrary ring, say L_i, is conjugate to a matrix in $U(L_i)$.

The group of units of the algebra $(K + \mathfrak{p}_{Ci})/F_n$ is equal to $K^* \times \mathrm{GL}_\infty(L_i) = K^* \times (U(L_i) \ltimes E_\infty(L_i))$. The algebra epimorphism $\psi_{Ci} \colon K + \mathfrak{p}_{Ci} \to (K + \mathfrak{p}_{Ci})/F_n$, $a \mapsto a + F_n$ induces the exact sequence of groups

$$1 \to (1 + F_n)^* \to (1 + \mathfrak{p}_{Ci})^* \xrightarrow{\psi_{Ci}} \mathrm{GL}_\infty(L_i) = U(L_i) \ltimes E_\infty(L_i), \tag{3.7}$$

which yields the short exact sequence of groups

$$1 \to (1 + F_n)^* \to U_{Ci}(K) \ltimes E_\infty(\mathbb{S}_1(i)) \to U(K) \ltimes E_\infty(L_i) \to 1, \tag{3.8}$$

since $(1+F_n)^* \subseteq E_\infty(\mathbb{S}_1(i))$, by Proposition 3.12 (1). Recall that $U_{Ci}(K) \ltimes E_\infty(\mathbb{S}_1(i)) \subseteq (1+\mathfrak{p}_{Ci})^*$. In fact, equality holds.

Proposition 3.10. *$(1+\mathfrak{p}_{Ci})^* = U_{Ci}(K) \ltimes E_\infty(\mathbb{S}_1(i))$ and the image $\mathrm{im}(\psi_{Ci}) = U(K) \ltimes E_\infty(L_i)$ is a normal subgroup of $\mathrm{GL}_\infty(L_i)$ for all $i = 1, \dots, n$.*

Proof. In view of the exact sequences (3.7) and (3.8), it suffices to show that the image of the map ψ_{Ci} in (3.7) is equal to $U(K) \ltimes E_\infty(L_i)$, which is a normal subgroup of $\mathrm{GL}_\infty(L_i)$. Since $\psi_{Ci}(U_{Ci}(K) \ltimes E_\infty(\mathbb{S}_1(i))) = U(K) \ltimes E_\infty(L_i)$ and

$$U(L_i) = U(K) \times \left\{ x_i^m \prod_{j\neq i} E_{00}(j) + 1 - \prod_{j\neq i} E_{00}(j) \;\middle|\; m \in \mathbb{Z} \right\},$$

this is equivalent to showing that if $\psi_{Ci}(u) = x_i^m \prod_{j\neq i} E_{00}(j) + 1 - \prod_{j\neq i} E_{00}(j)$ for some element $u \in (1+\mathfrak{p}_{Ci})^*$ and an integer $m \in \mathbb{Z}$, then $m = 0$. Let $u(m) := v_i(m)\prod_{j\neq i} E_{00}(j) + 1 - \prod_{j\neq i} E_{00}(j)$, where

$$v_i(m) := \begin{cases} x_i^m & \text{if } m \geqslant 0, \\ y_i^{|m|} & \text{if } m < 0. \end{cases}$$

Then $u(m) \in 1 + \mathfrak{p}_{Ci}$ and $\psi_{Ci}(u(m)) = \psi_{Ci}(u)$. Hence, $u(m) = u + f_m$ for some element $f_m \in F_n$. Note that

$$u(m) = \begin{cases} u(1)^m & \text{if } m \geqslant 0, \\ u(-1)^{|m|} & \text{if } m < 0 \end{cases}$$

and, by (3.3) and (3.4), $\mathrm{ind}(u(m)) = -m$. By Corollary 3.7 (2),

$$0 = \mathrm{ind}(u) = \mathrm{ind}(u + f_m) = \mathrm{ind}(u(m)) = -m$$

and so $m = 0$, as required. □

Combining Proposition 3.9 (1) and Proposition 3.10, we have the next corollary.

Corollary 3.11.

$$\begin{aligned}(1+\mathfrak{a}_{n,n-1})^* &= \Theta'_{n,n-1} \times_{\mathrm{ex}} \left({}^{\mathrm{set}}\prod_{i=1}^{n} (1+\mathfrak{p}_{Ci})^* \right) \\ &\simeq \Theta'_{n,n-1} \times_{\mathrm{ex}} \left({}^{\mathrm{set}}\prod_{i=1}^{n} U_{Ci}(K) \ltimes E_\infty(\mathbb{S}_1(i)) \right) \\ &\simeq \langle \theta_{n,1} \rangle \ltimes \cdots \ltimes \langle \theta_{n,n-1} \rangle \ltimes \left({}^{\mathrm{set}}\prod_{i=1}^{n} U_{Ci}(K) \ltimes E_\infty(\mathbb{S}_1(i)) \right).\end{aligned}$$

Using Corollary 3.11, we can write down explicit generators for the group $(1+\mathfrak{a}_{n,n-1})^*$ (see Theorem 4.5 where explicit generators are given for all the groups $(1+\mathfrak{a}_{n,s})^*$). By Proposition 3.10, the sequence (3.7) can be completed to the exact sequence of group homomorphisms

$$1 \to (1+F_n)^* \to (1+\mathfrak{p}_{Ci})^* \xrightarrow{\psi_{Ci}} \mathrm{GL}_\infty(L_i) \xrightarrow{\deg_{x_i}} \mathbb{Z} \to 1, \tag{3.9}$$

where $\deg_{x_i}(x_i^m \prod_{j\neq i} E_{00}(j) + 1 - \prod_{j\neq i} E_{00}(j)) = m$.

For elements g and h of a group, $[g,h] := ghg^{-1}h^{-1}$ is their *group commutator*.

Proposition 3.12.

(1) $(1+F_n)^* \subseteq E_\infty(\mathbb{S}_1(i))$ *for all* $i = 1,\dots,n$, *where* $E_\infty(\mathbb{S}_1(i))$ *is the subgroup of* $(1+\mathfrak{p}_{Ci})^*$ *generated by all the elementary matrices* $1+aE_{\alpha\beta}(Ci)$, *where* $a \in \mathbb{S}_1(i)$, $\alpha,\beta \in \mathbb{N}^{n-1}$ *and* $\alpha \neq \beta$, *and*

$$E_{\alpha\beta} := E_{\alpha\beta}(Ci) := \prod_{j\neq i} E_{\alpha_j\beta_j}(j).$$

(2) *For all* $i \neq j$, $E_\infty(\mathbb{S}_1(i)) \cap E_\infty(\mathbb{S}_1(j)) = (1+F_n)^*$. *In particular,* $\bigcap_{i=1}^n E_\infty(\mathbb{S}_1(i)) = (1+F_n)^*$.

Proof. (1) In view of symmetry of the indices $1,\dots,n$, it suffices to show that the inclusion holds for, say, $i=n$, i.e. $(1+F_n)^* \subseteq E_\infty(\mathbb{S}_1(n))$. Since $(1+F_n)^* \simeq \mathrm{GL}_\infty(K)$, the group $(1+F_n)^*$ is generated by two sorts of elements: $a = 1+\lambda E_{\alpha\beta}E_{kl}(n)$, where $\lambda \in K$ and $(\alpha_1,\dots,\alpha_{n-1},k) \neq (\beta_1,\dots,\beta_{n-1},l)$, and $b = 1+\lambda E_{00}$, where $\lambda \in K\setminus\{-1\}$ and $E_{00} := \prod_{i=1}^n E_{00}(i)$.

First, let us show that $a \in E_\infty(\mathbb{S}_1(n))$. If $\alpha \neq \beta$, then the inclusion obviously holds since $a = 1+(\lambda E_{kl}(n))E_{\alpha\beta}$ and $\lambda E_{kl}(n) \in \mathbb{S}_1(i)$. If $\alpha = \beta$, i.e. $a = 1+\lambda E_{\alpha\alpha}E_{kl}(n)$, then necessarily $k \neq l$ since $(\alpha_1,\dots,\alpha_{n-1},k) \neq (\alpha_1,\dots,\alpha_{n-1},l)$. For each element $\gamma \in \mathbb{N}^{n-1}$ such that $\gamma \neq \alpha$, the elements $1+E_{\alpha\gamma}E_{kk}(n)$ and $1+\lambda E_{\gamma\alpha}E_{kl}(n)$ belong to the group $E_\infty(\mathbb{S}_1(n))$ and so do their group commutators

$$[1+E_{\alpha\gamma}E_{kk}(n), 1+\lambda E_{\gamma\alpha}E_{kl}(2)] = 1+\lambda E_{\alpha\alpha}E_{kl}(n). \tag{3.10}$$

Therefore, all the generators a belong to the group $E_\infty(\mathbb{S}_1(n))$.

It remains to prove that $b \in E_\infty(\mathbb{S}_1(n))$. In the 2×2 matrix ring $M_2(\mathbb{S}_1(n))$ with entries in the algebra $\mathbb{S}_1(n)$ we have the equality, for all scalars $\lambda \in K\setminus\{-1\}$,

$$\begin{pmatrix} 1 & 0 \\ -\dfrac{y_n}{1+\lambda} & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda x_n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y_n & 1 \end{pmatrix}\begin{pmatrix} 1 & -\lambda x_n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \dfrac{\lambda^2 x_n}{1+\lambda} \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1+\lambda & 0 \\ 0 & \dfrac{1}{1+\lambda} \end{pmatrix}\begin{pmatrix} 1-\dfrac{\lambda E_{00}(n)}{1+\lambda} & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.11}$$

This can be checked by direct multiplication using the equalities $y_nx_n = 1$, $x_ny_n = 1-E_{00}(n)$, $y_nE_{00}(n) = 0$ and $E_{00}(n)x_n = 0$ that hold in the algebra $\mathbb{S}_1(n)$. If we replace the matrix ring $M_2(\mathbb{S}_1(n)) = \bigoplus_{i,j=0}^1 E_{ij}\mathbb{S}_1(n)$ by its isomorphic copy $M_2' := \bigoplus_{i,j=0}^1 eE_{ij}(n-1)\mathbb{S}_1(n)$, where

$$e := \begin{cases} \prod_{i=1}^{n-2} E_{00}(i) & \text{if } n \geqslant 3, \\ 1 & \text{if } n = 2, \end{cases}$$

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then the equality (3.11) can be seen as an equality in the ring M'_2. In this case, the first six matrices in the equality belong to the group $E_\infty(\mathbb{S}_1(n))$. Therefore, the last matrix

$$c = \begin{pmatrix} 1 - \dfrac{\lambda E_{00}(n)}{1+\lambda} & 0 \\ 0 & 1 \end{pmatrix}$$

belongs to the group $E_\infty(\mathbb{S}_1(n))$ and as an element of the group $E_\infty(\mathbb{S}_1(n))$ it can be written as

$$\begin{aligned} c &= eE_{00}(n-1)\left(1 - \frac{\lambda}{1+\lambda}E_{00}(n)\right) + 1 - eE_{00}(n-1) \\ &= 1 - \frac{\lambda}{1+\lambda}\prod_{i=0}^{n} E_{00}(i) \in (1+F_n)^*. \end{aligned}$$

Since the map $\varphi\colon K\setminus\{-1\} \to K\setminus\{-1\}$, $\lambda \mapsto -\lambda/(1+\lambda)$ is a bijection ($\varphi^{-1} = \varphi$), all the elements b belong to the group $E_\infty(\mathbb{S}_1(n))$. The proof of the first statement is complete.

(2) By statement (1), for all $i \neq j$,

$$(1+F_n)^* \subseteq E_\infty(\mathbb{S}_1(i)) \cap E_\infty(\mathbb{S}_1(j)) \subseteq (1+\mathfrak{p}_{Ci})^* \cap (1+\mathfrak{p}_{Cj})^* = (1+\mathfrak{p}_{Ci}\cap\mathfrak{p}_{Cj})^* = (1+F_n)^*$$

and so statement (2) is obvious. □

4. The structure of the groups $\mathbb{S}_n^*$ and G_n, and their generators

In this section a proof of Theorem 2.6 is given and the groups $\mathbb{S}_n^*$, $(1+\mathfrak{a}_n)^*$ and G_n and their generators are found explicitly (see Theorems 4.1, 4.2, 4.5 and 4.6).

Proof of Theorem 2.6. To prove the theorem we use induction on n. The initial step when $n = 2$ follows from Corollary 3.11 as in this case there is only one option, $(n, s) = (2, 1)$. So, let $n > 2$ and suppose that the theorem holds for all pairs (n', s'), $s' = 1, \dots, n'-1$, such that $n' < n$. For the number n, we use a second downward induction on $s = 1, \dots, n-1$ starting with $s = n-1$. In this case, i.e. $(n, s) = (n, n-1)$, the theorem holds as it follows from Corollary 3.11. So, let $s < n-1$ and suppose that the statement is true for all pairs (n, s') with $s' = s+1, \dots, n-1$. For each number $i = 1, \dots, n$ the algebra $\mathbb{S}_{Ci} \otimes K(x_i)$ is isomorphic to the algebra $\mathbb{S}_{n-1}$ but over the field $K(x_i)$ of rational functions. By the induction on n, the theorem holds for the algebra $\mathbb{S}_{Ci} \otimes K(x_i)$. In order to stress that we consider the algebra $\mathbb{S}_{Ci}$ over the field $K(x_i)$ rather than K we add the subscript 'Ci' to all the notation introduced for the algebra $\mathbb{S}_{Ci}$ but over the field K. For example, $\mathfrak{a}_{n-1,s,Ci} \otimes K(x_i)$ stands for the ideal $\mathfrak{a}_{n-1,s}$ of the algebra $\mathbb{S}_{Ci}$ but over the field $K(x_i)$, etc.

For each number $i = 1, \dots, n$ and for each number $s = 1, \dots, n-2$ the composition of the two algebra homomorphisms

$$\mathbb{S}_n \to \mathbb{S}_n/\mathfrak{p}_i \simeq \mathbb{S}_{Ci} \otimes K[x_i, x_i^{-1}] \to \mathbb{S}_{Ci} \otimes K(x_i)$$

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induces the group homomorphism $(1+\mathfrak{a}_{n,s})^* \to (1+\mathfrak{a}_{n-1,s,Ci}\otimes K(x_i))^*$. This homomorphism yields the commutative diagram where all the maps are obvious (and natural):

$$\begin{array}{ccc}
(1+\mathfrak{a}_{n,s})^* & \longrightarrow & (1+\mathfrak{a}_{n-1,s,Ci}\otimes K(x_i))^* \\
\downarrow & & \downarrow \\
\dfrac{(1+\mathfrak{a}_{n,s})^*}{\Gamma_{n,s}} & \longrightarrow & \dfrac{(1+\mathfrak{a}_{n-1,s,Ci}\otimes K(x_i))^*}{\Gamma_{n-1,s,Ci}} \\
\downarrow{\scriptstyle \bar{\psi}_{n,s}} & & \downarrow{\scriptstyle \bar{\psi}_{n-1,s,Ci}} \\
\dfrac{\prod_{|I|=s}(1+\bar{\mathfrak{p}}_I)^*}{\bar{\Gamma}_{n,s}} \simeq \mathbb{X}_{n,s} \simeq \mathbb{X}'_{n,s} & \xrightarrow{\varphi_{n,s,i}} & \dfrac{\prod'(1+\bar{\mathfrak{p}}_{I,Ci}\otimes K(x_i))^*}{\bar{\Gamma}_{n-1,s,Ci}} \simeq \mathbb{X}_{n-1,s,Ci} \simeq \mathbb{X}'_{n-1,s,Ci}
\end{array}$$

where $\prod' := \prod_{\{I:\ |I|=s,\ i\notin I\}}$ and the map $\varphi_{n,s,i}\colon \mathbb{X}'_{n,s} \to \mathbb{X}'_{n-1,s,Ci}$ is given by the rule

$$\varphi_{n,s,i}((j,I)) = \begin{cases} (j,I) & \text{if } i \notin I \cup j, \\ 0 & \text{otherwise.} \end{cases}$$

This is obvious. By the induction on n, we have the equality $\psi'_{n-1,s,Ci}((1+\mathfrak{a}_{n-1,s,Ci}\otimes K(x_i))^*) = \psi'_{n-1,s,Ci}(\Theta_{n-1,s,Ci})$ for each $s=1,\dots,n-2$. Then, by the commutative diagram above,

$$\varphi_{n,s,i}\psi'_{n,s}((1+\mathfrak{a}_{n,s})^*) \subseteq \psi'_{n-1,s,Ci}((1+\mathfrak{a}_{n-1,s,Ci}\otimes K(x_i))^*) = \psi'_{n-1,s,Ci}(\Theta_{n-1,s,Ci}). \tag{4.1}$$

It follows from the definition of the map $\varphi_{n,s,i}$ that

$$\varphi_{n,s,i}(\mathbb{Y}'_{n,s}) \subseteq \mathbb{Y}'_{n-1,s,Ci}. \tag{4.2}$$

Summarizing, for each $i=1,\dots,n$, by (2.19) and (2.21), there is the map

$$\varphi_{n,s,i}\colon \mathbb{X}'_{n,s} = \psi'_{n,s}(\Theta_{n,s}) \oplus \mathbb{Y}'_{n,s} \to \mathbb{X}'_{n-1,s,Ci} = \psi'_{n-1,s,Ci}(\Theta_{n-1,s,Ci}) \oplus \mathbb{Y}'_{n-1,s,Ci}$$

satisfying (4.1) and (4.2). The group homomorphism

$$\varphi_{n,s} := \prod_{i=1}^{n} \varphi_{n,s,i}\colon \mathbb{X}'_{n,s} \to \prod_{i=1}^{n} \mathbb{X}'_{n-1,s,Ci}$$

is a monomorphism since it has trivial kernel: $\ker(\varphi_{n,s}) = \oplus\{\mathbb{Z}(j,I) \mid \forall i \in I \cup j\}$, where the pairs (j,I) in the direct sum are such that $i \in I \cup j$ for all $i=1,\dots,n$ (see the definition of the map $\varphi_{n,s,i}$), i.e. $I \cup j = \{1,\dots,n\}$, but the number of elements in the set $I \cup j$ is $s+1 < n-1+1 = n$, a contradiction. This means that $\ker(\varphi_{n,s}) = 0$. Let $u \in (1+\mathfrak{a}_{n,s})^*$. Then $\psi'_{n,s}(u) = a+b$ for unique elements $a \in \psi'_{n,s}(\Theta_{n,s})$ and $b \in \mathbb{Y}'_{n,s}$. By (4.1) and (4.2), $\varphi_{n,s,i}(b) = 0$ for all $i=1,\dots,n$, i.e. $\varphi_{n,s}(b) = 0$ and so $b=0$ since the map $\varphi_{n,s}$ is a monomorphism. This proves that $\psi'_{n,s}((1+\mathfrak{a}_{n,s})^*) = \psi_{n,s}(\Theta_{n,s})$. By induction, the theorem holds. The proof of Theorem 2.6 is complete. □

For each number $s = 1, \dots, n-1$, consider the following subsets of the group $(1+\mathfrak{a}_{n,s})^*$:

$$\mathbb{E}_{n,s} := \prod_{|I|=s} U_I(K) \ltimes E_\infty(\mathbb{S}_{CI}) \quad \text{and} \quad \mathbb{P}_{n,s} := \prod_{|I|=s} (1+\mathfrak{p}_I)^*, \tag{4.3}$$

the products of subgroups of $(1+\mathfrak{a}_{n,s})^*$ in an arbitrary order that is fixed for each s.

Theorem 4.1.

(1) $(1+\mathfrak{a}_n)^* = \Theta_{n,1}\Gamma_{n,1} = \Theta_{n,1}\mathbb{E}_{n,1}\Theta_{n,2}\mathbb{E}_{n,2}\cdots\Theta_{n,n-1}\mathbb{E}_{n,n-1}$. *Moreover, for* $s = 1, \dots, n-1$,

$$(1+\mathfrak{a}_{n,s})^* = \Theta_{n,s}\Gamma_{n,s} = \Theta_{n,s}\mathbb{E}_{n,s}\Theta_{n,s+1}\mathbb{E}_{n,s+1}\cdots\Theta_{n,n-1}\mathbb{E}_{n,n-1}.$$

(2) $(1+\mathfrak{a}_n)^* = \Theta_{n,1}\Upsilon_{n,1} = \Theta_{n,1}\mathbb{P}_{n,1}\Theta_{n,2}\mathbb{P}_{n,2}\cdots\Theta_{n,n-1}\mathbb{P}_{n,n-1}$. *Moreover, for* $s = 1, \dots, n-1$,

$$(1+\mathfrak{a}_{n,s})^* = \Theta_{n,s}\Upsilon_{n,s} = \Theta_{n,s}\mathbb{P}_{n,s}\Theta_{n,s+1}\mathbb{P}_{n,s+1}\cdots\Theta_{n,n-1}\mathbb{P}_{n,n-1}.$$

Proof. (1) By Theorem 2.6 and Corollary 3.11,

$$\begin{aligned}(1+\mathfrak{a}_{n,s})^* &= \Theta_{n,s}\Gamma_{n,s}\\ &= \Theta_{n,s}\prod_{|I|=s} U_I(K) \ltimes E_\infty(\mathbb{S}_{CI})\cdot(1+\mathfrak{a}_{n,s-1})^*\\ &= \Theta_{n,s}\mathbb{E}_{n,s}(1+\mathfrak{a}_{n,s-1})^*\\ &= \Theta_{n,s}\mathbb{E}_{n,s}\Theta_{n,s-1}\mathbb{E}_{n,s-1}(1+\mathfrak{a}_{n,s-2})^*\\ &= \Theta_{n,s}\mathbb{E}_{n,s}\cdots\Theta_{n,n-2}\mathbb{E}_{n,n-2}(1+\mathfrak{a}_{n,n-1})^*\\ &= \Theta_{n,s}\mathbb{E}_{n,s}\Theta_{n,s+1}\mathbb{E}_{n,s+1}\cdots\Theta_{n,n-1}\mathbb{E}_{n,n-1}.\end{aligned}$$

(2) Since $(1+\mathfrak{a}_{n,s})^* = \Theta_{n,s}\Gamma_{n,s} \subseteq \Theta_{n,s}\Upsilon_{n,s} \subseteq (1+\mathfrak{a}_{n,s})^*$, we see that

$$\begin{aligned}(1+\mathfrak{a}_{n,s})^* &= \Theta_{n,s}\Upsilon_{n,s}\\ &= \Theta_{n,s}\prod_{|I|=s}(1+\mathfrak{p}_I)^*\cdot(1+\mathfrak{a}_{n,s-1})^*\\ &= \Theta_{n,s}\mathbb{P}_{n,s}(1+\mathfrak{a}_{n,s-1})^*\\ &= \Theta_{n,s}\mathbb{P}_{n,s}\Theta_{n,s-1}\mathbb{P}_{n,s-1}(1+\mathfrak{a}_{n,s-2})^*\\ &= \Theta_{n,s}\mathbb{P}_{n,s}\cdots\Theta_{n,n-2}\mathbb{P}_{n,n-2}(1+\mathfrak{a}_{n,n-1})^*\\ &= \Theta_{n,s}\mathbb{P}_{n,s}\Theta_{n,s+1}\mathbb{P}_{n,s+1}\cdots\Theta_{n,n-1}\mathbb{P}_{n,n-1},\end{aligned}$$

by Corollary 3.11. □

Using Lemma 2.5, we can strengthen Theorem 4.1.

Theorem 4.2.

(1) $(1+\mathfrak{a}_n)^* = \Theta'_{n,1}\mathbb{E}_{n,1}\Theta'_{n,2}\mathbb{E}_{n,2}\cdots\Theta'_{n,n-1}\mathbb{E}_{n,n-1}$. *Moreover, for* $s=1,\dots,n-1$, $(1+\mathfrak{a}_{n,s})^* = \Theta'_{n,s}\mathbb{E}_{n,s}\Theta'_{n,s+1}\mathbb{E}_{n,s+1}\cdots\Theta'_{n,n-1}\mathbb{E}_{n,n-1}$.

(2) $(1+\mathfrak{a}_n)^* = \Theta'_{n,1}\mathbb{P}_{n,1}\Theta'_{n,2}\mathbb{P}_{n,2}\cdots\Theta'_{n,n-1}\mathbb{P}_{n,n-1}$. *Moreover, for* $s=1,\dots,n-1$, $(1+\mathfrak{a}_{n,s})^* = \Theta'_{n,s}\mathbb{P}_{n,s}\Theta'_{n,s+1}\mathbb{P}_{n,s+1}\cdots\Theta'_{n,n-1}\mathbb{P}_{n,n-1}$.

Proof. The statements follow from Lemma 2.5, (2.22) and Theorem 4.1: repeat the proof of Theorem 4.1 replacing $\Theta_{n,t}$ by $\Theta'_{n,t}$ everywhere for all t. □

By Theorem 4.1 (1) and Lemma 2.5, the group $\Delta_{n,s}$ is a free abelian group of rank $\binom{n}{s+1}s$ for $s=1,\dots,n-1$:

$$\Delta_{n,s} := (1+\mathfrak{a}_{n,s})^*/\Gamma_{n,s} = \Theta'_{n,s}\Gamma_{n,s}/\Gamma_{n,s} \simeq \prod_{|J|=s+1}\prod_{j\in J\backslash \max(J)}\langle\theta_{\max(J),j}\rangle \simeq \mathbb{Z}^{\binom{n}{s+1}s}, \tag{4.4}$$

where the double product is the direct product of groups.

Corollary 4.3. $\mathcal{Z}_{n,s}\simeq \mathbb{Y}'_{n,s}\simeq \mathbb{Z}^{\binom{n}{s+1}}$ *for* $s=1,\dots,n-1$ *(see (2.10)).*

Proof. Recall that $\psi'_{n,s}((1+\mathfrak{a}_{n,s})^*) = \psi'_{n,s}(\Theta_{n,s})$ (see Theorem 2.6), $\mathbb{X}'_{n,s} = \mathbb{K}'_{n,s}\oplus \mathbb{Y}'_{n,s}$ and $\psi'_{n,s}(\Theta_{n,s}) = \mathbb{K}'_{n,s}$, by (2.21). Then

$$\begin{aligned}\mathcal{Z}_{n,s} &= \frac{\prod_{|I|=s}(1+\bar{\mathfrak{p}}_I)^*}{\psi_{n,s}((1+\mathfrak{a}_{n,s})^*)}\\ &\simeq \frac{\prod_{|I|=s}(1+\bar{\mathfrak{p}}_I)^*/\bar{\Gamma}_{n,s}}{\psi_{n,s}((1+\mathfrak{a}_{n,s})^*)/\bar{\Gamma}_{n,s}}\\ &\simeq \frac{\mathbb{X}'_{n,s}}{\psi'_{n,s}((1+\mathfrak{a}_{n,s})^*)}\\ &\simeq \frac{\mathbb{K}'_{n,s}\oplus\mathbb{Y}'_{n,s}}{\psi'_{n,s}(\Theta_{n,s})} = \frac{\mathbb{K}'_{n,s}\oplus\mathbb{Y}'_{n,s}}{\mathbb{K}'_{n,s}}\\ &\simeq \mathbb{Y}'_{n,s}\simeq \mathbb{Z}^{\binom{n}{s+1}}.\end{aligned}$$

□

Theorem 4.4. $\Upsilon_{n,s} = \Gamma_{n,s}$ *for all* $s=1,\dots,n$. *In particular, the groups* $\Gamma_{n,s}$ *are* G_n*-invariant (hence, normal) subgroups of* $\mathbb{S}_n^*$ *(since* $\Upsilon_{n,s}$ *are too).*

Proof. Since $\Upsilon_{n,n} = \Gamma_{n,n} = (1+F_n)^*$, we can assume that $s\neq n$. By Theorem 4.1 and Lemma 2.5, $(1+\mathfrak{a}_{n,s})^* = \Theta_{n,s}\Gamma_{n,s} = \Theta'_{n,s}\Gamma_{n,s}$ for $s=1,\dots,n-1$ and the last product is exact. Since $\Gamma_{n,s}\subseteq\Upsilon_{n,s}$, we have the equality $(1+\mathfrak{a}_{n,s})^* = \Theta'_{n,s}\Upsilon_{n,s}$. So, in order to show that the equality $\Gamma_{n,s} = \Upsilon_{n,s}$ holds, it suffices to prove that $\Theta'_{n,s}\cap\Upsilon_{n,s} = \{1\}$. To prove this equality, first we use an induction on $n\geqslant 2$ and then, for a fixed n, we use a second downward induction on $s=1,\dots,n-1$, starting with $s=n-1$. For $n=2$, there is a single option to consider, $(n,s)=(2,1)$. In this case the equality holds by Corollary 3.11.

Let $n > 2$ and suppose that equality holds for all pairs (n', s) with $n' < n$. For $(n, n-1)$, the equality is true by Corollary 3.11. Suppose that $s < n-1$ and that the equality holds for all pairs (n, s') with $s' = s+1, \dots, n-1$. Suppose that $\Theta'_{n,s} \cap \Upsilon_{n,s} \neq \{1\}$. We seek a contradiction. Choose an element, say u, from the intersection such that $u \neq 1$. Then the element u is a unique product $u = \prod_{|J|=s+1} \prod_{j \in J \setminus \max(J)} \theta_{\max(J),j}(J)^{n(j,J)}$, where $n(j, J) \in \mathbb{Z}$. Since $u \neq 1$, $n(j, J) \neq 0$ for some pair (j, J). Since $|J| = s+1 < n$, the complement CJ of the set J is a non-empty set. Let f be the composition of the obvious algebra homomorphisms

$$\mathbb{S}_n \to \mathbb{S}_n / \sum_{i \in CJ} \mathfrak{p}_i \simeq \mathbb{S}_J \otimes L_{CJ} \to \mathbb{S}_J \otimes Q_{CJ},$$

where Q_{CJ} is the field of fractions of the Laurent polynomial algebra L_{CJ}. The algebra $\mathbb{S}_J \otimes Q_{CJ}$ is isomorphic to the algebra $\mathbb{S}_{s+1}$ but over the field Q_{CJ}. Let $\Theta'_{s+1,s,J}$ and $\Gamma_{s+1,s,J}$ be the corresponding $\Theta'_{s+1,s}$ and $\Gamma_{s+1,s}$ for the algebra $\mathbb{S}_J \otimes Q_{CJ} \simeq \mathbb{S}_{s+1} \otimes Q_{CJ}$ (over the field Q_{CJ}). Since $f(\Gamma_{n,s}) \subseteq \Gamma_{s+1,s,J}$, $f(\Theta'_{n,s}) \subseteq \Theta'_{s+1,s,J} \cdot U = \prod_{k \in J \setminus \max(J)} \theta_{\max(J),k}(J)^{n(k,J)} \cdot U$, where $U := \prod_{l \in J} U_{J \setminus l}(Q_{CJ}) \subseteq \Gamma_{s+1,s,J}$, using the induction on n, the inclusion $f(u) \in f(\Theta'_{n,s}) \cap f(\Gamma_{n,s})$ yields $n(j, J) = 0$, a contradiction. Therefore, $\Theta'_{n,s} \cap \Upsilon_{n,s} = \{1\}$ and the statements of the theorem hold. $\square$

By (4.4) and Theorem 4.4,

$$(1 + \mathfrak{a}_{n,s})^* / \Upsilon_{n,s} = (1 + \mathfrak{a}_{n,s})^* / \Gamma_{n,s} \simeq \mathbb{Z}^{\binom{n}{s+1} s}. \tag{4.5}$$

The next theorem gives explicit generators for the groups $\mathbb{S}_n^*$, $(1 + \mathfrak{a}_n)^*$ and $(1 + \mathfrak{a}_{n,s})^*$.

Theorem 4.5.

(1) *The group $(1 + \mathfrak{a}_n)^*$ is generated by the following elements:*

- (a) $\theta_{\max(J),j}(J)$, *where* $j \in J \setminus \max(J)$ *and* $|J| = 2, \dots, n$;
- (b) $1 + x_i^t E_{0\alpha}(I)$, $1 + x_i^t E_{\alpha 0}(I)$, $1 + y_i^t E_{0\alpha}(I)$ *and* $1 + y_i^t E_{\alpha 0}(I)$, *where* $t \in \mathbb{N} \setminus \{0\}$, $i \notin I$, $|I| = 1, \dots, n-1$ *and* $\alpha \in \mathbb{N}^I \setminus \{0\}$;
- (c) $1 + (\lambda - 1) E_{00}(I)$, $1 + E_{0\alpha}(I)$ *and* $1 + E_{\alpha 0}(I)$, *where* $\lambda \in K^*$, $I \neq \emptyset$ *and* $\alpha \in \mathbb{N}^I \setminus \{0\}$.

(2) *For $s = 1, \dots, n-1$, the group $(1 + \mathfrak{a}_{n,s})^*$ is generated by the following elements:*

- (a) $\theta_{\max(J),j}(J)$, *where* $j \in J \setminus \max(J)$ *and* $|J| = s+1, \dots, n$;
- (b) $1 + x_i^t E_{0\alpha}(I)$, $1 + x_i^t E_{\alpha 0}(I)$, $1 + y_i^t E_{0\alpha}(I)$ *and* $1 + y_i^t E_{\alpha 0}(I)$, *where* $t \in \mathbb{N} \setminus \{0\}$, $i \notin I$, $|I| = s, \dots, n-1$ *and* $\alpha \in \mathbb{N}^I \setminus \{0\}$;
- (c) $1 + (\lambda - 1) E_{00}(I)$, $1 + E_{0\alpha}(I)$ *and* $1 + E_{\alpha 0}(I)$, *where* $\lambda \in K^*$, $|I| = s, \dots, n$ *and* $\alpha \in \mathbb{N}^I \setminus \{0\}$.

For $s = n$, the group $(1 + \mathfrak{a}_{n,n})^* = (1 + F_n)^*$ is generated by the elements $1 + (\lambda - 1) E_{00}(I)$, $1 + E_{0\alpha}(I)$ and $1 + E_{\alpha 0}(I)$, where $\lambda \in K^*$, $I = \{1, \dots, n\}$ and $\alpha \in \mathbb{N}^n \setminus \{0\}$.

(3) *The group $\mathbb{S}_n^* = K^* \times (1 + \mathfrak{a}_n)^*$ is generated by the elements from statement (1) and K^*.*

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Proof. (1) Statement (1) is a special case of statement (2) when $s = 1$.

(2) The statement is obvious for $s = n$ (by (4.6), (4.7) and (4.8), where $I = \{1, \ldots, n\}$). So, let $s = 1, \ldots, n-1$. By Theorem 4.2 (1), the group $(1+\mathfrak{a}_{n,s})^*$ is generated by the sets $\Theta'_{n,t}$ and $\mathbb{E}_{n,t}$, where $t = 1, \ldots, n-1$. Each element of any of the sets $\Theta'_{n,t}$ is a product of elements from (a). Recall that $\mathbb{E}_{n,t} := \prod_{|I|=t} U_I(K) \ltimes E_\infty(\mathbb{S}_{CI})$. Each element of any of the groups $U_I(K)$ is a product of elements from (c). For each $i = 1, \ldots, n$, the algebra $\mathbb{S}_1(i)$ is the direct sum $\bigoplus_{j\geqslant 1} K y_i^j \oplus K \oplus \bigoplus_{j \geqslant 1} K x_i^j \oplus F(i)$ (see (2.4)). By a straightforward computation,

$$[1 + aE_{\alpha\beta}(I), 1 + bE_{\beta\gamma}(I)] = 1 + abE_{\alpha\gamma}(I) \tag{4.6}$$

for all $a, b \in \mathbb{S}_{CI}$ and distinct $\alpha, \beta, \gamma \in \mathbb{N}^I$, where $[u, v] = uvu^{-1}v^{-1}$ is the (group) commutator of elements u and v. In this paper the commutator stands for the group commutator (unless it is stated otherwise). For all $\lambda \in K^*$, I with $|I| = s, \ldots, n$ and $\alpha \in \mathbb{N}^I \setminus \{0\}$,

$$(1 + (\lambda - 1)E_{00}(I)) \cdot (1 + E_{0\alpha}(I)) \cdot (1 + (\lambda - 1)E_{00}(I))^{-1} = 1 + \lambda E_{0\alpha}(I), \tag{4.7}$$

$$(1 + (\lambda - 1)E_{00}(I))^{-1} \cdot (1 + E_{\alpha 0}(I)) \cdot (1 + (\lambda - 1)E_{00}(I)) = 1 + \lambda E_{\alpha 0}(I). \tag{4.8}$$

It follows from (4.6), (4.7) and (4.8) that each element of each of the sets $E_\infty(\mathbb{S}_{CI})$ is a product of elements from (b) and (c). The proof of statement (1) is complete.

(3) Statement (3) is obvious. □

The next theorem presents explicit generators for the group G_n.

Theorem 4.6. *Let $J_s := \{1, \ldots, s\}$, where $s = 1, \ldots, n$. The group $G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$ is generated by the transpositions (ij), where $i < j$; the elements $t_{(\lambda,1,\ldots,1)} : x_1 \mapsto \lambda x_1$, $y_1 \mapsto \lambda^{-1} y_1$, $x_k \mapsto x_k$, $y_k \mapsto y_k$, $k = 2, \ldots, n$; and the inner automorphisms ω_u, where u belongs to the following sets:*

(1) *$\theta_{s,1}(J_s)$, $s = 2, \ldots, n$;*

(2) *$1 + x_n^t E_{0\alpha}(J_s)$, $1 + x_n^t E_{\alpha 0}(J_s)$, $1 + y_n^t E_{0\alpha}(J_s)$ and $1 + y_n^t E_{\alpha 0}(J_s)$, where $t \in \mathbb{N} \setminus \{0\}$, $s = 1, \ldots, n-1$ and $\alpha \in \mathbb{N}^s \setminus \{0\}$;*

(3) *$1 + (\lambda - 1)E_{00}(J_s)$, $1 + E_{0\alpha}(J_s)$ and $1 + E_{\alpha 0}(J_s)$, where $\lambda \in K^*$, $s = 1, \ldots, n$ and $\alpha \in \mathbb{N}^s \setminus \{0\}$.*

Proof. The group $G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$ (see Theorem 1.5 (3)) is generated by its three subgroups S_n, $\mathbb{T}^n$ and $\mathrm{Inn}(\mathbb{S}_n) = \{\omega_v \mid v \in (1 + \mathfrak{a}_n)^*\}$. The transpositions generate the symmetric group S_n. Then, by conjugating,

$$(1i) t_{(\lambda,1,\ldots,1)} (1i)^{-1} = t_{(1,\ldots,1,\lambda,1,\ldots,1)} \quad (\lambda \text{ is in the } i\text{th position}),$$

we obtain generators for the torus $\mathbb{T}^n$. Similarly, by conjugating the elements of the sets (1), (2) and (3) (i.e. using $s\omega_v s^{-1} = \omega_{s(v)}$ for all $s \in S_n$), we obtain all the elements from the sets (a), (b) and (c) of Theorem 4.5 when we identify the groups $\mathrm{Inn}(\mathbb{S}_n)$ and $(1 + \mathfrak{a}_n)^*$ via $\omega_v \leftrightarrow v$. Now the theorem is obvious. □

5. The commutants of the groups G_n and $\mathbb{S}_n$, and an analogue of the Jacobian homomorphism

In this section the groups $[G_n, G_n]$ and $G_n/[G_n, G_n]$ are found (see Theorem 5.4) and they are used to show the uniqueness of an analogue $\mathbb{J}_n$ (see (1.2)) of the Jacobian homomorphism for $n > 2$, and in finding the exotic Jacobians $\mathbb{J}_n^{\mathrm{ex}}$ for $n = 1, 2$.

5.1. The groups $[G_n, G_n]$ and $G_n/[G_n, G_n]$

The subgroup of a group G generated by all the commutators $[a, b] := aba^{-1}b^{-1}$, where $a, b \in G$, is called the *commutant* (or the *commutator subgroup*) of the group G and is denoted either by $[G, G]$ or $G^{(1)}$. The commutant is the least normal subgroup G' of G such that the factor group G/G' is abelian. If $\varphi\colon G \to H$ is a group homomorphism, then $\varphi([G, G]) \subseteq [H, H]$. If, in addition, the group H is abelian, then $[G, G] \subseteq \ker(\varphi)$. To find the commutant of a group is a technical process, especially if the group is large. In the next two easy lemmas we collect patterns that appear in finding the commutant of the group G_n. Their repeated applications make arguments short.

Lemma 5.1.

(1) *The commutant $[A \ltimes B, A \ltimes B]$ of a skew product $A \ltimes B$ of two groups is equal to $[A, A] \ltimes ([A, B] \cdot [B, B])$, where $[A, B]$ is the subgroup of B generated by all the commutators $[a, b] := aba^{-1}b$ for $a \in A$ and $b \in B$. Hence, $B \cap [A \ltimes B, A \ltimes B] = [A, B] \cdot [B, B]$ and*

$$\frac{A \ltimes B}{[A \ltimes B, A \ltimes B]} \simeq \frac{A}{[A, A]} \times \frac{B}{[A, B] \cdot [B, B]}.$$

(2) *If, in addition, the group B is a direct product of groups $\prod_{i=1}^{m} B_i$ such that $aB_ia^{-1} \subseteq B_i$ for all elements $a \in A$ and $i = 1, \dots, m$, then $[A \ltimes B, A \ltimes B] = [A, A] \ltimes \prod_{i=1}^{m}([A, B_i][B_i, B_i])$.*

Proof. (1) Note that $[a, b] = \omega_a(b)b^{-1}$, where $\omega_a(b) = aba^{-1}$. For $a \in A$ and $b, c \in B$,

$$\begin{aligned} c[a, b] &= c\omega_a(b)b^{-1} \\ &= \omega_a(\omega_{a^{-1}}(c)b)(\omega_{a^{-1}}(c)b)^{-1}\omega_{a^{-1}}(c)bb^{-1} \\ &= \omega_a(\omega_{a^{-1}}(c)b)(\omega_{a^{-1}}(c)b)^{-1} \cdot \omega_{a^{-1}}(c) \\ &= [a, \omega_{a^{-1}}(c)b] \cdot \omega_{a^{-1}}(c). \end{aligned}$$

It follows from these equalities (when, in addition, we choose $c \in [B, B]$) that the subgroup of B that is generated by its two subgroups, $[A, B]$ and $[B, B]$, is equal to their set theoretic product $[A, B][B, B] := \{ef \mid e \in [A, B],\ f \in [B, B]\}$. Then the subgroup of $C := [A \ltimes B, A \ltimes B]$ that is generated by its three subgroups $[A, A]$, $[A, B]$ and $[B, B]$ is equal to the right-hand side, say R, of the equality of statement (1). It remains to prove that $C \subseteq R$. This inclusion follows from the fact that, for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$,

$$[a_1b_1, a_2b_2] = \omega_{a_1}([b_1, a_2])\omega_{a_1a_2}([b_1, b_2])[a_1, a_2]\omega_{a_2}([a_1, b_2]), \tag{5.1}$$

which follows from the equalities $[ab,c]=\omega_a([b,c])[a,c]$ and $[a,b]^{-1}=[b,a]$:

$$\begin{aligned}[a_1b_1,a_2b_2]&=\omega_{a_1}([b_1,a_2b_2])[a_1,a_2b_2]\\&=([a_2b_2,a_1]\omega_{a_1}([a_2b_2,b_1]))^{-1}\\&=(\omega_{a_2}([b_2,a_1])[a_2,a_1]\omega_{a_1}(\omega_{a_2}([b_2,b_1])[a_2,b_1])^{-1}\\&=\omega_{a_1}([b_1,a_2])\omega_{a_1a_2}([b_1,b_2])[a_1,a_2]\omega_{a_2}([a_1,b_2]).\end{aligned}$$

(2) By statement (1), it suffices to show that $[A,\prod_{i=1}^m B_i]=\prod_{i=1}^m [A,B_i]$. The general case follows easily from the case in which $m=2$ (by induction). The case $m=2$ follows from (5.1), where we put $b_1=1$, $a_1\in A$, $a_2\in B_1$ and $b_2\in B_2$. □

Lemma 5.2.

(1) *Let $\varphi\colon G\to H$ be a group epimorphism such that* $\ker(\varphi)\subseteq[G,G]$*. Then* $[G,G]=\varphi^{-1}([H,H])$.

(2) *Let N be a normal subgroup of a group such that $N\subseteq[G,G]$ and the factor group G/N is abelian. Then $N=[G,G]$.*

Proof. (1) Since φ is an epimorphism with $\ker(\varphi)\subseteq[G,G]$, it follows that the inclusion $\varphi^{-1}([H,H])\subseteq[G,G]$ is obvious. Then the composition of the group epimorphisms $G\xrightarrow{\varphi}H\to H/[H,H]$ and the fact that the group $H/[H,H]$ is abelian yield the opposite inclusion $\varphi^{-1}([H,H])\supseteq[G,G]$.

(2) Applying statement (1) to the group epimorphism $\varphi\colon G\to G/N$ we get statement (2): $[G,G]=\varphi^{-1}([G/N,G/N])=\varphi^{-1}(e)=\ker(\varphi)$. □

For all transpositions $(ij)\in S_n$ and elements $t_{(\lambda_1,\dots,\lambda_n)}\in\mathbb{T}^n$,

$$[(ij),t_{(\lambda_1,\dots,\lambda_n)}]=t_{(1,\dots,1,\lambda_i^{-1}\lambda_j,1,\dots,1,\lambda_j^{-1}\lambda_i,1,\dots,1)},\tag{5.2}$$

where the elements $\lambda_i^{-1}\lambda_j$ and $\lambda_j^{-1}\lambda_i$ are in the ith and jth place, respectively.

Lemma 5.3. *For each natural number $n\geqslant 2$, $[S_n\ltimes\mathbb{T}^n,S_n\ltimes\mathbb{T}^n]=[S_n,S_n]\ltimes\mathbb{T}_1^n$, where $\mathbb{T}_1^n:=\{t_{(\lambda_1,\dots,\lambda_n)}\in\mathbb{T}^n\mid\prod_{i=1}^n\lambda_i=1\}$.*

Proof. Let R and L be the right-hand side and the left-hand side of the equality, respectively. By Theorem 5.1 (1), (5.1) and (5.2), $R\supseteq L$. To prove the reverse inclusion, consider two group epimorphisms:

$$\begin{aligned}\varphi\colon S_n\ltimes\mathbb{T}^n\to K^*,\ (\sigma,t_{(\lambda_1,\dots,\lambda_n)})\mapsto\prod_{i=1}^n\lambda_i,\\ \psi\colon S_n\ltimes\mathbb{T}^n\to S_n\ltimes\mathbb{T}^n/\mathbb{T}^n\simeq S_n,\ (\sigma,t_\lambda)\mapsto\sigma.\end{aligned}$$

Then $R\subseteq\ker(\varphi)=S_n\ltimes\mathbb{T}_1^n$ and $R\subseteq\psi^{-1}([S_n,S_n])=[S_n,S_n]\ltimes\mathbb{T}^n$, and hence $R\subseteq(S_n\ltimes\mathbb{T}_1^n)\cap([S_n,S_n]\ltimes\mathbb{T}^n)=L$, as required. □

Let J be a subset of the set $\{1,\dots,n\}$ that contains at least two elements, let i and j be two distinct elements of the set J and let $\lambda \in K^*$. By multiplying out, we see that (recall that $\mu_I(x_j) = x_j E_{00}(I) + 1 - E_{00}(I)$)

$$\mu_{J\backslash i}(y_i) e_{J\backslash j} = e_{J\backslash j}\mu_{J\backslash i}(x_i) = e_{J\backslash j} - e_J, \tag{5.3}$$
$$\mu_{J\backslash i}(y_i) e_J = e_J \mu_{J\backslash i}(x_i) = 0, \tag{5.4}$$
$$\mu_{J\backslash j}(x_j y_j) = 1 - e_J, \tag{5.5}$$
$$e_J \mu_{J\backslash j}(\lambda) = \mu_{J\backslash j}(\lambda) e_J = \lambda e_J. \tag{5.6}$$

Note that (where $\lambda \in K^*$)

$$[\theta_{ij}(J), \mu_{J\backslash j}(\lambda)] = \mu_J(\lambda^{-1}) \tag{5.7}$$

since (by direct computations, consider the four cases as in (2.16))

$$[\theta_{ij}(J), \mu_{J\backslash j}(\lambda)] * x^\alpha = \begin{cases} \lambda^{-1}x^\alpha & \text{if } \forall k \in J\colon \alpha_k = 0, \\ x^\alpha & \text{otherwise.} \end{cases}$$

Alternatively, using the equalities (5.3), (5.4), (5.5) and (5.6), we can show directly that (5.7) holds:

$$\begin{aligned}
&[\theta_{ij}(J), \mu_{J\backslash j}(\lambda)] \\
&\quad = \theta_{ij}(J)\mu_{J\backslash j}(\lambda)\theta_{ji}(J)\mu_{J\backslash j}(\lambda^{-1}) \\
&\quad = \mu_{J\backslash i}(y_i)\cdot\mu_{J\backslash j}(x_j)\mu_{J\backslash j}(\lambda)\mu_{J\backslash j}(y_j)\cdot\mu_{J\backslash i}(x_i)\mu_{J\backslash j}(\lambda^{-1}) \\
&\quad = \mu_{J\backslash i}(y_i)\cdot\mu_{J\backslash j}(x_j y_j)\cdot\mu_{J\backslash j}(\lambda)\cdot\mu_{J\backslash i}(x_i)\mu_{J\backslash j}(\lambda^{-1}) \\
&\quad = \mu_{J\backslash i}(y_i)\cdot(1 - e_J)\cdot\mu_{J\backslash j}(\lambda)\cdot\mu_{J\backslash i}(x_i)\mu_{J\backslash j}(\lambda^{-1}) && \text{(by (5.5))} \\
&\quad = (1 + (\lambda - 1)\mu_{J\backslash i}(y_i)e_{J\backslash j}\mu_{J\backslash i}(x_i))\cdot\mu_{J\backslash j}(\lambda^{-1}) && \text{(by (5.4))} \\
&\quad = (1 + (\lambda - 1)(e_{J\backslash j} - e_J)\mu_{J\backslash i}(x_i))\cdot\mu_{J\backslash j}(\lambda^{-1}) && \text{(by (5.3))} \\
&\quad = (1 + (\lambda - 1)(e_{J\backslash j} - e_J))\cdot\mu_{J\backslash j}(\lambda^{-1}) && \text{(by (5.3) and (5.4))} \\
&\quad = (\mu_{J\backslash j}(\lambda) + (1 - \lambda)e_J)\cdot\mu_{J\backslash j}(\lambda^{-1}) = 1 + (1 - \lambda)\lambda^{-1}e_J && \text{(by (5.6))} \\
&\quad = 1 + (\lambda^{-1} - 1)e_J = \mu_J(\lambda^{-1}).
\end{aligned}$$

By taking the inverse of both sides of (5.7) and using the fact that $[a,b]^{-1} = [b,a]$, we have the equality

$$[\mu_{J\backslash j}(\lambda), \theta_{ij}(J)] = \mu_J(\lambda). \tag{5.8}$$

Let J be a subset of the set $\{1,\dots,n\}$. If i and j are distinct elements of the set J (hence, $|J| \geqslant 2$), then, for all elements $s \in S_n$,

$$s\omega_{\theta_{ij}(J)}s^{-1} = \omega_{\theta_{s(i)s(j)}(s(J))}, \tag{5.9}$$
$$[(ij), \omega_{\theta_{ij}(J)}] = \omega_{\theta_{ij}(J)^{-2}}. \tag{5.10}$$

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The equality (5.9) is obvious and the equality (5.10) follows from (5.9) and (2.17):

$$\begin{aligned}[(ij), \omega_{\theta_{ij}(J)}] &= (ij)\omega_{\theta_{ij}(J)}(ij)^{-1}\omega^{-1}_{\theta_{ij}(J)} \\ &= \omega_{\theta_{ji}(J)}\omega_{\theta_{ij}(J)^{-1}} \\ &= \omega_{\theta_{ij}(J)^{-1}}\omega_{\theta_{ij}(J)^{-1}} \\ &= \omega_{\theta_{ij}(J)^{-2}}.\end{aligned}$$

If i, j and k are distinct elements of the set J (hence, $|J| \geqslant 3$), then

$$[(ik), \omega_{\theta_{ij}(J)}] = \omega_{\theta_{ki}(J)}. \tag{5.11}$$

In more detail,

$$\begin{aligned}[(ik), \omega_{\theta_{ij}(J)}] &= (ik)\omega_{\theta_{ij}(J)}(ik)^{-1}\omega^{-1}_{\theta_{ij}(J)} \\ &= \omega_{\theta_{kj}(J)}\omega_{\theta_{ji}(J)} && \text{(by (5.9) and (2.17))} \\ &= \omega_{\theta_{ki}(J)} && \text{(by (2.18))}.\end{aligned}$$

By (5.11), if $n > 2$, then the current group Θ_n belongs to the commutant $[G_n, G_n]$, but for $n = 2$ this is not true (see Theorem 5.4 (1)) and this is the reason for existence of the exotic 'Jacobian' homomorphism $\mathbb{J}_2^{\text{ex}}$.

Let $\theta_{ij} := \theta_{ij}(\{i, j\})$ and $\mu_j(\lambda) := \mu_{\{j\}}(\lambda)$, where $\lambda \in K^*$. Then

$$[t_{(1,\dots,1,\lambda_i,1,\dots,1)}, \omega_{\theta_{ij}}] = \omega_{\mu_j(\lambda_i^{-1})}, \tag{5.12}$$

where the scalar $\lambda_i \in K^*$ is in the ith position. In more detail,

$$[t_{(1,\dots,1,\lambda_i,1,\dots,1)}, \omega_{\theta_{ij}}] = \omega_{\mu_j(\lambda_i^{-1}y_i)\mu_i(x_j)} \cdot \omega_{\theta_{ij}^{-1}} = \omega_{\mu_j(\lambda_i^{-1})\theta_{ij}\theta_{ij}^{-1}} = \omega_{\mu_j(\lambda_i^{-1})}.$$

Theorem 5.4. *Let $\theta := \theta_{12}(\{1, 2\})$ and let $\mathcal{N}_2 := \{\omega_u \mid u \in \langle\theta^2\rangle \cdot \prod_{|I|=1} U_I(K) \ltimes E_I(\mathbb{S}_{CI})\} \subseteq G_2$. Then*

(1)

$$[G_n, G_n] = \begin{cases} \{\omega_u \mid u \in E_\infty(K)\} & \textit{if } n = 1, \\ \mathbb{T}_1^1 \ltimes \mathcal{N}_2 & \textit{if } n = 2, \\ [S_n, S_n] \ltimes \mathbb{T}_1^n \ltimes \operatorname{Inn}(\mathbb{S}_n) & \textit{if } n > 2; \end{cases}$$

(2)

$$G_n/[G_n, G_n] \simeq \begin{cases} K^* \times K^* & \textit{if } n = 1, \\ \mathbb{Z}/2\mathbb{Z} \times K^* \times \mathbb{Z}/2\mathbb{Z} & \textit{if } n = 2, \\ \mathbb{Z}/2\mathbb{Z} \times K^* & \textit{if } n > 2. \end{cases}$$

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Proof. Recall that $(1+\mathfrak{a}_n)^* \simeq \mathrm{Inn}(\mathbb{S}_n)$, $u \leftrightarrow \omega_u$ (see Theorem 1.5 (3)). To save on notation we identify these two groups by the isomorphism above. Then $G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n) = S_n \ltimes \mathbb{T}^n \ltimes (1+\mathfrak{a}_n)^*$.

We first prove the case in which $n=1$. By [**7**, Theorem 4.1],

$$G_1 \simeq \mathbb{T}^1 \ltimes (1+F)^* \simeq \mathbb{T}^1 \ltimes (U(K) \ltimes E_\infty(K)) = (\mathbb{T}^1 \times U(K)) \ltimes E_\infty(K).$$

Since $[E_\infty(K), E_\infty(K)] = E_\infty(K)$ (hence, $E_\infty(K) \subseteq [G_1, G_1]$) and the factor group $G_1/E_\infty(K) \simeq \mathbb{T}^1 \times U(K)$ is abelian, by Lemma 5.2 (2), $[G_1, G_1] = \{\omega_u \mid u \in E_\infty(K)\}$. Hence, $G_1/[G_1, G_1] \simeq \mathbb{T}^1 \times U(K) \simeq K^* \times K^*$.

Let $n \geqslant 2$. Note that $E_\infty(\mathbb{S}_{CI}) = [E_\infty(\mathbb{S}_{CI}), E_\infty(\mathbb{S}_{CI})] \subseteq [G_n, G_n]$ for all non-empty subsets I of the set $\{1, \dots, n\}$. It follows from (5.8), (5.11) and Theorem 4.2 (1) that

$$(1+\mathfrak{a}_{n,2})^* \subseteq [G_n, G_n].$$

We now prove the case of $n=2$. By (5.10), $\theta^2 \in [G_2, G_2]$. By (5.12), $\prod_{|I|=1} U_I(K) \ltimes E_I(\mathbb{S}_{CI}) \subseteq [G_2, G_2]$. By Theorem 4.2 (1) and (2.10),

$$\bar{G}_2 := G_2/(1+\mathfrak{a}_{2,2})^* \simeq S_2 \ltimes \mathbb{T}^2 \ltimes \langle\theta\rangle \ltimes \prod_{|I|=1} U_I(K) \ltimes E_I(L_{CI}).$$

Note that for the group commutator $[(12), \omega_\theta] = \omega_{\theta^{-2}}$ (by (5.10)) and, for all elements $t_\lambda \in \mathbb{T}^2$,

$$[t_\lambda, \omega_\theta] \equiv \omega_{\mu_2(\lambda_1^{-1})\mu_1(\lambda_2)} \mod (1+\mathfrak{a}_{2,2})^*.$$

Indeed,

$$\begin{aligned} [t_\lambda, \omega_\theta] &\equiv \omega_{\mu_2(\lambda_1^{-1})\theta\mu_1(\lambda_2)} \omega_{\theta^{-1}} \\ &\equiv \omega_{\mu_2(\lambda_1^{-1})\theta\mu_1(\lambda_2)\theta^{-1}} \\ &\equiv \omega_{\mu_2(\lambda_1^{-1})\mu_1(\lambda_2)\theta\theta^{-1}} \\ &\equiv \omega_{\mu_2(\lambda_1^{-1})\mu_1(\lambda_2)} \mod (1+\mathfrak{a}_{2,2})^*. \end{aligned}$$

It follows that the group $N := \langle\theta^2\rangle \ltimes \prod_{|I|=1} U_I(K) \ltimes E_I(L_{CI})$ is a normal subgroup of $\bar{G}_2$, $N \subseteq [\bar{G}_2, \bar{G}_2]$ and $\bar{G}_2/N \simeq (S_2 \ltimes \mathbb{T}^2) \times (\langle\theta\rangle/\langle\theta^2\rangle)$. By Lemma 5.3, $[\bar{G}_2/N, \bar{G}_2/N] = [S_2 \ltimes \mathbb{T}^2, S_2 \ltimes \mathbb{T}^2] = \mathbb{T}^2_1$. Then, by Lemma 5.2 (1) and Proposition 3.12 (1), statement (1) follows. Then, by Lemma 5.1 (1),

$$\frac{G_2}{[G_2, G_2]} \simeq \frac{\bar{G}_2/N}{[\bar{G}_2/N, \bar{G}_2/N]} \simeq S_2 \times \frac{\mathbb{T}^2}{\mathbb{T}^2_1} \times \frac{\langle\theta\rangle}{\langle\theta^2\rangle} \simeq \mathbb{Z}_2 \times K^* \times \mathbb{Z}_2.$$

Finally, we prove the case in which $n > 2$. By Theorem 4.2 (1), $(1+\mathfrak{a}_{n,1})^* = \Theta'_{n,1}\mathbb{E}_{n,1} \cdots \Theta'_{n,n-1}\mathbb{E}_{n,n-1}$. By (5.11), $\Theta'_{n,s} \subseteq [G_n, G_n]$ for all $s = 1, \dots, n-1$. By (5.8), $\mathbb{E}_{n,s} \subseteq [G_n, G_n]$ for all $s = 2, \dots, n-1$ and, by (5.12), $\mathbb{E}_{n,1} \subseteq [G_n, G_n]$. Therefore, $(1+\mathfrak{a}_{n,1})^* \subseteq [G_n, G_n]$. Then the factor group $\bar{G}_n := G_n/(1+\mathfrak{a}_{n,1})^*$ is isomorphic to the

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group $S_n \ltimes \mathbb{T}^n$. By Lemma 5.3, $[\bar{G}_n, \bar{G}_n] = [S_n, S_n] \ltimes \mathbb{T}_1^n$ and statement (1) follows, by Lemma 5.2 (1). By Lemma 5.2 (1),

$$\frac{G_n}{[G_n, G_n]} \simeq \frac{\bar{G}_n}{[\bar{G}_n, \bar{G}_n]} \simeq \frac{S_n}{[S_n, S_n]} \times \frac{\mathbb{T}^n}{\mathbb{T}_1^n} \simeq \mathbb{Z}_2 \times K^*.$$

The proof of the theorem is complete. □

Recall that $\mathrm{aff}_n := S_n \ltimes \mathbb{T}^n$.

Corollary 5.5.

(1)

$$\frac{\mathrm{Inn}(\mathbb{S}_n)}{\mathrm{Inn}(\mathbb{S}_n) \cap [G_n, G_n]} \simeq \begin{cases} K^* & \textit{if } n = 1, \\ \mathbb{Z}_2 & \textit{if } n = 2, \\ 0 & \textit{if } n > 2. \end{cases}$$

(2)

$$\frac{\mathrm{aff}_n}{[\mathrm{aff}_n, \mathrm{aff}_n]} \simeq \begin{cases} K^* & \textit{if } n = 1, \\ \mathbb{Z}_2 \times K^* & \textit{if } n > 1. \end{cases}$$

(3)

$$\frac{G_n}{[G_n, G_n]} \simeq \frac{\mathrm{aff}_n}{[\mathrm{aff}_n, \mathrm{aff}_n]} \times \frac{\mathrm{Inn}(\mathbb{S}_n)}{\mathrm{Inn}(\mathbb{S}_n) \cap [G_n, G_n]} \simeq \frac{\mathrm{aff}_n}{[\mathrm{aff}_n, \mathrm{aff}_n]} \times \begin{cases} K^* & \textit{if } n = 1, \\ \mathbb{Z}_2 & \textit{if } n = 2, \\ 0 & \textit{if } n > 2. \end{cases}$$

Proof. (1) We keep the notation of the proof of Theorem 5.4 (in particular, we identify the groups $\mathrm{Inn}(\mathbb{S}_n)$ and $(1 + \mathfrak{a}_n)^*$), as above. For $n = 1$, $\mathrm{Inn}(\mathbb{S}_1) = U(K) \ltimes E_\infty(K)$ and $[G_1, G_1] = E_\infty(K)$ (see Theorem 5.4 (1)) and the statement follows.

For $n = 2$, by Theorem 5.4 (1), $\mathrm{Inn}(\mathbb{S}_n)/\,\mathrm{Inn}(\mathbb{S}_n) \cap [G_n, G_n] \simeq \langle\theta\rangle/\langle\theta^2\rangle \simeq \mathbb{Z}_2$.

For $n = 3$, by Theorem 5.4 (1), $\mathrm{Inn}(\mathbb{S}_n) \subseteq [G_n, G_n]$.

(2) For $n = 1$, $\mathrm{aff}_1 = \mathbb{T}^1$ and statement (2) is obvious. For $n > 1$, statement (2) follows from Lemma 5.3: $\mathrm{aff}_n / [\mathrm{aff}_n, \mathrm{aff}_n] \simeq (S_n/[S_n, S_n]) \times (\mathbb{T}^n/\mathbb{T}_1^n) \simeq \mathbb{Z}_2 \times K^*$.

(3) Since $G_n = \mathrm{aff}_n \ltimes \mathrm{Inn}(\mathbb{S}_n)$, the first isomorphism follows from Lemma 5.1 (1) and then the second isomorphism follows from statement (1). □

5.2. An analogue of the polynomial Jacobian homomorphism

We keep the notation of the introduction. We want to find an analogue of the polynomial Jacobian homomorphism (1.1) for the algebra $\mathbb{S}_n$. The algebra $\mathbb{S}_n$ is non-commutative and non-Noetherian, with trivial centre, i.e. $Z(\mathbb{S}_n) = K$ [**4**, Proposition 4.1], and there are no obvious 'partial' derivatives for the algebra $\mathbb{S}_n$. So, in order to find the analogue, we first define the Jacobian homomorphism in invariant group-theoretic terms,

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i.e. we select natural properties/conditions that uniquely determine $\mathcal{J}$. Then, for the algebra $\mathbb{S}_n$, the conditions obtained uniquely determine an analogue of the Jacobian homomorphism for $n \geqslant 3$ but for $n = 1, 2$, where there are exactly two of them.

The group $\mathcal{P}_n = \Sigma_n \times_{\mathrm{ex}} \mathrm{Aff}_n$ is an exact product of its two subgroups, where $\mathrm{Aff}_n := \{\sigma_{A,a} \colon x \mapsto Ax + a \mid A \in \mathrm{GL}_n(K),\ a \in K^n\}$ is the *affine group* and

$$\Sigma_n := \{\sigma \in \mathcal{P}_n \mid \sigma(x_i) \equiv x_i \bmod (x_1, \dots, x_n)^2,\ i = 1, \dots, n\}$$

is the *Jacobian* group, where $(x_1, \dots, x_n)$ is the maximal ideal of the polynomial algebra P_n. Recall that an exact product means that each element $\sigma \in \mathcal{P}_n$ is a *unique* product $\sigma = \xi \cdot \sigma_{A,a}$, where $\sigma_{A,a} \in \mathrm{Aff}_n$ and $\xi \in \Sigma_n$. Indeed, $\sigma \colon x \mapsto a + A(x + \cdots)$, where the three dots mean higher terms, and so $\sigma = \xi \sigma_{A,a}$, where $\xi \colon x \mapsto x + \cdots$. The Jacobian homomorphism $\mathcal{J}_n$ is determined by its restriction to the affine subgroup, since $\mathcal{J}_n(\xi) = 1$ for all $\xi \in \Sigma_n$ (trivial), and

$$\mathcal{J}_n(\sigma) = \mathcal{J}_n(\sigma_{A,a}) = \det(A). \tag{5.13}$$

The group $G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$ has a similar structure to the group $\mathcal{P}_n$. The subgroup $\mathrm{aff}_n := S_n \ltimes \mathbb{T}^n$ is an affine part of the group G_n and the subgroup $\mathrm{Inn}(\mathbb{S}_n)$ plays the role of the Jacobian subgroup Σ_n due to the following corollary.

Corollary 5.6 (Bavula [7, Corollary 5.5]).

$$\mathrm{Inn}(\mathbb{S}_n) = \{\sigma \in G_n \mid \sigma(x_i) \equiv x_i \bmod \mathfrak{p}_i,\ \sigma(y_i) \equiv y_i \bmod \mathfrak{p}_i\ \forall i\}.$$

Definition 5.7. An analogue $\mathbb{J}_n$ of the polynomial Jacobian homomorphism $\mathcal{J}_n$ is a group homomorphism $\mathbb{J}_n \colon G_n \to K^*$ that acts on the affine subgroup aff_n as in the polynomial case (i.e. it sends the affine automorphism to its Jacobian).

There is at least one such map that is given in (1.2) and (1.3).

Theorem 5.8.

(1) *For $n > 2$ the analogue $\mathbb{J}_n$ of the polynomial Jacobian homomorphism $\mathcal{J}_n$ is unique and given in (1.2) and (1.3).*

(2) *For $n = 1, 2$ there is another one $\mathbb{J}_n^{\mathrm{ex}}$, the so-called exotic Jacobian homomorphism, given by the following rule.*

   (a) *For $n = 1$, $\sigma = t_\lambda \cdot \omega_u \in G_1 = \mathbb{T}^1 \ltimes \{\omega_v \mid v \in (1+F)^*\}$, where $t_\lambda \in \mathbb{T}^1$ and $u \in (1+F)^* \simeq \mathrm{GL}_\infty(K)$, and $\mathbb{J}_1^{\mathrm{ex}}(\sigma) = \lambda \cdot \det(u)$. The homomorphisms $\mathbb{J}_1$ and $\mathbb{J}_1^{\mathrm{ex}}$ are algebraically independent characters of the group G_1.*

   (b) *For $n = 2$, $\sigma = s t_\lambda \omega_{\theta^i} \xi \in G_2$, where $s \in S_2$, $t_\lambda \in \mathbb{T}^2$, $i \in \{0, 1\}$ and $\xi \in \mathcal{N}_2$, $\mathbb{J}_2^{\mathrm{ex}}(\sigma) = (-1)^i \operatorname{sgn}(s) \lambda_1 \lambda_2$. Note that $(\mathbb{J}_2^{\mathrm{ex}})^2 = \mathbb{J}_2^2$.*

Proof. (1) Statement (1) follows from the fact that $G_n/[G_n, G_n] \simeq \mathrm{aff}_n/[\mathrm{aff}_n, \mathrm{aff}_n]$ (see Corollary 5.5 (3)).

(2) Statement (2) follows from Corollary 5.5 (3). □

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