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A PATH GUESSING GAME WITH WAGERING

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We consider a two-player game in which the first player (the Guesser) tries to guess, edge-by-edge, the path that second player (the Chooser) takes through a directed graph. At each step, the Guesser makes a wager as to the correctness of her guess and receives a payoff proportional to her wager if she is correct. We derive optimal strategies for both players for various classes of graphs, and we describe the Markov-chain dynamics of the game under optimal play. These results are applied to the infinite-duration Lying Oracle Game, in which the Guesser must use information provided by an unreliable Oracle to predict the outcome of a coin toss.

1. INTRODUCTION

In this article we study a two-player zero-sum game in which Player I (the Guesser) tries to guess, edge-by-edge, the path that Player II (the Chooser) takes through a directed graph. At each step, the Guesser makes a wager as to the correctness of her guess and receives a payoff proportional to her wager if she is correct. Optimal strategies for both players are derived for various classes of graphs, and the Markov-chain dynamics of the game are analyzed.

The Path Guessing Game studied here is a generalization of the Lying Oracle Game [1,2]. In the Lying Oracle Game, an Oracle makes a sequence of n statements, at most k of which can be lies, and a Guesser makes bets on whether the Oracle's next statement will be a lie or not. We will see that the Lying Oracle Game is equivalent to our Path Guessing Game on a certain graph whose maximum outdegree is 2. Ravikumar [5] demonstrated a reciprocal relationship between the Lying Oracle problem and the continuous version of Ulam's Liar Game. In that game, a Questioner tries to find a subset of smallest measure that contains an unknown number in [0, 1] by

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asking a Responder n questions about the number's location in the interval. Again the Responder may lie up to k times. Under optimal play, the measure of the Questioner's subset is the reciprocal of the Bettor's fortune in the Lying Oracle Game. In [6], Rivest, Mayer, Kleitman, Winklemann, and Spencer used this game to analyze binary search in the presence of errors.

In addition to its intrinsic interest, the Path Guessing Game provides a context in which new questions about the Lying Oracle Game can be asked and answered. For instance, what are the optimal strategies for the *infinite-duration* Lying Oracle Game, in which no block of n statements can contain more than k lies? Questions of this sort are taken up in the last section of this article, after a general analysis of the Path Guessing Game has been conducted.

To describe the Path Guessing Game precisely, let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. Call a vertex $j \in \mathcal{V}$ a *terminal node* if it has outdegree 0, and assume that each terminal node j has been assigned a positive *value* v_j. Both players know these values. At a typical stage of the game, the players are at some nonterminal node i of the graph. The Guesser wagers some fraction $w \in [0, 1]$ of her current fortune F against the claim that she can guess which of the n_i possible successor nodes of i the Chooser will choose as the node to which the game next moves. The Chooser, knowing w, chooses this successor node. If the Guesser guesses correctly, then she receives a payoff of $(n_i - 1)wF$ from the Chooser. (The factor $n_i - 1$ weights the Guesser's payoff by her odds against being correct in the "average case.") If she is wrong, she pays wF to the Chooser. The case $n_i = 1$ is special because the Guesser can always guess correctly. In this case we suppose that she can bet her whole fortune and double it with certainty. Thus the Guesser's fortune evolves via the mapping

$$F \longmapsto \begin{cases} (1 + (n_i - 1)w)\,F & \text{if } n_i \geq 2 \text{ and the Guesser is correct} \\ (1 - w)\,F & \text{if } n_i \geq 2 \text{ and the Guesser is incorrect} \\ 2F & \text{if } n_i = 1. \end{cases} \tag{1}$$

Eventually, the game reaches some terminal node of the graph, say j, at which point the the Guesser's fortune is multiplied by v_j, and the game is over. The goal of the Guesser is to maximize her expected fortune, whereas the goal of the Chooser is to minimize the Guesser's expected fortune. Thus, the Chooser has an incentive to steer the path through the graph to end at a node where v_j is small. On the other hand, he cannot be so naive as to steer the path with certainty toward the terminal node of least value, since that will make it easy for the Guesser to make successful wagers along the path. We are interested in the optimal strategies for the players in this game and the dynamics of play under the optimal strategies.

Most of our notation is standard. Random variables and matrixes are denoted with uppercase letters; constants and vectors are lowercase. We will write $i \to j$ to indicate that there is a directed edge from node i to node j in the graph $\mathcal{G}$. The outdegree of vertex i will be denoted by n or n_i. The symbol $\mathbb{1}$ will indicate either a vector or matrix, all of whose entries are 1. The dimensions should be clear from the contex. For readability, the Guesser (Player I) will be consistently referred to as "she," whereas the Chooser (Player II) will be referred to as "he."

2. THE GAME ON TREES

In this section we investigate the Path Guessing Game when the graph $\mathcal{G}$ is a tree. First, we will consider the simple case when $\mathcal{G}$ is a fan (i.e., a tree of height 1, with one root node and n terminal nodes). The results for this case will then be extended to a general finite tree via a straightforward induction on the height of the tree.

Let $\mathcal{G}$ be a fan with $n \geq 2$ leaves. We assume each leaf j has been assigned a positive value v_j. The players are initially located at the root node of $\mathcal{G}$, and the Guesser's initial fortune is \$1. The Chooser's play consists of selecting the destination node, so his strategy set is $\Sigma_c = \{j : 1 \leq j \leq n\}$. The Guesser must choose both a wager and a guess as to the destination node, so her strategy set is $\Sigma_g = \{(j, w) : 1 \leq j \leq n \text{ and } w \in [0, 1]\}$.

First, regard the wager $w \in [0, 1]$ as given. Then the Path Guessing Game on $\mathcal{G}$ is equivalent to a zero-sum game whose payoff matrix to the Guesser is given by

$$A = \begin{pmatrix} (1 + (n-1)w)\, v_1 & (1-w)\, v_2 & \cdots & (1-w)\, v_n \\ (1-w)\, v_1 & (1 + (n-1)w)\, v_2 & \cdots & (1-w)\, v_n \\ \vdots & \vdots & \ddots & \vdots \\ (1-w)\, v_1 & (1-w)\, v_2 & \cdots & (1 + (n-1)w)\, v_n \end{pmatrix}.$$

Consider the mixed strategy p for the Chooser in which he visits a leaf node with a probability that is inversely proportional to its value:

$$p^{\mathrm{T}} = \frac{1}{\sum_{j=1}^{n} v_j^{-1}} \left(v_1^{-1}, v_2^{-1}, \ldots, v_n^{-1}\right)^{\mathrm{T}}, \tag{2}$$

It is straightforward to verify that $Ap = H\mathbb{1}$, where $H = n / \sum_j v_j^{-1}$ is the harmonic mean of the values. This implies that if the Chooser adopts strategy (2), then the Guesser's expected fortune is equal to H regardless of which strategy she employs. Next consider the mixed strategy q for the Guesser defined by

$$q = w^{-1} p - n^{-1}\left(w^{-1} - 1\right)\mathbb{1}, \tag{3}$$

where p is defined by (2). The vector q is nonnegative if and only if

$$w \in [w_c, 1], \tag{4}$$

where w_c is the *critical wager* defined by

$$w_c = 1 - n \min_j p_j. \tag{5}$$

In this case it is easy to verify that $q^{\mathrm{T}} A = H\mathbb{1}^{\mathrm{T}}$, implying that if the Guesser adopts (3), then her expected fortune is again equal to H regardless of which strategy the Chooser employs. Moreover, it is straightforward to show that if the Chooser (Guesser) uses a strategy other than (2) [(3), (4)], then the Guesser (Chooser) has a strategy that increases (decreases) the Guesser's expected fortune. Thus, when $w \geq w_c$, strategies (2) and (3) constitute an equilibrium for the game with payoff matrix A. The value of the game is $H = q^{\mathrm{T}} A p$.

Now, consider the game in which the Guesser chooses both her wager w and her guess and the Chooser chooses the destination node j (knowing w). If the Guesser were to choose $w < w_c$, then the Chooser could use the strategy

$$p^{\mathrm{T}} = \frac{1}{\sum_{j=2}^{n} v_j^{-1}} \left(0, v_2^{-1}, \ldots, v_n^{-1}\right)^{\mathrm{T}},$$

where we have assumed without loss of generality that V_1 is the maximum of all of the values. With this strategy, it is easy to verify that the Guesser's conditional expected fortunes are now smaller than the harmonic mean of the values: $Ap < H\mathbb{1}$. Thus, the Guesser should always wager an amount greater than or equal to the critical wager w_c. We have proven that (2)–(4) characterize the optimal strategies for the Path Guessing Game on a fan.

Now, let $\mathcal{G}$ be a finite rooted tree. Note that if play has progressed to the point where the players are at a node i whose children are all leaves, then at that point they are playing the game on a fan. If the outdegree of i is at least 2, the optimal strategies are given by (2)–(4) and the value of node i is $v_i = H_i$, the harmonic mean of the values of the children of i. If the outdegree of i is 1, then the Guesser will double her fortune at i, and so the value of node i is $v_i = 2v_j$, where j is the sole child of i. Likewise, values can be assigned to all of the interior nodes of the tree, all the way up to the root. Playing in accordance with these values is optimal, by a straightforward induction on the height of the tree. These ideas are summarized in the following theorem.

THEOREM 1 (Optimal Play on Trees): *Let $\mathcal{G}$ be a finite rooted tree in which each leaf node ℓ has been assigned a positive value v_ℓ. Define the values of all the other nodes i of $\mathcal{G}$ by*

$$v_i = \begin{cases} 2v_j & \text{if } n_i = 1 \text{ and } i \to j \\ n_i / \left(\sum_{j:i\to j} v_j^{-1}\right) & \text{if } n_i \geq 2. \end{cases}$$

At a node i of outdegree $n_i \geq 2$, the optimal strategy for the Chooser is given by

$$p_{i,j} = \begin{cases} v_j^{-1} / \left(\sum_{k:i\to k} v_k^{-1}\right) & \text{if } i \to j \\ 0 & \text{otherwise,} \end{cases}$$

whereas the optimal strategies for the Guesser are characterized by

$$w_i \in [w_c, 1]$$

and

$$q_{i,j} = \begin{cases} \left(n_i p_{i,j} - (1-w)\right) / (n_i w_i) & \text{if } i \to j \\ 0 & \text{otherwise,} \end{cases}$$

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where w_c is the critical wager defined by

$$w_c = 1 - n_i \min_{j:i \to j} p_{i,j}.$$

At a node of outdegree 1*, the Guesser should wager everything and is guaranteed to double her fortune. Under optimal play, the Guesser's fortune F at the end of the game satisfies $E[F] = v_0$, where v_0 is the value of the root node of the tree.*

3. THE GAME ON TERMINATING GRAPHS

We will call a connected digraph in which every nonterminal node has a directed path to some terminal node a *terminating graph*. In this section we assume that $\mathcal{G}$ is a terminating graph with N vertices, in which each terminal node j has been preassigned a positive value v_j. We will derive optimal play in the Path Guessing Game on $\mathcal{G}$ by first assigning appropriate values to the nonterminal nodes and then proving that playing the game in accordance with these values is optimal.

Let s be a positive integer, and consider the truncated *s-step game*, which is identical to the nontruncated game except that if the players have not reached a terminal node by step s, the game is simply stopped. The s-step game can be completely described by N_{NT} *path trees*, where N_{NT} is the number of nonterminal nodes of $\mathcal{G}$. The path tree corresponding to a nonterminal vertex i represents all the paths emanating from i whose lengths are at most s. Terminal nodes in the original graph $\mathcal{G}$ will be leaf nodes in the path trees. The nonterminal nodes that are reachable in s steps from node i will also appear as leaf nodes in the path tree corresponding to i. For the path trees in this truncated game, terminal nodes retain the values assigned to them in the nontruncated game. Leaf nodes of the path trees that correspond to nonterminal nodes are assigned the value of 1, in keeping with the idea that the game simply stops if one of these nodes is reached. The s-step game is thus reduced to a game on the path trees of length s, which can be solved by propagating the values up the path trees using Theorem 1.

The propagation of the values for the s-step game can be summarized nicely using a matrix approach. Let N_{NT} be the number of nonterminal nodes and let N_{T} be the number of terminal nodes. Number the nodes so that the nonterminal nodes are numbered first, followed by the terminal nodes. Consider first the path trees for a 1-step game on $\mathcal{G}$. Let u_0 denote the vector of the reciprocal values of the leaf nodes. In keeping with our numbering convention, u_0 is a partitioned vector of the form

$$u_0 = \left(\frac{\mathbb{1}}{u_{\text{T}}} \right), \tag{6}$$

where $\mathbb{1}$ is the $N_{\text{NT}} \times 1$ vector of ones and u_{T} is the $N_{\text{T}} \times 1$ vector containing the preassigned reciprocal values of the terminal nodes of $\mathcal{G}$. By Theorem 1, the reciprocal

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values of the N_{NT} root nodes of the path trees for the 1-step game are given by

$$u_1 = Mu_0,$$

where M is the $N \times N$ *propagation matrix* $M = (m_{i,j} : i,j \in \mathcal{V})$ defined by

$$m_{i,j} = \begin{cases} 1 & \text{if } i \text{ is a terminal node and } i = j \\ 1/2 & \text{if } n_i = 1 \text{ and } i \to j \\ 1/n_i & \text{if } n_i \geq 2 \text{ and } i \to j \\ 0 & \text{otherwise.} \end{cases} \qquad (7)$$

Inductively, the reciprocal values for the root nodes of the path trees for the s-step game are given by

$$u_s = M^s u_0. \qquad (8)$$

Our first goal is to prove that $\lim_{s\to\infty} u_s$ exists. In accordance with our numbering convention, the partitioned form of the propagation matrix M is

$$M = \left[\begin{array}{c|c} A & B \\ \hline 0 & I \end{array}\right], \qquad (9)$$

where A is $N_{NT} \times N_{NT}$, B is $N_{NT} \times N_T$, the zero submatrix is $N_T \times N_{NT}$, and the identity submatrix is $N_T \times N_T$. It follows that the powers of M are of the form

$$M^s = \left[\begin{array}{c|c} A^s & \left(\sum_{i=0}^{s-1} A^i\right) B \\ \hline 0 & I \end{array}\right]. \qquad (10)$$

The next lemma establishes that limiting values exist and are strictly positive and that the limiting reciprocal value vector u satisfies $Mu = u$.

LEMMA 1 (Limiting Values): *Let $\mathcal{G}$ be a terminating graph and let M be the associated propagation matrix defined by* (7). *Then* $\lim_{s\to\infty} M^s$ *exists, with*

$$\lim_{s\to\infty} M^s = \left[\begin{array}{c|c} 0 & (I-A)^{-1}B \\ \hline 0 & I \end{array}\right], \qquad (11)$$

where A and B are the submatrixes defined by (9). *In particular, the limiting reciprocal value vector* $u = \lim_{s\to\infty} M^s u_0$ *exists, satisfies $Mu = u$, and is strictly positive. Partitioning the limiting reciprocal value vector consistently with the partition of M,*

$$u = \left(\frac{u_{NT}}{u_T}\right), \qquad (12)$$

the limiting reciprocal values u_{NT} of the nonterminal nodes are related to the values u_T of the terminal nodes by

$$u_{NT} = (I-A)^{-1} Bu_T. \qquad (13)$$

PROOF: Note that if all of the eigenvalues of A were less than 1 in absolute value, we would immediately conclude that $\lim_{s\to\infty}\sum_{i=0}^{s-1} A^i = (I-A)^{-1}$ and (11) would follow. We claim that in fact all of the eigenvalues of A *are* strictly less than 1 in absolute value. To see this, first observe that A is a nonnegative substochastic matrix, so by the standard Perron–Frobenius theory of nonnegative matrixes (see [4], for instance), the maximal eigenvalue r of A is nonnegative and less than or equal to 1 and $|\lambda| \le r$ for any other eigenvalue λ of A. We claim that $r < 1$. Suppose, by way of contradiction, that $r = 1$. Then there is an associated nonnegative eigenvector x, which we may assume without loss of generality has been scaled so that its maximum component is equal to 1. Now, the equation $Ax = x$ along with definition (7) imply the following:

1. If the nonterminal node i of $\mathcal{G}$ has outdegree 1 (i.e., $n_i = 1$) and if the (sole) child of i is a terminal node j, then $x_i = 0$.
2. If the nonterminal node i has $n_i = 1$ and if its (sole) child is a nonterminal node j, then $x_i = x_j/2$.
3. If the nonterminal node i has $n_i \ge 2$, then

$$x_i = n_i^{-1} \sum_{j\in J(i)} x_j, \tag{14}$$

where $J(i)$ is the set of {nonterminal nodes j such that $m_{i,j} \neq 0$}.

Items 1 and 2 imply that entries of x corresponding to nonterminal nodes of outdegree 1 are all strictly less than 1. Now consider the remaining entries of x, which correspond to nonterminal nodes i of outdegree 2 or more. Let d_i denote the distance from nonterminal node i to the nearest terminal node. Item 1 says that if $d_i = 1$, then $x_i = 0$. Now, suppose $d_i = 2$. Then there is a directed edge from node i to some node j with $d_j = 1$. Thus, $x_j < 1$, and therefore $x_i < 1$ by (14) (since the cardinality of $J(i)$ is at most n_i). So any nonterminal node i with $d_i = 2$ must have $x_i < 1$. An easy induction shows that any nonterminal node i that is at a finite distance from some terminal node must have $x_i < 1$. However, in a terminating graph, *every* nonterminal node is at a finite distance from some terminal node. Therefore, $x_i < 1$ for *all* nonterminal nodes i, contradicting our scaling of x so that its maximum component was 1. Thus, $r < 1$ as claimed, and we have (11).

From (8) and (11) it follows that the limiting reciprocal value $u = \lim_{s\to\infty} u_s$ exists and satisfies

$$u = \left[\begin{array}{c|c} 0 & (I-A)^{-1}B \\ \hline 0 & I \end{array}\right] u_0. \tag{15}$$

Equation (13) follows immediately. Using (13) and (9), it is easy to see that the limiting vector u is a right eigenvector of the propagation matrix, corresponding to the maximal eigenvalue $r = 1$.

It remains to show that $u > 0$. We claim that no row of $(I-A)^{-1}B$ consists entirely of zeros. To see this, note that each nonterminal node i is connected to some

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terminal node k by a path in $\mathcal{G}$ of some length $s > 0$. The node j immediately preceeding k in this path must be a nonterminal node. It follows that the i, j entry in A^{s-1} is nonzero, as is the j, k entry in B. Therefore, row i of $A^{s-1}B$ cannot consist entirely of zeros. Since $(I - A)^{-1} B$ is a sum of such (nonnegative) terms, its ith row cannot consist entirely of zeros either, establishing the claim. Now, the preassigned reciprocal value vector u_{T} is positive by assumption, so by (13) we conclude that u_{NT} is strictly positive. This completes the proof of the lemma. ■

Now, we turn to the full nontruncated game. The strategies associated with the limiting values of Lemma 1 will be referred to as the *limiting strategies* for the game on $\mathcal{G}$. For the Chooser, the limiting strategy is

$$p_{i,j} = \text{Prob}\,(X_{t+1} = j \mid X_t = i) = \begin{cases} 1 & \text{if } n_i = 1 \text{ and } i \to j \\ u_j / \left(\sum_{k:i\to k} u_k\right) & \text{if } n_i \geq 2 \text{ and } i \to j \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

where X_t denotes the vertex occupied by the players at time t. For the Guesser, the limiting strategies are characterized by

$$w_i \begin{cases} = 1 & \text{if } n_i = 1 \\ \in [w_c, 1] & \text{if } n_i \geq 2 \end{cases} \tag{17}$$

and

$$q_{i,j} = \text{Prob}\,(G_{t+1} = j \mid X_t = i) = \begin{cases} 1 & \text{if } n_i = 1 \text{ and } i \to j \\ \left(n_i p_{i,j} - (1 - w)\right) / (n_i w_i) & \text{if } n_i \geq 2 \text{ and } i \to j \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

where G_{t+1} is the Guesser's prediction of where the Chooser will go on the next step and w_c is the critical wager at node i given by

$$w_c = 1 - n_i \min_{j:i\to j} p_{i,j}.$$

Let $p = \left(p_{i,j} : i, j \in \mathcal{V}\right)$ denote the transition probability matrix corresponding to the Chooser's limiting strategy (16). These are the transition probabilities for the Markov chain that describes the players' position in the graph as a function of time. Strictly speaking, these probabilities only make sense if i is a nonterminal node. However, we can think of each terminal node as having an attached loop leading back to itself, with the understanding that all wagering and guessing stops as soon as the players reach a terminal node. Then $p_{i,i} = 1$ if i is a terminal node, which makes the

Markov chain X_t well defined for all t. The end of actual play in the game is given by the *hitting time* T of the chain to the set of terminal nodes:

$$T = \min\{t \geq 0 : X_t \text{ is a terminal node}\}.$$

The next theorem gives the basic properties of the game under the limiting strategies.

THEOREM 2 (Limiting Strategies): *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a terminating graph in which each terminal node has been assigned a positive value. Let M be the propagation matrix for $\mathcal{G}$ defined by* (7) *and let $v = (v_i : i \in \mathcal{V})$ be the vector of limiting values. Then under the limiting strategies, the Chooser's transition probability matrix P is diagonally similar to the propagation matrix:*

$$P = VMV^{-1}, \tag{19}$$

where $V = \operatorname{diag}(v)$ is the diagonal matrix of limiting values. Under the limiting strategies, the game duration T is finite with probability 1*, and the Guesser's expected fortune at the end of the game is equal to the value of the node at which play began:*

$$E\left[F_T \mid X_0 = i\right] = v_i. \tag{20}$$

PROOF: Because the vector u of limiting reciprocal values satisfies $Mu = u$, (16) reduces to

$$p_{i,j} = \frac{m_{i,j}\, u_j}{u_i} = \frac{m_{i,j}\, v_i}{v_j}$$

for *all* i and j, and (19) follows. Because the limiting reciprocal value vector u is strictly positive, it follows from (16) that $p_{i,j} > 0$ whenever $i \to j$ in $\mathcal{G}$. Therefore, the hitting time T is finite with probability 1 under the limiting strategies.

For (20), note first that it is obviously true if i is a terminal node. If i is nonterminal, then conditioning on the position of the players at time $t = 1$ gives

$$E\left[F_T \mid X_0 = i\right] = \begin{cases} 2E\left[F_T \mid X_0 = j\right] & \text{if } n_i = 1 \text{ and } i \to j \\ \displaystyle\sum_{j: i \to j} p_{i,j} E\left[F_T \mid X_0 = i, X_1 = j\right] & \text{if } n_i \geq 2. \end{cases} \tag{21}$$

Now, in the second case we have

$$\begin{aligned} &E\left[F_T \mid X_0 = i, X_1 = j\right] \\ &\quad = \left[q_{i,j}\left(1 + (n_i - 1)w_i\right) + \left(1 - q_{i,j}\right)\left(1 - w_i\right)\right] E\left[F_T \mid X_0 = j\right] \\ &\quad = \left(w_i\left(n_i q_{i,j} - 1\right) + 1\right) E\left[F_T \mid X_0 = j\right] \\ &\quad = n_i p_{i,j} E\left[F_T \mid X_0 = j\right] \\ &\quad = \frac{v_i}{v_j} E\left[F_T \mid X_0 = j\right], \end{aligned}$$

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so that (21) becomes

$$E\left[F_T \mid X_0 = i\right] = \begin{cases} 2E\left[F_T \mid X_0 = j\right] & \text{if } n_i = 1 \text{ and } i \to j \\ \sum_{j:i\to j} m_{i,j} \frac{v_i^2}{v_j^2} E\left[F_T \mid X_0 = j\right] & \text{if } n_i \geq 2 \end{cases} \quad \textbf{(22)}$$

This is recognized as the matrix equation

$$\zeta = V^2 M V^{-2} \zeta,$$

where $\zeta(i) = E\left[F_T \mid X_0 = i\right]$. So the vector $z = V^{-2}\zeta$ satisfies $Mz = z$, and using the partition in (9), we get

$$Az_{\mathrm{NT}} + Bz_{\mathrm{T}} = z_{\mathrm{NT}}, \quad \textbf{(23)}$$

where we have partitioned z consistent with the partition of M in (9). We know that for *terminal* nodes i we have $\zeta(i) = v_i$, and it follows that $z_{\mathrm{T}} = V_{\mathrm{T}}^{-2}\zeta_{\mathrm{T}} = u_{\mathrm{T}}$. Therefore, the solution z_{NT} of (23) is $z_{\mathrm{NT}} = (I - A)^{-1} Bu_{\mathrm{T}}$, which equals u_{NT} by (13). Thus, $z = u$, and $\zeta = V^2 z = v$. This completes the proof. ■

As might be expected, the limiting strategies are in fact optimal for the game on terminating graphs.

THEOREM 3 (Optimal Play on Terminating Graphs): *Let $\mathcal{G}$ be a terminating graph in which each terminal node has been assigned a positive value. Then the limiting strategies* (16)*,* (18)*, and* (17) *are optimal.*

PROOF: It is clear that we can restrict our attention to strategies that are *purely positional*, in the sense that at every time the players are at a given vertex i, they play the same strategy. We will continue to use E to denote expected values under the limiting strategies, whereas E^* will denote expected values under general (but fixed) purely positional strategies. Suppose both players are playing a purely positional strategy, possibly different from the limiting strategies defined by (16), (18), and (17). Define $v^*(i)$ by

$$v^*(i) = \limsup_{t\to\infty} E^*\left[F_t | X_0 = i\right]. \quad \textbf{(24)}$$

Note that for terminal vertices i, $v^*(i)$ equals the preassigned value V_i. We first claim that if the players are playing optimally, then $0 < v^*(i) < \infty$ for all nonterminal vertices i. Indeed, $v^*(i) > 0$ because the Guesser can always elect to bet zero at every vertex, whereas $v^*(i) < \infty$ because the Chooser can always elect to take the shortest route from vertex i to a terminal vertex.

Next, we claim that if the Chooser is playing optimally, then T, the hitting time to the set of terminal states, must be finite with probability 1. To see this, suppose by way of contradiction that $P^*(T = \infty) > 0$. Then there must exist a nonterminal vertex i that is part of a strongly connected subgraph $\mathcal{G}^*$ of $\mathcal{G}$ that can be visited infinitely often by the players. Without loss of generality, there is an edge from i to a vertex j not in $\mathcal{G}^*$ that leads to some terminal node z and such that the Chooser's probability of selecting the edge from i to j is zero. (Otherwise, T would be finite with probability 1.) The outdegree n_i of i is therefore at least 2. Consider the following strategy for the Guesser: When at vertex i, the Guesser bets one-half of her fortune on the node k that the Chooser is most likely to select; when at any other vertex in $\mathcal{G}^*$, the Guesser bets nothing. Because the Chooser goes from i to j with probability 0, the probability $p^*_{i,k}$ that he goes to the node k must satisfy

$$p^*_{i,k} \geq \frac{1}{n_i - 1}.$$

Therefore, the Guesser's expected fortune after playing at vertex i (as a proportion of her current fortune) are

$$\begin{aligned}\frac{1}{2}\left(1 - p^*_{i,k}\right) + \left(1 + \frac{1}{2}\left(n_i - 1\right)\right) p^*_{i,k} &= \frac{1}{2}\left(p^*_{i,k} n_i + 1\right) \\ &\geq \frac{1}{2}\left(\frac{n_i}{n_i - 1} + 1\right) \\ &> 1.\end{aligned}$$

Thus, the Guesser increases her fortune on average each time the players are at vertex i. Since there is a positive probability this will happen infinitely often, $v^*(i)$ as defined by (24) will be infinite. However, this contradicts the claim proven earlier that $v^*(i)$ is finite if the Chooser is playing optimally. Therefore, the claim that $P^*(T = \infty) = 0$ is established.

Now that T is finite with probability 1, it follows that $v^*(i)$ as defined by (24) satisfies

$$v^*(i) = E^*\left[F_T | X_0 = i\right],$$

provided that the players are playing optimally. Therefore, given that play starts at node i, the game on $\mathcal{G}$ is equivalent to the game on the fan whose leaves have the values $v^*(j)$, where $i \to j$. Optimal play in this game is given by Theorem 1. Since this is true for each vertex i in $\mathcal{G}$, the reciprocal value vector $u^* = \left(u^*(i) : i \in \mathcal{V},\ u^*(i) = [v^*(i)]^{-1}\right)$ must satisfy $Mu^* = u^*$, where M is the propagation matrix (7) for $\mathcal{G}$. However, u^* must agree with u, the reciprocal value vector for the limiting strategy, on the terminal nodes, since the values of those are preassigned. It follows now from the proof of Lemma 1 that the nonterminal values u^*_{NT} must satisfy

$$u^*_{\mathrm{NT}} = (I - A)^{-1} B u_{\mathrm{T}}$$

and, therefore, $u^* = u$. This completes the proof. ∎

For what terminating graphs $\mathcal{G}$ is the Path Guessing Game fair, in the sense that the Guesser's expected fortune at the end of the game is equal to the \$1 with which she started out?

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THEOREM 4: *Let $\mathcal{G}$ be a terminating graph. Then the Path Guessing Game on $\mathcal{G}$ is fair if and only if each terminal node has a value of* 1 *and each nonterminal node has outdegree at least* 2.

PROOF: The game will be fair if and only if the value of every node is 1. From (13), this is equivalent to $\mathbb{1} = (I - A)^{-1} B\mathbb{1}$, and it follows from this that for each nonterminal i, the ith row sum of A plus the ith row sum of B must equal 1. By the construction of the propagation matrix, this can happen if and only if $\mathcal{G}$ has no nonterminal node of outdegree 1. ■

4. STRONGLY CONNECTED GRAPHS AND THE DISCOUNTED GAME

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a strongly connected digraph on N vertices. The rules for the Path Guessing Game on $\mathcal{G}$ are identical to those for a terminating graph, the only difference being that there are no terminal nodes in a strongly connected graph. As in the previous section, we will derive optimal play on $\mathcal{G}$ by considering the truncated game on $\mathcal{G}$ obtained by the stopping the Path Guessing Game after s steps.

We are interested in the values of the root nodes of the path trees for the s-step game, as s goes to infinity. Consider first the one-step game on $\mathcal{G}$. Leaf nodes of the path trees of height 1 are assigned the value 1, and by Theorem 1, values of the root nodes of the path trees are given by

$$u_1 = M\mathbb{1},$$

where u_1 is the vector of reciprocal values, $\mathbb{1}$ is the vector of all ones, and $M = \left(m_{i,j} : i,j \in \mathcal{V}\right)$ is the $N \times N$ propagation matrix given by

$$m_{i,j} = \begin{cases} 1/2 & \text{if } n_i = 1 \text{ and } i \to j \\ 1/n_i & \text{if } n_i \geq 2 \text{ and } i \to j \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

where n_i is the outdegree of vertex i. Inductively, reciprocal values for the root nodes of the path trees for the s-step game are given by

$$u_s = M^s\mathbb{1}.$$

In contrast to the case of terminating graphs, limiting values for the s-step game might be infinite. For instance, if $\mathcal{G}$ is a 2-cycle, then $v_s(i) = 2^s$ for $i = 1, 2$. However, recall from Theorem 1 that optimal strategies depend only on the *ratios* of values. We will be able to show that limiting ratios continue to exist in the strongly connected case and that the corresponding strategies are optimal.

Toward this end, consider the following *discounted game* on the strongly connected graph $\mathcal{G}$: Play proceeds just as before, but after each payoff, the Guesser's fortune is multiplied by a fixed *discount factor* $d \in (0, 1]$. Intuitively, one can think of the discounted game as modeling a situation in which the real value of money is decreasing with time, as in an inflationary economy. (In this case, the reciprocal of the discount rate would be the inflation rate.) It is clear that optimal strategies for the discounted game are the same as for the undiscounted game.

For the discounted 1-step game, the values of all the leaf nodes are d. Reciprocal values are therefore $1/d$, and so the reciprocal values of the root nodes in the discounted game are

$$\tilde{u}_1 = M\left(d^{-1}\mathbb{1}\right) = d^{-1}M\mathbb{1}.$$

Inductively, values for the root nodes of the path trees for the discounted s-step game are

$$\tilde{u}_s = d^{-s}M\mathbb{1}, \tag{26}$$

and the idea is to determine the value of the discount factor d that makes the limiting reciprocal values finite and nonzero.

It turns out that the correct choice is to make the discount factor equal to the largest positive eigenvalue of the propagation matrix M. Before proving this, we will make one more assumption on the graph $\mathcal{G}$. In addition to being strongly connected, we will assume that $\mathcal{G}$ is *aperiodic*. For a strongly connected graph, aperiodicity means that the greatest common divisor of the lengths of all the cycles in $\mathcal{G}$ is 1. (Note that any strongly connected graph that contains a loop is aperiodic.) Aperiodicity rules out certain cyclic phenomena that, although not unduly hard to characterize, serve mainly to cloud the important issues. The next lemma collects the properties of the propagation matrix M that we will need.

LEMMA 2: *Let $\mathcal{G}$ be a strongly connected aperiodic digraph, with associated propagation matrix M given by* (25). *Then we have the following:*

1. *M has a positive maximal eigenvalue r, with the property that any other eigenvalue λ of M satisfies $|\lambda| < r$.*
2. *There are strictly positive right and left eigenvectors x and y respectively associated with the maximal eigenvalue r.*
3. *The right and left eigenspaces of M associated with r (and containing x and y, respectively) each have dimension* 1.
4. *No other eigenvector of M is positive.*
5. *The maximal eigenvalue r satisfies $\frac{1}{2} \le r \le 1$.*

PROOF: Since $\mathcal{G}$ is strongly connected and aperiodic, it follows that M is irreducible and primitive, and properties 1 through 4 follow from the standard Perron–Frobenius theory of nonnegative matrixes (see [4], for instance). Property 5 follows from the fact that all of the row sums of M are between $\frac{1}{2}$ and 1. ■

We are now in a position to prove that limiting values exist and are strictly positive for the discounted game.

LEMMA 3 (Limiting Values for the Discounted Game): *Let $\mathcal{G}$ be a strongly connected aperiodic digraph, with associated propagation matrix M given by* (25). *Let r be the maximal eigenvalue of M, with associated positive right and left eigenvectors x and y, respectively. Then we have*

$$\lim_{s\to\infty} r^{-s}M^s = \frac{xy^{\mathrm{T}}}{x^{\mathrm{T}}y}, \tag{27}$$

where the limit is a strictly positive matrix. Moreover, the limiting reciprocal value vector $\tilde{u}$ for the discounted game (with discount factor $d = r$) given by

$$\tilde{u} = \lim_{s\to\infty} r^{-s}M^s\mathbb{1} = \frac{xy^{\mathrm{T}}}{x^{\mathrm{T}}y}\mathbb{1} \tag{28}$$

is a positive right eigenvector of M corresponding to r:

$$M\tilde{u} = r\tilde{u}. \tag{29}$$

PROOF: Equation (27) is easily derived from Lemma 2 and the Jordan canonical form for M (see [3], for instance). Equation (28) follows directly from (26) and (27). Finally,

$$\begin{aligned} M\tilde{u} &= M\lim_{s\to\infty} r^{-s}M^s\mathbb{1} \\ &= r\lim_{s\to\infty} r^{-(s+1)}M^{s+1}\mathbb{1} \\ &= r\tilde{u}, \end{aligned}$$

and $\tilde{u}$ is positive because both x and y are. ■

From this point onward, when we refer to the "discounted game," it is understood that the discount factor is the maximal eigenvalue of M. Additionally, for ease of notation we drop the tilde and simply use u to refer to the limiting reciprocal values for the discounted game.

Turning to the nontruncated infinite duration game, we now explore strategies corresponding to the limiting values. For the Chooser, the transition probabilities are

$$\begin{aligned} p_{i,j} &= \mathrm{Prob}\,(X_{t+1} = j \mid X_t = i) \\ &= \begin{cases} 1 & \text{if } n_i = 1 \text{ and } i \to j \\ u_j/\left(\sum_{k:i\to k} u_k\right) & \text{if } n_i \geq 2 \text{ and } i \to j, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{30}$$

whereas the Guesser has guessing probabilities

$$\begin{aligned} q_{i,j} &= \mathrm{Prob}\,(G_{t+1} = j \mid X_t = i) \\ &= \begin{cases} 1 & \text{if } n_i = 1 \text{ and } i \to j \\ \frac{p_{i,j}-\beta p_{i,\min}}{1-n_i\beta p_{i,\min}} & \text{if } n_i \geq 2 \text{ and } i \to j \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{31}$$

and wagers

$$w_i = \begin{cases} 1 & \text{if } n_i = 1 \\ 1 - n_i \beta p_{i,\min} & \text{if } n_i \geq 2 \end{cases} \tag{32}$$

when at node i. Here again β is arbitrary in $(0, 1]$, and $p_{i,\min}$ is the probability of the vertex least likely to be chosen by the Chooser:

$$p_{i,\min} = \min\{p_{i,j} : j \in \mathcal{V}, i \to j\}.$$

We will refer to the strategies given by (30)–(32) as the *limiting strategies* for the players. The next theorem gives the basic properties of the game under the limiting strategies.

THEOREM 5 (Limiting Strategies): *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a strongly connected aperiodic graph. Let M be the propagation matrix for $\mathcal{G}$ defined by* (25). *Let r be the maximal eigenvalue of M and let $v = (v_i : i \in \mathcal{V})$ be the vector of limiting values. Then under the limiting strategies* (30)–(32)*, the Chooser's transition probability matrix P is diagonally similar to the propagation matrix:*

$$P = \frac{1}{r} V M V^{-1}, \tag{33}$$

where $V = \operatorname{diag}(v)$ is the diagonal matrix of limiting values. Under the limiting strategies, the Guesser's fortune satisfies

$$\lim_{t\to\infty} r^t E\left[F_t \mid X_0 = i\right] = c\, v_i, \tag{34}$$

where c is a positive constant that depends only on $\mathcal{G}$.

PROOF: Because the limiting reciprocal value vector u satisfies $Mu = ru$, (30) reduces to

$$p_{i,j} = \frac{1}{r}\frac{m_{i,j}\, u_j}{u_i} = \frac{1}{r}\frac{m_{i,j}\, v_i}{v_j}$$

for all i and j, and (33) follows.

For (34), start by considering a vertex i whose outdegree n_i is at least 2. Then for any adjacent vertex j, we have

$$\begin{aligned} E\left[F_1 \mid X_0 = i, X_1 = j\right] &= (1 + (n_i - 1)w_i)\, q_{i,j} + (1 - w_i)\left(1 - q_{i,j}\right) \\ &= w_i\left(n_i q_{i,j} - 1\right) + 1 \\ &= n_i\left(p_{i,j} - \beta p_{i,\min}\right) - \left(1 - n_i \beta p_{i,\min}\right) + 1 \\ &= n_i p_{i,j} \\ &= \frac{1}{r}\frac{v_i}{v_j}. \end{aligned}$$

Therefore,

$$v_i = E\left[F_1 \mid X_0 = i, X_1 = j\right] r\, v_j. \tag{35}$$

In fact, (35) holds for vertices i of outdegree 1 as well. To see this, note that if i has outdegree 1 and $i \to j$, then by (25) and $Mu = ru$ we see that

$$v_i = 2rv_j.$$

However, since i has outdegree 1, $F_1 = 2$ with probability 1, so in fact the last equation is equivalent to (35). Thus, (35) holds for *all* vertices i and j such that $i \to j$. Now, using the Markov property, we have

$$\begin{aligned} E\left[F_t \mid X_0 = i\right] &= \sum_j E\left[F_t \mid X_0 = i, X_1 = j\right] p_{i,j} \\ &= \sum_j E\left[F_1 \mid X_0 = i, X_1 = j\right] E\left[F'_{t-1} \mid X_0 = j\right] p_{i,j} \\ &= \sum_j \frac{1}{r}\frac{v_i}{v_j} E\left[F_{t-1} \mid X_0 = j\right] p_{i,j} \\ &= \sum_j \frac{1}{r^2}\frac{v_i^2}{v_j^2} m_{i,j} E\left[F_{t-1} \mid X_0 = j\right] \end{aligned}$$

The matrix form of this equation is

$$\zeta_t = r^{-2}V^2MV^{-2}\zeta_{t-1}.$$

where $\zeta_t(i) = E\left[F_t \mid X_0 = i\right]$. Since $\zeta_0 = \mathbb{1}$, we have

$$\zeta_t = r^{-2t}V^2M^tV^{-2}\mathbb{1}, \tag{36}$$

and so by Lemma 3,

$$\begin{aligned} \lim_{t\to\infty} r^t\zeta_t &= r^{-t}V^2M^tV^{-2}\mathbb{1} \\ &= V^2\, \frac{xy^{\mathrm{T}}}{x^{\mathrm{T}}y}\, V^{-2}\mathbb{1}. \end{aligned}$$

To see that the limit is a multiple of the limiting value vector V, denote $G = r^{-2}V^2MV^{-2}$, so that

$$\zeta_t = G^t\mathbb{1}. \tag{37}$$

Note that G has the same pattern of zero and nonzero entries as M and, hence, is irreducible and primitive. We claim that $1/r$ is the maximal eigenvalue of G and that the limiting value vector V is an associated eigenvector. To see this, first observe that

$$GV = r^{-2}V^2MV^{-2}v = r^{-2}V^2Mu = r^{-2}V^2ru = r^{-1}v,$$

so that $1/r$ is indeed an eigenvalue of G, with associated eigenvector v. To see that $1/r$ is the maximal eigenvalue of G, note that if (λ, x) is any eigenpair for G, then

$Gx = \lambda x$ implies that $M(V^{-2}x) = r^2\lambda(V^{-2}x)$, so that $r^2\lambda$ is an eigenvalue of M, with associated eigenvector $V^{-2}x$. However, r is the *maximal* eigenvalue of M, so we must have $\left|r^2\lambda\right| \le r$, which implies $|\lambda| \le 1/r$, which means that $1/r$ is the maximal eigenvalue of G. Now, since G is primitive with maximal eigenvalue $1/r$, it follows that $r^t G^t \mathbb{1}$ converges to some multiple of the associated eigenvector v. By (37) this means that

$$\lim_{t\to\infty} r^t E\left[F_t \mid X_0 = i\right] = c\, v_i.$$

This completes the proof. ■

Because $\mathcal{G}$ is strongly connected and aperiodic, the random walk on $\mathcal{G}$ that arises from the limiting strategies is ergodic and has a unique invariant measure μ. Under this invariant measure, the *discounted* fortune process

$$D_t = r^t F_t \tag{38}$$

is stationary. Let $E_\mu[\cdot]$ denote the expectation operator under the invariant measure (i.e., assuming that the initial position X_0 of the players has distribution μ). The *steady-state fortunes* defined by

$$\eta_j = E_\mu\left[D_t \mid X_t = j\right], \qquad j \in \mathcal{V},$$

represent the Guesser's average discounted fortune when located at vertex j. By stationarity, they do not depend on t. The next theorem characterizes the invariant measure and steady-state fortunes.

THEOREM 6: *Let $\mathcal{G}$ be a strongly connected aperiodic graph with propagation matrix M and associated maximal eigenvalue r. Then the invariant measure μ for the position process $(X_t : t \ge 0)$ from the limiting strategies is the entrywise product of appropriately scaled right and left eigenvectors x and y respectively of M associated with r:*

$$\mu = \frac{1}{x^{\mathrm{T}}y}\left(x_i y_i : i \in \mathcal{V}\right). \tag{39}$$

The steady-state fortunes are of the form

$$E_\mu\left[D_t \mid X_t = j\right] = c\,\frac{\mu_j}{v_j}, \tag{40}$$

where c is a constant that depends on $\mathcal{G}$.

PROOF: The invariant measure μ satisfies $p^{\mathrm{T}}\mu = \mu$. However, $P = r^{-1}VMV^{-1}$, so the invariant measure must satisfy $M^{\mathrm{T}}V\mu = rV\mu$. By primitivity, $V\mu = cy$, where y is the (unique) left eigenvector of M correponding to r. Thus,

$$\mu = cV^{-1}y = c\left(u_i y_i : i \in \mathcal{V}\right),$$

and (39) follows because the reciprocal value vector u is a right eigenvector of M.

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For (40), by conditioning on the position of the players at time $t-1$, one sees that

$$E_\mu\left[D_t \mid X_t = j\right] = \sum_{i:i\to j} \frac{v_i}{v_j} E_\mu\left[D_t \mid X_t = i\right] p_{i,j} = \sum_{i:i\to j} \frac{1}{r}\frac{v_i^2}{v_j^2} m_{i,j} E_\mu\left[D_t \mid X_t = i\right].$$

The matrix form of this equation is

$$r\eta^{\mathrm{T}} = \eta^{\mathrm{T}} V^2 M V^{-2},$$

where $\eta_j = E_\mu\left[D_t \mid X_t = j\right]$. By the primitivity of M we have $V^2\eta = cy$, and (40) now follows. ■

THEOREM 7 (Optimal Play on Strongly Connected Graphs): *Let $\mathcal{G}$ be a strongly connected aperiodic graph. Then the limiting strategies* (33), (31), *and* (32) *are optimal for the infinite-duration Path Guessing Game on $\mathcal{G}$.*

PROOF SKETCH: As in the proof of Theorem 3, we can restrict attention to purely positional strategies. For any choice of purely positional strategies by the Players, there will exist some discount factor d such that the limits

$$\lim_{s\to\infty} d^t E\left[F_t \mid X_0 = i\right] = v_i^\star$$

exist and are nonzero. For this choice of strategies, the discounted game starting at node i is therefore equivalent to the game on the fan with root node i, whose leaves are the set of vertices j such that $i \to j$ in $\mathcal{G}$ and with vertex j having the value $d\, v_j^\star$. By the results of Section 2, the players are playing optimally if and only if $Mu^\star = du^\star$, where $u^\star$ is the vector of reciprocals of the entries of $v^\star$. However, M is irreducible and primitive, so its *only* positive eigenvector is the reciprocal value vector u of Lemma 3. Thus, $u^* = u$, and it follows that the optimal strategies are the same as the limiting strategies. ■

For what strongly connected aperiodic graphs $\mathcal{G}$ is the Path Guessing Game fair, in the sense that the Guesser's expected fortune at the end of the game is equal to the \$1 with which she started out?

THEOREM 8: *Let $\mathcal{G}$ be a strongly connected aperiodic graph. Then the Path Guessing Game on $\mathcal{G}$ is fair if and only if each vertex of $\mathcal{G}$ has outdegree at least 2.*

PROOF: If each vertex has outdegree 2 or more, then the propagation matrix M is stochastic, implying that $r = 1$ and $u = \mathbb{1}$, and it follows from (36) that the game is fair: $E\left[F_t \mid X_0 = i\right] = 1$ for all $i \in \mathcal{V}$. On the other hand, if there exists a vertex i of outdegree 1, then M is strictly substochastic and in fact the ith component of $M\mathbb{1}$ equals $1/2$. By primitivity there exists an integer $t_0 > 0$ such that M^{t_0} is strictly positive. It follows that that *all* of the components of $M^{t_0+1}\mathbb{1}$ are strictly less than one and, hence $\lim_{t\to\infty} M^t\mathbb{1} = 0$. Therefore, the maximal eigenvalue r of M must be strictly less than 1, therefore, the Guesser's expected fortune under optimal play grows without bound. ■

5. THE LYING ORACLE GAME

In this section we apply the previous results to the Lying Oracle Game. As described in [1], this is a two-player game between an Oracle and a Bettor. The game is based on a sequence of n tosses of a fair coin. The Oracle has perfect foreknowledge of each toss but might choose to lie to the Bettor in up to k of the n tosses. Before each toss, the Bettor privately writes down a guess as to whether she thinks the Oracle will lie or tell the truth on this toss. The Bettor also announces a wager on her guess. The Oracle then announces his (possibly mendacious) prediction. Finally, the coin is tossed and the outcome is recorded. If the Bettor is correct, she wins the amount of her wager; if she is incorrect, she loses the amount of her wager. Play continues in this fashion until the n of tosses are completed.

The Lying Oracle Game has been shown to have a reciprocal relationship with the continuous version of Ulam's search problem [5], which has been used as a model of binary search in the presence of errors [6]. In Ulam's search problem, a questioner is searching for a number in the interval [0, 1] chosen by a responder. The questioner asks the responder n questions about the number's location in the interval, but the responder can lie up to k times. The questioner seeks to find a subset of smallest measure that contains the chosen number, whereas the responder seeks to maximize the measure of that subset. Under optimal play, the measure of the questioner's subset is the reciprocal of the Bettor's fortune in the Lying Oracle Game. In spite of this striking result, it is unknown whether there is any type of formal equivalence between these two games [5].

The Lying Oracle Game can be generalized in several ways. In [2] the authors consider the game when the coin is not fair. The "at most k lies in n tosses" rule can be generalized to a set of arbitrary "lying patterns." Here, we make the point that the original Lying Oracle Game (with a fair coin) can be seen as a special case of the Path Guessing Game. Moreover, the results from the previous sections can be used to derive optimal play in the Lying Oracle Game when the number of coin tosses is infinite.

Example 1: Consider the game $\mathcal{G}_{n,1}$ in which the Oracle can lie at most one time in any block of n predictions. The graph for the game is shown in Figure 1, where the edge labels specify where the players move depending on whether the Oracle tells the truth or lies. The players are located initially at vertex 1.

Note that since the outdegree of each vertex is 2 or less, our payoff rule (1) coincides with the payoff rule for the Lying Oracle Game. The propagation matrix is $M = \frac{1}{2}A$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

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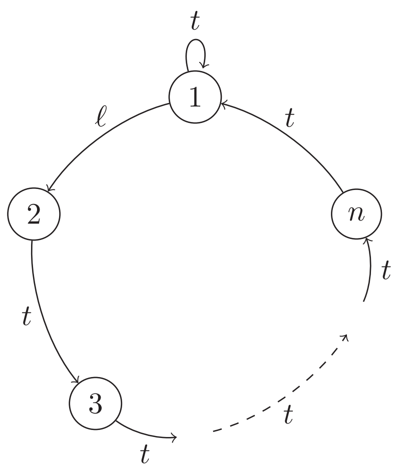

FIGURE 1. The game $\mathcal{G}_{n,1}$.

The maximal eigenvalue of M is $r = \frac{1}{2}\lambda$, where $\lambda = \lambda(n)$ is the largest positive solution of

$$0 - \det(\lambda I - A) = \lambda^n - \lambda^{n-1} - 1.$$

The row sums of A tell us that $\lambda \in [1, 2]$, and it is not hard to show that as n approaches infinity, λ decreases to 1 monotonically. Thus, r approaches 1/2, as we would expect. The right eigenvector of M corresponding to r is

$$x = \left(\lambda^{n-1}, 1, \lambda, \lambda^2, \ldots, \lambda^{n-3}, \lambda^{n-2}\right),$$

and the left eigenvector corresponding to r is

$$y = \left(\lambda^{n-1}, \lambda^{n-2}, \lambda^{n-3}, \ldots, \lambda, 1\right).$$

The Oracle's optimal strategy at node 1 (the only nontrivial node) is

$$p_{1,1} = \lambda^{-1}, \quad p_{1,2} = \lambda^{-n}.$$

In other words, the Oracle tells the truth at node 1 with probability λ^{-1} and lies with probability $1 - \lambda^{-1} = \lambda^{-n}$. Note that as n increases without bound, the probability that the Oracle tells the truth when he is at node 1 approaches 1:

$$\lim_{n\to\infty} p_{1,1} = \lim_{n\to\infty} \lambda^{-1} = 1.$$

The Bettor's minimum risk ($w = w_c$) optimal strategy is

$$q_{1,1} = 1, \quad q_{1,2} = 0,$$
$$w_{1,1} = w_{1,2} = \lambda^{-1} - \lambda^{-n}.$$

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Under optimal play, the players perform a random walk through the graph, with invariant measure

$$\mu = \frac{1}{\lambda^n + n - 1} (\lambda^n, 1, 1, \dots, 1)\,.$$

Observe that the fraction of time that the Oracle lies under optimal play is

$$\mu_2 = \frac{1}{\lambda^n + n - 1},$$

which increases to $1/n$ as n approaches infinity. Thus, under optimal play, the Oracle lies slightly less often than the rules allow.

We remark that if the Oracle's set of allowed "lying patterns" can be expressed in the form of a finite set of finite *forbidden patterns*—sequences of truths and lies that the Oracle must avoid—then the Lying Oracle Game (with a fair coin) is equivalent to the Path Guessing Game on a certain finite graph. All games in which the Oracle can lie at most k times in any block of n statements fall into this class, and as such, they can be analyzed using the results of Section 4. Our approach also leads to new variants of the Lying Oracle Game in which the Oracle is allowed to end the game under *certain conditions*, rather than after a specified number of tosses. For instance, we can consider a game in which the Oracle can lie at most k times in any block of n tosses and can stop the game after any toss on which he told the truth. In these games, the Guesser tries to predict whether the Oracle will lie, tell the truth, or stop the game. Such games correspond to a Path Guessing Game on a terminating graph.

Example 2: Consider the game $\mathcal{G}^*_{n,1}$ in which the Oracle can lie at most one time in any block of n tosses and can stop the game on the first round or after any round on which he told the truth. The terminating graph representing this game is shown in Figure 2.

There are n nonterminal nodes and one terminal node. Using the results of Section 3, we find that under optimal play, the probability $p_{i,n+1}$ that the Oracle stops the game from node i is given by

$$p_{i,n+1} = \begin{cases} \dfrac{2^n - 1}{3\left(2^{n-1} + 2^{n-2} - 1\right)} & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ \dfrac{2^n - 1}{2^{n+1} - 2^{i-2} - 2} & \text{if } 3 \le i \le n. \end{cases}$$

It is easy to show that if n and i go to infinity concurrently such that i/n approaches $x \in (0, 1)$, then $p_{i,n+1}$ approaches $1/2$. We also have $p_{1,n+1}$ approaching $4/9$ as n approaches infinity. Thus, even for large n, the game duration is likely to be very short.

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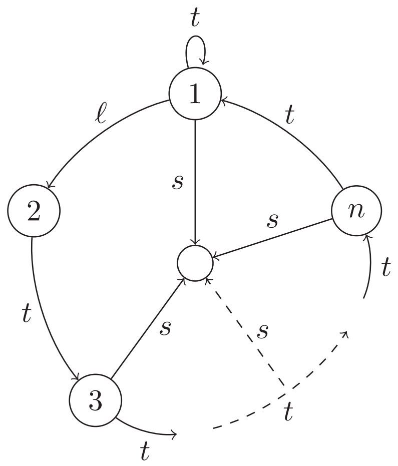

FIGURE 2. The game $\mathcal{G}^*_{n,1}$ from Example 2.

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