Math. Struct. in Comp. Science (1997), *vol.* 7, *pp.* 283–302. Printed in the United Kingdom

An axiomatization of partial *n*-place operations†

MARCO FORTI[1]‡ FURIO HONSELL[2] and MARINA LENISA[3]

[1] *Dip. di Matematica Applicata 'U. Dini', Università di Pisa, Italy.* forti@dm.unipi.it.
[2] *Dip. di Matematica e Informatica, Università di Udine, Italy.* honsell@dimi.uniud.it.
[3] *Dip. di Informatica, Università di Pisa, Italy.* lenisa@di.unipi.it.

Received 16 October 1995; revised 15 September 1996

We propose a general theory of partial *n*-place operations based solely on the primitive notion of the *application of a (possibly partial) operation to n objects*. This theory is strongly selfdescriptive in that the fundamental manipulations of operations, that is, *application, composition, abstraction, union, intersection* and so on, are themselves internal operations. We give several applications of this theory, including implementations of partial *n*-ary λ-calculus, and other operation description languages. We investigate the issue of *extensionality* and give *weakly extensional* models of the theory.

1. Introduction

In this paper we propose a tentative axiomatization of the *primitive notion* of *operation*. Our axiomatization builds solely on the intuition of an operation as an object that *acts* (operates) on one, or more, objects and possibly produces a *result*. Hence, our notion encompasses both *total unary* operations, as in ordinary λ-calculus or Combinatory Logic, as well as *n-place partial* operations.

Our theory is intended to unify general notions of operation and application introduced and first investigated in Schönfinkel (1924), Curry (1929), Church (1932) and Von Neumann (1928). Since this theory is also conceived to be the 'operational part' of a foundational theory of Computer Science and Mathematics, it should be highly *self-descriptive* and *open-ended*. Hence the most relevant operations acting on operations are themselves first class objects of our theory, for example *application*, *abstraction* and *composition*. Moreover, this theory allows for the possible introduction ('engrafting') of qualitatively new notions, both of a mathematical character and other kinds, for example, categories and functors, collections and sets, predicates, algorithms, variables, and so on. Therefore we do not assume that *all* objects are operations.

Our axiomatization is presented in the *semiformal* style normally used in basic foundational theories. However, it can be adequately formalized in first-order logic, by means of a suitable choice of primitive predicates. It could be conceived as an 'engrafting' of the

† This work was partially supported by 40% and 60% MURST grants, CNR, EEC Science MASK, BRA Types 6453 contracts. The first author is a member of GNSAGA of CNR.
‡ The main results of this paper were communicated by this author at the 10^{th} International Conference of Logic, Methodology and Philosophy of Science – Florence, August 1995.

concept of operation in the Basic Theory TB of De Giorgi *et al.* (1994), or rather in the theory TBCS of Forti and Honsell (1994). In fact it originates within the Foundational Programme that has been developed by Ennio De Giorgi and his group since the late seventies at the Scuola Normale Superiore in Pisa (Italy).

Technically, we draw inspiration from the work of D. Scott, G. Plotkin and G. Longo on the categorical and denotational interpretations of λ-calculus (Scott 1975; Plotkin 1975; Longo 1983; Asperti and Longo 1991). Our approach is somewhat more general in that we do not assume that all operations act on all the objects of some appropriate *type*, as do morphisms in Cartesian Closed Categories. Our approach is close to that of Feferman (1974). We assume that all operations are objects, hence any operation is subject to the action of many other operations, which is different from what happens in theories *à la* Gödel–Bernays–Von Neumann (Von Neumann 1928) and *à la* Frege–Aczel (Frege 1903; Aczel 1980).

In Section 2 we introduce the basic axioms constituting the theory Oper and isolate its subtheory Comb. In Section 3 we study the theory Comb: we give several examples of noteworthy operations and show that our theory is powerful enough to accommodate Combinatory Logic. In Section 4 we study the theory Oper and show how to implement an expressive operation description language. In Section 5 we present further axioms that postulate various degrees of extensionality. In Section 6 we give models of these theories. Final remarks and directions for future work appear in Section 7.

Many ideas appearing in this paper have been presented by the authors at the Seminar on the Foundations of Mathematics at the Scuola Normale Superiore in Pisa in the years 1994 and 1995 (Lenzi 1994, 1995). The authors are grateful to all the participants of the seminar and in particular to E. De Giorgi, G. Lenzi and V. M. Tortorelli for helpful discussions. The authors are also grateful to the referees for useful remarks.

2. The fundamental axioms

We assume the natural numbers as primitive, in order to deal naturally with the notion of n-place operation. The only other primitive notions that we consider are 'f is an n-place operation' and 'y is the result of applying the n-place operation f to the objects $x_1, \ldots, x_n$'. For convenience we introduce the following notation:

— if f is an n-place operation, we write† $f \in \Omega_n$;
— if $f \in \Omega_n$ and y is the result of applying f to the objects $x_1, \ldots, x_n$, we write $f \Updownarrow x_1 \ldots x_n y$.

The unusual notation $\Updownarrow$ originates from the theory TB of De Giorgi *et al.* (1994), see also Forti and Honsell (1994). In the theory TB objects are classified according to their *relational complexity*, called *arity*, which corresponds to the 'dimension of the graph'. Hence the arity of a 1-place (or *simple*) operation is two, the arity of a 2-place (*binary*) operation is three, *etc.* In general, n-place operations are particular $n+1$-ary objects. In the theory TB the $\Updownarrow$ notation is used for the *fundamental relation* *Rfond* n+1, which

† In the theory TB of De Giorgi *et al.* (1994) this can be substantiated by postulating the existence of the collections Ω_n for each positive $n \in \mathbb{N}$; in our context this is merely a notation.

expresses the *action* of $(n+1)$-ary objects. In this context this is a mere notation, which could have been replaced, *e.g.*, by $f(x_1 \dots x_n) \simeq y$, following Feferman (1974). We did not use the latter notation in order to emphasize the fact that we do not deal with a theory of *pseudoterms*, as in Feferman (1974). In fact, all our terms denote *existing* objects. Therefore, we freely use the primitive notion of *identity* on *objects*, rather than a suitably defined *equivalence* on *terms*. When we write $fx_1 \dots x_n = y$, we mean that y is the unique object z such that $f \Updownarrow x_1 \dots x_n z$.

We begin by postulating the *uniqueness* of *arity* and the *functionality* of operations.

Axiom Op.**1.** (Arity and Functionality)

1 If $n \neq m$ and $f \in \Omega_n$, then $f \notin \Omega_m$
2 Let $f \in \Omega_n$, if $f \Updownarrow x_1 \dots x_n y$ and $f \Updownarrow x_1 \dots x_n z$, then $y = z$.

If $f \Updownarrow x_1 \dots x_n y$ we shall refer to the *unique* such y as $fx_1 \dots x_n$.

We postulate the existence of basic *combinatorial* operations, namely *projections, applications, compositions* and *abstractions.*

Axiom Op.**2.** (Projections and Empty operations)
For any $0 \leqslant i \leqslant n$ there exists an operation $\Pi_{in} \in \Omega_n$, such that:

$$\forall x_1 \dots x_n \; \nexists y. \; \Pi_{0n} \Updownarrow x_1 \dots x_n y,$$

$$\forall x_1 \dots x_n. \;\; \Pi_{in} \Updownarrow x_1 \dots x_n x_i \text{ , if } 0 < i \leqslant n.$$

Axiom Op.**3.** (Application or Evaluation)
For any $n > 0$ there exists an operation $App_n \in \Omega_{n+1}$, such that for any $f \in \Omega_n$

$$\forall x_1 \dots x_n y. \;\; App_n \Updownarrow f x_1 \dots x_n y \iff f \Updownarrow x_1 \dots x_n y.$$

Axiom Op.**4.** (Generalized Composition)
For any $n > 0$ and $m > 0$ there exists an operation $Comp_{nm} \in \Omega_{n+1}$.
If $g \in \Omega_n$ and $f_1, \dots, f_n \in \Omega_m$, there exists $h \in \Omega_m$ such that $Comp_{nm} \Updownarrow g f_1 \dots f_n h$, and

$$\forall x_1 \dots x_m y. \;\; h \Updownarrow x_1 \dots x_m y \iff$$

$$\exists z_1 \dots z_n \forall i. f_i \Updownarrow x_1 \dots x_m z_i \text{ and } g \Updownarrow z_1 \dots z_n y.$$

Axiom Op.**5.** (Currification, Abstraction, Separation or Parametrization)
For any $n > 0$ and $m > 0$ there exists an operation $Cur_{nm} \in \Omega_1$. If $f \in \Omega_{n+m}$, there exists $g \in \Omega_n$ such that $Cur_{nm} \Updownarrow fg$, and

$$\forall x_1 \dots x_n \exists h \in \Omega_m. \;\; g \Updownarrow x_1 \dots x_n h,$$

$$\forall y_1 \dots y_m z. \;\; h \Updownarrow y_1 \dots y_m z \iff f \Updownarrow x_1 \dots x_n y_1 \dots y_m z.$$

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

It is worth noticing that the operation $Cur_{nm}f$ of Axiom Op.5 is everywhere defined, no matter what arguments it is fed. This has the somewhat unpleasent consequence of forcing an everywhere undefined $(n+m)$-place operation to have a total currification. $Cur_{nm}f$ could have been defined differently to ensure that 'small' operations have 'small' currifications. For example, one could have postulated that the currification cannot assume empty values, as in the following axiom.

Axiom Op.5′. For any $n > 0$ and $m > 0$ there exists an operation $Cur'_{nm} \in \Omega_1$.
If $f \in \Omega_{n+m}$, there exists $g \in \Omega_n$ such that $Cur'_{nm} \Updownarrow fg$ and

$$\forall x_1 \dots x_n (\exists h \in \Omega_m .\ g \Updownarrow x_1 \dots x_n h \iff$$

$$\exists y_1 \dots y_m z.\ \ f \Updownarrow x_1 \dots x_n y_1 \dots y_m z)$$

$$\forall x_1 \dots x_n y_1 \dots y_m z.\ \ (f \Updownarrow x_1 \dots x_n y_1 \dots y_m z \iff$$

$$\exists h.\ \ (g \Updownarrow x_1 \dots x_n h \ \ \&\ \ h \Updownarrow y_1 \dots y_m z)).$$

We also introduce four postulates inspired by the logical operators of Predicate Calculus.

Axiom Op.**6.** (Equality and Inequality Tests)
There exist operations $Eq, Neq \in \Omega_2$.

$$\forall xyz.\ Eq \Updownarrow xyz \iff x = y \ \&\ z = 1$$

$$\forall xyz.\ Neq \Updownarrow xyz \iff x \neq y \ \&\ z = 1.$$

Axiom Op.**7.** (Union or Disjunction)
For each $n > 0$ there exists an operation $Bun_n \in \Omega_2$.
If $f, g \in \Omega_n$, there exists $h \in \Omega_n$ such that $Bun_n \Updownarrow fgh$ and

$$\forall x_1 \dots x_n y.\ \ h \Updownarrow x_1 \dots x_n y \implies f \Updownarrow x_1 \dots x_n y \vee g \Updownarrow x_1 \dots x_n y,$$

$$\forall x_1 \dots x_n.\ \ (\exists y.\ \ f \Updownarrow x_1 \dots x_n y \vee g \Updownarrow x_1 \dots x_n y) \implies (\exists y.\ \ h \Updownarrow x_1 \dots x_n y).$$

Axiom Op.**8.** (Universal Quantification)
For each $n > 0$ there exists an operation $\forall_n \in \Omega_1$.
If $f \in \Omega_{n+1}$, there exists $g \in \Omega_n$ such that $\forall_n \Updownarrow fg$ and

$$\forall x_1 \dots x_n y.\ (g \Updownarrow x_1 \dots x_n y \iff \forall x_{n+1}.\ f \Updownarrow x_1 \dots x_n x_{n+1} y).$$

Axiom Op.**9.** (Existential Quantification or Hilbert's ϵ)
For each $n > 0$ there exists an operation $\exists_n \in \Omega_1$.
If $f \in \Omega_{n+1}$, there exists $g \in \Omega_n$ such that $\exists_n \Updownarrow fg$ and

$$\forall x_1 \dots x_n y.\ (g \Updownarrow x_1 \dots x_n y \implies \exists x_{n+1}.\ f \Updownarrow x_1 \dots x_n x_{n+1} y)$$

$$\forall x_1 \dots x_n.\ (\exists x_{n+1} y.\ f \Updownarrow x_1 \dots x_n x_{n+1} y \implies \exists y.\ g \Updownarrow x_1 \dots x_n y).$$

We call Oper the theory consisting of the axioms Op.1–9, and Comb its subtheory consisting only of the the axioms Op.1–5.

Given the *intensional* character of the notion of operation, we judge it inappropriate to introduce in a theory of operations the axiom of *full extensionality* Ext given by the following axiom.

Axiom Ext. Let $f, g \in \Omega_n$, then $f = g$ if and only if

$$\forall x_1 \dots x_n y.\ f \Updownarrow x_1 \dots x_n y \Longleftrightarrow g \Updownarrow x_1 \dots x_n y.$$

In effect, Axiom Ext is consistent with Comb only, and not with Oper, as we shall see in Section 6. It is therefore useful to introduce the relation of *extensional equivalence* between n-place operations:

Definition 2.1. Let $f, g \in \Omega_n$, then $f \cong g$ if and only if

$$\forall x_1 \dots x_n y.\ f \Updownarrow x_1 \dots x_n y \Longleftrightarrow g \Updownarrow x_1 \dots x_n y.$$

3. Examples and applications: the theory Comb

In this section we derive some interesting consequences of the theory Comb.

3.1. *Constants*

The operation $K_n = Cur_{1n}\Pi_{1n+1}$ produces all n-place *constant* functions. That is, for all objects $y, x_1, \dots, x_n$, $K_n y \Updownarrow x_1 \dots x_n y$.

3.2. *Dummy arguments*

The operation $D_{mn} = Cur_{1\,m+n}(Comp_{n+1\,n+m+1} App_n \Pi_{1\,n+m+1} \dots \Pi_{n+1\,n+m+1})$ adds m *dummy arguments* to any n-place operation f. That is,

$$\forall x_1 \dots x_n y_1 \dots y_m z.\ f \Updownarrow x_1 \dots x_n z \Longleftrightarrow D_{mn} f \Updownarrow x_1 \dots x_n y_1 \dots y_m z.$$

3.3. *Generalized substitution*

For all $m, n > 0$ the operation

$$S_{mn} = Cur_{m+1\,n}(Comp_{m+1\,k} App_m (A_0)(A_1) \dots (A_m)),$$

where $k = m + n + 1$, and

$$\begin{aligned}
A_0 &= Comp_{n+1\,k} App_m \Pi_{1\,k}\, \Pi_{m+2\,k} \dots \Pi_{k\,k}, \\
A_1 &= Comp_{n+1\,k} App_m \Pi_{2\,k} \Pi_{m+2\,k} \dots \Pi_{k\,k}, \\
&\ \vdots \\
A_m &= Comp_{n+1\,k} App_m \Pi_{m+1\,k}\, \Pi_{m+2\,k} \dots \Pi_{k\,k}
\end{aligned}$$

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

provides a generalized substitution *à la* Schönfinkel. Namely, for all $g_0, g_1, \dots, g_m$, we have that $S_{mn}g_0g_1 \dots g_m$ is defined and is an n-place operation. Moreover, if $g_0, g_1, \dots, g_m \in \Omega_n$,

$$\forall x_1 \dots x_n y.\ S_{mn}g_0g_1 \dots g_m \Updownarrow x_1 \dots x_n y \iff$$

$$\exists z_0 z_1 \dots z_m.\ (g_0 \Updownarrow x_1 \dots x_n z_0\ \&\ g_1 \Updownarrow x_1 \dots x_n z_1\ \&\ \dots \\ \&\ g_m \Updownarrow x_1 \dots x_n z_m\ \&\ z_0 \Updownarrow z_1 \dots z_m y).$$

The operation $Cur_{11}S_{11}$ generalizes, to the case of partial 1-place operations, Schönfinkel's classical combinator S. It is interesting to state the following theorem.

Theorem 3.1. The theory Comb is equivalent to the theory asserting the existence of the operations Π_{in} together with the operations K_n and S_{mn} for $m, n > 0$.

Proof. We need to define operations corresponding to App_n, $Comp_{nm}$ and Cur_{nm} using Π_{in}, K_n and S_{mn}. For example, we can put

$$App'_n = S_{n\,n+1}\Pi_{1\,n+1} \dots \Pi_{n+1\,n+1}$$

$$Comp'_{nm} = S_{n+1\,n+1}(K_{n+1}S_{nm})(S_{1\,n+1}(K_{n+1}K_m)\Pi_{1\,n+1})G_1 \dots G_n,$$

where $G_i = S_{1\,n+1}(K_{n+1}S_{mm})(S_{1\,n+1}(K_{n+1}K_m)\Pi_{i+1\,n+1})$.
One can operate similarly for Cur_{nm}. □

3.4. *Compiling λ-calculus*

Any term of the λ-calculus, say M, whose free variables appear in the list $x_1, \dots, x_n$, can be *compiled* as an n-place operation $\|M\| \in \Omega_n$. For convenience, we assume that the bound variables of M do not appear in the list $x_1, \dots, x_n$. By induction on the complexity of the λ-term M, we define

— $\|x_i\|_n = \Pi_{in}$ for $i \leqslant n$
— $\|MN\|_n = Comp_{2n}App_1\, \|M\|_n\, \|N\|_n$
— $\|\lambda z.M\|_n = Cur_{n1}\, \|M[x_{n+1}/z]\|_{n+1}$,

where $M[y/z]$ denotes 'capture avoiding substitution'.

In order to deal with closed λ-terms we would like to extend the above definition to the case $n = 0$. This can be done unproblematically for all *values* in the sense of Plotkin (1975), *i.e.*, terms starting with an abstraction, by putting $\|\lambda z.M\|_0 = \|M[x_1/z]\|_1$. The natural definition of $\|MN\|_0$ would be $App_1\, \|M\|_0\, \|N\|_0$. But the latter expression need not be defined when MN is not 'valuable'. To claim that $\|M\|_0$ is defined for all closed terms would be a further assumption on the Universe.

3.5. *Recursion theorem*

Put

$$\begin{aligned} A &= \|x_1(x_2x_2)x_3\|_3 = Comp_{23}App_1(Comp_{23}App_1\Pi_{13}(Comp_{23}App_1\Pi_{23}\Pi_{23}))\Pi_{33}, \\ B &= \|\lambda x_2x_3.x_1(x_2x_2)x_3\|_1 = Cur_{11}(Cur_{21}A), \\ Y &= \|(\lambda x_2x_3.x_1(x_2x_2)x_3)(\lambda x_2x_3.x_1(x_2x_2)x_3)\|_1 = Comp_{21}App_1BB. \end{aligned}$$

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

Then the following theorem holds.

Theorem 3.2. There exists an operation $Y \in \Omega_1$ such that

$$\forall f \in \Omega_1\ \exists g \in \Omega_1.\ Y \Updownarrow fg\ \&\ \forall xy.\ (g \Updownarrow xy \Longleftrightarrow \exists h.\ f \Updownarrow gh\ \&\ h \Updownarrow xy).$$

3.6. *A partial n-ary λ-calculus*

In this context, it is natural to generalize the language of ordinary λ-calculus to allow for *simultaneous* n-ary abstractions and applications. Hence we introduce the language Λ_ω of partial n-ary λ-calculus as follows

$$M ::= x \mid c \mid M^\ulcorner M_1 \dots M_k{}^\urcorner \mid \lambda^\ulcorner x_1 \dots x_k{}^\urcorner.M,\ \text{for all } k > 0,$$

where the variables $x_1, \dots, x_k$ are all different.

The language CL_ω of partial n-ary Combinatory Logic is the sublanguage of Λ_ω obtained by omitting the last clause.

The compilation introduced in Subsection 3.4 can be extended to terms of these languages, given an interpretation of constants. The compilation function $\| \ \|_n$ is defined on terms whose free variables appear in the list $x_1, \dots, x_n$ and whose bound variables do not appear in it.

— $\|x_i\|_n = \Pi_{in}$ for $i \leqslant n$
— $\|c\|_n = K_n \overline{c}$
— $\|M^\ulcorner M_1 \dots M_k{}^\urcorner\|_n = Comp_{k+1\,n} App_k\, \|M\|_n\, \|M_1\|_n \dots \|M_k\|_n$†
— $\|\lambda^\ulcorner z_1 \dots z_k{}^\urcorner.M\|_n = Cur_{nk}\, \|M[x_{n+i}/z_i]\|_{n+k}$.

As was the case in Subsection 3.4, the natural definition of $\| \ \|_0$ may be *undefined* in the case of applicative terms.

The following theorems show that the definition is *adequate*.

Lemma 3.1. Let M be a term of CL_ω whose free variables are in the list $x_1, \dots, x_{k+n}$, and let $N_1, \dots, N_n$ be terms of Λ_ω whose free variables are in the list $x_1, \dots, x_k$. Then

$$Comp_{k+n\,k}\, \|M[N_1/x_{k+1}, \dots, N_n/x_{k+n}]\|_{k+n} \Pi_{1k} \dots \Pi_{kk}\, \|N_1\|_k \dots \|N_n\|_k \ \cong$$

$$Comp_{k+n\,k}\, \|M\|_{k+n} \Pi_{1k} \dots \Pi_{kk}\, \|N_1\|_k \dots \|N_n\|_k.$$

Proof. The proof is by induction on the structure of M. The base cases are immediate. The induction step necessitates only simple rewritings using *weak associativity* of *Comp* twice, namely

$$Comp_{n,k} f (Comp_{m,k} h_1 g_1 \dots g_m) \dots (Comp_{m,k} h_n g_1 \dots g_m) \cong$$

$$Comp_{m,k} (Comp_{n,m} f h_1 \dots h_n) g_1 \dots g_m,$$

for all $f \in \Omega_n$, $h_1, \dots, h_n \in \Omega_m$ and $g_1, \dots, g_m \in \Omega_k$. □

† Alternatively, one could have put
— $\|M^\ulcorner M_1 \dots M_k{}^\urcorner\|_n = S_{kn}\, \|M\|_n\, \|M_1\|_n \dots \|M_k\|_n$.

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

Theorem 3.3. Let M be a term of CL_ω whose free variables are in the list $x_1, \ldots, x_n$ and let $N_1, \ldots, N_n$ be terms of Λ_ω whose free variables are in the list $x_1, \ldots, x_k$. Then, for all $a_1, \ldots, a_k$ such that the $\|N_i\|_k$ are defined,

$$\forall y. \left(\|M[N_1/x_1, \ldots, N_n/x_n]\|_k \Updownarrow a_1 \ldots a_k y \iff (Comp_{nk} \|M\|_n \|N_1\|_k \ldots \|N_n\|_k) \Updownarrow a_1 \ldots a_k y \right).$$

In particular,

$$\|M[N_1/x_1, \ldots, N_n/x_n]\|_k \cong Comp_{nk} \|M\|_n \|N_1\|_k \ldots \|N_n\|_k$$

if

— either, all the variables $x_1, \ldots, x_n$ occur *indeed* in M
— or, all the functions $\|N_i\|_k$ are *total* functions.

Proof. The proof is similar to the proof of Lemma 3.1. □

Theorem 3.4. Let M be a term of CL_ω whose free variables are in the list $x_1, \ldots, x_{k+n}$, and let $N_1, \ldots, N_n$ be terms of Λ_ω whose free variables are in the list $x_1, \ldots, x_k$. Then

$$\|(\lambda^\ulcorner x_{k+1} \ldots x_{k+n}{}^\urcorner.M)^\ulcorner N_1 \ldots N_n{}^\urcorner\|_k \cong Comp_{k+nk} \|M[N_1/x_{k+1}, \ldots, N_n/x_{k+n}]\|_{k+n} \Pi_{1k} \ldots \Pi_{kk} \|N_1\|_k \ldots \|N_n\|_k.$$

Proof. We need the following fact:

(†) for all $g_0 \in \Omega_{n+k}$ and all $g_1, \ldots, g_n \in \Omega_k$,

$$Comp_{n+1\,k}\, App_n(Cur_{k\,n}\, g_0) g_1 \ldots g_n \cong Comp_{n+k\,k}\, g_0 \Pi_{1k} \ldots \Pi_{kk} g_1 \ldots g_n.$$

Now, by definition,

$$\begin{aligned} &\|(\lambda^\ulcorner x_{k+1} \ldots x_{k+n}{}^\urcorner.M)^\ulcorner N_1 \ldots N_n{}^\urcorner\|_k \\ &\qquad = Comp_{n+1\,k} App_n \|\lambda^\ulcorner x_{k+1} \ldots x_{k+n}{}^\urcorner.M\|_k \|N_1\|_k \ldots \|N_n\|_k \\ &\qquad = Comp_{n+1\,k} App_n(Cur_{kn} \|M\|_{n+k}) \|N_1\|_k \ldots \|N_n\|_k. \end{aligned}$$

Using (†), the last expression is extensionally equivalent to

$$Comp_{n+k\,k} \|M\|_{n+k} \Pi_{1k} \ldots \Pi_{kk} \|N_1\|_k \ldots \|N_n\|_k,$$

which, by Lemma 3.1, is extensionally equivalent to

$$Comp_{n+k\,k} \|M[N_1/x_{k+1}, \ldots, N_n/x_{k+n}]\|_{k+n} \Pi_{1k} \ldots \Pi_{kk} \|N_1\|_k \ldots \|N_n\|_k.$$ □

4. Examples and applications: the theory Oper

In this section we derive some interesting consequences of the theory Oper.

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

4.1. *Intersection*

For each $n > 0$ there exists an operation $Bint_n \in \Omega_2$, such that for all f, g, h if $Bint_n \Updownarrow fgh$, then $f, g, h \in \Omega_n$ and

$$\forall x_1 \dots x_n y.\ \ (f \Updownarrow x_1 \dots x_n y\ \&\ g \Updownarrow x_1 \dots x_n y \iff h \Updownarrow x_1 \dots x_n y).$$

Simply take

$$\|Comp_{2n}\ulcorner\Pi_{12}\ x_1\ (Comp_{2n}\ulcorner Eq\ x_1\ x_2\urcorner)\urcorner\|_2,$$

where basic operations are taken *autonymously*.

4.2. *Definition by cases*

For each $n > 0$ there exists an operation $Def_n \in \Omega_4$ such that if $Def_n \Updownarrow f_0\ f_1\ g\ a\ h$, then $f_0, f_1, g, h \in \Omega_n$ and

$$\forall x_1 \dots x_n y.(h \Updownarrow x_1 \dots x_n y \iff$$

$$(f_0 \Updownarrow x_1 \dots x_n y\ \&\ g \Updownarrow x_1 \dots x_n a)\ \vee (f_1 \Updownarrow x_1 \dots x_n y\ \&\ \exists z \neq a.\ g \Updownarrow x_1 \dots x_n z)).$$

Simply take $\|Comp_{2n}\ulcorner Bun_n\ A_0\ A_1\urcorner\|_4$, where

$$\begin{aligned} A_0 &= Comp_{2n}\ulcorner\Pi_{12}\ x_0\ (Comp_{2n}\ulcorner Eq\ (K_n x_4)\ x_3\urcorner)\urcorner \\ A_1 &= Comp_{2n}\ulcorner\Pi_{12}\ x_1\ (Comp_{2n}\ulcorner Neq\ (K_n x_4)\ x_3\urcorner)\urcorner. \end{aligned}$$

4.3. *Singleton*

For each $n > 0$ there exists an operation $Sing_n \in \Omega_{n+1}$. For all objects $x_1, \dots, x_n, y$, there exists $h \in \Omega_n$ such that $Sing_n \Updownarrow x_1 \dots x_n yh$ and

$$\forall z_1, \dots, z_n, w.\ \ (h \Updownarrow z_1 \dots z_n w \iff x_1 = z_1, \dots, x_n = z_n, y = w).$$

Simply take

$$A = Cur_{n+1\,n}(Bint_{2n+1}\ E_1(Bint_{2n+1} E_2(\dots(Bint_{2n+1} E_{n-1}\ E_n)\dots))),$$

where $E_i = \|Eq\ \ulcorner x_i x_{n+1+i}\urcorner\|_{2n+1}$ and then put

$$Sing_n = Comp_{2\,n+1} App_1\ \|K_{n+1}\ x_{n+1}\|_{n+1} A.$$

4.4. *Inversion*

There exists an operation $Inv \in \Omega_1$. If $f \in \Omega_1$, there exists $g \in \Omega_1$ such that $Inv \Updownarrow fg$ and

$$\forall y.((\exists z.g \Updownarrow yz) \iff (\exists x.f \Updownarrow xy))$$

$$\forall yz.(g \Updownarrow yz \implies f \Updownarrow zy).$$

Simply take

$$Cur_{11}(\exists_3\ \|Eq\ulcorner(x_1\ulcorner x_3\urcorner)\ x_2\urcorner\|_3).$$

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

4.5. *Totality test*

There exists an operation $Tot \in \Omega_1$, such that for any $f \in \Omega_1$, $Tot \Updownarrow f1$ if and only if

$$\forall x \exists y. f \Updownarrow x\ y.$$

Simply take

$$\forall_2(\exists_3 \|Eq^{\ulcorner}(x_1{}^{\ulcorner}x_2{}^{\urcorner})\ x_3{}^{\urcorner}\|_3).$$

4.6. *A functional programming language*

Following the line of Feferman (1984), we extend the language Λ_ω to a (non-executable) functional programming language FL_ω by adding the syntactic class of formulæ and a new clause in the definition of terms. Namely, put

$M, N ::= \ldots \mid$ `if` Φ `then` M `else` N
$\Phi, \Psi :: M = N \mid \Phi \ \&\ \Psi \mid \Phi \vee \Psi \mid \neg\Phi \mid \forall x.\Phi \mid \exists x.\Phi.$

The compilation of FL_ω in Oper is motivated by the natural 'three-valued'-logic interpretation of connectives and quantifiers:

— $\|$`if` Φ `then` M `else` $N\|_n = Def_n \|M\|_n \|N\|_n \|\Phi\|_n\ 0$;
— $\|M = N\|_n = Bun_n F_1 F_2$, where
  $F_1 = Comp_{2n}\ Eq\ \|M\|_n\ \ \|N\|_n$ and
  $F_2 = (Comp_{11}(K_n 1)\ Comp_{2n}\ Neq\ \|M\|_n\ \ \|N\|_n)$;
— $\|\neg\Phi\|_n = Comp_{1n} Sc\ \|\Phi\|_n$;
— $\|\Phi\ \&\ \Psi\|_n = Bun_n G_1 G_2$, where
  $G_1 = Bint_n\ \|\Phi\|_n\ \|\Psi\|_n$ and
  $G_2 = Bun_n(Neg_n\ \|\Phi\|_n)\ (Neg_n\ \|\Psi\|_n)$;
— $\|\Phi \vee \Psi\|_n = Bun_n H_1 H_2$, where
  $H_1 = Bint_n\ \|\Phi\|_n\ \|\Psi\|_n$ and
  $H_2 = Bun_n(Pos_n\ \|\Phi\|_n)\ (Pos_n\ \|\Psi\|_n)$;
— $\|\forall x_{n+1}.\Phi\|_n = Bun_n(\forall_n(Pos_n\ \|\Phi\|_{n+1}))(\exists_n(Neg_n\ \|\Phi\|_{n+1}))$;
— $\|\exists x_{n+1}.\Phi\|_n = Bun_n(\exists_n(Pos_n\ \|\Phi\|_{n+1}))(\forall_n(Neg_n\ \|\Phi\|_{n+1}))$.

The operation $Sc = Bun_1(Sing_1 0\ 1)(Sing_1 1\ 0)$ exchanges the 'truth values' 0 and 1, while

$$\begin{aligned} Pos_n &= \|(Comp_{1n} Eq(K_n 1)x_1)\|_1, \text{ and} \\ Neg_n &= \|(Comp_{1n} Sc(Comp_{1n} Eq(K_n 0)x_1))\|_1 \end{aligned}$$

yield the restrictions of a given n-ary operation to n-tuples, which are mapped respectively to 1 and 0.

Finally, we extend the adequacy Theorem 3.4 to FL_ω as follows.

Theorem 4.1. Let M be a term of FL_ω not containing abstractions whose free variables are in the list $x_1, \ldots x_{k+n}$, and let $N_1, \ldots N_n$ be terms of FL_ω whose free variables are in the list $x_1, \ldots x_k$. Then

$$\|(\lambda^{\ulcorner} x_{k+1} \ldots x_{k+n}{}^{\urcorner}.M)^{\ulcorner} N_1 \ldots N_n{}^{\urcorner}\|_k \cong$$

$$Comp_{k+nk}\ \|M[N_1/x_{k+1}, \ldots, N_n/x_{k+n}]\|_{k+n} \Pi_{1k} \ldots \Pi_{kk}\ \|N_1\|_k \ldots \|N_n\|_k.$$

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

5. Extensionality axioms

So far we have only been able to show *extensional equivalence* of operations. This approach is closer, in spirit, to Combinatory Logic rather than λ-calculus. The following theorem shows that any sufficiently rich theory of operations is *non-extensional* in a very strong sense, *i.e.*, it is inconsistent with the axiom of *selection* Sel:

Axiom Sel. For all n there exists an operation $\epsilon_n \in \Omega_1$ such that for all $f \in \Omega_n$

$$\epsilon_n f \cong f \quad \text{and} \quad \forall g \in \Omega_n (f \cong g \;\Rightarrow\; \epsilon_n f = \epsilon_n g).$$

Theorem 5.1. Comb + Sel + 'there exists $Sing_1$' is inconsistent.

Proof. Put $f = Comp_{11}(Sing_1(\epsilon_1 \Pi_{01})(K_1 0))\epsilon_1$. Then f is defined on x if and only if x is an empty operation, and then $fx = K_1 0$. Now using the operation Y of Theorem 3.2, we have that Yf is defined on x if and only if f is defined on Yf and $f(Yf)$ is defined on x. This is clearly impossible since f is defined only on empty operations. Hence Yf is empty, but then $f(Yf) = K_1 0$, which cannot be extensionally equal to Yf, contrary to Theorem 3.2. □

We do not elaborate on the consequences of the Axiom Sel, we only remark that the Recursion Theorem 3.2 can be strengthened by asserting the existence of an operator $\overline{Y}$ such that

$$\overline{Y} f = \begin{cases} \epsilon_1(f(\overline{Y} f)) & \text{if } f(\overline{Y} f) \text{ exists and belongs to } \Omega_1 \\ \epsilon_1 \Pi_{01} & \text{otherwise} \end{cases}$$

Simply take $\overline{Y} = \|\epsilon_1 \ulcorner Y \ulcorner Comp_{11} \ulcorner x_1 \epsilon_1 \urcorner\urcorner\urcorner\|_1$.

As in Feferman (1974), we can show that the Theory Oper is inconsistent even with the 'full' extensionality of the sole 'total' operations.

Theorem 5.2. Comb + 'there exists Def_1' $\Longrightarrow$ 'there exist two different extensionally equivalent total operations'.

Proof. Clearly, $Def_1(K_1(K_1 1))(K_1(K_1 0))(\Pi_{11})(K_1 0)$ is a total operation with no fixed points. However, if total operations are extensional, Yf gives a fixed point of f, whenever f is a total operation assuming total operations as values. □

Hence, since we have to give up general extensionality axioms, it is interesting to investigate lists of 'milder' extensionality axioms, corresponding to the usual 'algebraic' properties of the fundamental operations. For example, one could postulate associativity of composition, neutrality of projections and idempotency, commutativity and associativity of union and intersection. All these axioms postulate the equality of otherwise only extensionally equivalent operations.

Axiom MExt.1. (Neutrality of Projections)
For all $f \in \Omega_n$ $\quad Comp_{nn} f \Pi_{1n} \ldots \Pi_{nn} = f$.

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

Axiom MExt.2. (Associativity of Composition)
For all $f \in \Omega_n$, $h_1, \ldots, h_n \in \Omega_m$, $g_1, \ldots, g_m \in \Omega_k$

$$Comp_{nk} f(Comp_{mk} h_1 g_1 \ldots g_m) \ldots (Comp_{mk} h_n g_1 \ldots g_m) =$$

$$Comp_{mk}(Comp_{nm} f h_1 \ldots h_n) g_1 \ldots g_m.$$

Axiom MExt.3. (Union) For all $f, g, h \in \Omega_n$
$Bun_n f f = f$, $Bun_n f g = Bun_n g f$ and $Bun_n(Bun_n f g)h = Bun_n f(Bun_n g h)$.

Axiom MExt.4. (Intersection) For all $f, g, h \in \Omega_n$
$Bint_n f f = f$, $Bint_n f g = Bint_n g f$ and $Bint_n(Bint_n f g)h = Bint_n f(Bint_n g h)$.

We can also consider axioms corresponding to the 'full n-ary version of the β-axiom of λ_v-calculus', namely

Axiom MExt.5. (β-reductions)
(i) For all $f \in \Omega_{n+m}$

$$Comp_{m+1\, n+m} App_m F \Pi_{n+1\, m+n} \ldots \Pi_{m+n\, m+n} = f,$$

where $F = Comp_{n\, n+m}(Cur_{n\, m} f)\Pi_{1\, m+n} \ldots \Pi_{n\, m+n}$.
(ii) For all $f \in \Omega_{n+m}$, $g_1, \ldots, g_m \in \Omega_n$

$$Comp_{m+1\, n} App_m (Cur_{n\, m} f) g_1 \ldots g_m = Comp_{m+n\, n} f \Pi_{1\, n} \ldots \Pi_{n\, n} g_1 \ldots g_m.$$

6. Some models

In this section we sketch the constructions of two models that yield the consistency of the theories:

$$\begin{aligned} \mathsf{T}_1 &= \mathsf{Comb} + \mathsf{Ext} \\ \mathsf{T}_2 &= \mathsf{Oper} + \mathsf{MExt.}\ 1 - 5. \end{aligned}$$

6.1. *A model for* T_1

A model for the theory T_1 can be obtained by a simple generalization of standard techniques of denotational semantics. Namely, take the initial solution of the recursive domain equation

$$D \simeq \bigoplus_{n \geqslant 1} [D^n \rightarrow D_\perp] \oplus A,$$

in the category of (possibly bottomless) $C.P.O.$'s and Scott continuous functions, where $\oplus$ denotes the disjoint sum constructor, D^n denotes the $C.P.O.$ consisting of all n-tuples of elements of D, $[\rightarrow]$ denotes the Scott continuous function space constructor, $(\,)_\perp$ denotes the 'lifted' space constructor, A is a 'flat' $C.P.O.$ of atoms, and $\bigoplus_{n \geqslant 1} D_n$ denotes the infinite disjoint union of the D_n's.

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

Once we fix an isomorphism $\imath : D \longrightarrow \bigoplus_{n\geqslant 1}[D^n \to D_\perp] \oplus A$, the interpretations are all natural. Namely,

— $f \in \Omega_n$ means that $\imath(f) \in [D^n \to D_\perp]$
— $f \Updownarrow x_1 \dots x_n y$ means that $\imath(f)(x_1,\dots,x_n) = y$ and $y \in D$
— $\Pi_{in} = \imath^{-1}(\pi_{in})$, where $\pi_{in} \in [D^n \to D_\perp]$ is the i^{th}-projection
— $App_n = \imath^{-1}(eval_n)$, where $eval_n \in [D^{n+1} \to D_\perp]$ is defined by

$$eval_n(x_0x_1\dots x_n) = \begin{cases} \imath(x_0)(x_1\dots x_n) & \text{if } \imath(x_0) \in [D^n \to D_\perp] \\ \perp & \text{otherwise} \end{cases}$$

— $Comp_{nm} = \imath^{-1}(\gamma_{nm})$ and $Cur_{nm} = \imath^{-1}(\kappa_{nm})$ for suitable Scott continuous functions $\gamma_{nm} \in [D^{n+1} \to D_\perp]$ and $\kappa_{nm} \in [D \to D_\perp]$.

The Axiom Ext is clearly valid in D, since $\imath$ is injective.

Moreover, one can immediately see that every countable subset of D is the *codomain* of a suitable unary operation in D. However, as usual in partially ordered models, points cannot be freely mapped in D because of the continuity constraints. In particular, the following 'axiom of finite displacement' (or 'finite separability' or 'discreteness' (Flagg and Myhill 1989)) fails in the model D.

Axiom FinDisp. (Finite Displacement)
Given n different points $x_1,\dots,x_n$ and arbitrary points $y_1,\dots,y_n$ there exists an operation $f \in \Omega_1$ such that $f \Updownarrow x_i y_i$ for $i = 1,\dots,n$.

We conjecture that a model for theory $\mathsf{T}_1 + \mathsf{FinDisp}$ can be obtained using a suitable term model over the language $\Lambda_\omega(\Delta)$. This language is obtained by adding to Λ_ω an infinite sequence Δ of constants, whose intended meaning is that of unary 'finite displacement operations'. Such a construction should exploit a natural generalization of the machinery of δ-reductions of ordinary λ-calculus, see *e.g.*, Barendregt (1984). A similar idea has been successfully exploited in Plotkin (1995).

6.2. *A model for* T_2

We now proceed to sketch a model of the theory T_2, starting from a model $\mathscr{M}$ of $\mathsf{ZF}^-\mathsf{U}$ + 'there exist universe-many urelements'. We fix an external well-ordering of the universe in order to determine the values of the non-deterministic operations $\exists_n$ and Bun_n. Urelements are partitioned in ω classes Ω_n, equinumerous with the universe. They will be 'activated' as operations by a suitable transfinite induction. We operate in various stages:

1. to each functional $n+1$-ary graph G in $\mathscr{M}$ we associate a fixed urelement u_G in Ω_n;
2. to each of the basic operations Π_{in}, App_n, Cur_{nm}, $Comp_{nm}$, Eq, Neq, Bun_n, $Bint_n$, $\forall_n$, $\exists_n$ we associate new urelements $u_{\Pi_{in}}$, u_{App_n}, $u_{Cur_{nm}}$, $u_{Comp_{nm}}$, u_{Eq}, u_{Neq}, u_{Bun_n}, u_{Bint_n}, $u_{\forall_n}$, $u_{\exists_n}$ in the appropriate Ω_k;
3. to each possible value of $Cur_{nm}, \forall_n, \exists_n$ we assign new urelements denoted by $u_{Cur_{nm}f}$, $u_{\forall_n f}$, $u_{\exists_n f}$ in the appropriate Ω_k;

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

4. for each $f \in \Omega_n$ and each s-tuple $(x_1, \ldots, x_s) \in \mathscr{M}^s$, $s < n$, we associate a new urelement $u_{f,x_1,\ldots,x_s}$ in Ω_{n-s}, whose intended meaning is the value of $(Cur_{s,n-s}f)$ applied to $x_1, \ldots, x_s$;

5. we fix a subclass Y_n of Ω_n of the same size as Ω_n and we associate to each finite subset E, of size at least 2, of $\Omega_n \setminus Y_n$ a new urelement $u_{\cup E} \in Y_n$, whose intended meaning is the 'union' of the corresponding operations;

6. we fix a subclass Z_n of Ω_n of the same size as Ω_n and we associate to each finite subset E, of size at least 2, of $\Omega_n \setminus Z_n$ a new urelement $u_{\cap E} \in Z_n$ whose intended meaning is the 'intersection' of the corresponding operations;

7. in order to deal with the intended values of $Comp_{mn}$ we need to take care of iterated applications of composition, since we want to model a composition operator satisfying MExt. 1, 2, 5. To this end we associate different urelements only to 'irreducibile' well-formed terms of a suitable language for representing iterated compositions. For each n we fix a subclass X_n of Ω_n of the same size as Ω_n. The terms of the language are defined inductively, starting from a class of constants standing for the elements of $\bigcup_{n>0}(\Omega_n \setminus X_n)$, as the expressions of the form $u_0[t_1 \ldots t_n]$ where $u_0 \in (\Omega_n \setminus X_n)$ and $t_1, \ldots, t_n$ are terms. One can naturally introduce a notion of 'order' on such terms: each constant from $\Omega_n \setminus X_n$ has order n and $u_0[t_1 \ldots t_n]$ is a term of order m if and only if all the t_i's have order m. Well-formed terms are those having an order. The reduction rules on well-formed terms are:

   — $f[u_{\Pi_{1n}} \ldots u_{\Pi_{nn}}] \rightsquigarrow f$.
   — $u_{App_m}[u_{Cur_{nm}f}[u_{\Pi_{1\,m+n}} \ldots u_{\Pi_{n\,m+n}}]u_{\Pi_{n+1\,m+n}} \ldots u_{\Pi_{m+n\,m+n}}] \rightsquigarrow f$
   — $u_{App_m}[u_{Cur_{nm}f}\, g_1 \ldots g_m] \rightsquigarrow f[u_{\Pi_{1\,n}} \ldots u_{\Pi_{nn}} g_1 \ldots g_m]$.

   The set of irreducible well-formed terms of order n is put into one-to-one correspondence with X_n. If s is a term of order n, we denote by $s\star(t_1, \ldots, t_n)$ the term obtained from s by replacing each occurrence of a non-applied constant c in s by $c[t_1 \ldots t_n]$.

8. We can now 'activate' the urelements of Ω by inductively defining on ordinals the sets

$$A_\alpha = \{(f,\ x_1,\ \ldots,\ x_n,\ y) \mid f \in \Omega_n \quad f \Updownarrow x_1 \ldots x_n y \quad \text{at level } \alpha\}.$$

   At level 0 we put

   — $(u_G,\ x_1,\ \ldots,\ x_n,\ y) \in A_0$ if and only if G is functional and $(x_1, \ldots, x_n, y) \in G$
   — $(u_{\Pi_{in}},\ x_1,\ \ldots,\ x_n,\ x_i) \in A_0$ for all $(x_1, \ldots, x_n) \in \mathscr{M}$
   — $(u_{eq},\ x,\ x,\ 1) \in A_0$ for all $x \in \mathscr{M}$
   — $(u_{neq},\ x,\ y,\ 1) \in A_0$ for all $x \neq y \in \mathscr{M}$
   — $(u_{\forall_n},\ f,\ u_{\forall_n f}) \in A_0$ for all $f \in \Omega_{n+1}$
   — $(u_{\exists_n},\ f,\ u_{\exists_n f}) \in A_0$ for all $f \in \Omega_{n+1}$
   — $(u_{Cur_{nm}},\ f,\ u_{Cur_{nm}f}) \in A_0$ for all $f \in \Omega_{n+m}$
   — $(u_{Cur_{nm}f},\ x_1,\ \ldots,\ x_n,\ u_{f,\ x_1,\ \ldots,\ x_n}) \in A_0$ for all $f \in \Omega_{n+m}$ and $x_1,\ \ldots,\ x_n \in \mathscr{M}$

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

— $(u_{Bun_n}, f, g, u_{\cup E}) \in A_0$ for $f \neq g \in \Omega_n$ where

$$E = \begin{cases} \{f, g\} & \text{if } f, g \notin Y_n \\ \{f\} \cup G & \text{if } f \notin Y_n \text{ and } g = u_{\cup G} \\ \{g\} \cup F & \text{if } g \notin Y_n \text{ and } f = u_{\cup F} \\ F \cup G & \text{if } f = u_{\cup F} \text{ and } g = u_{\cup G} \end{cases}$$

and $(u_{Bun_n}, f, f, f) \in A_0$ for $f \in \Omega_n$

— $(u_{Bint_n}, f, g, u_{\cap E}) \in A_0$ for $f \neq g \in \Omega_n$ where

$$E = \begin{cases} \{f, g\} & \text{if } f, g \notin Z_n \\ \{f\} \cup G & \text{if } f \notin Y_n \text{ and } g = u_{\cap G} \\ \{g\} \cup F & \text{if } g \notin Y_n \text{ and } f = u_{\cap F} \\ F \cup G & \text{if } f = u_{\cap F} \text{ and } g = u_{\cap G} \end{cases}$$

and $(u_{Bint_n}, f, f, f) \in A_0$ for $f \in \Omega_n$

— $(u_{Comp_{nm}}, f, g_1, \dots, g_n, u_{v(t)}) \in A_0$ for $f \in \Omega_n$, $g_1, \dots, g_n \in \Omega_m$ where $v(t)$ is the 'normal form' of the term

$$t = \begin{cases} f[g_1 \dots g_n] & \text{if } f \notin X_n \\ s \star [g_1 \dots g_n] & \text{if } f \in X_n \text{ and } f = u_s. \end{cases}$$

At level $\alpha + 1$ we put

— $A_\alpha \subseteq A_{\alpha+1}$

— $(u_{App_n}, f, x_1, \dots, x_n, y) \in A_{\alpha+1}$
if $f \in \Omega_n$ and $(f, x_1, \dots, x_n, y) \in A_\alpha$

— $(u_{\forall_n f}, x_1, \dots, x_n, y) \in A_{\alpha+1}$
if $f \in \Omega_{n+1}$ and $\forall z \in \mathscr{M}.(f, x_1, \dots, x_n, z, y) \in A_\alpha$

— $(u_{\exists_n f}, x_1, \dots, x_n, y) \in A_{\alpha+1}$
if $f \in \Omega_{n+1}$ and for no y $(u_{\exists_n f}, x_1, \dots, x_n, y) \in A_\alpha$ and and y is the 'least' element such that $\exists z \in \mathscr{M}.(f, x_1, \dots, x_n, z, y) \in A_\alpha$

— $(u_{f, x_1, \dots, x_n}, x_{n+1}, \dots, x_{n+m}, y) \in A_{\alpha+1}$
if $f \in \Omega_{n+m}$ and $(f, x_1, \dots, x_{n+m}, y) \in A_\alpha$

— $(u_{\cup E}, x_1, \dots, x_n, y) \in A_{\alpha+1}$
if $E \subseteq \Omega_n$ and for no y $(u_{\cup E}, x_1, \dots, x_n, y) \in A_\alpha$ and y is the least element such that $\exists f \in E.(f, x_1, \dots, x_n, y) \in A_\alpha$

— $(u_{\cap E}, x_1, \dots, x_n, y) \in A_{\alpha+1}$
if $E \subseteq \Omega_n$ and $\forall f \in E.(f, x_1, \dots, x_n, y) \in A_\alpha$

— $(u_{v(t)}, x_1, \dots, x_n, y) \in A_{\alpha+1}$
if $u_{v(t)} \in X_n$, $v(t) = u_0[t_1 \dots t_k]$ and, for $i = 1, \dots, k$,
$\exists z_i \in \mathscr{M}.(u_{v(t_i)}, x_1, \dots, x_n, z_i) \in A_\alpha)$ and $(u_0, z_1, \dots, z_k, y) \in A_\alpha$.

At limit λ we put $A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$.

Finally, we extend $\mathscr{M}$ to a model of Oper by putting

$$f \Updownarrow x_1 \dots x_n y \iff (f, x_1, \dots, x_n, y) \in A_v,$$

where v is the least ordinal such that $A_v = A_{v+1}$.

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

The model $\mathscr{M}$ thus defined satisfies T_2 by construction and also the following axiom concerning 'graph' operations.

Axiom GrOp. (Graph Operations)
Given any functional set of $n+1$-tuples G, there exists an operation $g \in \Omega_n$ such that $g \Updownarrow x_1 \dots x_n y$ if and only if $(x_1, \dots, x_n, y) \in G$.

7. Final remarks

7.1. *Alternative axiomatizations*

7.1.1. In axiomatizing a theory of operations, different primitive notions can be taken. For instance, one can focus on 1-place operations only. Two alternatives then arise:

— directly code $(n+1)$-place operations as *total* 1-place operations, whose values are n-place operations;

— first introduce a primitive notion of n-tuple, and then code n-place operations as operations acting only on n-tuples.

The n-place approach taken in this paper is, in our view, much more natural. Moreover, it allows us to represent both alternatives above.

The first alternative can be encoded by putting $Op_1 = \Omega_1$ and

$$Op_{n+1} = \{f \in \Omega_1 \mid f \text{ is total and } \forall x.fx \in Op_n\}.$$

The second alternative can be encoded by defining a notion of n-tuple as a fixed 1-place operation defined on $[1, 2, \dots, n]$, and then by putting

$$\Omega^{(n)} = \{f \in \Omega_1 \mid f \text{ is defined only on } n\text{-tuples}\}.$$

7.1.2. The theory Oper can be formalized as a first-order theory in various ways. A direct approach is that of introducing countably many unary predicates $\{A_n\}_{n\geqslant 1}$ representing the Ω_n's and countably many predicates $\{B_n\}_{n\geqslant 1}$, such that B_n has arity $n+2$ and $B_n f x_1 \dots x_n y$ represents the relation $f \Updownarrow x_1 \dots x_n y$. Then, by introducing suitable sequences of constants, one can formalize directly *each* axiom of Oper as a *sequence* of axioms. Following this approach, however, we assume implicitly that the natural numbers $\mathbb{N}$ 'live' in the metatheory.

Alternately, one can formalize the notion of *natural number* within the theory, thus allowing for a *finite axiomatization*, albeit possibly also capturing *non-standard natural numbers*. In this case, however, the presentation of the theory should be substantially modified, so as to use only a *single* predicate for all $\Updownarrow$ relations. For example, one could introduce an internal notion of *tuple*.

7.2. *Extensions of the theory* Oper

7.2.1. Although natural, the axioms MExt of Section 5 were given just by way of example. In fact, many more 'mild' extensionality properties could have been 'forced' in the model $\mathscr{M}$ of Section 6.2, using the same techniques: *e.g.*,

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

- for all $f \in \Omega_k$, $Cur_{1k}App_k f = f = Comp_{1k}\Pi_{11}f$
- for all $h \in \Omega_{n+k+s}$ and $f_1, \ldots, f_{n+k} \in \Omega_n$

$$Comp_{n+k\,n}(Cur_{n+k\,s}h)f_1 \ldots f_{n+k} =$$

$$Cur_{ns}(Comp_{n+k+s\,n+s}hf'_1 \ldots f'_{n+k}\Pi_{n+1\,n+s} \ldots \Pi_{n+s\,n+s}),$$

where $f'_i = Comp_{n\,n+s}f_i\Pi_{1\,n+s} \ldots \Pi_{n\,n+s}$.

We conjecture that suitable 'mild' extensionality axioms could be added consistently to Oper, so as to obtain the following strengthening of Theorems 3.4 and 4.1:

Theorem 7.1. Let M be a term of FL_ω whose free variables are in the list $x_1, \ldots x_{k+n}$ and let $N_1, \ldots N_n$ be terms of FL_ω whose free variables are in the list $x_1, \ldots x_k$, then

$$\|(\lambda^\ulcorner x_{k+1} \ldots x_{k+n}{}^\urcorner.M)^\ulcorner N_1 \ldots N_n{}^\urcorner\|_k =$$

$$Comp_{k+n\,k}\,\|M[N_1/x_{k+1}, \ldots, N_n/x_{k+n}]\|_{k+n}\Pi_{1k} \ldots \Pi_{kk}\,\|N_1\|_k \ldots \|N_n\|_k.$$

7.2.2. Generalizing the inductive technique used in the construction of the model $\mathscr{M}$ of Section 6.2, one could add to $\mathscr{M}$ many more operations. For example, in the style of Feferman (1974), one could add 'selectors for total operations' satisfying the n-place counterpart of the axiom $(\tilde{E})$ of Feferman (1974), namely:

Axiom E*. For each $n > 0$ there exists an operation $\epsilon_n^{tot} \in \Omega_1$ such that

$$\exists g.\epsilon_n^{tot} \Updownarrow fg \iff (f \in \Omega_n \quad \& \quad \forall x_1 \ldots x_n \exists y.f \Updownarrow x_1 \ldots x_n y)$$

$$\forall fg \in \Omega_n(\epsilon_n^{tot}f \cong f \quad \& \quad f \cong g \Longrightarrow \epsilon_n^{tot}f = \epsilon_n^{tot}g).$$

7.3. *Essential non-determinacy of union and existential quantification*

It is interesting to note that the *non-deterministic* nature of the operations Bun_n and $\exists_n$ is in general *unavoidable*. For instance, it is inconsistent to assume that Bun_n 'flatly' picks the value of the first argument whenever possible, *i.e.*, if $f \Updownarrow x_1 \ldots x_n y$, then $Bun_n fg \Updownarrow x_1 \ldots x_n y$. In fact, assuming this, one can define, given $f \in \Omega_1$, an operation g whose domain is the *complement* of that of f. Simply take

$$g = Bint_1(K_1 0)(Bun_1(Comp_{1\,1}(K_1 1)f)(K_1 0)).$$

Then 'Curry's paradox' can be derived at once.

7.4. *Comparison with related work*

7.4.1. Notice that the natural structure of 'operations as computations', consisting of all *partial recursive functions*, is a model of all axioms of Oper but Op.8.

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

7.4.2. We shall not carry out here a detailed comparison of the theory Comb with existing work on 'unary' *partial* λ-calculus. We just point out that in our system *partiality* is forced at the very beginning, because 'arities' are taken seriously, and so the operations App_n are *necessarily* partial.

7.4.3. In comparing the theory Oper to that of Feferman (1974), it is more natural to consider the n-ary counterparts of the *comprehension scheme* $(\tilde{C})$ and of the *selection scheme* $(\tilde{S})$, namely:

Axiom $\mathsf{C}^\star$. $\exists f \in \Omega_n\ ((\forall u_1 \ldots u_n \exists! y.\ \phi(u_1, \ldots, u_n, y) \to \exists y.\ f \Updownarrow u_1 \ldots u_n y)\ \&$
$(\forall u_1 \ldots u_n \forall y.\ f \Updownarrow u_1 \ldots u_n y \to \phi(u_1, \ldots, u_n, y)))$

Axiom $\mathsf{S}^\star$. $\exists f \in \Omega_n\ ((\forall u_1 \ldots u_n \exists y.\ \phi(u_1, \ldots, u_n, y) \to \exists y.\ f \Updownarrow u_1 \ldots u_n y)\ \&$
$(\forall u_1 \ldots u_n \forall y.\ f \Updownarrow u_1 \ldots u_n y \to \phi(u_1, \ldots, u_n, y)))$

where $\phi(u_1, \ldots, u_n, y)$ is a formula of the appropriate language (for example that of Subsection 7.1.2), whose free variables are among $u_1, \ldots, u_n, y$, and which is *monotonic* in the sense of Feferman (1974).

It is worth noticing that this n-ary formulation of the axioms avoids an elaborate encoding of the explicit dependence of f on the parameters.

One can easily see that the schema $\mathsf{S}^\star$, and hence also the schema $\mathsf{C}^\star$, hold in any model of the theory Oper. In fact, any *monotonic* formula is equivalent to a prenex disjunctive normal form, all whose atomic subformulæ are *positive*. Hence, by induction on the structure of such formulæ, one can easily prove that, for any monotonic formula $\phi(u_1, \ldots, u_n, y)$, there exists $g \in \Omega_{n+1}$ such that

$$\forall u_1 \ldots u_n u_{n+1} y\ (g \Updownarrow u_1 \ldots u_n\ u_{n+1}\ y\ \leftrightarrow\ (\phi(u_1, \ldots, u_n, y)\ \wedge\ u_{n+1} = y))\ .$$

Then the operation $f \in \Omega_n$ defined by $f = \exists_n\ g$ satisfies the instantiation of the schema $\mathsf{S}^\star$ to the formula ϕ.

7.5. *A non-reductionist foundational theory*

As remarked in the Introduction, the theory of operations presented in this paper was originally conceived as the theory of operations of the Basic Theory TB of De Giorgi *et al.* (1994) and Forti and Honsell (1994). The theory TB is a general foundational theory for Mathematics, Logic and Computer Science, which is a significant step in the Foundational Programme of Ennio De Giorgi (Clavelli *et al.* 1988; Forti and Honsell 1989; De Giorgi *et al.* 1994; Lenzi 1994, 1995). This Foundational Programme is informed by the following principles:

— *Non-reductionism*: the fact that there are many kinds of qualitatively different objects and concepts should be taken seriously. For instance the intuitive notion of *operation* brings about the *non-extensional* concept of computation process, which escapes any description of operations in terms of graphs only. Similarly, conceiving *collections* as truth-valued operations forces unnecessary commitments on the definition of collection,

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

and does not make apparent their intrinsic extensionality. Taking *natural numbers* as primitives saves us from having to fix priorities among different implementations, such as Von Neumann Ordinals (Von Neumann 1928), Church Numerals (Church 1932), etc..

— *Self-description*: the most relevant operations and relations that a foundational theory utilizes should themselves be objects of the theory. For example, in the present paper, *application*, *abstraction* and *composition* are first class objects, similar to the main operations on collections (*union, relative complement, cartesian product*, and so on) in Forti and Honsell (1994).

— *Open-endedness*: a foundational theory should be open to extensions. The introduction of qualitatively new notions, both of a mathematical character and other kinds, should always be possible. A foundational theory should be a framework suitable for accommodating most of the classical and modern theories arising in Mathematics, Logic, Computer Science, and possibly other sciences (Economics, Linguistics, *etc.*). Any sufficiently clear concept should be 'engraftable' (*innestabile*) in a natural way in it. For example, the concept of *variable* in classical Mathematical Physics and Economics is engrafted in De Giorgi *et al.* (1994). Also, metamathematical notions, such as *formula, proposition*, and *interpretation*, are engrafted, in the same style, in (De Giorgi *et al.* 1995), by introducing suitable kinds of objects together with relations and operations acting on them.

In this view, the theory TBCS of Forti and Honsell (1994), integrated with the theory Oper, presented in this paper, appears as a very rich foundational theory, which can be taken as a natural, general basis for the developments of Set Theory, Mathematics and Theoretical Computer Science. In fact, it deals simultaneously with the concepts of collections and sets, functions and correlations, natural numbers and operations, in a highly self-descriptive way. The theory TBCS + Oper could be interestingly compared to those of Flagg and Myhill (1989), Grue (1992) and Berline and Grue (to appear).

A model for the full theory TBCS + Oper can be obtained by carrying out the construction of Section 6.2, starting from the model $\mathscr{V}$ of the Theory TBCS given in Forti and Honsell (1994), and then suitably extending the inductive definitions of *R_fond h* and *R_univ h*.

References

Aczel, P. (1980) Frege Structures and the Notions of Propositions, Truth and Sets. In: Barwise, J. *et al.* (eds.) *The Kleene Symposium*, North-Holland 31–59.

Asperti, A. and Longo, G. (1991) *Categories, Types, and Structures*, MIT Press.

Barendregt, H. (1984) *The Lambda Calculus*, North-Holland.

Berline, C. and Grue, K. (to appear) A simple semantic consistency proof for map theory based on zeta-denotational semantics. *Theoretical Computer Science*.

Church, A. (1932) A Set of Postulates for the Foundation of Logic. *Ann. Math.* **33** 346–366. Second Paper, *ibid.* (1933) **34** 839–864.

Clavelli, M., De Giorgi, E., Forti, M. and Tortorelli, V. M. (1988) A self-reference oriented theory for the Foundations of Mathematics. In: *Analyse Mathématique et applications - Contributions en l'honneur de Jacques-Louis Lions*, Gauthier-Villars 67–115.

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press

Curry, H. B. (1929) An Analysis of Logical Substitution. *Am. J. Math.* **51** 363–384.

Curry, H. B. (1930) Grundlagen der kombinatorischen Logik. *Am. J. Math.* **52** 509–536, 789–834.

De Giorgi, E. (1990) Rapporti tra matematica e altre forme del sapere umano. *Atti Acc. Naz. Lincei Rend. Adunanze Solenni* **9** 9–15.

De Giorgi, E., Forti, M. and Lenzi, G. (1994) Una proposta di Teorie Base dei Fondamenti della Matematica. *Rend. Mat. Acc. Lincei* (9) **5** 11–22.

De Giorgi, E., Forti, M. and Lenzi, G. (1994) Introduzione delle variabli nel quadro delle Teorie Base dei Fondamenti della Matematica. *Rend. Mat. Acc. Lincei* (9) **5** 117–128.

De Giorgi, E., Forti, M., Lenzi, G. and Tortorelli, V. M. (1995) Calcolo dei Predicati e Concetti Metateorici in una Teoria Base dei Fondamenti della Matematica. *Rend. Mat. Acc. Lincei* (9) **6** 79–92.

Feferman, S. (1974) Non-extensional type-free theories of partial operations and classifications, I. In: Diller, J. *et al.* (eds.) Proof Theory Symposium. *Springer-Verlag Lecture Notes in Mathematics* **500** 73–118.

Feferman, S. (1984) Toward Useful Type-Free Theories, I. *J. Symb. Logic* **49** 75–111.

Flagg, R. and Myhill, J. (1989) A Type-Free System Extending ZFC. *Ann. Pure Appl. Logic* **43** 79–97.

Forti, M. and Honsell, F. (1989) Models of Selfdescriptive Set Theories. In: Colombini, F. *et al.* (eds.) *Partial Differential Equations and the Calculus of Variations, Essays in Honor of Ennio De Giorgi*, Birkhäuser 473–518.

Forti, M. and Honsell, F. (1994) Collections and sets within the Basic Theories for the Foundations of Mathematics. Technical Report **11** Ist. Mat. Appl. 'U.Dini', Pisa.

Frege, G. (1903) *Grundgesetze der Arithmetic, begriffsschriftlich abgeleitet*, Breslau.

Grue, K. (1992) Map Theory. *Theoretical Computer Science* **102** (1) 1–133.

Jech, T. J. (1978) *Set theory*, Academic Press.

Lenzi, G. (editor) (1994,1995) Scuola Normale Superiore, Seminario sui Fondamenti della Matematica del prof. E. De Giorgi, aa.aa. 1993-94, 1994-95. Typeset Lecture Notes, Pisa.

Longo, G. (1983) Set-theoretical models of λ-calculus: theories, expansions, isomorphisms. *Ann. Pure Appl. Logic* **24** 153–188.

Plotkin, G. (1975) Call-by-value, call-by-name and the λ-calculus *Theor. Comp. Sci.* **1** 125–159.

Plotkin, G. (1995) *On a question of Friedman* (manuscript).

Quine, W. V. O. (1937) New Foundations for Mathematical Logic. *Amer. Math. Monthly* **44** 70–80.

Schönfinkel, M. (1924) Über die Bausteine der mathematischen Logik *Math. Ann.* **92** 305–316.

Scott, D. (1975) Some Philosophical issues concerning Theories of Combinators. In: Böhm, C. (ed.) λ-Calculus and Computer Science. *Springer-Verlag Lecture Notes in Computer Science* **37** 346–366.

Scott, D. (1976) Data Types as Lattices. *SIAM J. Comp.* **5** 522–587.

Von Neumann, J. (1928) Die Axiomatisierung der Mengenlehre. *Math. Zeitsch.* **27** 669–752.

https://doi.org/10.1017/S0960129597002272 Published online by Cambridge University Press