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General synthetic domain theory – a logical approach[†]

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Synthetic domain theory (SDT) is a version of Domain Theory where 'all functions are continuous'. Following the original suggestion of Dana Scott, several approaches to SDT have been developed that are *logical* or *categorical*, *axiomatic* or *model-oriented* in character and that are either *specialised* towards Scott domains or aim at providing a *general* theory axiomatising the structure common to the various notions of domains studied so far.

In Reus and Streicher (1993), Reus (1995) and Reus (1998), we have developed a logical and axiomatic version of SDT, which is special in the sense that it captures the essence of Domain Theory *à la* Scott but rules out, for example, Stable Domain Theory, as it requires order on function spaces to be pointwise. In this article we will give a *logical* and *axiomatic* account of a *general* SDT with the aim of grasping the structure common to all notions of domains.

As in *loc.cit.*, the underlying logic is a sufficiently expressive version of *constructive type theory*. We start with a few basic axioms giving rise to a core theory on top of which we study various notions of predomains (such as, for example, complete and well-complete S-spaces (Longley and Simpson 1997)), define the appropriate notion of domain and verify the usual induction principles of domain theory.

Although each domain carries a logically definable 'specialization order', we avoid order-theoretic notions as much as possible in the formulation of axioms and theorems. The reason is that the order on function spaces cannot be required to be pointwise, as this would rule out the model of stable domains *à la* Berry.

The consequent use of logical language – understood as the internal language of some categorical model of type theory – avoids the irritating coexistence of the *internal* and the *external* view pervading purely categorical approaches. Therefore, the paper is aimed at providing an *elementary introduction to synthetic domain theory*, albeit requiring some knowledge of basic type theory.

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1. Introduction

1.1. *Origins of synthetic domain theory*

In various lectures in the 1980's, Dana Scott strongly promoted the idea of using intuitionistic higher order logic or set theory as an adequate logical framework for axiomatising Domain Theory as an *Extensional Theory of Computation*.

The key idea behind traditional Domain Theory *à la* Scott (Scott 1970; Scott 1972; Scott 1993) and Eršhov (Eršhov 1973; Eršhov 1977) is to approximate the notion of *computability* by the notion of *continuity*. This amounts to a systematic axiomatic generalisation of the well-known Myhill–Shepherdson Theorem saying that *all effective operations are continuous with respect to the finite information topology* – see, for example, Rogers (1967).

When working in classical set theory – where *functions* are identified with *functional relations* – one has to live with the coexistence of full and continuous function spaces of domains (which both carry a domain structure). Although one may have classical theories of domains with continuous function spaces only (such as, for example, LCF (Paulson 1987; Regensburger 1994)), in such theories one cannot have the intuitively appealing *Axiom of Unique Choice (AUC)* stating that any functional relation is traced by a continuous function.

But it is consistent with *intuitionistic* logic to claim that *all functions between domains are continuous* and even that *all functions between domains are computable*. Moreover, there are *intended models* for this situation, namely the various *realisability models*. These might be considered as the standard model built relative to some *partial combinatory algebra* (*pca*) and providing an untyped *model of computation*. A prominent example is the Kleene algebra of Gödel numbers of partial recursive functions.

Nevertheless, until the end of the 1980's there was not much activity seriously taking up Scott's suggestion of developing Domain Theory in a purely axiomatic way in an intuitionistic setting. A notable exception is the Ph. D. Thesis of Giuseppe Rosolini (Rosolini 1986), which introduced the basic notion of synthetic domain theory: the 'r.e. subobject classifier' Σ – see also Rosolini (1987). Using Σ, one may define for every intuitionistic set a relation of 'observational inequality' corresponding to the 'information ordering' of classical domain theory. But, in contrast with classical domain theory, this 'information ordering' appears as a derived notion and not as a structure component supporting the view that 'domains are certain sets' and *not* a 'certain kind of structure', be it order-theoretic or topological.

This line of research was taken up again by Wesley Phoa in his Ph. D. Thesis (Phoa 1990), where he gave a detailed account of Domain Theory in the Effective Topos, *i.e.*, the model built on Kleene realisability.

Whereas these approaches were model based, in 1989, Martin Hyland (Hyland 1991) and Paul Taylor (Taylor 1991) independently introduced two different, but essentially equivalent, formulations of a synthetic theory of domains. Their idea was to define *predomains* as so-called Σ-*replete objects*, that is, the objects of the least internal full reflective subcategory (of the ambient category of intuitionistic sets) containing the r.e. subobject classifier Σ. This property of Σ-repleteness can be rephrased as closure under 'all generalised limit processes', *cf.* Streicher (1998). Both Hyland and Taylor strongly advertised the use of the so-called 'internal language' of a topos or a model of type theory, which amounts to jumping back and forth between the diagrammatic language of category theory and the language of higher order logic.

1.2. *The logical approach to SDT*

The starting point of our Logical Approach to SDT is that *an axiomatic theory of domains has to be developed in a purely logical way* if it should be suitable as a *logical framework for the verification of functional programs.*

Of course, this does not mean avoiding categorical notions and diagrams altogether, but understanding them in the sense of the internal language. In order to make this possible, we have to assume the existence of an *internal full subcategory* Set† of the ambient category that establishes a model of the type theory in use. From a type-theoretic point of view, such an internal full subcategory is nothing but an appropriate *type-theoretic universe* Set.

Fortunately, realisability models – even over arbitrary partial combinatory algebras – provide such universes, namely the full internal subcategory of partial equivalence relations (*per*s). Moreover, these universes are even *impredicative* in the sense that they are closed under arbitrary (internal) products of families whose index set need not itself be an element of the universe Set. However, impredicativity of Set is not needed for developing a substantial part of general SDT, though it is necessary to require that the universe Set is closed under products of families whose index set is in Set. But the pleasant aspect of impredicativity of Set is that (pre)domains will provide a model of *polymorphic* λ*-calculus*.

However, for sheaf models of SDT, see, for example, Fiore and Rosolini (1997), there arises the problem that it is not even obvious whether they only contain predicative universes (one definitely knows that Grothendieck toposes cannot contain nontrivial impredicative universes). This seems to have the consequence that for sheaf models not all concepts and arguments can be expressed in the internal language, which certainly is a drawback when one desires SDT to provide a convenient *logic* of domains. But we do not worry too much about this problem because we consider the realisability models as the intended models of our axiomatisation anyway. The reason is that in realisability models, functions between data types are precisely the algorithmic ones – in the sense of 'algorithmic' as given by the underlying partial combinatory algebra.

In our previous draft paper (Reus and Streicher 1993), we coined the term 'naive synthetic domain theory' for the logical approach to SDT in analogy with the term 'naive synthetic differential geometry' as introduced by Lavendhomme in his lecture notes (Lavendhomme 1987). An earlier published account of 'naive synthetic differential geometry' can be found in the first part of the book Kock (1981). There he develops an impressive part of differential calculus in the informal language of intuitionistic higher order logic or set theory from a few axioms (inconsistent with classical logic) without any reference to specific models of the axioms.

† Not to be confused with the category of classical sets usually employed in category theory.

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In this paper, of which an extended abstract has already appeared in Reus and Streicher (1997), our aim is to give an analogous account of Naive SDT, though we probably will not achieve the extent of clarity and elegance exemplified by the work of Kock and Lavendhomme. Due to the fact that SDT is more categorical in nature than synthetic differential geometry, we will often use the diagrammatic language of category theory, but we emphasise that all diagrams have to be understood as living in the universe Set. However, see Reus (1995) and Reus (1998) for a 'diagram-free' formulation of a synthetic theory of Domains *à la* Scott. This theory has been fully formalised and verified mechanically in Reus (1996) using (an extension of) the LEGO Proof Checker.

1.3. *The key ideas underlying synthetic domain theory*

The basic idea underlying synthetic differential geometry is to replace the 'analytic' formulation of differential calculus in terms of the 'ε-δ-language' by a 'synthetic' formulation in terms of 'infinitesimals' whose existence is ensured by appropriate axioms, which, however, contradict classical logic. Analogously, the basic idea of synthetic domain theory is to replace the traditional 'analytic' formulation of domain theory in terms of sets endowed with some order-theoretic or topological structure by a new 'synthetic' formulation where domains are simply *sets with certain properties* from which one may derive (analogues of) the required order-theoretic or topological structure.

1.3.1. *Special synthetic domain theory à la Scott* The starting point for achieving such a 'synthetic' formulation is the observation that there is already in classical domain theory a one-to-one-correspondence between Scott open subsets of a domain D and the continuous maps from D to Σ – the 2-element lattice also known as *Sierpinski space*. Thus, the *topological structure of all domains* is *concentrated into one single domain* Σ, since for an arbitrary domain D, the topology on D is not provided as *some additional structure* but is already given by the collection of maps from D to Σ (forming itself a domain Σ^D).

In the Kleene realisability model these 'open sets' appear as the *semi-decidable predicates* constituting the most general form of experiment applicable to a computational object. Here the elements $\top$ and $\bot$ of Σ correspond to the propositions expressing *termination* and *divergence* of computations, respectively. In other words, Σ can be considered as the set of those propositions corresponding to Σ^0_1-sentences. As semi-decidable predicates on a set X are closed under finite conjunctions and disjunctions and existential quantification over $\mathbb{N}$, they can be considered as a so-called 'natural topology' on X. Notice, however, that the 'open sets of the natural topology' on X will, in general, *not be closed under arbitrary unions* but only under unions of $\mathbb{N}$-indexed families.

Now, the starting point of synthetic domain theory is to postulate *axiomatically* a distinguished set Σ associating with every set X its 'natural topology' whose 'open sets' are given by the functions from X to Σ[†]. This situation was axiomatised by Taylor

[†] This idea goes back to work of Eršhov who developed domain theory in parallel with Dana Scott (Eršhov 1977).

in Taylor (1991) and has subsequently been developed in Reus and Streicher (1993), Reus (1995) and Reus (1998).

1.3.2. *General synthetic domain theory* The assumption that the Σ-predicates on a set X, that is, the $p : X \to \Sigma$, form a weak topology on X is an assumption that is typical for domain theory *à la* Scott. But already the requirement that Σ-predicates are closed under finite disjunctions entails the existence of the Σ-predicate

$$por_\Sigma \triangleq \lambda u, v : \Sigma.\, u \vee v,$$

which *cannot be computed sequentially*. The requirement of closure under finite disjunctions would rule out all models where functions between domains are *sequential*. But Longley has shown in his Thesis (Longley 1994) that in the realisabilty model over the Böhm tree model of untyped λ-calculus, the Σ-predicate por_Σ does not exist (essentially because all λ-definable functions are sequential).

Thus there is a need for a *more general axiomatic setting* that does not rule out interesting models but is strong enough for developing the relevant part of basic domain theory that is needed for the verification of functional programs.

A fairly general categorical axiomatisation has been developed by Rosolini and presented in Rosolini (1995). However, it does not employ the internal language and only gives a core theory. It is not clear under which additional axioms, for example, the domain theoretic induction principles can be derived from his axioms. Universes are not used either.

A detailed account of domain theory in *arbitrary realisability models* has been given in Longley and Simpson (1997). Their exposition is *purely model based* and refrains from developing their theory axiomatically. In Simpson (1996) a version of the well-complete S-spaces in intuitionistic set theory is developed, which is more reminiscent of our approach, though we use type theory instead of set theory.

Our logical approach to *general synthetic domain theory* is heavily based on and inspired by the above mentioned work. One could say that we aim at an *axiomatic logical reformulation* of the work of Longley and Simpson inspired by the categorical axiomatisation of Rosolini. Our axiomatisation has been chosen in such a way that, on the one hand, it is much more restrictive than that in Rosolini (1995) and, on the other hand, is valid in all realisability models as considered in Longley and Simpson (1997) (recently $\mathscr{P}\omega$-realisability models have been advocated in Birkedal *et al.* (1998)). The main purpose of our logical approach is to demonstrate that the *consequent use of the internal type-theoretic language* facilitates a simpler presentation avoiding both complicated external category-theoretic arguments as in Rosolini (1995) and complicated explicit constructions in realisability models as in Longley and Simpson (1997). This is also achieved in Simpson (1996) using intuitionistic set theory. Our type theoretic formulation – at least in our opinion – still appears to be simpler and allows for comparison with such other variants as Reus (1995) and Reus and Streicher (1993).

We now give a brief sketch of the ideas underlying our approach to general synthetic domain theory based on the above mentioned work.

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First, we assume a set S together with a distinguished element $\top \in S$ such that for all $u, v \in S$ we have that $u = v$ iff $u = \top \Leftrightarrow v = \top$. This allows us to consider S as a subobject of Prop modulo '$\Leftrightarrow$' via the embedding $\mathsf{def}: S \to \mathsf{Prop}$ defined as $\mathsf{def}(u) \triangleq (u = \top)$. Furthermore, we assume that there is an object $\bot \in S$ with $\neg\mathsf{def}(\bot)$.

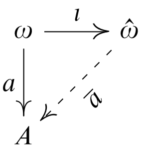
General SDT has to be 'order-free' in the sense that one should avoid using the information ordering – though it can always be defined – as much as possible because in arbitrary realisability models one cannot expect it to be well behaved for the following reasons, *cf.* Longley and Simpson (1997).

(1) The order on function spaces need not be pointwise.
(2) Not all ascending $\mathbb{N}$-indexed chains will have limits.

However, to ensure the existence of a canonical fixpoint for all endomaps $f: A \to A$, we need some notion of 'ω-chain'. This notion must be (in general) more restrictive than the notion of an ascending $\mathbb{N}$-indexed chain but, nevertheless, has to include the so-called 'Kleene chains' $(f^n(\bot))_{n\in\mathbb{N}}$ for all endomaps $f: A \to A$.

Now the key idea of Rosolini (1995) and Longley and Simpson (1997) was to define ω-chains in A as maps $a: \omega \to A$, where ω is the initial algebra for the lifting functor L derived from the dominance S. Then the closure of a set A under 'limits of ω-chains' is expressed by the requirement that for any $a: \omega \to A$ there exists a *unique extension* $\overline{a}: \hat{\omega} \to A$ of a along the canonical map $\iota: \omega \to \hat{\omega}$ from the initial L-algebra ω to the terminal L-coalgebra $\hat{\omega}$ (which is required to exist).

Diagrammatically, the situation looks as follows:



Accordingly, the *analogue* of Scott continuity, that is, the property that any map between domains preserves suprema of ascending chains, amounts to the requirement that for any $f: A \to B$ and $a: \omega \to A$ we have $\overline{f \circ a} = f \circ \overline{a}$.

The intuition underlying this reformulation of *completeness* and *continuity* is the following. In analogy with the category of posets and monotonic maps, the set ω corresponds to the *ordinal* ω, and $\hat{\omega}$ corresponds to the ordered set obtained from ω by adding a greatest element ∞ (that is, the ordinal $\omega + 1$). The *limit* of an ω-chain a is then defined as $\overline{a}(\infty)$. As $\overline{f \circ a} = f \circ \overline{a}$, we then get

$$\overline{f \circ a}(\infty) = (f \circ \overline{a})(\infty) = f(\overline{a}(\infty)),$$

that is, that f preserves limits of ω-chains.

It has been shown (for example, in Rosolini (1995)) that applying this construction to the Kleene chain for an endomap $f: A \to A$ gives rise to a canonical fixpoint of f.

Notice that *prima facie* the Kleene chain for an endomap f is a map $c_f: \mathbb{N} \to A$ and not an ω-chain $a_f: \omega \to A$. However, one can define a_f as the unique L-algebra morphism from the initial L-algebra ω to the L-algebra $\alpha \circ Lf$ where $f: A \to A$ and $\alpha: LA \to A$ is a *focal* L-algebra structure on A, that is, $\alpha \circ \eta_A = id_A$. The focal L-algebra structure provides A with the required domain structure. This map a_f is related to the Kleene chain $c_f(n) = f^n(\bot)$ via the canonical map $\mathsf{step}: \mathbb{N} \to \omega$ as $a_f \circ \mathsf{step} = c_f$.

In our generalised setting, the analogue of the traditional ascending $\mathbb{N}$-indexed chains are those $c: \mathbb{N} \to A$ for which there exists an extension along step, that is, for which there exists a (unique) $a: \omega \to A$ with $a \circ \mathsf{step} = c$. It has been shown in Longley and Simpson (1997) that there are realisability models where there exist ascending $\mathbb{N}$-indexed chains that cannot be extended along step.

1.4. *Survey of contents*

In Section 2 we recall the type-theory on which we base our logical approach to general SDT and postulate some logical axioms that appear natural but are not usually assumed in constructive type theories.

The notions of $\mathscr{L}$-completeness and S-separable sets, *i.e.*, sets the elements of which are separable by S-predicates, also called S-spaces, are introduced in Section 3, and their closure properties are proved.

In Section 4 we give a type-theoretic (re)formulation of the so-called Dominance Axiom (originally introduced by Rosolini in the context of toposes). From this we derive the lifting monad L and show the existence of an initial L-algebra $\phi: L\,\omega \to \omega$ and a terminal L-coalgebra $\upsilon: \hat{\omega} \to L\,\hat{\omega}$. Furthermore, we characterise those $\mathscr{L}$ for which the $\mathscr{L}$-complete sets are closed under lifting.

In Section 5 the notion of complete set is introduced to provide a synthetic analogue of closure under 'limits of ω-chains'. We go on to study two extremal classes of predomains, a class of complete S-spaces closed under lifting and containing the set $S = L\,1$, that is, the well-complete S-spaces of Longley and Simpson, and the replete sets of Hyland and Taylor.

In Section 6 we introduce domains, *i.e.*, predomains with a 'focal L-algebra structure', which constitute an analogue of complete partial orders with a bottom element. We introduce two further axioms guaranteeing the uniqueness of focal L-algebra structures on S-spaces.

In Section 7 we present a synthetic analogue of Kleene's fixpoint construction to provide any endomap of a domain with a canonical fixpoint.

Analogues of the common induction principles for these canonical fixpoints are verified in Section 8. For this purpose, we introduce an appropriate synthetic version of the classical notion of 'admissible predicate'. In particular, we verify the principle of Park Induction, from which it follows that for a map $f: A \to A$ on a domain A its canonical fixpoint is actually the least fixpoint with respect to the information order on A.

In Section 9 we discuss S-separatedness (a generalised variant of Markov's Principle), which is equivalent to the requirement that equality on domains is $\neg\neg$-closed and, therefore, is valid in all realisability models. The advantage of separatedness of S is that two of our axioms become derivable and, moreover, it supports proof by case analysis.

Finally, in Section 10, we treat the solutions of domain equations. In particular, following a suggestion of Simpson, we show how one may solve domain equations if the type-theoretic universe Set is impredicative, *i.e.*, closed under arbitrary dependent products.

2. The logic

The logic underlying our formulation of SDT is an *extension* of the Extended Calculus of Constructions (ECC) as described in Luo (1994) and implemented within the LEGO system described, for example, in Luo and Pollack (1992).

We will now describe the basic constructs of our extension of ECC. For a comprehensive investigation of ECC, see Luo (1994), and for a detailed description of our extension, see the first author's Thesis, Reus (1995).

There is an infinite hierarchy of type universes $\mathsf{Prop} \subset \mathsf{Set} \subset \mathsf{Type}_0 \subset \mathsf{Type}_1 \subset \cdots \subset \mathsf{Type}_n \subset \ldots$ with $\mathsf{Prop} \in \mathsf{Type}_0$, $\mathsf{Set} \in \mathsf{Type}_0$ and $\mathsf{Type}_0 \in \mathsf{Type}_1 \in \cdots \in \mathsf{Type}_n \in \ldots$. Notice that we do not assume that $\mathsf{Prop} \in \mathsf{Set}$. The elements of Prop will be called 'propositions' and the elements of Set will be called 'sets'. Note that in this paper we will never use universes higher than Type_0, but higher universes are useful for expressing deliverables.

The universes Set and Type_i have *predicative* dependent products where a universe $\mathscr{U}$ is closed under predicative dependent products iff $\Pi x{:}A.\,B(x) \in \mathscr{U}$ whenever $A \in \mathscr{U}$ and $B{:}A \to \mathscr{U}$. The universe Prop even has *impredicative* products in the sense that $\Pi x{:}A.\,B(x) \in \mathsf{Prop}$ whenever $A \in \mathsf{Type}_i$ for some i and $B{:}A \to \mathsf{Prop}$. We usually write $\forall x{:}A.\,P(x)$ instead of $\Pi x{:}A.\,B(x)$ if $B{:}A \to \mathsf{Prop}$.

Furthermore, we assume that Prop, Set and Type_i are closed under strong dependent sums, where a universe $\mathscr{U}$ is closed under strong dependent sums iff $\Sigma x{:}A.\,B(x) \in \mathscr{U}$ whenever $A \in \mathscr{U}$ and $B{:}A \to \mathscr{U}$.

The universe Prop is used for representing propositions as types, and the universe Set is where our data types and, in particular, our domains will live. Therefore, we require the universe Set to be closed under definitions of arbitrary inductive types and families (in the sense of Dybjer (1994)). Thus, in particular, in Set we have a type of natural numbers, and Set is closed under binary products and sums. Moreover, as a particular case of inductive families we have *identity types in Set, which, however, we postulate to live already in the universe Prop*, that is, $\mathsf{Id}_A(t,s) \in \mathsf{Prop}$ whenever $A \in \mathsf{Set}$ and $t, s \in A$, which states that t and s are equal elements of A. Instead of $\mathsf{Id}_A(t,s)$, we usually write $t = s$.

The impredicativity of Prop allows one to define predicates by quantification over predicates. Accordingly, inductively defined predicates can be expressed as usual (in higher order logic) as least predicates satisfying some closure conditions.

Notice that our extension of ECC is different from the original ECC in that we have a universe Set, which in many aspects is like the universe Type_0 of the original ECC. The main difference is that Set does not contain Prop as an element. This appears to be natural as the type Prop of all propositions should not be considered as a data type. However, we assume $\mathsf{Prop} \subset \mathsf{Set}$, since proof objects will appear as parts of data objects when dealing with subsets (which will be explained in more detail below).

In general, Set is not assumed to be impredicative, though it is consistent to assume it[†]. Whenever it appears to be useful to assume impredicativity of Set (as, for example, in Section 10), we state this assumption explicitly.

For our purposes we assume the following additional axioms, where we write $\exists(!)$ for (unique) existential quantification definable in terms of $\forall$ (see, for example, Luo (1994)). Furthermore, we usually write $A \Rightarrow B$ for $A \to B$ if A and B are propositions. Note that throughout this paper we will introduce further axioms 'by need', but, for convenience, a quick reference list of all axioms is given in Appendix A. Note also that by postulating the axioms below, we require that the corresponding type is inhabited and that Axiom 3 is not a proposition but a set because of the use of the Σ-type.

Axiom 1. The universe Prop of propositions is 'proof-irrelevant', that is,

$$\forall P:\mathsf{Prop}.\, \forall p, q:P.\ p = q\ .$$

Axiom 2. Functions between sets are extensional, that is,

$$\forall A:\mathsf{Set}.\, \forall B:A \to \mathsf{Set}.\, \forall f, g:\Pi x:A.\, B(x).\ (\forall a:A.f(a) = g(a)) \Rightarrow f = g\ .$$

Axiom 3. Functional relations are required to be tracked by functions, that is, we postulate the Axiom of Unique Choice (AUC)

$$\begin{aligned}&\Pi A:\mathsf{Set}.\, \Pi B:A \to \mathsf{Set}.\, \Pi P:\Pi x:A.\, B(x) \to \mathsf{Prop}.\\ &\qquad (\forall x:A.\, \exists!\, y:B(x).\, P\ x\ y)) \to \Sigma f:\Pi x:A.\, B(x).\, \forall a:A.\, P\ a\ f(a)\ .\end{aligned}$$

We are aware of the fact that these axioms cannot be endowed with computational meaning in the sense of Martin-Löf type theory. But that is irrelevant for our purposes, as we do not intend to extract algorithms from (existence) proofs (which would be hopeless anyway because of Axiom 1). Instead, we employ type-theoretic language, as it appears to be convenient to have dependent types and proof objects available. In the AUC we have already benefitted from this by using a strong Σ-type instead of a second-order defined existential quantifier, which allows us to get the function f by projection on the first component.

Σ-types are also convenient for expressing subsets in the presence of Axiom 1. If A is a set and $P:A \to \mathsf{Prop}$, we define the corresponding subset as

$$\{\, x \in A \mid P(x)\,\} \triangleq \Sigma x:A.\, P(x),$$

which is in Set, since $\mathsf{Prop} \subset \mathsf{Set}$. Moreover, the first projection

$$\pi_0 : \{x \in A \mid P(x)\} \to A$$

is an injective map from $\{\, x \in A \mid P(x)\,\}$ to A, since, by Axiom 1, $P(\pi_0(z))$ contains at most one element.

[†] It would lead to inconsistency via Girard's Paradox only if we assumed also that $\mathsf{Prop} \in \mathsf{Set}$, which we do not. In realisability models one can always ensure that Set is impredicative by interpreting it as the set of all *per*s (partial equivalence relations), where a per is realised by all elements of the partial combinatory algebra under consideration. In this case the universe Prop will be interpreted as the subtype of those sets that contain at most one element, *i.e.*, those pers R where xRy if xRx and yRy. See Reus (1995) and Reus (1998) for a detailed description of these models and the verifications of our axioms.

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Following the practice of topos theory, we will use informal set-theoretic notation instead of the formal type theoretic language, which makes sense, as we have subset types and set comprehension available. Thus, by *abus de langage* we (almost always) omit the inclusion maps $\pi_0 : \{ x \in A \mid P(x) \} \to A$ for subset types. Notice that the type theory described above is closely related to the internal language of toposes, see, for example, Lambek and Scott (1980) and Phoa (1990). But it is different from the logic of a topos in the following two aspects. On the one hand, in a topos there need not exist a universe Set containing the natural numbers object $\mathbb{N}$. On the other hand, our type theory, as opposed to the logic of a topos, does not support the following 'extensionality principle for propositions'

$$\forall P : \mathsf{Prop}. (P \Leftrightarrow Q) \Rightarrow P = Q,$$

which claims that equivalent propositions are equal. However, it still is the case that predicates P and Q on a set A are logically equivalent iff their inclusions into A via first projection are isomorphic as subobjects of A.

It turns out that in our subsequent development of SDT the extensionality principle for propositions is never needed. And that is good, as it is not known to be consistent with the assumption that Set is impredicative or with other subsequent axioms that are in conflict with classical set theory.

Notice that in the following all diagrams have to be understood as living in Set (or sometimes in some universe Type_i), which appears as a small category from the point of view of the type theory in use. In Set (considered as a small category) an equaliser of maps f and g from A to B is constructed as usual, that is, $\{x \in A \mid f(x) = g(x)\}$. Accordingly, we can construct pullbacks in Set as usual. Furthermore, a map $f : A \to B$ in Set can be factorised as a surjection e followed by an injection m in the following way: let $I \triangleq \{ y \in B \mid \exists x{:}A.\, y = f(x) \}$, $m \triangleq \pi_0 : I \to B$, and $e : A \to I$ be the unique map[†] with $f = m \circ e$.

3. $\mathscr{L}$-Completeness and S-spaces

3.1. *$\mathscr{L}$-Completeness*

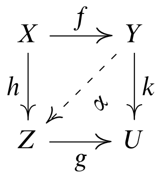
The 'order-free' analogue of closure under suprema of ω-chains will be that for any ω-chain $a : \omega \to A$ there is a unique extension $\overline{a} : \hat{\omega} \to A$ of a along $\iota : \omega \to \hat{\omega}$, that is, $a = \overline{a} \circ \iota$.

This and related closure properties can be formulated quite elegantly using the following notion of *orthogonality*, which goes back to Freyd and Kelly (1972).

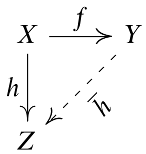
Definition 3.1. Let $f : X \to Y$ and $g : Z \to U$. One says that f is *orthogonal* to g, and writes $f \perp g$, iff for all $h : X \to Z$ and $k : Y \to U$ with $k \circ f = g \circ h$ there is a unique map

[†] More explicitly, $e(x) = \langle f(x), \langle x, \mathsf{r}_B(f(x)) \rangle \rangle$ where $\mathsf{r}_B(f(x)) \in f(x) = f(x)$ is the canonical proof object.

$\alpha\colon Y \to Z$ (called ‘fill-in’) with $\alpha \circ f = h$ and $g \circ \alpha = k$, that is, diagrammatically,



If $U = 1$, we write $f \perp Z$ for $f \perp !_Z$, that is, when any $h\colon X \to Z$ uniquely extends to a map $\overline{h}\colon Y \to Z$ along f,



Using the notion of orthogonality, we define an abstract notion of completeness relative to a class $\mathscr{L}$ of maps in some universe.

Definition 3.2. Let $\mathscr{L}$ be a class of maps, that is, a predicate on

$$\textstyle\sum X, Y : \mathscr{U}.\ (X \to Y)$$

where $\mathscr{U}$ is any of the universes Set or Type_i. A type $A \in \mathscr{U}$ is called $\mathscr{L}$-complete iff $e \perp A$ for all $e \in \mathscr{L}$, that is, $(X, Y, e) \in \mathscr{L}$ where $e\colon X \to Y$.

This is the general pattern, which will be instantiated later by specific classes $\mathscr{L}$ giving rise to notions of predomain as, for example, well complete or replete objects.

Remark 3.1. Note that for the rest of this paper we will mostly consider the case $\mathscr{U} = \mathsf{Set}$ and, accordingly, speak about $\mathscr{L}$-complete sets, well-complete sets, complete sets, and so on. In principle, however, we could also use any other of the universes Type_n.

In any case, the important point is that *our type theory allows us to quantify over* $\mathscr{L}$, as $\mathscr{L}$ is contained in some universe.

The $\mathscr{L}$-complete sets satisfy many closure properties even without any further assumptions about $\mathscr{L}$. First we consider closure under dependent products.

Theorem 3.2. Dependent products preserve $\mathscr{L}$-completeness.

Proof. Let X be a type and $B\colon X \to \mathscr{U}$ be a family of $\mathscr{L}$-complete types in $\mathscr{U}$. In order to show that $\Pi x{:}X.\, B(x)$ is $\mathscr{L}$-complete, assume that $(Y, Z, e) \in \mathscr{L}$ and $a\colon Y \to \Pi x{:}X.\, B(x)$. We have to show that there exists a unique $\overline{a}\colon Z \to \Pi x{:}X.\, B(x)$ with $\overline{a} \circ e = a$.

Let $\pi_x\colon \Pi x{:}X.\, B(x) \to B(x)$ denote the obvious projection, then by completeness of $B(x)$ for any $x \in X$ the map $\pi_x \circ a$ has a unique extension $\overline{\pi_x \circ a}$ along e. Thus, define $\overline{a}$ by the conditions $\pi_x \circ \overline{a} = \overline{\pi_x \circ a}$ for all $x \in X$.

As $\pi_x \circ \overline{a} \circ e = \overline{\pi_x \circ a} \circ e = \pi_x \circ a$, by extensionality, we get $\overline{a} \circ e = a$.

For uniqueness assume $b\colon Z \to \Pi x{:}X.\, B(x)$ with $b \circ e = a$. Then, $\pi_x \circ b \circ e = \pi_x \circ a$. Thus, by uniqueness of $\overline{\pi_x \circ a}$, we get $\pi_x \circ b = \overline{\pi_x \circ a} = \pi_x \circ \overline{a}$. Hence, by extensionality we conclude that $b = \overline{a}$. □

Next we characterise the complete subobjects of complete types.

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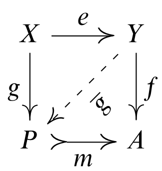
Lemma 3.3. Let A be $\mathscr{L}$-complete and $m: P \rightarrowtail A$ be a subobject of A. Then P is $\mathscr{L}$-complete iff m is *$\mathscr{L}$-closed*, that is, $e \perp m$ for all $e \in \mathscr{L}$.

Proof. $\Rightarrow$: Assume that P is $\mathscr{L}$-complete and $(X, Y, e) \in \mathscr{L}$. To show that $e \perp m$, assume that the outer rectangle of the following diagram commutes



By $\mathscr{L}$-completeness of P, there exists a unique $\overline{g}: Y \to P$ with $\overline{g} \circ e = g$. Therefore, we also have

$$m \circ \overline{g} \circ e = m \circ g = f \circ e,$$

from which it follows that $m \circ \overline{g} = f$, since A is $\mathscr{L}$-complete by assumption. If $\alpha: Y \to P$ with $\alpha \circ e = g$ and $f = m \circ \alpha$, then $\alpha = \overline{g}$ as m is monic. This proves the uniqueness of $\overline{g}$.
$\Leftarrow$: Assume that m is $\mathscr{L}$-closed. For completeness of P, assume that $(X, Y, e) \in \mathscr{L}$ and $g: X \to P$. By $\mathscr{L}$-completeness of A, there exists a unique $f: Y \to A$ with $f \circ e = m \circ g$. By $\mathscr{L}$-closedness of m, we have $e \perp m$ and, therefore, there exists a map $\overline{g}: Y \to P$ with $\overline{g} \circ e = g$. For uniqueness of $\overline{g}$, assume that $h: Y \to P$ with $h \circ e = g$. Then we have

$$m \circ h \circ e = m \circ g = m \circ \overline{g} \circ e,$$

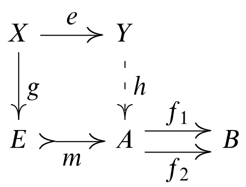
which implies $m \circ h = m \circ \overline{g}$ by $\mathscr{L}$-completeness of A and, therefore, $h = \overline{g}$ as m is monic by assumption. □

Remark 3.4. The previous lemma justifies the following *abus de langage*: for subobjects $m: P \rightarrowtail A$ of an $\mathscr{L}$-complete set A we often say 'P is an $\mathscr{L}$-closed subobject (of A)' instead of the more explicit phrase 'm is an $\mathscr{L}$-closed subobject (of A)' when P is $\mathscr{L}$-complete.

Next we prove that $\mathscr{L}$-complete sets are closed under equalisers.

Theorem 3.5. Let $f_1, f_2: A \to B$ be maps between $\mathscr{L}$-complete types. If $m: E \rightarrowtail A$ is an equaliser of f_1 and f_2, then E is $\mathscr{L}$-complete.

Proof. Let $m: E \rightarrowtail A$ be an equaliser of f_1 and f_2. Let $(X, Y, e) \in \mathscr{L}$ and $g: X \to E$. Then, by $\mathscr{L}$-completeness of A, there is a unique h such that the following diagram commutes



Therefore, we get

$$f_1 \circ h \circ e = f_1 \circ m \circ g = f_2 \circ m \circ g = f_2 \circ h \circ e,$$

which entails $f_1 \circ h = f_2 \circ h$ by $\mathscr{L}$-completeness of B. Thus, as m is an equaliser, there is

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a map $\overline{g}: Y \to E$ with $m \circ \overline{g} = h$. Now we have

$$m \circ \overline{g} \circ e = h \circ e = m \circ g,$$

from which it follows that $\overline{g} \circ e = g$, since m is monic.

For uniqueness of $\overline{g}$, assume that $k \circ e = g$. Then we get

$$m \circ k \circ e = m \circ g = m \circ \overline{g} \circ e,$$

entailing $m \circ k = m \circ \overline{g}$ by $\mathscr{L}$-completeness of A, and, therefore, $k = \overline{g}$, since m is monic. □

Remark 3.6. It follows that $\mathscr{L}$-complete sets are closed under retracts.

3.2. *S-spaces*

Next we define the notion of S-*space* as an abstract version of T_0-spaces where each point is determined uniquely by its collection of open neighbourhoods. This notion has been considered in the specific context of the effective topos for the 'r.e.- subobject-classifier' Σ by Rosolini (Rosolini 1986; Rosolini 1987), and later by Phoa (Phoa 1990).

In this subsection we assume S to be an arbitrary set, that is, $S \in \mathsf{Set}$.

Notation. We write $S(\_)$ for $S^{(\_)}$ and $S^n(\_)$ for the n-times application of the contravariant functor $S(\_) : \mathscr{U} \to \mathscr{U}$, the morphism part of which is given by $S(f : A \to B) = \lambda h{:}S(B).\, h \circ f$.

Definition 3.3. For any type X, let the map $\varepsilon_X : X \to S^2(X)$ be defined as

$$\varepsilon_X(x) = \lambda p{:}S(X).\, p(x).$$

Remark 3.7. Notice that ε is a natural transformation from the identity functor on $\mathscr{U}$ to $S^2(\_) : \mathscr{U} \to \mathscr{U}$ forming the unit of the so-called 'continuation monad', whose multiplication is given by $\mu_X \triangleq S(\varepsilon_{S(X)})$.

Lemma 3.8. For any type X

$$S(\varepsilon_X) \circ \varepsilon_{S(X)} = id_{S(X)}.$$

In other words, $\varepsilon_{S(X)}$ is a split mono.

Proof. Let $p \in S(X)$ and $x \in X$. Then,

$$(S(\varepsilon_X) \circ \varepsilon_{S(X)})(p)(x) = S(\varepsilon_X)(\varepsilon_{S(X)}(p))(x) = \varepsilon_{S(X)}(p)(\varepsilon_X(x)) = \varepsilon_X(x)(p) = p(x)\ .$$

□

We will call a set $X \in \mathsf{Set}$ an S-*space* iff every $x \in X$ is determined uniquely by $\varepsilon_X(x)$.

Definition 3.4. An S-*space* (or S-*separable* space) is a set X such that ε_X is monic.

We immediately get plenty of nontrivial examples of S-spaces.

Lemma 3.9. The following propositions hold:

(1) The set $S(A)$ is an S-space for all sets A.

(2) $S \cong S(1)$ is an S-space.

(3) If Set is impredicative, that is, Set is closed under arbitrary dependent products, then $S(X)$ is an S-space for every type X.

Proof. The proof follows immediately from Lemma 3.8. □

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3.2.1. *A representation theorem for S-spaces* We will now prove that a set is an S-space iff it is a subset of some power of S.

Theorem 3.10. A set $A \in \mathsf{Set}$ is an S-space iff it is a subobject of $S(I)$ for some $I \in \mathsf{Set}$.

Proof. $\Rightarrow$: If $\varepsilon_A : A \to S^2(A)$ is monic, then A is a subobject of $S^2(A)$.
$\Leftarrow$: Suppose that $m : A \rightarrowtail S(I)$ is monic. By naturality of ε, we get that

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon_A} & S^2(A) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle S^2(m)} \\ S(I) & \underset{\varepsilon_{S(I)}}{\rightarrowtail} & S^3(I) \end{array}$$

where $\varepsilon_{S(I)}$ is monic by Lemma 3.8. Thus, ε_A is monic and A is an S-space. □

3.2.2. *Closure properties of S-spaces* Some important closure properties of S-spaces follow immediately from the above representation theorem for S-spaces.

Corollary 3.11. S-spaces are closed under subobjects and, therefore, in particular, under equalisers and retracts.

Next we consider closure of S-spaces under dependent products. For this we first need the following lemmas.

Lemma 3.12. Let X be a type and $B : X \to \mathsf{Set}$ with $B(x)$ an S-space for all $x \in X$. Then $\Pi x{:}X.\, B(x)$ appears as a subobject of $S^2(\Pi x{:}X.\, B(x))$ via $\varepsilon_{\Pi x:X.\, B(x)}$.

Proof. Let $Z \triangleq \Pi x{:}X.\, B(x)$. In order to show that ε_Z is a mono, assume that $f, g \in Z$ and $\phi(f) = \phi(g)$ for all $\phi \in S(Z)$. For all $x \in X$ and $p \in S(B(x))$ for the map

$$P_{x,p} : Z \to S \quad \text{with} \quad P_{x,p}(h) \triangleq p(h(x))$$

we have

$$p(f(x)) = P_{x,p}(f) = P_{x,p}(g) = p(g(x)).$$

Thus, $f(x) = g(x)$ as $B(x)$ is an S-space by assumption. By extensionality for dependent functions, it follows that $f = g$. Thus, $\varepsilon_Z : Z \rightarrowtail S^2(Z)$ is monic. □

Now from the lemma it follows that S-spaces are appropriately closed under dependent products.

Proposition 3.13. If $A \in \mathsf{Set}$ and $B : A \to \mathsf{Set}$ with $B(x)$ an S-space for all $x \in A$, then $\Pi x{:}A.\, B(x)$ is an S-space, too.

Moreover, if Set is an impredicative universe, that is, Set is closed under *arbitrary* dependent products, then for all types X and families $B : X \to \mathsf{Set}$ with $B(x)$ an S-space for all $x \in X$ the dependent product $\Pi x{:}X.\, B(x)$ is an S-space, too.

Proof. The proof follows from Lemma 3.12. □

3.2.3. *A representation theorem for $\mathscr{L}$-complete S-spaces* Next we prove a representation theorem for $\mathscr{L}$-complete S-spaces under the assumption that S is $\mathscr{L}$-complete.

Theorem 3.14. If S is $\mathscr{L}$-complete, a set is an $\mathscr{L}$-complete S-space iff it is an $\mathscr{L}$-closed subobject of some power of S by a set.

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Proof. $\Rightarrow$: If A is an $\mathscr{L}$-complete S-space, then $\varepsilon_A : A \rightarrowtail S^2(A)$ is a mono, which is $\mathscr{L}$-closed by Lemma 3.3, as A is $\mathscr{L}$-complete by assumption. Thus A is an $\mathscr{L}$-closed subobject of $S^2(A)$ via ε_A.
$\Leftarrow$: If $I \in \mathsf{Set}$ and $m : A \rightarrowtail S(I)$ is an $\mathscr{L}$-closed subobject of $S(I)$, then A is an S-space by the representation theorem for S-spaces and it is $\mathscr{L}$-complete by Lemma 3.3, as $S(I)$ is $\mathscr{L}$-complete, since S is by assumption and Corollary 3.13, and m is $\mathscr{L}$-closed by assumption. □

4. Initial and terminal lift algebra

4.1. *The dominance S*

In the previous section we made no assumptions about S beyond being a set. Now we will postulate that S corresponds to a set of propositions providing a notion of 'well-behaved' subobject. Following Rosolini, who first introduced this notion in his Thesis (Rosolini 1986) for the particular case of toposes, see also Rosolini (1987), we will call S a *dominance*.

In contrast with toposes, in our more general type-theoretic setting, equivalent propositions need not be equal in general. Accordingly, we cannot consider S simply as a subset of Prop, but we have to require that elements of S are equal iff their associated propositions are logically equivalent. That means that, morally, we consider S as a subobject of Prop *modulo logical equivalence* although this quotient itself will not exist *qua* type in general[†]. Moreover, we have, of course, to make explicit by an inclusion map how we consider S as embedded into (this fictitious quotient of) Prop. This will be achieved by associating with every $u \in S$ the proposition $u = \top$, where $\top$ is a distinguished element of S (corresponding to the true proposition $\top = \top$).

Axiom 4. There is a distinguished set $S \in \mathsf{Set}$ together with distinguished elements $\top, \bot \in S$ such that

$$\neg(\bot = \top)$$

and

$$\forall u, v{:}S.\, [u = \top \Leftrightarrow v = \top] \Rightarrow u = v,$$

from which it follows that the map $\mathsf{def} : S \to \mathsf{Prop}$ given by

$$\mathsf{def}(u) \triangleq (u = \top) \in \mathsf{Prop}$$

is an inclusion allowing S to be considered as a subset of Prop that reflects logical equivalence in Prop to equality in S.

Moreover, S is required to be a *dominance*, that is, for all $u \in S$ and $v : S^{\mathsf{def}(u)}$ there exists a (necessarily unique) $u \angle v \in S$ with

$$\mathsf{def}(u \angle v) \Leftrightarrow \exists p{:}\mathsf{def}(u).\, \mathsf{def}(v(p)),$$

which provides a 'dependent conjunction' on S.

[†] Typically, in realisability models, *i.e.*, categories of assemblies on a pca, as in Longley and Simpson (1997), the quotient of Prop by logical equivalence does not exist as the latter is not $\neg\neg$-closed in general.

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Remark 4.1. Notice that $\angle$ provides us with an operation $\wedge_S : S \times S \to S$ given by

$$u \wedge_S v \triangleq u \angle \lambda p{:}\mathsf{def}(u).\, v$$

satisfying

$$\mathsf{def}(u \wedge_S v) \Leftrightarrow \mathsf{def}(u) \wedge \mathsf{def}(v).$$

Accordingly, we write $u \wedge v$ for $u \wedge_S v$.

A mono $m{:}X \rightarrowtail Y$ is called an S-*mono* or S-*subobject of* Y iff there exists a (necessarily unique) map $\chi{:}Y \to S$ with $\chi(y) = \top \Leftrightarrow \exists x{:}X.y = m(x)$ for all $y \in Y$. Such a unique χ will be called the *classifying map for* m.

Remark 4.2. Notice that S-monos are closed under composition, since S is required to be a dominance. If $m{:}Y \rightarrowtail X$ and $n{:}Z \rightarrowtail Y$ are S-monos classified by $\varphi{:}X \to S$ and $\psi{:}Y \to S$, respectively, then $m \circ n$ is classified by

$$\chi(x) \triangleq \varphi(x) \angle \lambda u{:}\mathsf{def}(\varphi(x)).\, \psi(\mathsf{y}(x,u)),$$

where $\mathsf{y}(x,u)$ is the unique $y \in Y$ with $x = m(y)$ that exists by (AUC).

The following lemma will be useful later when we discuss the notion of ‘information ordering’.

Lemma 4.3. For all $u \in S$ we have $u = \bot$ iff $\neg\,(u = \top)$, that is, $u = \bot$ iff $\neg\mathsf{def}(u)$. Moreover, we have $\neg\neg(u = \top \vee u = \bot)$ for all $u \in S$.

Proof. By Axiom 4, we have for all $u \in S$ that

$$(\bot = \top \Leftrightarrow u = \top) \Leftrightarrow u = \bot,$$

and therefore

$$\neg\,(u = \top) \;\Leftrightarrow u = \bot,$$

as $\neg\,(\bot = \top)$ by Axiom 4.

For the second claim, notice that $\neg(u = \top \vee u = \bot)$ is equivalent to $\neg(u = \top) \wedge \neg(u = \bot)$, which in turn is equivalent to $(u = \bot) \wedge \neg(u = \bot)$ (by the previous argument). As $(u = \bot) \wedge \neg(u = \bot)$ is evidently contradictory, we get that $\neg\neg(u = \top \vee u = \bot)$. □

4.2. *Lifting*

From the dominance we will now construct a *lifting operation* that allows one to classify partial maps whose domain of definition is given by an S-valued predicate. Note that – just as for the notion of $\mathscr{L}$-completeness – we could consider lifting for types of all universes, but for the purposes of this paper it suffices to have available a lifting operation for sets.

Definition 4.1. The lifting functor $L{:}\mathsf{Set} \to \mathsf{Set}$ is defined as follows.
For $A \in \mathsf{Set}$

$$LA \triangleq \textstyle\sum u{:}S.\, A^{\mathsf{def}(u)},$$

and for $f{:}A \to B$ in Set

$$L(f) \triangleq \lambda z{:}LA.\, \langle \pi_0(z), \lambda p{:}\mathsf{def}(\pi_0(z)).\, f(\pi_1(z)(p)) \rangle.$$

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Moreover, let η be the natural transformation from the identity functor on Set to L given by

$$\eta_A \triangleq \lambda x{:}A.\,\langle \top, \lambda p{:}\mathsf{def}(\top).\,x\rangle$$

for all $A \in \mathsf{Set}$.

Notice that $L0 \cong 1$ and $L1 \cong S$, where 0 is the empty set and 1 is the singleton set containing precisely the element $*$.

Convention: Following common practice, in the following we will not distinguish between $a \in A \in \mathsf{Set}$ and the function $1 \to A$ sending $*$ to a.

Definition 4.2. For $A \in \mathsf{Set}$, let $\bot_A \in LA$ be defined as

$$\bot_A \triangleq \langle \bot, ?_A\rangle,$$

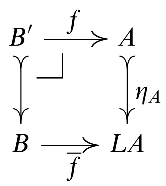
where $?_A$ is the unique map from $\mathsf{def}(\bot) \cong 0$ to A, and let

$$c_A \triangleq [\bot_A, \eta_A] : 1 + A \to LA$$

be the map sending $*$ to $\bot_A$ and $a \in A$ to $\eta_A(a)$.

Notice that c is a natural transformation from $1 + \_$ to L.

Remark 4.4. For any $A \in \mathsf{Set}$, the map $\eta_A : A \to LA$ *classifies* partial maps into A whose domain of definition is given by an S-predicate. More precisely, for any $B \in \mathsf{Set}$, $p : B \to S$ and $f : B' \triangleq \{\, x \in B \mid \mathsf{def}(p(x))\,\} \to A$, there is a *unique* map $\overline{f} : B \to LA$ such that the following diagram is a pullback



where $\overline{f}$ is given explicitly as

$$\overline{f}(y) = \langle\, p(y), \lambda u{:}\mathsf{def}(p(y)).\, f(\langle y, u\rangle)\,\rangle.$$

Composition of two partial maps is achieved as usual via a pullback construction using the fact that S-monos compose (*cf.* Remark 4.1).

4.3. *Initial L-algebra and terminal L-coalgebra*

We now show that there exists an initial algebra and a terminal coalgebra for the lifting functor L, see, for example, Freyd (1991; 1992), where one can find a detailed discussion of these notions for arbitrary endofunctors on an arbitrary category.

Recall that an L-*algebra* is a map $\alpha : LA \to A$. An L-algebra morphism from $\alpha : LA \to A$ to $\beta : LB \to B$ is a map $h : A \to B$ with $\beta \circ Lh = h \circ \alpha$. As $LA = \sum u{:}S.\, A^{\mathsf{def}(u)}$, a map $\alpha : LA \to A$ gives rise to an S-indexed family $\alpha_u : A^{\mathsf{def}(u)} \to A$ for $u \in S$, where $\alpha_u(f) \triangleq \alpha(\langle u, f\rangle)$. Conversely, any S-indexed family $(\alpha_u : A^{\mathsf{def}(u)} \to A \mid u \in S)$ gives rise to a map $\alpha : LA \to A$, where $\alpha(\langle u, f\rangle) \triangleq \alpha_u(f)$. Under this correspondence, L-algebras are nothing but algebras in Set for the signature where for any $u \in S$ there is a $\mathsf{def}(u)$-ary

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operation. Accordingly, a map $h:A \to B$ is an L-algebra morphism from $\alpha:LA \to A$ to $\beta:LB \to B$ iff

$$h(\alpha_u(a)) = \beta_u(h \circ a)$$

for all $u \in S$ and $a \in A^{\mathsf{def}(u)}$.

Dually, an L-coalgebra is a map $\alpha:A \to LA$ and an L-coalgebra morphism from $\alpha:A \to LA$ to $\beta:B \to LB$ is a map $h:A \to B$ such that $\beta \circ h = Lh \circ \alpha$. As $LA = \sum u{:}S.\, A^{\mathsf{def}(u)}$, a map $\alpha:A \to LA$ gives rise to a map $\alpha^{(1)} \triangleq \pi_0 \circ \alpha : A \to S$ and a family $\alpha_a^{(2)} \triangleq \pi_1(\alpha(a)) : A^{\mathsf{def}(\alpha^{(1)}(a))}$ for $a \in A$. Conversely, a map $\alpha^{(1)}:A \to S$ together with a family $(\alpha_a^{(2)}:A^{\mathsf{def}(\alpha^{(1)}(a))} \mid a \in A)$ gives rise to a map $\alpha:A \to LA$ with $\alpha(a) \triangleq \langle \alpha^{(1)}(a), \alpha_a^{(2)} \rangle$. Accordingly, a map $h:A \to B$ is an L-coalgebra morphism from $\alpha:A \to LA$ to $\beta:B \to LB$ iff the equalities

$$\alpha^{(1)}(a) = \beta^{(1)}(h(a)) \qquad h \circ \alpha_a^{(2)} = \beta_{h(a)}^{(2)}$$

hold for all $a \in A$.

Theorem 4.5. In Set there exist a terminal L-coalgebra $\upsilon:\hat{\omega} \to L\hat{\omega}$ and an initial L–algebra $\phi:L\omega \to \omega$. The latter can be constructed as the least sub-L-algebra of υ^{-1}. Thus, the unique L–algebra morphism $\iota:\omega \rightarrowtail \hat{\omega}$ is monic.

Proof. Following Jibladze (1997), we first construct the terminal L-coalgebra $\upsilon:\hat{\omega} \to L\,\hat{\omega}$. We define $\hat{\omega}$ as the subset of $S^{\mathbb{N}}$ consisting of all $f \in S^{\mathbb{N}}$ with $\mathsf{def}(f(n{+}1)) \Rightarrow \mathsf{def}(f(n))$ for all $n \in \mathbb{N}$. We define $\upsilon:\hat{\omega} \to L\,\hat{\omega}$ as

$$\upsilon(f) \triangleq \langle f(0), \lambda z{:}\mathsf{def}(f(0)).\ \lambda n{:}\mathbb{N}.\, f(n+1) \rangle$$

for every $f \in \hat{\omega}$.

Suppose $\alpha:A \to LA$ is some L-coalgebra. An L-coalgebra morphism from α to υ is given by a map $h:A \to \hat{\omega} \subseteq S^{\mathbb{N}}$ such that

$$\alpha^{(1)}(a) = h(a)(0)$$

$$h \circ \alpha_a^{(2)} = \lambda z{:}\mathsf{def}(\alpha^{(1)}(a)).\ \lambda n{:}\mathbb{N}.\, h(a)(n+1)$$

for all $a \in A$.

By a twist of arguments such maps are in one-to-one-correspondence with maps $\tilde{h}:\mathbb{N} \to S^A$ satisfying the primitive recursive equations

$$\tilde{h}(0) = \alpha^{(1)}$$

$$\tilde{h}(n+1) = \lambda a{:}A.\, \alpha^{(1)}(a) \angle (\tilde{h}(n) \circ \alpha_a^{(2)}),$$

as it can be shown by induction over $\mathbb{N}$ that for any such $\tilde{h}$ we have $\lambda n{:}\mathbb{N}.\, \tilde{h}(n)(a) \in \hat{\omega}$ for all $a \in A$. Thus, there exists exactly one L-coalgebra morphism from α to υ, since $\tilde{h}$ is uniquely determined by its defining primitive recursive equations.

Thus, $\upsilon:\hat{\omega} \to L\hat{\omega}$ is a terminal L-coalgebra. Therefore, the map υ is an isomorphism as this is the case for all terminal coalgebras, see, for example, Freyd (1991; 1992). More explicitly, we have

$$\upsilon^{-1}(\langle u, o \rangle)(0) = u$$

$$\upsilon^{-1}(\langle u, o \rangle)(n+1) = u \angle \lambda p{:}\mathsf{def}(u).\, o(u)(n)$$

for all $u \in S$, $o \in \hat{\omega}^{\mathsf{def}(u)}$ and $n \in \mathbb{N}$.

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We will now construct an initial L-algebra as the least sub-L-algebra of the L-algebra $\upsilon^{-1}: L\hat{\omega} \to \hat{\omega}$. Again, this construction was originally given in Jibladze (1997), though our version of the proof makes use of our previous observation that L-algebras can be considered as algebras in the sense of universal algebra.

Let ω be the least subset P of $\hat{\omega}$ closed under all operations υ_u^{-1}, that is, $\upsilon_u^{-1}(f) \in P$ for all $u \in S$ and $f \in P^{\mathsf{def}(u)}$. Thus, υ^{-1} restricts to a map $\phi: L\omega \to \omega$, which we show to be an initial L-algebra.

Let $\alpha: LA \to A$ be an L-algebra. As ϕ is the least sub-L-algebra of υ^{-1}, there is at most one L-algebra morphism from ϕ to α. To prove this, suppose $h_1, h_2: \omega \to A$ are L-algebra morphisms from ϕ to α. As $E \triangleq \{x \in \omega \mid h_1(x) = h_2(x)\}$ gives rise to a sub-L-algebra of $\hat{\omega}$, and ω is the least such, it follows that $\omega = E$. Thus, $h_1(x) = h_2(x)$ for all $x \in \omega$, that is, $h_1 = h_2$.

Next we show the existence of such an L-algebra morphism. Let G be the least subset $R \subseteq \omega \times A$ such that for all $u \in S$, $o \in \omega^{\mathsf{def}(u)}$ and $g \in A^{\mathsf{def}(u)}$ it holds that $\langle \phi_u(o), \alpha_u(g)\rangle \in R$ whenever $\langle o(p), g(p)\rangle \in R$ for all $p \in \mathsf{def}(u)$. By definition of G, it is the graph of an L-algebra morphism from ϕ to α provided G is the graph of a function from ω to A.

By the Axiom of Unique Choice, it suffices to prove that G is functional, that is, that for all $f \in \omega$ there exists a unique $a \in A$ with $\langle f, a\rangle \in G$. We prove functionality of G by induction over ω.

Suppose that $u \in S$ and $o \in \omega^{\mathsf{def}(u)}$ such that for all $p \in \mathsf{def}(u)$ there is a unique $a_p \in A$ with $\langle o(p), a_p\rangle \in G$. We must show that there exists a unique $a \in A$ such that $\langle \phi_u, a\rangle \in G$. By the Axiom of Unique Choice, there is a (unique) map $g \in A^{\mathsf{def}(u)}$ with $g(p) = a_p$, that is, we have $\langle o(p), g(p)\rangle \in G$ for all $p \in \mathsf{def}(u)$. Now, from the closure property of the inductively defined set G, it follows that $\langle \phi_u(o), \alpha_u(g)\rangle \in G$. Thus, there exists an $a \in A$ with $\langle \phi_u(o), a\rangle \in G$.

For uniqueness, suppose that $a \in A$ with $\langle \phi_u(o), a\rangle \in G$. Then, because of the inductive definition of G, it follows that $\langle \phi_u(o), a\rangle = \langle \phi_{u'}(o'), \alpha_{u'}(g')\rangle$ for some $u' \in S$, $o' \in \omega^{\mathsf{def}(u')}$ and $g' \in A^{\mathsf{def}(u')}$ with $\langle o'(p), g'(p)\rangle \in G$ for all $p \in \mathsf{def}(u')$. Thus, we have $\phi\,\langle u, o\rangle = \phi_u(o) = \phi_{u'}(o') = \phi\,\langle u', o'\rangle$. As ϕ is obtained as the restriction of the isomorphism υ^{-1} to ω and, therefore, ϕ is one-to-one, it follows that $u = u'$ and $o = o'$. But then, for all $p \in \mathsf{def}(u)$ we have $\langle o(p), g'(p)\rangle \in G$ and, also, $\langle o(p), g(p)\rangle \in G$. From the induction hypothesis, it follows that $g(p) = g'(p)$ for all $p \in \mathsf{def}(u)$. Thus, we have $g = g'$ and, therefore, $a = \alpha_{u'}(g') = \alpha_u(g)$ also, establishing the uniqueness of $\alpha_u(g)$. □

From an old observation by Lambek, the structure maps of initial algebras and terminal coalgebras are isomorphisms. Therefore, the following definition makes sense.

Definition 4.3. The maps

$$\sigma \triangleq \phi \circ \eta_\omega : \omega \to \omega$$

$$\hat{\sigma} \triangleq \upsilon^{-1} \circ \eta_{\hat{\omega}} : \hat{\omega} \to \hat{\omega}$$

are called *successor maps* on ω and $\hat{\omega}$, respectively.

From these definitions and naturality of η it is obvious that

$$\hat{\sigma} \circ \iota = \iota \circ \sigma,$$

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as

$$\hat{\sigma} \circ \imath = \upsilon^{-1} \circ \eta_{\hat{\omega}} \circ \imath = \upsilon^{-1} \circ L\,\imath \circ \eta_{\omega} = \imath \circ \phi \circ \eta_{\omega} = \imath \circ \sigma.$$

Definition 4.4. Whenever $\alpha: LA \to A$ is an L-algebra, let $\mathsf{step}_\alpha : \mathbb{N} \to A$ be the unique map satisfying

$$\begin{array}{ccccc} 1+\mathbb{N} & & \xrightarrow{[0,s]} & & \mathbb{N} \\ {\scriptstyle 1+\mathsf{step}_\alpha}\downarrow & & & & \downarrow{\scriptstyle \mathsf{step}_\alpha} \\ 1+A & \xrightarrow[c_A]{} & LA & \xrightarrow[\alpha]{} & A \end{array}$$

where $s(n) = n+1$. The uniqueness of step_α follows from the fact that $[0,s]$ is, by definition, the initial algebra for the functor $1 + \_$. We write step for $\mathsf{step}_\phi : \mathbb{N} \to \omega$ and $\widehat{\mathsf{step}}$ for $\mathsf{step}_{\upsilon^{-1}} : \mathbb{N} \to \hat{\omega}$. Notice that $\widehat{\mathsf{step}} = \imath \circ \mathsf{step}$.

Next we define an object $\infty \in \hat{\omega}$, intuitively corresponding to the 'limit' of the sequence $(\widehat{\mathsf{step}}(n))_{n\in\mathbb{N}}$ in $\hat{\omega}$. This object $\infty \in \hat{\omega}$ will play a key role in the construction of canonical fixpoints for endofunctions on domains, *cf.* Crole and Pitts (1992).

Definition 4.5. Let $\infty: 1 \to \hat{\omega}$ be the unique map satisfying

$$\begin{array}{ccc} 1 & \xrightarrow{\eta_1} & L1 \\ {\scriptstyle \infty}\downarrow & & \downarrow{\scriptstyle L\infty} \\ \hat{\omega} & \xrightarrow[\upsilon]{} & L\hat{\omega} \end{array}$$

Lemma 4.6. The object ∞ is a fixpoint of $\hat{\sigma}$, that is, $\hat{\sigma} \circ \infty = \infty$, and, moreover, ∞ is an equalizer of $id_{\hat{\omega}}$ and $\hat{\sigma}$.

Proof. That ∞ is a fixpoint can be seen as follows:

$$\begin{array}{rcll} \hat{\sigma} \circ \infty & = & \upsilon^{-1} \circ \eta_{\hat{\omega}} \circ \infty & \\ & = & \upsilon^{-1} \circ L\infty \circ \eta_1 & \text{(by naturality of } \eta) \\ & = & \upsilon^{-1} \circ \upsilon \circ \infty & \\ & = & \infty\ . & \end{array}$$

To show that $\infty: 1 \to \hat{\omega}$ is an equalizer of $id_{\hat{\omega}}$ and $\hat{\sigma}$, assume that $f: B \to \hat{\omega}$ with $\hat{\sigma} \circ f = f$. We show that for the unique map $!_B: B \to 1$ we have $\infty \circ !_B = f$. For this it suffices to show that both f and $\infty \circ !_B$ are L-coalgebra morphisms from η_B to υ.

The map $!_B$ is an L-coalgebra morphism from η_B to η_1 by naturality of η, and, by definition, ∞ is an L-coalgebra morphism from η_1 to υ. Thus, $\infty \circ !_B$ is an L-coalgebra morphism from η_B to υ. From $f = \hat{\sigma} \circ f = \upsilon^{-1} \circ \eta_{\hat{\omega}} \circ f$ it follows that $\upsilon \circ f = \eta_{\hat{\omega}} \circ f = Lf \circ \eta_B$ and, hence, f is an L-coalgebra morphism from η_B to υ. □

4.4. *Closure under lifting*

In this subsection we will characterise those classes $\mathscr{L}$ of maps in Set such that $\mathscr{L}$-complete sets are closed under lifting. Moreover, we show S-spaces to be closed under lifting.

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4.4.1. *Closure under lifting for $\mathscr{L}$-complete sets* We first give a necessary condition for a class $\mathscr{L}$ of maps in Set such that $\mathscr{L}$-complete sets are closed under lifting.

Lemma 4.7. If $\mathscr{L}$ is a class of maps in Set such that $\mathscr{L}$-complete sets are closed under lifting, then S is $\mathscr{L}$-complete.

Proof. As 1 is $\mathscr{L}$-complete and $\mathscr{L}$-complete sets are closed under lifting, it follows that $L\,1$ is $\mathscr{L}$-complete. As $S \cong L\,1$, we get that S is $\mathscr{L}$-complete, that is, we have $e \perp S$ for all e in $\mathscr{L}$. □

Maps e with $e \perp S$ were originally introduced by Martin Hyland in Hyland (1991) and called S-equable maps. We prefer to call them S-*iso*s. As closure under lifting is sort of a minimal requirement for a class of predomains given as $\mathscr{L}$-complete sets, the previous lemma says that one may restrict attention to classes $\mathscr{L}$ of S-isos. A detailed explanation of S-isos as 'generalised limit process' has been given in Streicher (1998). Similarly, predomains will be classes of sets *closed* under a collection of 'generalised limit process' as given by a class $\mathscr{L}$ of S-isos.

The notion of S-iso and some other useful notions are collected in the following definition.

Definition 4.6. A map $e{:}X \to Y$ in Set is called an S-*iso* iff $e \perp S$. Accordingly, a map e in Set is called an S-*epi* iff $S(e)$ is monic, that is, $p \circ e = q \circ e \Rightarrow p = q$ for all $p, q \in Y \to S$.

The following lemma will be crucial for giving a necessary and sufficient characterisation of those $\mathscr{L}$ for which $\mathscr{L}$-complete sets are closed under lifting.

Notation: In the following, we write f^*g for the pullback of g along f.

Lemma 4.8. If $\mathscr{L}$ is a class of S-isos and $A \in$ Set, then $L\,A$ is $\mathscr{L}$-complete iff $m^*e \perp A$ for all $e \in \mathscr{L}$ and S-subobjects m of the codomain of e.

Proof. $\Rightarrow$: Assume that LA is $\mathscr{L}$-complete. Let $e{:}X \to Y$ be an S-iso in $\mathscr{L}$ and $m{:}Z \rightarrowtail Y$ be an S-subobject of Y. Let $n{:}m^*X \rightarrowtail X$ and $m^*e{:}m^*X \to Z$ be a pullback cone over e and m. Notice that n is also an S-mono, as these are preserved by pullbacks along arbitrary morphisms.

We have to show that $m^*e \perp A$. To do this, let $g{:}m^*X \to A$. We have to construct a unique extension $\overline{g}$ of g along m^*e.

As L classifies partial maps whose domain of definition is an S-subobject, *cf.* Remark 4.2, there is a unique map $f{:}X \to L\,A$ such that n and g is a pullback cone over f and η_A. By completeness of LA, there is a unique extension $\overline{f}$ of f along e. Now let $m'{:}P \rightarrowtail Y$ and $h{:}P \to A$ be a pullback cone over $\overline{f}$ and η_A, that is, $\overline{f}$ classifies the partial map (m', h) from Y to A. Notice that m' is an S-mono, since it arises as the pullback of the S-mono η_A classified by $L(!_A)$.

We next show that m and m' are isomorphic subobjects of Y. Let $\chi{:}Y \to S$ be the classifying map for m. Then $\chi \circ e$ classifies n, since n is the pullback of m along e. The map $L(!_A) \circ f$ is also a classifier for n, since $L(!_A)$ classifies η_A and n arises as the pullback of

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η_A along f. By uniqueness of classifying maps (for n), we get that

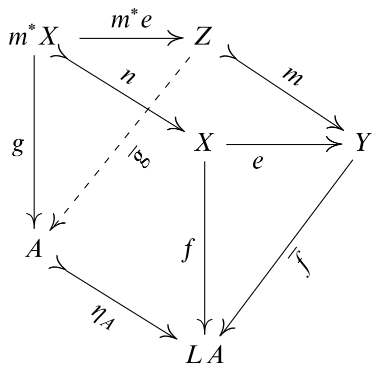
$$\chi \circ e = L(!_A) \circ f = L(!_A) \circ \overline{f} \circ e \; .$$

As S is $\mathscr{L}$-complete (from the assumption that all maps in $\mathscr{L}$ are S-isos), we get that $\chi = L(!_A) \circ \overline{f}$. But, as χ classifies m and $L(!_A) \circ \overline{f}$ classifies m', we get that m and m' are isomorphic subobjects of Y. Thus, without loss of generality, we may assume that $m = m'$.

We now choose $\overline{g} \triangleq h$ as the desired extension of g along m^*e. That $g = \overline{g} \circ m^*e$ follows from the following equational reasoning since η_A is monic

$$\eta_A \circ g = f \circ n = \overline{f} \circ e \circ n = \overline{f} \circ m \circ m^*e = \eta_A \circ \overline{g} \circ m^*e \; .$$

The situation is visualized in the following diagram, where all squares are pullbacks:



We still have to show the uniqueness of $\overline{g}$. Suppose $h' \circ m^*e = g$. Let $f' : Y \to LA$ classify the partial map (m, h'). Then $f' \circ e$ classifies the partial map (n, g), and, therefore, by uniqueness of classifying maps, it follows that $f = f' \circ e$. As e is an S-iso and LA is complete, we get that $f' = \overline{f}$, and, therefore, the partial maps (m, h') and $(m, \overline{g})$ are isomorphic, from which it follows that $\overline{g} = h'$.

$\Leftarrow$: Assume that $m^*e \perp A$ for all $e : X \to Y$ in $\mathscr{L}$ and S-monos $m : Z \rightarrowtail Y$. We have to show that LA is $\mathscr{L}$-complete.

Let $e : X \to Y$ be map in $\mathscr{L}$ and $f : X \to LA$. We are looking for a unique extension $\overline{f} : Y \to LA$ along e. Let $n : f^*A \rightarrowtail X$ and $g : f^*A \to A$ be a pullback cone over f and η_A. Notice that n is an S-mono since it appears as a pullback of the S-mono η_A. Let $\chi : X \to S$ be the classifying map for n. As e is an S-iso, there exists a unique extension $\overline{\chi}$ of χ along e. Let m be the S-mono classified by $\overline{\chi}$ and $\alpha : f^*A \to Z$ be the mediating arrow with $m \circ \alpha = e \circ n$ (and $!_Z \circ \alpha = !_{f^*A}$). Obviously, α appears as pullback of e along m and, therefore, we write m^*e for α. By assumption, there exists a unique extension $\overline{g}$ of g along m^*e. Let $\overline{f} : Y \to LA$ classify the partial map $(m, \overline{g})$. Then all squares in the

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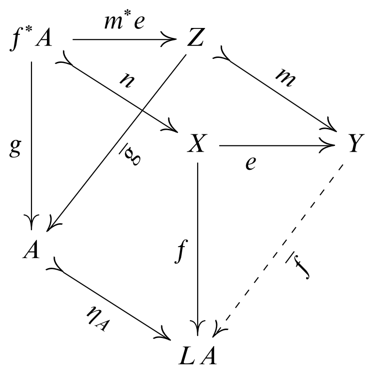
following diagram are pullback squares.



As both $\overline{f} \circ e$ and f classify the partial map (n, g) it follows that $f = \overline{f} \circ e$.

For uniqueness of $\overline{f}$, assume that $f = h \circ e$. Let (m', g') be the partial map classified by h. As $L(!_A) \circ \overline{f} \circ e = L(!_A) \circ f = L(!_A) \circ h \circ e$ and e is an S-iso, we get that $L(!_A) \circ \overline{f} = L(!_A) \circ h$, and, therefore, classify the same S-mono m. Thus, without loss of generality we may assume that $m = m'$ and, therefore, $g = g' \circ m^*e$. But then $\overline{g} = g'$, since by assumption $m^*e \perp A$. Thus, we have $\overline{f} = h$ as both maps classify the partial map $(m, \overline{g})$. □

From this it follows immediately that lifting reflects $\mathscr{L}$-completeness provided $\mathscr{L}$ contains only S-isos.

Corollary 4.9. Suppose that $\mathscr{L}$ contains only S-isos. If LA is $\mathscr{L}$-complete, then A is $\mathscr{L}$-complete too.

Proof. If LA is $\mathscr{L}$-complete, then by Lemma 4.8 it follows that $m^*e \perp A$ for all $e : X \to Y$ in $\mathscr{L}$ and S-subobjects m of Y. As all identity maps are S-monos, it follows that $e \perp A$ for all e in $\mathscr{L}$, that is, that A is $\mathscr{L}$-complete. □

From Lemma 4.8 we now obtain the desired characterisation of those $\mathscr{L}$ for which $\mathscr{L}$-complete sets are closed under lifting.

Theorem 4.10. Let $\mathscr{L}$ be a class of maps in Set. Then $\mathscr{L}$-complete sets are closed under lifting iff $\mathscr{L}$ contains only S-isos and for all $\mathscr{L}$-complete A, we have that $m^*e \perp A$ whenever e is a map in $\mathscr{L}$ and m is an S-subobject of the codomain of e.

Proof. $\Rightarrow$: From Lemma 4.7 it follows that all maps in $\mathscr{L}$ must be S-isos. The second condition is a consequence of Lemma 4.8.
$\Leftarrow$: This follows from Lemma 4.8. □

This characterisation gives rise to the following condition, which is sufficient to ensure that $\mathscr{L}$-complete sets are closed under lifting.

Corollary 4.11. If $\mathscr{L}$ contains only S-isos and is closed under pullbacks along S-monos, then $\mathscr{L}$-complete sets are closed under lifting.

Next we show a lemma that will be essential later for proving that S-replete (and well-complete) sets are closed under lifting.

Lemma 4.12. S-isos are stable under pullbacks along S-monos.

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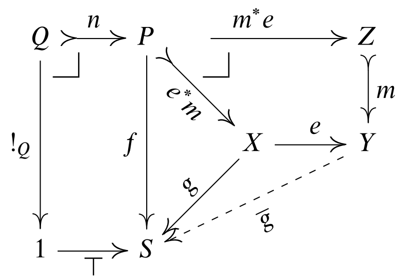
Proof. Suppose that $e{:}X \to Y$ is an S-iso and that $m{:}Z \rightarrowtail Y$ is an S-mono. Let $m^*e{:}P \to Z$ and $e^*m{:}P \rightarrowtail X$ be the pullback cone over m and e. Notice that e^*m is an S-mono, too. For $f{:}P \to S$ we have to construct a unique extension $\overline{f}$ of f along m^*e. Let $n{:}Q \rightarrowtail P$ be the S-mono classified by f and $g{:}X \to S$ be the classifier for the S-mono $e^*m \circ n$. As $g \circ e^*m$ classifies n and classifying maps are unique, we have $f = g \circ e^*m$. Since e is an S-iso, there is a unique map $\overline{g}{:}Y \to S$ with $\overline{g} \circ e = g$. Choosing $\overline{f} \triangleq \overline{g} \circ m$, we obviously get an extension of f along m^*e.

The construction of $\overline{g}$ is illustrated by the following diagram



To show uniqueness of $\overline{f}$, suppose there is an $h{:}Z \to S$ such that $h \circ m^*e = f$. Let $m'{:}Z' \rightarrowtail Z$ be classified by h, and k be the classifier of $m \circ m'$. Then $k \circ m$ classifies m'. Thus $h = k \circ m$. Then the pullback of m' along m^*e is n. Thus, the pullback of $m \circ m'$ along e is n. Then $k \circ e$ classifies $e^*m \circ n$, from which it follows by uniqueness of classifiers that $g = k \circ e$. As e is an S-iso $k = \overline{g}$ and, therefore, $h = k \circ m = \overline{g} \circ m = \overline{f}$. □

4.4.2. *Closure under lifting for S-spaces*

Theorem 4.13. S-spaces are closed under lifting.

Proof. Let A be an S-space. Then LA appears as a retract of

$$S_A \triangleq \{\phi \in S^2(A) \mid \phi(\lambda x{:}A.\top) = \top \Rightarrow \exists a{:}A.\, \phi = \varepsilon_A(a)\} \subseteq S^2(A)$$

via the maps $i{:}LA \to S_A$ and $j{:}S_A \to LA$ defined as

$$i(\langle s, a\rangle) \triangleq \lambda p{:}S(A).\, s \angle p \circ a$$

$$j(\phi) \triangleq \langle \phi(\lambda x{:}A.\top), \mathsf{get}(\phi)\rangle,$$

where $\mathsf{get}(\phi)$ is the function such that $\mathsf{get}(\phi)(u)$ is the unique $a \in A$ with $\phi = \varepsilon_A(a)$ for every $u \in \mathsf{def}(\phi(\lambda x{:}A.\top))$. The existence of such an a is guaranteed by $\phi \in S_A$, and it is unique since A is an S-space. By the closure properties of S-spaces, LA is an S-space. Note that for the construction of get, one needs the Axiom of Unique Choice (AUC). □

5. Predomains

In classical domain theory, in order to explain the meaning of recursively defined elements of data types, one uses *complete partial orders*, that is, partially ordered sets where every ascending chain has a limit (that is, a supremum). In this section we will give an 'order-free' analogue of this completeness requirement that makes sense for arbitrary sets (and types).

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We also investigate some stronger notions of completeness guaranteeing closure under lifting. The least restrictive notion of that kind is the notion of well-completeness introduced by Longley and Simpson (1997), whereas the most restrictive notion of that kind is S-repleteness as introduced by Hyland and Taylor in Hyland (1991) and Taylor (1991). It will turn out that all S-replete sets will be S-spaces.

In the following, we will use the term predomains to mean well-complete S-spaces, since they form the widest class of complete S-spaces that is closed under lifting.

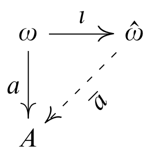
5.1. *Complete sets*

Now we introduce an 'order-free' analogue of the requirement that all ascending chains have limits. But, in the vein of staying 'order-free', we consider so-called ω-chains instead of $\mathbb{N}$-indexed ascending chains where an ω-chain in a set A is a map $a : \omega \to A$.

Although any ω-chain $a : \omega \to A$ induces an ascending chain $a \circ \mathsf{step} : \mathbb{N} \to A$, it is not the case that for arbitrary realisability models every ascending chain $c : \mathbb{N} \to A$ can be obtained as $a \circ \mathsf{step}$ for some ω-chain $a : \omega \to A$ – see Longley and Simpson (1997) for examples of S-spaces where all ω-chains have limits but some ascending $\mathbb{N}$-chains do not.

In the following we restrict attention to completeness for sets simply because that is sufficient for our purposes, though completeness could be defined for all types.

Definition 5.1. A set A is called *complete* iff $\iota \perp A$, that is, for all ω-chains $a : \omega \to A$ there is a unique map $\overline{a} : \hat{\omega} \to A$ with $\overline{a} \circ \iota = a$



Thus, the complete sets are the $\mathscr{L}$-complete sets for $\mathscr{L} = \{\iota\}$.

The following axiom states that S is complete.

Axiom 5. The map $\iota : \omega \to \hat{\omega}$ is an S-iso, that is, $S(\iota)$ is an isomorphism.

Remark 5.1. Notice that our definition of 'completeness' is an internal version of the external notion of completeness introduced in Longley and Simpson (1997) for realisability models.

Notice also that the above definition of 'completeness' is nothing but a weakening of Hyland and Taylor's notion of 'replete object' (Hyland 1991; Taylor 1991) (*cf.* also Section 5.3, Definition 5.3). Hyland (1991) showed that the S-replete sets are the least class of sets containing S and being closed under arbitrary dependent products and equalisers.

5.1.1. *Closure properties of complete sets* First we consider closure under dependent products.

Theorem 5.2. If $A \in \mathsf{Set}$ and $B : A \to \mathsf{Set}$ is a family of complete sets, then $\Pi x{:}A.\, B(x)$ is a complete set, too. Moreover, if Set is an impredicative universe, then complete S-spaces are closed under arbitrary dependent products.

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Proof. The proof follows immediately from Theorem 3.2 and Corollary 3.13 by choosing $\mathscr{L} \triangleq \{\imath\}$. □

Similarly, it follows that complete sets are closed under equalisers.

Theorem 5.3. Let $f_1, f_2 : A \to B$ be maps between complete sets. Then the equaliser $m : E \rightarrowtail A$ of f_1 and f_2 is complete, too.

Proof. The proof follows immediately from Theorem 3.5 and Axiom 5 by choosing $\mathscr{L} \triangleq \{\imath\}$. □

5.1.2. *A representation theorem for complete S-spaces*

Theorem 5.4. A set is a complete S-space iff it is a complete subobject of some power of S by a set.

Proof. The proof follows immediately from Theorem 3.14 by putting $\mathscr{L} \triangleq \{\imath\}$. □

5.2. *Well-complete sets*

The main problem with complete S-spaces is that they are not (known to be) closed under lifting. For this reason one has to look for a more restrictive notion, which nevertheless satisfies all the closure properties of complete sets. Again, we just consider well-complete *sets*, though the notion would make sense for arbitrary types, too.

Definition 5.2. A set A is called *well-complete* iff its lifting LA is orthogonal to $\imath$, that is, iff LA is complete.

The following lemma identifies well-completeness as an instance of $\mathscr{L}$-completeness.

Lemma 5.5. A set A is well-complete iff $m^*\imath \perp A$ for all S-subobjects m of $\hat{\omega}$. Thus, a set is well-complete iff it is $\mathscr{L}$-complete for $\mathscr{L} \triangleq \{\, m^*\imath \mid m\ S\text{-subobject of } \hat{\omega} \,\}$.

Proof. The proof follows straightforwardly from Lemma 4.8 and Axiom 5. □

From this it follows that complete subsumes well-complete.

Corollary 5.6. Any well-complete set is complete.

Proof. The proof follows from Corollary 4.9 and Axiom 5. □

Theorem 5.7. Well-complete sets are closed under equalisers, lifting and dependent products of families indexed by a set. Moreover, if Set is impredicative, well-complete sets are closed even under arbitrary dependent products.

Proof. By Lemma 5.5, the well-complete sets are the $\mathscr{L}$-complete sets for $\mathscr{L} \triangleq \{\, m^*\imath \mid m\ S\text{-subobject of } \hat{\omega} \,\}$. Then, from Theorems 3.2 and 3.5 one gets closure under dependent products and equalisers, respectively. Closure under lifting is a consequence of Corollary 4.11 and Lemma 4.12. □

Corollary 5.8. The well-complete sets form the largest collection of complete sets closed under lifting.

Proof. Suppose that $\mathscr{C}$ is a collection of complete sets that is closed under lifting. If $A \in \mathscr{C}$, then $LA \in \mathscr{C}$. Thus, LA is complete and, therefore, A is well-complete. □

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Corollary 5.9. Well-complete S-spaces are closed under equalisers, lifting and dependent products of families indexed by sets. If Set is impredicative, well-complete S-spaces are closed even under arbitrary dependent products.

Proof. The proof follows immediately from Theorem 5.7 and the closure properties of S-spaces (Corollaries 3.11 and 3.13). □

Lemma 5.10. S is well-complete, that is, $\iota \perp LS$.

Proof. The set $S \cong L1$ is complete by Axiom 5, thus 1 is well- complete. Therefore, $S \cong L1$ is well-complete by closure under lifting (*cf.* Theorem 5.7). □

Remark 5.11. Axiom 5 is equivalent to S being well-complete because $S \cong L1$ is a retract of LS (*cf.* Remark 3.6). Notice that the proof of Lemma 5.10 uses Axiom 5 and thus cannot be employed to prove the equivalence.

5.2.1. *A representation theorem for well-complete S-spaces*

Theorem 5.12. A set is a well-complete S-space iff it is a well-complete subobject of some power of S by a set.

Proof. The proof follows immediately from Theorem 3.14 by taking for $\mathscr{L}$ the collection of all pullbacks of ι along S-subobjects of $\hat{\omega}$. □

5.3. *S-replete sets*

The notion of S-replete object or set was originally introduced in Hyland (1991) and Taylor (1991) in the context of a more restrictive axiomatic setting tailored towards a domain theory *à la Scott*. But their definition makes sense in our more general setting too.

In the previous subsection we have defined a set A to be 'complete' iff $\iota \perp A$, that is, iff A is closed under limits of ω-chains, whereas we will now call a set A to be S-*replete* or simply *replete* iff $e \perp A$ *for all* S-isos e. Streicher (1998) explained the sense in which S-isos can be considered as 'generalised limit processes' and that, accordingly, S-repleteness of a set may be understood as being 'closed under all generalised limit processes' (as given by S-isos).

Definition 5.3. A set A is called S-*replete* iff $e \perp A$ for all S-isos e.

Remark 5.13. Obviously, this is again an instance of $\mathscr{L}$-completeness by taking for $\mathscr{L}$ the collection of *all* S-isos. This observation is crucial for obtaining the subsequent closure properties. Note also that S is S-replete by definition.

Lemma 5.14. Dependent products and equalisers preserve S-repleteness. If Set is impredicative, S-replete sets are closed under arbitrary dependent products.

Proof. The proof follows from Theorem 3.2 and Theorem 3.5. □

Lemma 5.15. The S-replete sets are closed under lifting.

Proof. The proof follows from Corollary 4.11 and Lemma 4.12. □

Next we show that S-replete sets are already S-spaces. However, we need an auxiliary lemma (due to P. Taylor) characterising S-isos first.

Lemma 5.16. A map $e: X \to Y$ is an S-iso iff e is an S-epi and $g \circ e = \varepsilon_X$ for some $g: Y \to S^2(X)$.

Proof. $\Rightarrow$: If e is an S-iso (that is, $S(e)$ is an iso), then $S(e)$ is monic, that is, e is an S-epi. A (unique) $g: Y \to S^2(X)$ with $\varepsilon_X = g \circ e$ exists, as $S^2(X)$ is S-replete by Remark 5.13 and Lemma 5.14.
$\Leftarrow$: Suppose that $S(e)$ is monic and $\varepsilon_X = g \circ e$ for some $g: Y \to S^2(X)$. Let $p: X \to S$. We have to show that p can be extended uniquely to a map $\overline{p}: Y \to S$ with $\overline{p} \circ e = p$. Uniqueness follows from the assumption that e is an S-epi. For existence, we define $\overline{p}: Y \to S$ as $\overline{p}(y) \triangleq (g\, y)\, p$, for which we have

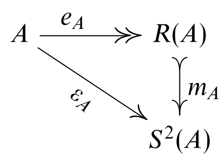
$$\overline{p}(e(x)) = g(e(x))(p) = (g \circ e)(x)(p) = (\varepsilon_X(x))\, p = p(x),$$

as desired. □

Next we show that S-replete sets are automatically S-spaces.

Theorem 5.17. All S-replete sets are S-spaces.

Proof. Let A be S-replete. Let $\varepsilon_A = m_A \circ e_A$ be an epi-mono-factorisation of ε_A, that is, $e_A: A \to R(A)$ is epic and $m_A: R(A) \rightarrowtail S^2(A)$ is monic, as exhibited in the diagram



As e_A is epic, it is also an S-epi and, therefore, by Lemma 5.16 it follows that e_A is an S-iso. Thus, there is a unique map $i_A: R(A) \to A$ with $id_A = i_A \circ e_A$, as, by assumption, A is S-replete. So e_A is a split mono. As e_A is also epic, it follows that e_A is actually an isomorphism.

Thus, the map ε_A is monic and, therefore, A appears as a subobject of $S^2(A)$, from which it follows by the Representation Theorem that A is an S-space. □

This gives rise to the following observation.

Corollary 5.18. Any S-replete set is a well-complete (and thus complete) S-space.

Proof. By Theorem 5.17, A is an S-space. It is also well-complete by Theorem 4.10, since the S-isos are closed under pullbacks along S- subobjects (by Lemma 4.12). □

6. Domains

The well-complete and S-replete *S-spaces* studied so far are 'synthetic' analogues of 'classical' *predomains*, that is, partial orders where every ascending chain has a supremum. As we know from classical domain theory, for the purposes of fixpoint theory we have to consider *domains*, that is, predomains admitting a least element.

For S-spaces the synthetic analogue of the existence of a least element will be the existence of a so-called focal L-algebra structure. For arbitrary sets A there may be

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several different focal L-algebra structures on A. When we add two further axioms, namely Axioms 6 and 7 below, it turns out that for S-spaces, focal L-algebra structures are unique provided they exist. Such S-spaces will be called *pointed S-spaces.*

Accordingly, *domains will be pointed S-spaces that are well-complete.*

6.1. *Focal L-algebras and pointed S-spaces*

Definition 6.1. An L-algebra $\alpha : LA \to A$ is called *focal* iff $\alpha \circ \eta_A = id_A$. The element $\alpha(\bot_A)$ will be sometimes referred to as $\bot_\alpha$.

If $\alpha : LA \to A$ and $\beta : LB \to B$ are focal L-algebras then a *homomorphism from α to β* is a map $h : A \to B$ with

$$h \circ \alpha = \beta \circ Lh.$$

Notice that in general the focal L-algebra structure on a set A need not be unique.

We now formulate the final two axioms that guarantee the uniqueness of focal L-algebra structures for S-spaces and, moreover, are sufficient for proving all further results of this paper.

Axiom 6. For any set A, the map $c_A = [\bot_A, \eta_A] : 1 + A \to LA$ is an S-epi, that is,

$$p = q \Leftrightarrow p \circ c_A = q \circ c_A$$

for all $p, q : LA \to S$.

Axiom 6 means that S-predicates $p, q : LA \to S$ are equal if and only if $p \circ \eta_A = q \circ \eta_A$ and $p(\bot_A) = q(\bot_A)$, that is, when p and q have the same behaviour on $\bot_A$ and elements of the form $\eta_A(a)$.

Axiom 7. For all $u \in S$ and $f : S \to S$ if $f(u) = \top$, then $f(\top) = \top$.

In other words, for all endo-maps f of S we have $f(\top) = \top$ if $f(u) = \top$ for some $u \in S$.

From Axiom 6, we immediately get that any c_A is epic with respect to S-spaces.

Lemma 6.1. Let A be a set and B be an S-space. If $f_1, f_2 : LA \to B$ with $f_1 \circ c_A = f_2 \circ c_A$, then $f_1 = f_2$.

Proof. As B is an S-space, it suffices to prove that for all $p \in S(B)$ we have $p \circ f_1 = p \circ f_2$. Thus, by Axiom 6 it suffices to show that $p \circ f_1 \circ c_A = p \circ f_2 \circ c_A$. But this follows immediately from the assumption that $f_1 \circ c_A = f_2 \circ c_A$. □

Lemma 6.2. Let α be a focal L-algebra and $p \in S(A)$. Then for all $a \in A$ we have $p(a) = \top$ whenever $p(\alpha(\bot_A)) = \top$.

Proof. Let $a \in A$ and $h_a : S \to LA$ be the map defined as $h_a(u) \triangleq \langle u, \lambda p : \mathsf{def}(u).\, a \rangle$. Then $h_a(\bot) = \bot_A$ and $h_a(\top) = \eta_A(a)$. Thus, we have

$$(p \circ \alpha \circ h_a)(\bot) = p(\alpha(\bot_A)) = \top$$

$$(p \circ \alpha \circ h_a)(\top) = p(\alpha(\eta_A(a))) = p(a),$$

from which it follows by Axiom 7 that $p(a) = \top$. □

Theorem 6.3. An S-space A admits at most one focal L-algebra structure.

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Proof. Suppose that $\alpha_1 : LA \to A$ and $\alpha_2 : LA \to A$ are focal L-algebra structures on an S-space A.
By Lemma 6.1, for α_1 to equal α_2 it is sufficient to show that $\alpha_1 \circ c_A = \alpha_2 \circ c_A$, that is, it suffices to show that $\alpha_1 \circ \eta_A = \alpha_2 \circ \eta_A$ and $\alpha_1(\bot_A) = \alpha_2(\bot_A)$. As α_1 and α_2 are focal L-algebras, we have $\alpha_1 \circ \eta_A = id_A = \alpha_2 \circ \eta_A$. Thus, it remains to check that $\alpha_1(\bot_A) = \alpha_2(\bot_A)$.

As A is an S-space, it suffices to show that $p(\alpha_1(\bot_A)) = \top$ iff $p(\alpha_2(\bot_A)) = \top$. However, this follows immediately from Lemma 6.2 above. □

Lemma 6.4. Let A and B be S-spaces admitting (unique) focal L-algebra structures $\alpha : LA \to A$ and $\beta : LB \to B$, respectively. Then, for maps $f : A \to B$, the following two conditions are equivalent:

(1) f is an L-algebra morphism from α to β

(2) f is strict, that is, $f(\bot_\alpha) = \bot_\beta$.

Proof. The map $f : A \to B$ is an L-algebra morphism iff $f \circ \alpha = \beta \circ Lf$. By Lemma 6.1 this is equivalent to the following two requirements:

$$f \circ \alpha \circ \eta_A = \beta \circ Lf \circ \eta_A$$

$$f(\alpha(\bot_A)) = \beta(Lf(\bot_A))$$

The first requirement is valid anyway, as we have

$$f \circ \alpha \circ \eta_A = f = \beta \circ \eta_B \circ f = \beta \circ Lf \circ \eta_A,$$

and the second requirement is equivalent to $f(\alpha(\bot_A)) = \beta(\bot_B)$ as $Lf(\bot_A) = \bot_B$. □

Definition 6.2. A *pointed S-space* is an S-space A admitting a focal L-algebra structure denoted by α_A (as it is unique by Theorem 6.3 provided it exists). Furthermore, for a pointed S-space A, we write $\bot^A$ for $\alpha_A(\bot_A)$ and call it 'the bottom element of A'.

A function $f : A \to B$ between pointed S-spaces A and B will be called *strict* iff f preserves bottom elements, that is, $f(\bot^A) = \bot^B$.

6.2. *Closure properties of focal L-algebras and pointed S-spaces*

Theorem 6.5. Focal L-algebras are closed under dependent products. Moreover, these are also products in the category of focal L-algebras and strict maps.

Proof. Let $(\alpha_i : LA_i \to A_i \mid i \in I)$ be a family of focal L-algebras. Let $P \triangleq \Pi i{:}I.\, A_i$ be the (dependent) product of the underlying sets. Let $\varphi : LP \to P$ be the unique map with

$$\pi_i \circ \varphi \triangleq \alpha_i \circ L(\pi_i)$$

for all $i \in I$. We have $\varphi \circ \eta_P = id_P$, since for all $i \in I$ we have

$$\alpha_i \circ L(\pi_i) \circ \eta_P = \alpha_i \circ \eta_{A_i} \circ \pi_i = \pi_i.$$

Thus φ endows P with the structure of a focal L-algebra.

We still need to show that for any focal L-algebra $\beta : LB \to B$ and any family $(h_i : \beta \to \alpha_i \mid i \in I)$ of homomorphisms the unique map $\overline{h} : B \to P$ with $h_i = \pi_i \circ \overline{h}$ for all $i \in I$ is actually a homomorphism, that is, that $\varphi \circ L\overline{h} = \overline{h} \circ \beta$. But this is the case, since

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we have

$$\pi_i \circ \varphi \circ L\overline{h} = \alpha_i \circ L(\pi_i) \circ L\overline{h} = \alpha_i \circ L(\pi_i \circ \overline{h}) = \alpha_i \circ L h_i = h_i \circ \beta = \pi_i \circ \overline{h} \circ \beta$$

for all $i \in I$. □

Corollary 6.6. Pointed S-spaces are closed under dependent products of families indexed by a set. Moreover, if Set is impredicative, then pointed S-spaces are closed under all dependent products.

Proof. The proof follows immediately from the previous theorem and the closure properties of S-spaces. □

Theorem 6.7. Focal L-algebras are closed under equalisers of L-algebra morphisms. Moreover, these are also equalisers in the category of focal L-algebras and L-algebra morphisms. Thus, pointed S-spaces are closed under equalisers of strict maps.

Proof. Let $\alpha : LA \to A$ and $\beta : LB \to B$ be focal L-algebras, and $f, g : A \to B$ L-algebra-morphisms. Assume that $e : E \rightarrowtail A$ is an equaliser of f and g. First, notice that

$$f \circ \alpha \circ Le = \beta \circ Lf \circ Le = \beta \circ L(f \circ e) = \beta \circ L(g \circ e) = \beta \circ Lg \circ Le = g \circ \alpha \circ Le.$$

Thus, there is a unique map $h : LE \to E$ with $e \circ h = \alpha \circ Le$ endowing E with a focal L-algebra structure since we have

$$e \circ h \circ \eta_E = \alpha \circ Le \circ \eta_E = \alpha \circ \eta_A \circ e = e,$$

and since e is monic, this entails $h \circ \eta_E = id_E$. As $e \circ h = \alpha \circ Le$, we also get that $e : h \to \alpha$, that is, h is an L-algebra-morphism.

It remains to show that h has the universal property. Let $\gamma : LC \to C$ be a focal L-algebra and $k : \gamma \to \alpha$ be a homomorphism with $f \circ k = g \circ k$. Then there exists a unique $k' : C \to E$ with $k = e \circ k'$. We have

$$e \circ k' \circ \gamma = k \circ \gamma = \alpha \circ Lk = \alpha \circ L(e \circ k') = \alpha \circ Le \circ Lk' = e \circ h \circ Lk',$$

and as e is monic, $k' \circ \gamma = h \circ Lk'$, that is, $k' : \gamma \to h$.

The closure of focal S-spaces under equalisers of strict maps follows from the previous considerations and Lemma 6.4. □

Next we show that there are enough L-algebras.

Lemma 6.8. Let A be a set and let $\mu_A : LLA \to LA$ be the classifier of the partial map $(\eta_{LA} \circ \eta_A, id_A)$

$$\begin{array}{ccc} A & \xrightarrow{id_A} & A \\ {\scriptstyle \eta_{LA} \circ \eta_A} \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle \eta_A} \\ L^2 A & \xrightarrow[\mu_A]{} & LA \end{array}$$

then μ_A is a focal L-algebra on LA.

Proof. As both $\mu_A \circ \eta_{LA}$ and id_{LA} classify the partial map (η_A, id_A), it follows by uniqueness of classifying maps that $\mu_A \circ \eta_{LA} = id_{LA}$. □

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Remark 6.9. Both η and μ are components of natural transformations endowing L with a *monad structure*. Therefore, L is often called the *lifting monad*.

Using Axiom 6 it can be shown that any focal L-algebra is an Eilenberg–Moore algebra for the lifting monad. Hence, the category of pointed S-spaces with strict maps is isomorphic to the category of Eilenberg–Moore algebras for the lifting monad on S-spaces.

Corollary 6.10. If A is an S-space, then LA is a pointed S-space.

Proof. The proof follows immediately from the previous Lemma 6.8 and closure of S-spaces under lifting (Theorem 4.13). □

7. Fixpoints

We will next show that for any focal L-algebra structure $\alpha : LA \to A$ on a complete set A, any map $f : A \to A$ admits a *canonical fixpoint*, which can be constructed *à la Kleene* as in classical domain theory. Since well-complete and S-replete sets are all complete, the results of this section are valid for all notions of predomains considered in this paper.

The exposition is essentially based on Rosolini (1995).

Definition 7.1. Let $\alpha : LA \to A$ be a focal L-algebra. Let $f_\alpha \triangleq \alpha \circ L(f) : LA \to A$ and let $\mathsf{kl}_{\alpha,f} : \omega \to A$, – the *Kleene chain of f (with respect to α)* – be the unique map making the following diagram commute

$$\begin{array}{ccc}
L\omega & \xrightarrow{\ \phi\ } & \omega \\
{\scriptstyle L(\mathsf{kl}_{\alpha,f})}\downarrow & & \downarrow{\scriptstyle \mathsf{kl}_{\alpha,f}} \\
LA & \xrightarrow[\ f_\alpha\ ]{} & A
\end{array}$$

Lemma 7.1. Let $\alpha : LA \to A$ be a focal L-algebra and $f : A \to A$. Then

$$\mathsf{kl}_{\alpha,f}(\mathsf{step}(n)) = f^n(\bot_\alpha)$$

for all $n \in \mathbb{N}$.

Proof. The commuting diagram

$$\begin{array}{ccccc}
1+\mathbb{N} & & \xrightarrow[\ \simeq\ ]{[0,s]} & & \mathbb{N} \\
{\scriptstyle 1+\mathsf{step}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{step}} \\
1+\omega & \xrightarrow{\ c_\omega\ } & L\omega & \xrightarrow{\ \phi\ } & \omega \\
 & & {\scriptstyle L(\mathsf{kl}_{\alpha,f})}\downarrow & & \downarrow{\scriptstyle \mathsf{kl}_{\alpha,f}} \\
 & & LA & \xrightarrow[\ f_\alpha\ ]{} & A
\end{array}$$

gives rise to the equations

$$\mathsf{kl}_{\alpha,f}(\mathsf{step}(0)) = \bot_\alpha,$$

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as $\mathsf{kl}_{\alpha,f}(\mathsf{step}(0)) = f_\alpha(\bot_A) = (\alpha \circ Lf)(\bot_A) = \alpha(\bot_A) = \bot_\alpha$, and

$$\mathsf{kl}_{\alpha,f}(\mathsf{step}(n+1)) = f(\mathsf{kl}_{\alpha,f}(\mathsf{step}(n))),$$

as

$$\begin{aligned}
\mathsf{kl}_{\alpha,f}(\mathsf{step}(n+1)) &= (f_\alpha \circ L(\mathsf{kl}_{\alpha,f}) \circ \eta_\omega)(\mathsf{step}(n)) \\
&= (\alpha \circ Lf \circ L(\mathsf{kl}_{\alpha,f}) \circ \eta_\omega)(\mathsf{step}(n)) \\
&= (\alpha \circ \eta_A \circ f \circ \mathsf{kl}_{\alpha,f})(\mathsf{step}(n)) \\
&= (f \circ \mathsf{kl}_{\alpha,f})(\mathsf{step}(n)) \\
&= f(\mathsf{kl}_{\alpha,f}(\mathsf{step}(n))).
\end{aligned}$$

From these two equations it follows that $\mathsf{kl}_{\alpha,f}(\mathsf{step}(n)) = f^n(\bot_\alpha)$ by induction on n. □

Corollary 7.2. Let A be a complete set and $\alpha : LA \to A$ be a focal L-algebra. For any $f : A \to A$ let $\overline{\mathsf{kl}_{\alpha,f}}$ be the unique extension of the Kleene-chain $\mathsf{kl}_{\alpha,f}$ along ι, that is, $\mathsf{kl}_{\alpha,f} = \overline{\mathsf{kl}_{\alpha,f}} \circ \iota$. Then

$$\overline{\mathsf{kl}_{\alpha,f}}(\widehat{\mathsf{step}}(n)) = f^n(\bot_\alpha)$$

for all $n \in \mathbb{N}$.

Proof. The proof follows immediately by the previous lemma, since $\overline{\mathsf{kl}_{\alpha,f}} \circ \widehat{\mathsf{step}} = \overline{\mathsf{kl}_{\alpha,f}} \circ \iota \circ \mathsf{step} = \mathsf{kl}_{\alpha,f} \circ \mathsf{step}$. □

The proof of the following theorem follows Rosolini (1995).

Theorem 7.3. Let A be a complete set and $\alpha : LA \to A$ be a focal L-algebra structure on A. Then for any endomap $f : A \to A$, the element

$$\mathsf{fix}_\alpha(f) \triangleq \overline{\mathsf{kl}_{\alpha,f}}(\infty) \in A$$

is a fixpoint of f, that is,

$$f(\mathsf{fix}_\alpha(f)) = \mathsf{fix}_\alpha(f)$$

Proof. Recall that $\hat{\sigma} : \hat{\omega} \to \hat{\omega}$ has been defined as $\upsilon^{-1} \circ \eta_{\hat{\omega}} : \hat{\omega} \to \hat{\omega}$ and that $\hat{\sigma}(\infty) = \infty$ by Lemma 4.6.

Thus, in order to prove that $f(\mathsf{fix}_\alpha(f)) = \mathsf{fix}_\alpha(f)$, we shall show that

$$\overline{\mathsf{kl}_{\alpha,f}} \circ \hat{\sigma} = f \circ \overline{\mathsf{kl}_{\alpha,f}},$$

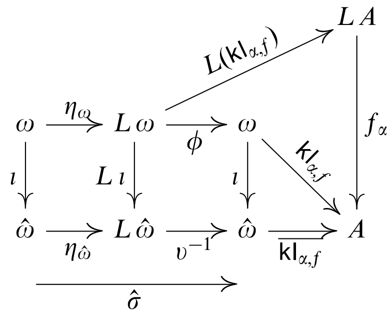
since this entails

$$f(\mathsf{fix}_\alpha(f)) = f(\overline{\mathsf{kl}_{\alpha,f}}(\infty)) = \overline{\mathsf{kl}_{\alpha,f}}(\hat{\sigma}(\infty)) = \overline{\mathsf{kl}_{\alpha,f}}(\infty)) = \mathsf{fix}_\alpha(f)\ .$$

As A is complete, it is sufficient to show

$$\overline{\mathsf{kl}_{\alpha,f}} \circ \hat{\sigma} \circ \iota = f \circ \overline{\mathsf{kl}_{\alpha,f}} \circ \iota,$$

which follows from the following commuting diagram



as

$$\overline{\mathsf{kl}_{\alpha,f}} \circ \hat{\sigma} \circ \iota = f_\alpha \circ L(\mathsf{kl}_{\alpha,f}) \circ \eta_\omega =$$

$$\alpha \circ L\,f \circ L(\mathsf{kl}_{\alpha,f}) \circ \eta_\omega = \alpha \circ L\,(f \circ \mathsf{kl}_{\alpha,f}) \circ \eta_\omega =$$

$$\alpha \circ \eta_A \circ (f \circ \mathsf{kl}_{\alpha,f}) = f \circ \mathsf{kl}_{\alpha,f} = f \circ \overline{\mathsf{kl}_{\alpha,f}} \circ \iota$$

by definition of f_α, naturality of η and focality of α. □

8. Domain-theoretic induction principles

In the previous section we have explained the meaning of recursive definitions of elements (typically recursively defined functions) by means of a canonical fixpoint construction for endofunctions on focal L-algebras. For the purposes of *program verification*, one also needs proof principles for these canonical fixpoints.

For example, if one considers the identity map id_A on a focal L-algebra $\alpha : L\,A \to A$, then all elements of A appear as fixpoints of id_A, but the canonical fixpoint construction *à la Kleene* picks out precisely one element of A. Of course, the Kleene chain $\mathsf{kl}_{\alpha,id_A}$ is constant with value $\perp_\alpha = \alpha(\perp_A)$ and so is $\overline{\mathsf{kl}_{\alpha,id_A}}$, from which we may conclude that $\mathsf{fix}_\alpha(id_A) = \perp_\alpha$. However, the Kleene chain will not usually be constant, so for arbitrary $f : A \to A$ it is more difficult to predict the behaviour of $\overline{\mathsf{kl}_{\alpha,f}}$ at ∞.

The traditional methods for verifying properties of recursively defined objects were called 'induction principles' because some of them *formally* resemble the scheme of induction over $\mathbb{N}$. Synthetic analogues of the induction principles that are known from classical domain theory (*Fixpoint Induction*, *Computational Induction* and *Park Induction*) will be proved in the following subsections. This is a somewhat delicate question since we deliberately want to avoid reference to order-theoretic notions and, accordingly, cannot employ the 'classical' characterisations of canonical fixpoints as *least fixpoints* or *order-theoretic suprema of the Kleene chain.*

8.1. *Fixpoint induction*

It is well known from classical domain theory that most of the induction principles are admitted only for predicates that are well-behaved with respect to limits, the so-called *admissible predicates.*

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Classically, a predicate P on a cpo A is *admissible* iff P is closed under order-theoretic suprema of ascending chains, that is, $P(\bigsqcup a)$ whenever $a: \mathbb{N} \to A$ is an ascending chain that is contained in P, that is, $P(a\,n)$ for all $n \in \mathbb{N}$. The straightforward 'synthetic' analogue of the 'classical' notion of admissibility would be the requirement that P is complete, that is, orthogonal to ι, because the key idea of the synthetic approach is to replace ascending $\mathbb{N}$-indexed chains by ω-chains.

Theorem 8.1. (*'Synthetic' Fixpoint Induction*) Let A be a complete set, $f: A \to A$, and $\alpha: LA \to A$ be a focal L-algebra. If $m: P \rightarrowtail A$ is complete and a subalgebra of f_α, that is, $f_\alpha \circ L(m)$ factors through m via some $\pi: LP \to P$ as shown in the following diagram

$$\begin{array}{ccc} LP & \overset{\pi}{\dashrightarrow} & P \\ {\scriptstyle Lm}\downarrow & & \downarrow{\scriptstyle m} \\ LA & \underset{f_\alpha}{\longrightarrow} & A \end{array}$$

then $\overline{\mathsf{kl}_{\alpha,f}}$ factors through m, and, therefore, $\mathsf{fix}_\alpha(f) \in P$.

Proof. By initiality of $\phi: L\omega \to \omega$, there exists a unique map $h: \omega \to P$ with $h \circ \phi = \pi \circ Lh$. Again by initiality of ϕ, we get that $\mathsf{kl}_{\alpha,f} = m \circ h$, pictorially

$$\begin{array}{ccc} L\,\omega & \overset{\phi}{\longrightarrow} & \omega \\ {\scriptstyle L\,h}\downarrow & & \downarrow{\scriptstyle h} \\ LP & \overset{\pi}{\longrightarrow} & P \\ {\scriptstyle Lm}\downarrow & & \downarrow{\scriptstyle m} \\ LA & \underset{f_\alpha}{\longrightarrow} & A \end{array}$$

As, by assumption, P is complete, there is a unique $\overline{h}: \hat{\omega} \to P$ with $h = \overline{h} \circ \iota$. So, finally, by completeness of A, $\overline{\mathsf{kl}_{\alpha,f}} = \overline{m \circ h} = m \circ \overline{h}$, which entails

$$\mathsf{fix}_\alpha(f) = \overline{\mathsf{kl}_{\alpha,f}}(\infty) = m(\overline{h}(\infty)),$$

from which it follows that $P(\mathsf{fix}_\alpha(f))$. □

We can do better, however. The above theorem can be specialized, since, when we consider focal S-spaces, it gives rise to fixpoint induction, as was pointed out by one of the referees.

Theorem 8.2. Let A be a complete focal S-space and $f: A \to A$. If P is a complete and focal set such that $m: P \rightarrowtail A$ is strict, then $\mathsf{fix}(f) \in P$ if $f \circ m$ factors through m.

Proof. Let $\alpha: LA \to A$ and $\pi_P: LP \to P$ be the unique focal L-algebra structures on the complete focal S-spaces A and P, respectively. Note that P is an S-space, as it is a subobject of A (*cf.* Corollary 3.11). Let $\tilde{f}: P \to P$ be the unique map with $f \circ m = m \circ \tilde{f}$, which exists because $f \circ m$ factors through m.

The theorem follows from the Synthetic Fixpoint Theorem above by setting $\pi \triangleq \pi_P \circ L\tilde{f}$.

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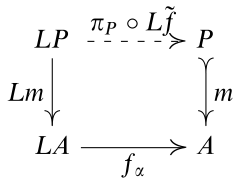
It simply remains to show that



commutes. But, since m, being strict, is an L-homomorphism from Lemma 6.4, the diagram commutes. □

Note that the well-known premisses of fixpoint-induction on A, that is,

$$P(\bot_A) \quad \text{and} \quad \forall a{:}A.P(a) \Rightarrow P(f(a))$$

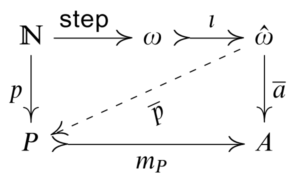
are expressed above by the requirement that $m{:}P \rightarrowtail A$ is strict and $m \circ f$ factors through m. As complete S-spaces enjoy good closure conditions, complete focal predicates on S-spaces appear as a good class of predicates for the purposes of program verification. For computational induction, however, a more restrictive class of predicates is needed, namely *admissible predicates.*

8.2. *Admissible predicates*

If in the classical definition of admissible predicate we replace 'ascending chain' by 'ω-chain', '$a(n)$' by '$a(\mathsf{step}(n))$' and '$\bigsqcup A$ is in P' by '$\overline{a}$ factors through P', we obtain the following notion of 'admissibility'.

Definition 8.1. Let A be a complete set. Then a subobject $m_P{:}P \rightarrowtail A$ is called *admissible* iff $\widehat{\mathsf{step}} \perp m_P$.

Thus $m_P{:}P \rightarrowtail A$ is admissible iff for all $a{:}\omega \to A$ the unique extension $\overline{a}{:}\hat{\omega} \to A$ factors through m_P, provided $a \circ \mathsf{step} = \overline{a} \circ \widehat{\mathsf{step}}$ factors through m_P, that is, there exists a (necessarily unique) map $\overline{p}{:}\hat{\omega} \to P$ such that



commutes. This is more akin to the 'classical' definition of 'admissible predicate', as it says that for an ω-chain $a{:}\omega \to A$ the image of $\overline{a}{:}\hat{\omega} \to A$ is contained in P iff the image of $a \circ \mathsf{step} = \overline{a} \circ \widehat{\mathsf{step}}$ is contained in P, that is, iff all 'finite approximations' $a(\mathsf{step}(n))$ satisfy predicate P.

Next we show that admissible predicates on complete sets are always complete.

Lemma 8.3. Let A be a complete set and $m_P{:}P \rightarrowtail A$ be an admissible subobject of A then P is complete, that is, $\iota \perp m_P$.

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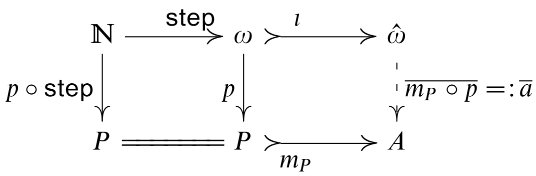
Proof. Let $p:\omega \to P$ and $a \triangleq m_P \circ p$. As A is complete, there exists a unique $\overline{a}:\hat{\omega} \to A$ with $\overline{a} \circ \iota = a$. It is obvious that $\overline{a} \circ \widehat{\mathsf{step}} = m_P \circ p \circ \mathsf{step}$.



Thus, by admissibility of P, there exists a (necessarily unique) map $\overline{p}:\hat{\omega} \to P$ such that $\overline{p} \circ \widehat{\mathsf{step}} = p \circ \mathsf{step}$ and $m_P \circ \overline{p} = \overline{a}$. Therefore, we get

$$m_P \circ \overline{p} \circ \iota = \overline{a} \circ \iota = a = m_P \circ p,$$

which entails $\overline{p} \circ \iota = p$ as m_P is monic.

For uniqueness of $\overline{p}$, assume that $\tilde{p}:\hat{\omega} \to P$ is a map with $\tilde{p} \circ \iota = p$. Then

$$m_P \circ \tilde{p} \circ \iota = m_P \circ p = a = \overline{a} \circ \iota,$$

which, by completeness of A, entails $m_P \circ \tilde{p} = \overline{a}$. Thus, we get $\tilde{p} = \overline{p}$, as m_P is monic and $m_P \circ \overline{p} = \overline{a}$. □

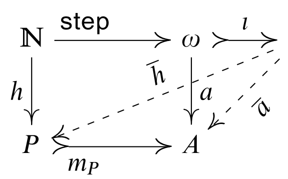
Furthermore, we can characterise admissible predicates on a complete set as the complete predicates that are orthogonal to step.

Lemma 8.4. A subobject $m_P:P \rightarrowtail A$ of a complete set A is admissible iff P is complete and $\mathsf{step} \perp m_P$.

Proof. Let $m_P:P \rightarrowtail A$ be a subobject of a complete set A.

$\Leftarrow$: If m_P is orthogonal to both ι and step, then m_P is also orthogonal to $\widehat{\mathsf{step}} = \iota \circ \mathsf{step}$ (by a well-known straightforward argument).

$\Rightarrow$: On the other hand, if m_P is admissible, then $\iota \perp m_P$ by Lemma 8.3. For $\mathsf{step} \perp m_P$, assume that $a:\omega \to A$ and $a \circ \mathsf{step}$ factors through m_P by some (necessarily unique) $h:\mathbb{N} \to P$. As A is complete, there exists a (necessarily unique) extension $\overline{a}:\hat{\omega} \to A$ with $\overline{a} \circ \iota = a$. Thus, we have $\overline{a} \circ \widehat{\mathsf{step}} = m_P \circ h$, and, therefore, by admissibility of m_P, there exists a (necessarily unique) map $\overline{h}:\hat{\omega} \to P$ making the diagram below commute.



Putting $k \triangleq \overline{h} \circ \iota$, we then get the existence of the desired mediating arrow k. This k is unique, since m_P is monic. □

For admissibility of equality predicates on complete S-spaces we need $\widehat{\mathsf{step}}$ to be an S-epi. As a preparation for this, we prove the following auxiliary lemma.

Lemma 8.5. S-epis are preserved by the lifting functor L.

Proof. Let $e:X \to Y$ be an S-epi. By definition of c and naturality of η, it can be readily shown that $L\,e \circ c_X = c_Y \circ (1 + e)$.

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Next, observe that $1+e$ is S-epic. To see this, assume that $[g_1, g_2] \circ (1+e) = [h_1, h_2] \circ (1+e)$ where $g_1, h_1 : 1 \to S$, and $g_2, h_2 : Y \to S$. Since

$$g_1 = [g_1, g_2] \circ (1+e) \circ \mathrm{inl} = [h_1, h_2] \circ (1+e) \circ \mathrm{inl} = h_1$$

$$g_2 \circ e = [g_1, g_2] \circ (1+e) \circ \mathrm{inr} = [h_1, h_2] \circ (1+e) \circ \mathrm{inr} = h_2 \circ e,$$

it follows that $g_2 = h_2$ since e is an S-epi, and, therefore, $[g_1, g_2] = [h_1, h_2]$.

Thus, as $1+e$ is S-epic and c_Y is S-epic (by Axiom 6), it follows that $L\,e \circ c_X = c_Y \circ (1+e)$ is S-epic. Thus, as c_X is S-epic too (by Axiom 6), we have that $L\,e$ is S-epic. □

The idea behind the proof of the next theorem is due to Simpson, *cf.* Simpson (1996), where, however, it was used for quite a different purpose.

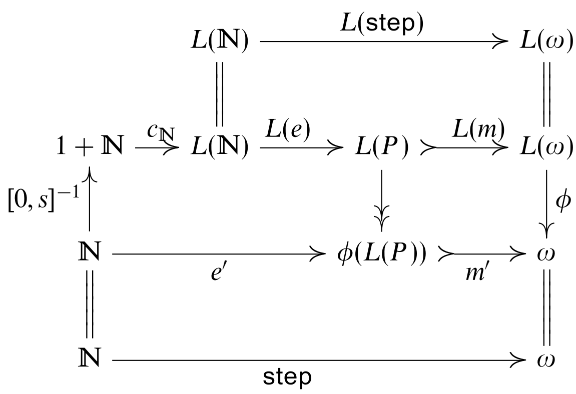
Theorem 8.6. The maps $\mathsf{step} : \mathbb{N} \to \omega$ and $\widehat{\mathsf{step}} : \mathbb{N} \to \hat{\omega}$ are S-epic.

Proof. As $\imath$ is an S-iso and $\widehat{\mathsf{step}} = \imath \circ \mathsf{step}$, it suffices to show that step is an S-epi. To do this, let $m : P \rightarrowtail \omega$ be the *greatest* subobject of ω through which step factors by an S-epi. Let $e : \mathbb{N} \to P$ be the unique map with $\mathsf{step} = m \circ e$, which, of course, is an S-epi. Notice that

$$\phi \circ L(\mathsf{step}) \circ c_{\mathbb{N}} = \phi \circ c_\omega \circ (1 + \mathsf{step}) = \mathsf{step} \circ [0, s],$$

as $L(\mathsf{step}) \circ c_{\mathbb{N}} = c_\omega \circ (1 + \mathsf{step})$ and $\phi \circ c_\omega \circ (1 + \mathsf{step}) = \mathsf{step} \circ [0, s]$ by definition of step (*cf.* Definition 4.4).

Now consider the following diagram:



where m' is the image of $\phi \circ L(m)$ and e' is the unique map with $\mathsf{step} = m' \circ e'$.

As both $c_{\mathbb{N}}$ and $L(e)$ are S-epis (by Axiom 6 and Lemma 8.5), the map e' is an S-epi, too, as it arises as a composition of S-epis. Therefore, m' is contained in m, from which it follows that P is a sub-L-algebra of ϕ via m. As ϕ is an initial L-algebra, the L-algebra morphism m is an isomorphism and, therefore, step is an S-epi. □

Now we are ready to establish the usual closure properties for admissible predicates.

Theorem 8.7. Equality on complete S-spaces is admissible.

Proof. Let A be a complete S-space. We have to show that the equality predicate on A given by the subobject $\delta_A \triangleq \langle id_A, id_A \rangle$ is orthogonal to $\widehat{\mathsf{step}}$.

To do this, let $f = \langle f_1, f_2 \rangle : \hat{\omega} \to A \times A$ and $g : \mathbb{N} \to A$ be maps such that the following

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diagram

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\widehat{\mathsf{step}}} & \hat{\omega} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f = \langle f_1, f_2 \rangle} \\ A & \xrightarrow[\delta_A]{} & A \times A \end{array}$$

commutes. Then $f_1 \circ \widehat{\mathsf{step}} = g = f_2 \circ \widehat{\mathsf{step}}$, which entails $f_1 = f_2$, as A is an S-space and $\widehat{\mathsf{step}}$ is an S-epi by Theorem 8.6. □

The following lemmas are direct consequences of orthogonality, as seen in the work of Freyd and Kelly.

Lemma 8.8. Admissible predicates on complete sets are closed under arbitrary intersections, and, therefore, in particular, under conjunction and universal quantification.

Proof. Let $(P_i)_{i \in I}$ be a family of admissible predicates on a complete set A. Let $\overline{a} : \hat{\omega} \to A$ be a map such that $\forall n{:}\mathbb{N}.\, \forall i{:}I.\, P_i(\overline{a}(\widehat{\mathsf{step}}(n)))$, that is, $\forall i{:}I.\, \forall n{:}\mathbb{N}.\, P_i(\overline{a}(\widehat{\mathsf{step}}(n)))$. As all P_i are admissible, it follows that $\forall i{:}I.\, \forall u{:}\hat{\omega}.\, P_i(\overline{a}(u))$, that is, $\forall u{:}\hat{\omega}.\, \forall i{:}I.\, P_i(\overline{a}(u))$. Thus, the predicate $\lambda x{:}A.\, \forall i{:}I.\, P_i(x)$ is admissible. □

As implication in type theory is a special case of universal quantification, we have the following corollary.

Corollary 8.9. Let P be an admissible predicate on a complete set A, and Q be an arbitrary proposition, then $\lambda x{:}A.\, Q \Rightarrow P(x)$ is an admissible predicate on A.

Proof. The proof follows immediately from Lemma 8.8 by the observation that

$$(Q \Rightarrow P(x)) \iff (\forall q{:}Q.\, P(x))$$

for all $x \in A$. □

Moreover, admissible predicates on complete sets are stable under substitution.

Lemma 8.10. Let $f : B \to A$ be a map between complete sets, and P be an admissible predicate on A. Then $P \circ f$ is an admissible predicate on B.

Proof. Let $\overline{b} : \hat{\omega} \to B$ with $(P \circ f)(\overline{b}(\widehat{\mathsf{step}}(n)))$ for all $n \in \mathbb{N}$. Then for $\overline{a} \triangleq f \circ \overline{b}$, we have $P(\overline{a}(\widehat{\mathsf{step}}(n)))$ for all $n \in \mathbb{N}$. By admissibility of P, it follows that $P(\overline{a}(u))$ for all $u \in \hat{\omega}$, that is, $(P \circ f)(\overline{b}(u))$ for all $u \in \hat{\omega}$. □

An important special case is the following.

Corollary 8.11. The binary predicate on S given by $(u \wedge v) = u$ is admissible.

Proof. The predicate on S given by $(x \wedge y) = x$ can be written more formally as $\lambda z : S \times S.\, (\pi_0(z) \wedge \pi_1(z)) = \pi_0(z)$, which, by the previous Lemma 8.10, is admissible as it is obtained by substitution along $\lambda z : S \times S.\, \langle \pi_0(z) \wedge \pi_1(z), \pi_0(z) \rangle$ from the equality predicate on S, which is admissible by Theorem 8.7. □

Definition 8.2. For any set A, the *information (pre)order* is the binary predicate on A given by

$$x \sqsubseteq_A y \overset{\triangle}{\iff} \forall p{:}S(A).\, (p(x) = \top) \Rightarrow (p(y) = \top)\ .$$

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Remark 8.12. Notice that in the light of the definition above, Axiom 7 ensures that $\top$ is the greatest element with respect to the partial preorder $\sqsubseteq_S$ on S, where $u \sqsubseteq_S v$ iff $f(u) = \top$ implies $f(v) = \top$ for all $f : S \to S$.

Finally, the 'information (pre)order' is an admissible predicate on complete sets.

Theorem 8.13. Let A be a complete set. Then the information preorder $\sqsubseteq_A$ is admissible.

Proof. First notice that for all $u, v \in S$ we have that $u = \top \Rightarrow v = \top$ is equivalent to $((u = \top) \wedge (v = \top)) \Leftrightarrow u = \top$, which in turn is equivalent to $(u \wedge v) = \top \Leftrightarrow u = \top$ since $(u \wedge v = \top) \Leftrightarrow ((u = \top) \wedge (v = \top))$. Because, by Axiom 4, $(u \wedge v = \top) \Leftrightarrow u = \top$ is equivalent to $u \wedge v = u$ and the latter is an admissible binary predicate on S by Corollary 8.11, we get that the binary predicate on S given by

$$(u = \top) \Rightarrow (v = \top)$$

is admissible.

Because admissible predicates are closed under substitution (by Lemma 8.10) and universal quantification (by Lemma 8.8), we get that the binary predicate $\sqsubseteq_A$ on A is admissible. □

8.3. *Computational induction*

We now prove the validity of computational induction for admissible predicates on complete sets.

Theorem 8.14. (*Computational Induction*) Let $\alpha : LA \to A$ be a focal L-algebra structure on a complete set A. Then, for any map $f : A \to A$ and admissible predicate P on A, we have $P(\mathsf{fix}_\alpha(f))$ if $P(f^n(\perp_\alpha))$ for all $n \in \mathbb{N}$.

Proof. By Lemma 7.1, we know that $\mathsf{kl}_{\alpha,f}(\mathsf{step}(n)) = f^n(\perp_\alpha)$. The assumption that $P(f^n(\perp_\alpha))$ for all $n \in \mathbb{N}$ expresses the fact that $\mathsf{kl}_{\alpha,f} \circ \mathsf{step} = \overline{\mathsf{kl}_{\alpha,f}} \circ \widehat{\mathsf{step}}$ factors through $m_P : P \rightarrowtail A$ (*cf.* Corollary 7.2). Thus, by admissibility of P, the map $\overline{\mathsf{kl}_{\alpha,f}}$ factors through m_P, and, therefore, $\mathsf{fix}_\alpha(f) = \overline{\mathsf{kl}_{\alpha,f}}(\infty) \in P$. □

As usual, we can also derive Fixpoint Induction from Computational Induction for admissible predicates.

Corollary 8.15. (*Fixpoint Induction for Admissible Predicates*) Let $\alpha : LA \to A$ be a focal L-algebra structure on a complete set A. Then for any map $f : A \to A$ and admissible predicate P on A, we have $P(\mathsf{fix}_\alpha(f))$ provided $P(\perp_\alpha)$ and $\forall x{:}A.\, P(x) \Rightarrow P(f(x))$.

Remark 8.16. Since admissible predicates are also complete, this theorem is superseded by Theorem 8.2, but it is slightly more general. It does not require A to be an S-space, but on the other hand only for S-spaces is equality admissible. Besides completeness and orthogonality to $\widehat{\mathsf{step}}$, there is a third, even stronger, notion of 'admissibility' with good closure properties, namely the so-called '*extremal monos*' of Taylor (1991), which are those monos orthogonal to *all* S-epis.

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8.4. *Park induction*

'Park induction' in classical domain theory states that the least fixpoint of a (continuous) endofunction $f : A \to A$ on a domain A is also the *least prefixpoint* of f. Of course, 'least' and 'prefixpoint' refer to the *information ordering* $\sqsubseteq_A$ on A (or simply $\sqsubseteq$ when A is clear from the context).

Notice that Park induction is needed in order to prove that a recursively defined function $f \triangleq \mathsf{fix}(\Phi) : A \to B$ *diverges* for some argument $a \in A$. Usually, this is achieved by exhibiting a function $g \in B^A$ with $\Phi(g) \sqsubseteq g$ and $g(a) = \bot_B$.

But before we can prove the validity of Park induction, we need the following observation.

Remark 8.17. Let $\alpha : LA \to A$ be a focal algebra structure on a set A. Then by Lemma 6.2, $\bot_\alpha \sqsubseteq a$ for all $a \in A$. Thus, $\bot_\alpha$ is a least element of A, which is unique if A is an S-space.

Theorem 8.18. (*Park Induction*) Let A be a complete set and $\alpha : LA \to A$ be a focal L-algebra. Then for all $f : A \to A$ and $a \in A$ with $f(a) \sqsubseteq_A a$, we have $\mathsf{fix}_\alpha(f) \sqsubseteq_A a$.

Proof. Suppose $f(a) \sqsubseteq_A a$. As the information ordering $\sqsubseteq_A$ is admissible by Theorem 8.13, the predicate $\lambda x{:}A.\, x \sqsubseteq_A a$ is admissible, and, therefore, we may use fixpoint induction to prove $\mathsf{fix}_\alpha(f) \sqsubseteq_A a$. By Remark 8.17, we have $\bot_\alpha \sqsubseteq a$.

If $x \sqsubseteq_A a$, then $f(x) \sqsubseteq_A f(a)$ since every map preserves the information ordering. From the assumption $f(a) \sqsubseteq_A a$, it follows that $f(x) \sqsubseteq_A a$. □

Thus, for a complete set A, we have that the canonical fixpoint $\mathsf{fix}_\alpha(f)$ is actually a least fixpoint (as it is even a least prefixpoint). Moreover, if A is an S-space, then $\mathsf{fix}_\alpha(f)$ is *the* least fixpoint of f that does not depend on the choice of α, since this is unique for an S-space provided it exists.

9. Separatedness of S and its consequences

In the intended models of our theory, namely the *realisability models*, a further principle is valid that we have not exploited yet, namely that equality on all types is $\neg\neg$-closed, that is,

$$\forall x, y : X.\ \neg\neg\ x = y \ \Rightarrow\ x = y$$

for all types X.

We will show that equality on S-spaces is already $\neg\neg$-closed if equality on S is $\neg\neg$-closed. But, first consider the following reformulation of the latter requirement.

Lemma 9.1. Equality on S is $\neg\neg$-closed iff def is $\neg\neg$-closed.

Proof. From Axiom 4, for all $u, v \in S$ we have $u = v$ iff $\mathsf{def}(u) \Leftrightarrow \mathsf{def}(v)$. Therefore, equality on S is $\neg\neg$-closed if def is $\neg\neg$-closed because $\neg\neg$-closure is stable under implication.

If equality on S is $\neg\neg$-closed, then $\mathsf{def}(u) = (u = \top)$ is $\neg\neg$-closed also. □

Thus, equality on S is $\neg\neg$-closed iff def embeds S into the $\neg\neg$-closed propositions of Prop, which is known as *Markov's Principle* for the case when S *is identified with the collection of all* Σ^0_1*-propositions asserting termination of computations.*

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As in our more general setting S corresponds to the class of propositions asserting termination of computations[†], it makes sense to refer to $\neg\neg$-closedness of def not as (generalised) *Markov's Principle* but as $(\neg\neg)$-*separatedness of* S, or S-*separatedness.*

Theorem 9.2. The S-space S is $\neg\neg$-separated iff the equality on any S-space is $\neg\neg$-closed.

Proof. For an S-space A, we have $x =_A y$ iff $\forall p{:}S(A).\, p(x) = p(y)$. But the latter proposition is $\neg\neg$-closed, since equality on S is $\neg\neg$-closed and $\neg\neg$-closed propositions are closed under universal quantification.

The other direction holds because S is an S-space. □

Thus, in the presence of S-separatedness, many interesting predicates on domains are $\neg\neg$-closed. This turns out to be useful, since for $\neg\neg$-closed predicates some forms of case analysis are valid, even constructively. These proof principles are needed in order to develop a more special synthetic domain theory *à la* Scott, as can be found in the first author's Thesis (Reus 1995; Reus 1998).

Lemma 9.3. If S is $\neg\neg$-separated, then for all $\neg\neg$-closed predicates P on S we have that $\forall u{:}S.\, P(u)$ iff ($P(\top)$ and $P(\bot)$).

Proof. Let P be a $\neg\neg$-closed predicate on S and $u \in S$. If $P(\top)$ and $P(\bot)$, then $(u = \top \vee u = \bot) \Rightarrow P(u)$. By applying contraposition twice, we get that

$$\neg\neg(u = \top \vee u = \bot) \Rightarrow \neg\neg P(u)\ ,$$

that is, $\neg\neg(u = \top \vee u = \bot) \Rightarrow P(u)$ by $\neg\neg$-closedness of P. But by Lemma 4.3, the premiss is true, and, thus, we may conclude $P(u)$. □

A similar result follows for lifted types.

Lemma 9.4. If S is $\neg\neg$-separated, then for any type X and $\neg\neg$-closed predicate P on LX we have that $\forall z{:}LX.\, P(z)$ iff $P(\bot_X)$ and $\forall x{:}X.\, P(\eta_X(x))$.

Proof. Let P be a $\neg\neg$-closed predicate on LX. We define a predicate Q on S by

$$Q(u) \stackrel{\triangle}{\iff} \forall f : X^{\mathsf{def}(u)}.\, P(\langle u, f\rangle),$$

which is $\neg\neg$-closed as $\neg\neg$-closed predicates are closed under substitution and universal quantification.

By definition of lifting, $\forall z : LX.\, P(z)$ is equivalent to

$$\forall u{:}S.\ \forall f : X^{\mathsf{def}(u)}.\ P\,(\langle u, f\rangle),$$

that is, iff $\forall u{:}S.\, Q(u)$. By Lemma 9.3, it suffices to prove $Q(\bot)$ and $Q(\top)$. The first condition, $Q(\bot)$, follows from the assumption $P(\bot_A)$ because there is only one map from $\mathsf{def}(\bot)$ to X (as $\mathsf{def}(\bot)$ is empty). The second condition, $Q(\top)$, follows from the assumption that $P(\eta_X(x))$ for all $x \in X$, since all maps from $\mathsf{def}(\top)$ to X are constant (as $\mathsf{def}(\top)$ contains precisely one element due to Axiom 1). □

From this lemma, Axiom 6 follows as an immediate corollary.

[†] But notice that in general S will not be closed under disjunction and existential quantification over $\mathbb{N}$. So in general not all Σ^0_1-propositions need to be equivalent to some $\mathsf{def}(u)$.

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Corollary 9.5. If S is $\neg\neg$-separated, then for all types X and maps $p, q : LX \to S$ we have $p = q$ iff $p \circ \eta_X = q \circ \eta_X$ and $p(\bot_X) = q(\bot_X)$, that is, c_X is an S-epi.

Proof. The proof follows immediately from Lemma 9.4 applied to the $\neg\neg$-closed predicate $\lambda z : LX.\, p(x) = q(x)$. □

Also, Axiom 7 becomes derivable in the presence of Markov's Principle.

Lemma 9.6. Let $f : S \to S$. If S is $\neg\neg$-separated, then $f(u) = \top$ implies $f(\top) = \top$ for all $u \in S$.

Proof. By Lemma 9.3, it suffices to consider the two cases $u = \top$ and $u = \bot$. The first case is trivial. For the second, suppose $f(\bot) = \top$. Employing the fixpoint theorem, there exists a fixpoint $\mathsf{fix}(f)$ such that $f(\mathsf{fix}(f)) = \mathsf{fix}(f)$, since S is a pointed complete space. From this it follows that $\neg(\mathsf{fix}(f) = \bot)$, since $\neg(\bot = \top)$. Thus, by Lemma 4.3, $\neg\neg(\mathsf{fix}(f) = \top)$. Finally, by S-separatedness, we obtain $\mathsf{fix}(f) = \top$, and, therefore, $f(\top) = f(\mathsf{fix}(f)) = \mathsf{fix}(f) = \top$. □

Remark 9.7. This nicely illustrates the power of separation for S, as in its presence it suffices to require just completeness of S beyond the purely logical Axioms 1–3 and a dominance structure in order to develop basic synthetic domain theory.

10. Domain equations

So far we have shown that well-complete (or replete) pointed S-spaces provide a suitable notion of domain as far as recursive definitions of objects and induction principles for them are concerned. However, in functional programming one is also interested in recursively defined types.

Recursive types are given by domain equations $D \cong F(D, D)$ where F is an internal *mixed-variance bifunctor*, that is, $F : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$, where $\mathscr{C}$ is the category of well-complete (or replete) pointed S-spaces and strict maps.

Following a suggestion of Longley and Simpson (1997) – under the assumption that the *universe* Set *is impredicative* – one may show the existence of solutions of domain equations as follows.

First one shows that $\mathscr{C}$ is *algebraically compact*, which means that for any internal functor $F : \mathscr{C} \to \mathscr{C}$ there exists an initial/terminal F-algebra, that is, an isomorphism $\alpha : FA \to A$ such that α is an initial F-algebra and $\alpha^{-1} : A \to FA$ is a terminal F-coalgebra. Since, by Freyd (1992), algebraically compact categories are closed under $(\_)^{\mathrm{op}}$ and $\times$, the category $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ is algebraically compact, too. Therefore, any internal mixed-variant functor $F : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ admits a canonical solution for $D \cong F(D, D)$ as given by the initial/terminal algebra for the functor $F^{\S} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ where $F^{\S}(\_{}_1, \_{}_2) = \langle F(\_{}_2, \_{}_1), F(\_{}_1, \_{}_2)\rangle$.

Theorem 10.1. Let Set be impredicative, then the category $\mathscr{C}$ of well-complete (or replete) pointed S-spaces (that is, domains) with strict maps is algebraically compact.

Proof. Let $F : \mathscr{C} \to \mathscr{C}$ be an internal functor. By Theorems 6.5 and 6.7, the category $\mathscr{C}$ is small complete. Accordingly, the category of F-algebras is small complete, too.

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The initial F-algebra $\alpha: FA \to A$ is constructed as the least sub-F-algebra of the weakly initial F-algebra $\phi: FP \to P$, where $P \triangleq \Pi_{\alpha:FA\to A}A$ and

$$\pi_\alpha \circ \phi = \alpha \circ F(\pi_\alpha)$$

for every F-algebra $\alpha: FA \to A$.

Next, we show that the initial F-algebra is the terminal F-coalgebra. Let $\beta: B \to FB$ be some F-coalgebra. Then let $h: B \to A$ be the canonical solution of the fixpoint equation

$$h = \alpha \circ Fh \circ \beta,$$

which exists†, since the set $[B \to_{st} A]$ of strict maps from B to A is a pointed well-complete S-space (see their closure properties). It remains to show that this $h: \beta \to \alpha^{-1}$ is unique. Suppose that $k: \beta \to \alpha^{-1}$. Let $e: A \to A$ be the canonical solution of $e = \alpha \circ Fe \circ \alpha^{-1}$. By initiality of α, any of its endomorphisms is equal to id_A, and, therefore, $e = id_A$. As $\bot_{B,A} = \bot_{A,A} \circ k$ and $f = g \circ k$ implies $\alpha \circ Ff \circ \beta = \alpha \circ Fg \circ Fk \circ \beta = \alpha \circ Fg \circ \alpha^{-1} \circ k$, it follows by fixpoint induction that $h = e \circ k$. Thus, we have $h = k$ as $e = id_A$. □

The above construction of an initial F-algebra uses impredicativity of the universe Set in an essential way, since the initial F-algebra is identified as the least sub-F-algebra of $\phi: FP \to P$, where P is the product of all carrier sets of F-algebras including ϕ itself.

This sort of 'cheating' is impossible in classical domain theory due to the absence of a non-trivial impredicative universe Set. Instead, canonical solutions of domain equations are constructed explicitly as inverse limits. This method was originally used by D. Scott to obtain models of the untyped λ-calculus and was later made more systematic by Plotkin and Smyth (Smyth and Plotkin 1982). This classical construction can be shown to work in a particular version of SDT suitable for Domain Theory *à la* Scott, and is essentially based on the axioms of Hyland (1991) and Taylor (1991). See Reus (1995) for a type-theoretic formalisation, which has been machine-checked in an extension of the LEGO system.

In general SDT, however, there arises the following problem with the usual inverse limit construction. Let $i_n: D_n \to D$, $q_n: D \to D_n$ be the embeddings and projections for the n-th approximation D_n to the inverse limit D. Then $\lambda n{:}\mathbb{N}.\, i_n \circ q_n$ is an ascending $\mathbb{N}$-indexed chain in $D \to D$ but its limit is not guaranteed to exist as it is not clear how to extend this chain to an $\hat{\omega}$-chain (along $\widehat{\mathsf{step}}$).

In Simpson (1996), however, Simpson has suggested how we can avoid this problem by considering inverse limits of ω-diagrams instead of $\mathbb{N}$-diagrams. The underlying logic of his approach is some version of Intuitionistic Set Theory (IST) in which he can endow the class of all domains with an L-algebra structure, which allows him to construct the ω-diagram whose inverse limit provides a solution for $D \cong F(D, D)$. It remains to be checked whether his set-theoretic construction together with its verification can also be expressed in type theory, but it seems as if there should be no severe problems.

† An alternative proof, suggested by one of the anonymous referees, is to apply the result of Hyland (1988), which states that any endofunctor on a small complete category has an initial algebra α as well as a terminal coalgebra β. Then it just remains to construct the homomorphism $\beta \to \alpha^{-1}$ by this fixpoint equation and the following uniqueness proof is obsolete.

11. Conclusion

We have presented a logical approach to general SDT avoiding any reference to external notions. Any realisability structure, as considered in Longley and Simpson (1997), provides a model of the logic and axioms employed in this paper. It is, however, not clear how the sheaf models of Fiore and Rosolini (Fiore and Rosolini 1997) can be extended to models of SDT in our sense. The main obstacle is to identify an appropriate type-theoretic universe Set in their sheaf models. (It is only known that such a universe cannot be impredicative.)

In Reus (1995), extensional S-spaces have been studied as an axiomatic analogue of the *extensional pers* of Freyd *et. al.* (Freyd *et al.* 1992) but under the stronger assumptions of Hyland (1991) and Taylor (1991), and Markov's Principle. We plan to investigate in a subsequent paper what the theory of extensional S-spaces looks like under the weaker assumptions of general SDT.

Furthermore, there is a need for a more detailed account of domain equations.

Finally, the extent to which our logical approach to general SDT may have an impact on methods for the verification of functional programs should be investigated.

Appendix A. Complete list of axioms

1. Proof irrelevance, that is, $\forall P : \mathsf{Prop}. \forall p, q : P.\ p = q$.
2. Extensionality.
3. Axiom of Unique Choice, that is,

$$\Pi A : \mathsf{Set}. \Pi B : A \to \mathsf{Set}. \Pi P : \Pi x{:}A. B(x) \to \mathsf{Prop}.$$
$$(\forall x{:}A. \exists !\, y{:}B(x). P\ x\ y) \to \Sigma f : \Pi x{:}A. B(x). \forall a{:}A. P\ a\ f(a)\ .$$

4. $S \in \mathsf{Set}$ with $\top$ a dominance, and where $S \subseteq \mathsf{Prop}$. Global elements in S are $\top$ and $\bot$. Equality on S is logical equivalence.
5. $S(\imath)$ is an isomorphism, where $\imath$ is the unique map from the initial to the terminal lift-algebra, or, equivalently, S is complete.
6. For any set A the map $c_A = [\bot_A, \eta_A] : 1 + A \to L\,A$ is an S-epi.
7. For all $u \in S$ and $f : S \to S$ if $f(u) = \top$, then $f(\top) = \top$.

Note that the last two axioms can be derived if one assumes separatedness of S, that is, that equality on S is $\neg\neg$-closed.

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