Mathematical Structures in Computer Science (2022), **32**, pp. 240–256
doi:10.1017/S0960129522000238

CAMBRIDGE UNIVERSITY PRESS

PAPER

On Petrie cycle and Petrie tour partitions of 3- and 4-regular plane graphs

Xin He[1*], Huaming Zhang[2] and Yijie Han[3]

[1]Department of Computer Science and Engineering, State University of New York at Buffalo, Buffalo, NY 14260, USA, [2]Computer Science Department, The University of Alabama in Huntsville, Huntsville, AL 35899, USA and [3]School of Computing and Engineering, University of Missouri-Kansas City, Kansas City, MO 64110, USA

*Corresponding author. Email: xinhe@buffalo.edu

(Received 31 December 2020; revised 1 July 2022; accepted 4 July 2022; first published online 30 August 2022)

Abstract

Given a plane graph $G = (V, E)$, a *Petrie tour* of G is a tour P of G that alternately turns left and right at each step. A *Petrie tour partition* of G is a collection $\mathscr{P} = \{P_1, \ldots, P_q\}$ of Petrie tours so that each edge of G is in exactly one tour $P_i \in \mathscr{P}$. A Petrie tour P is called a Petrie cycle if all its vertices are distinct. A *Petrie cycle partition* of G is a collection $\mathscr{C} = \{C_1, \ldots, C_p\}$ of Petrie cycles so that each vertex of G is in exactly one cycle $C_i \in \mathscr{C}$. In this paper, we study the properties of 3-regular plane graphs that have Petrie cycle partitions and 4-regular plane multi-graphs that have Petrie tour partitions. Given a 4-regular plane multi-graph $G = (V, E)$, a *3-regularization* of G is a 3-regular plane graph G_3 obtained from G by splitting every vertex $v \in V$ into two degree-3 vertices. G is called *Petrie partitionable* if it has a 3-regularization that has a Petrie cycle partition. The general version of this problem is motivated by a data compression method, *tristrip*, used in computer graphics. In this paper, we present a simple characterization of Petrie partitionable graphs and show that the problem of determining if G is Petrie partitionable is NP-complete.

Keywords: Plane graph; Petrie cycles; Petrie tours; left-right paths

1. Introduction

Throughout this paper, $G = (V, E)$ denotes an undirected connected plane graph, which may have multiple edges, but no self-loops. Given a vertex $v \in V$ and an edge $e = (u, v) \in E$, the *left-edge* of e (at v) is the edge $e_1 = (u_1, v)$ that follows e (at v) in clockwise (cw) direction, the *right-edge* of e (at v) is the edge $e_2 = (u_2, v)$ that follows e (at v) in counter-clockwise (ccw) direction.

A *walk* of G is a sequence $P = v_0e_1v_1e_2 \ldots e_kv_k$ where v_i are vertices of G (may be repeated) and $e_j = (v_{j-1}, v_j)$ are distinct edges of G. The *length* of P is k. If $v_0 = v_k$, P is called a *tour*. A walk (tour, respectively) consisting of distinct vertices is called a *path* (*cycle*, respectively). A walk is called a *Petrie walk* if the edge e_{i+1} is alternately the left- and the right-edge of e_i for $1 \le i < k$. A tour P is called a *Petrie tour* if it is a Petrie walk, and the alternating left- and right-edge condition also holds for e_{k-1}, e_k, and e_1. *Petrie path* and *Petrie cycle* are defined similarly. Petrie walks are also called *left-right paths* and studied in Shank (1975).

A *Petrie cycle partition* of G is a set $\mathscr{C} = \{C_1, \ldots, C_p\}$ of Petrie cycles such that each vertex of G is in exactly one $C_i \in \mathscr{C}$. If $\mathscr{C}$ consists of a single cycle C_1, C_1 is call a *Petrie Hamiltonian cycle* of G. The properties of graphs with Petrie Hamiltonian cycle have been studied in Fouquet et al. (1982), Ivanço and Jendroľ (1999), Ivančo et al. (1994).

A *Petrie tour partition* of G is a set $\mathscr{P} = \{P_1, \ldots, P_q\}$ of Petrie tours such that each edge of G is in exactly one $P_i \in \mathscr{P}$. If $\mathscr{P}$ consists of a single tour P_1, P_1 is call a *Petrie Eulerian tour* of G. The properties of graphs with Petrie Eulerian tours have been studied in Kidwell and Bruce Richter (1987), Žitnik (2002).

In this paper, we study the properties of 3-regular plane graphs that have Petrie cycle partitions and the properties of 4-regular plane graphs that have Petrie tour partitions. For reasons that will become clear later, the 3-regular plane graph contains no multiple edges, but 4-regular plane graphs may contain multiple edges. We describe a simple characterization for 3-regular plane graphs with Petrie cycle partitions. This extends the results on plane graphs with Petrie Hamiltonian cycles in Fouquet et al. (1982), Ivanço and Jendrol' (1999), Ivančo et al. (1994). We describe a simple characterization for 4-regular plane graphs with Petrie tour partitions. This extends the results on plane graphs with Petrie Eulerian tours in Kidwell and Bruce Richter (1987), Žitnik (2002).

We also study the properties of Petrie partitionable graphs (as defined in the abstract). The general version of this problem is motivated by a data compression method, *tristrip*, used in computer graphics. We present a nice characterization of such graphs and show that the problem of determining if G is Petrie partitionable is NP-complete.

The present paper is organized as follows. Section 2 introduces the definitions and the motivation of these problems in computer graphics. Section 3 discusses the Petrie cycle partition of 3-regular plane graphs. Section 4 considers the Petrie tour partition of 4-regular plane graphs. The results in Sections 3 and 4 are relatively easy generalizations of known results in Fouquet et al. (1982), Ivanço and Jendrol' (1999), Ivančo et al. (1994), Kidwell and Bruce Richter (1987), Žitnik (2002). To the best of our knowledge, they have not been published in literature. Since they are of independent interests and also needed by the discussion in Section 5, we include these results here. Section 5 describes a nice characterization of Petrie partitionable 4-regular plane graphs. In Section 6, we show the problem of determining if a input 4-regular plane graph is Petrie partitionable is NP-complete. Section 7 describes some open problems and concludes the paper.

2. Definitions and Motivations

In this section, we give definitions and preliminary results. We use standard terminology in Bondy and Murty (1979). Let $G = (V, E)$ be an undirected graph with n vertices and m edges. The *degree* of a vertex $v \in V$, denoted by $\deg(v)$, is the number of edges incident to v. G is called 3-regular (4-regular, respectively), if $\deg(v) = 3$ ($\deg(v) = 4$, respectively) for all $v \in V$. For a subset $E_1 \subseteq E$, the subgraph of G *induced by* E_1 consists of E_1 as its edge set and the set of the vertices incident to the edges in E_1 as its vertex set. G is *bipartite* if V can be partitioned into two subsets V_1 and V_2 such that no two vertices in V_1 are adjacent and no two vertices in V_2 are adjacent. A *k-vertex-coloring* of G is a coloring of V by k colors so that any two adjacent vertices have different colors. A *k-edge-coloring* of G is a coloring of E by k colors so that any two edges incident to the same vertex have different colors.

A *plane* graph G is a graph embedded in the plane without edge crossings (i.e. an embedded planar graph). The embedding of a plane graph G divides the plane into a number of regions called *faces*. The unbounded region is the *exterior face*. Other regions are *interior faces*. $\mathscr{F}$ denotes the set of the faces of G. For each face $F \in \mathscr{F}$, $\deg(F)$ is the number of edges on the boundary of F. It is well known that a plane graph G is bipartite if and only if $\deg(F)$ is even for all $F \in \mathscr{F}$ (Bondy and Murty 1979). A *k-face-coloring* of a plane graph G is a coloring of its faces by k colors so that any two faces sharing an edge as their common boundary have different colors.

A plane graph G is called a *triangulation* (*quadrangulation*, respectively), if $\deg(F) = 3$ ($\deg(F) = 4$, respectively) for all faces $F \in \mathscr{F}$. The *dual graph* $G^* = (V^*, E^*)$ of a plane graph

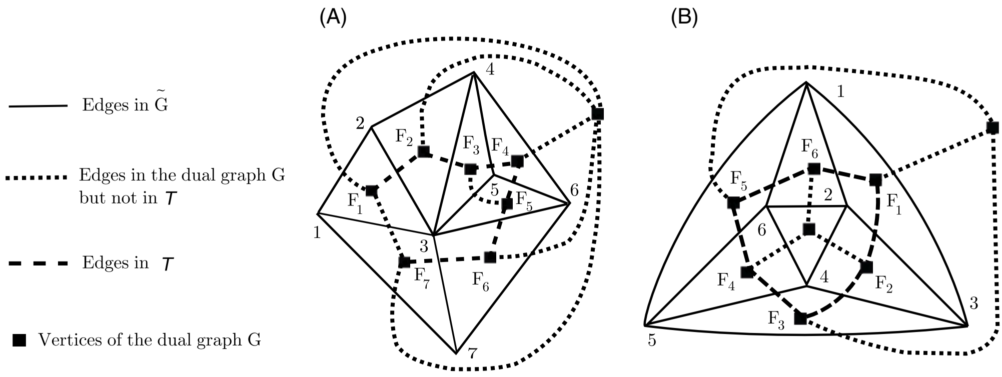

Figure 1. (A) A tristrip $\mathcal{T} = F_1F_2F_3F_4F_5F_6F_7$ represented by $\mathcal{S}_{\mathcal{T}} = 123456371$. (B) A tristrip cycle $\mathcal{T} = F_1F_2F_3F_4F_5F_6$ represented by $\mathcal{S}_{\mathcal{T}} = 123456$. (The solid thin lines are the edges of the triangular mesh $\tilde{G}$. The small black squares are the vertices of the dual graph G. The thick dashed lines are the edges of $\mathcal{T}$. The thick doted lines are the edges of G, not in $\mathcal{T}$).

$G = (V, E)$ is defined as follows: For each face F of G, V^* has a vertex v_F. For each edge e in G, G^* has an edge $e^* = (v_{F_1}, v_{F_2})$ where F_1 and F_2 are the two faces of G with e on their common boundary. e^* is called the *dual edge* of e. The mapping $e \leftrightarrow e^*$ is a one-to-one correspondence between E and E^*. G is a triangulation (quadrangulation, respectively) if and only if G^* is 3-regular (4-regular, respectively).

2.1 Motivation

The problem studied in this paper is motivated by a data compression technique used in computer graphics. 3D objects are often represented by *triangular mashes* in computer graphics. For our purpose, this is just a plane triangulation $\tilde{G} = (\tilde{V}, \tilde{E})$. Following the terms used in computer graphics, the members of the vertex set $\tilde{V} = \{1, 2, \ldots, n\}$ are called *points*. An important problem in computer graphics is how to represent $\tilde{G}$ efficiently. As a straightforward method, each face of $\tilde{G}$ can be represented by listing its three boundary points. If $\tilde{G}$ has N faces, this representation uses $3N$ points. For large 3D objects, this takes too much space. The *tristrips* representation of $\tilde{G}$ was discussed in Xiang et al. (1999). A tristrip is a sequence $\mathcal{T} = F_1F_2 \ldots F_t$ of faces in $\tilde{G}$, which can be represented by a sequence $S_{\mathcal{T}} = v_1v_2 \ldots v_{t+2}$ of points of $\tilde{G}$ in such a way that, for each i $(1 \le i \le t)$, the three points $v_iv_{i+1}v_{i+2}$ are the boundary points of the face F_i. An example of tristrip is shown in Figure 1A. A tristrip $\mathcal{T} = F_1 \ldots F_t$ is called a *tristrip cycle*, represented by the point sequence $S_{\mathcal{T}} = v_1v_2 \ldots v_t$, if both $\mathcal{T}$ and $S_{\mathcal{T}}$ are regarded as cyclic sequences and every three consecutive points $v_iv_{i+1}v_{i+2}$ $(1 \le i \le t)$ are the boundary points of the face F_i. (Here we define $t + 1 = 1$ and $t + 2 = 2$). An example of tristrip cycle is shown in Figure 1B. Thus, by using a tristrip, t faces in $\mathcal{T}$ are represented by $S_{\mathcal{T}}$ of $t + 2$ points (t points for a tristrip cycle).

If all faces of $\tilde{G}$ belong to one tristrip (or tristrip cycle), we can reduce the space for representing $\tilde{G}$ by a factor of 3 (Estkowski et al. 2002). However, a typical triangular mesh $\tilde{G}$ cannot be covered by one tristrip (or tristrip cycle). It is then a natural question: how to find the fewest disjoint tristrips (or tristrip cycles) that cover all faces of $\tilde{G}$? This minimization problem is known as *Stripification* problem in computer graphics. It was shown to be NP-complete in Estkowski et al. (2002). Various heuristic and exact (exponential time) algorithms have been studied in Porcu and Scateni (2003), Šíma (2005), Xiang et al. (1999).

The Stripification problem is closely related to the Petrie cycle partition problem as follows. Let $G = (V, E)$ be the dual graph of $\tilde{G}$. Clearly, G is 3-regular. For each face F of $\tilde{G}$, let v_F denote the

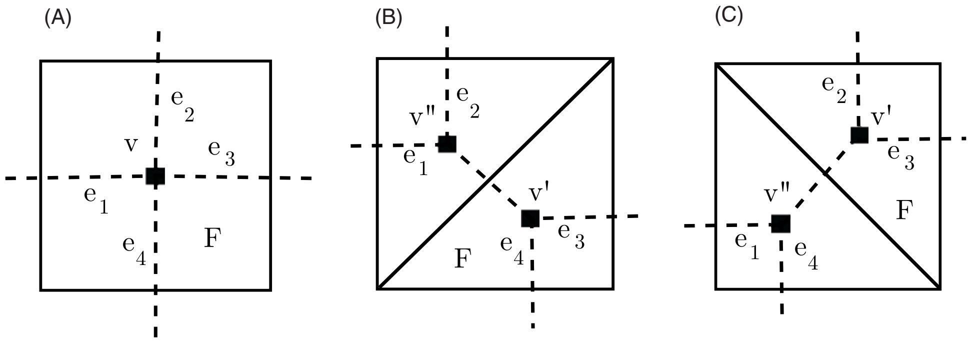

Figure 2. (A) A vertex v in G corresponding to a face F in $\tilde{G}$. (B) and (C) Two ways to split v.

vertex in G corresponding to F. Consider a sequence of faces $\mathcal{T} = F_1 \ldots F_t$ of $\tilde{G}$. It is easy to see that $\mathcal{T}$ is a tristrip (or tristrip cycle) of $\tilde{G}$ if and only if the corresponding sequence $v_{F_1} \ldots v_{F_t}$ is a Petrie path (or Petrie cycle) in G. (See Figure 1A and B.) Hence, the problem of finding a minimum tristrip cycle partition for the faces of $\tilde{G}$ is the same as the problem of finding a minimum Petrie cycle partition for G.

In computer graphics, 3D objects are also represented by *quadrangular meshes* (see Bommes et al. 2012, 2013; Dong et al. 2006). For our purpose, this is just a plane quadrangulation $\tilde{G} = (\tilde{V}, \tilde{E})$. If we add a chord into each face of $\tilde{G}$, it becomes a plane triangulation $\tilde{G}_3$ which is called a *triangular extension* of $\tilde{G}$. Since each face F of $\tilde{G}$ has degree 4, there are two ways to add a chord into F. If $\tilde{G}$ has $\tilde{f}$ faces, it has $2^{\tilde{f}}$ triangular extensions. One way to represent $\tilde{G}$ is first convert it to a plane triangular extension $\tilde{G}_3$ by adding chords into its faces and then represent $\tilde{G}_3$ by using tristrips or tristrip cycles (Estkowski et al. 2002). The question is: which of those $2^{\tilde{f}}$ triangular extensions can be partitioned into a minimum number of tristrips (or tristrip cycles)?

A special version of this problem is closely related to the Petrie tour partition problem. Consider the dual graph $G = (V, E)$ of $\tilde{G}$. Clearly, G is 4-regular. Consider a vertex $v \in V$ corresponding to a face F in $\tilde{G}$. Let e_1, e_2, e_3, e_4 be the four edges incident to v in cw order. The *split operation* at v splits v into two degree-3 vertices v' and v'' as shown in Figure 2. There are two ways to split v. They correspond to the two ways of adding a chord into the face F. Let G_3 be the 3-regular plane graph obtained by performing split operation at every vertex of G. We call G_3 a *3-regularization* of G. The edge (v', v'') of G_3 introduced by splitting a vertex $v \in V$ is denoted by $e(v)$ and called a *split edge* of G_3.

Suppose G has a Petrie tour partition $\mathcal{P} = \{P_1, \ldots, P_q\}$. Consider any vertex $v \in V$ with four incident edges $e_1, e_2, e_3, e_4 \in E$. Two tours P_i and P_j in $\mathcal{P}$ visit v (possibly $P_i = P_j$). We split v so that P_i and P_j are still tours after splitting. (See Figure 2B and C.) Do this at every vertex $v \in V$. Let G_3 be the resulting 3 regularization of G. It is easy to see that $\mathcal{P} = \{P_1, \ldots, P_q\}$ is a Petrie cycle partition of G_3. Thus, if G has a Petrie tour partition, then G has a 3-regularization G_3 with a Petrie cycle partition $\mathcal{P}$, where every edge e of G belongs to a Petrie cycle in $\mathcal{P}$. For its application in computer graphics, this restriction is not necessary. Figure 10A shows a 4-regular plane graph G which has no Petrie tour partition (as we will see later). However, it has a 3-regularization G_3 (shown in Figure 10B) which has a Petrie cycle partition with a single Petrie Hamiltonian cycle C. (Some edges of G are in C. Some are not). This motivates:

Definition 1. *A 4-regular plane graphs G is called* Petrie partitionable *if it has a 3-regularization with a Petrie cycle partition.*

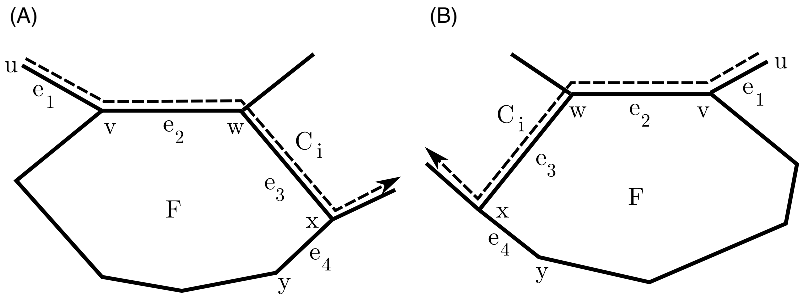

Figure 3. The proof of Lemma 2.

A main interest of this paper is to characterize the Petrie partitionable graphs. In computer graphics applications, the problem is to find a 3-regularization G' of G so that the faces of G' can be partitioned into tristrips and/or tristrip cycles. The NP-hardness result in Estkowski et al. (2002) suggests that this problem might be NP-hard also. The problem considered in this paper is a restricted version of the general problem: partition into Petrie cycles only. In contrast to the more general problem, this restriction leads to a simple characterization. The focus of our study is on the graph-theoretical properties of these problems. The insights obtained here may help to find more efficient algorithms for solving the general problem.

3. Characterization of 3-Regular Plane Graphs with Petrie Cycle Partition

In this section, $G=(V,E)$ always denotes a 3-regular connected plane simple graph (i.e. G has no self-loops nor parallel edges). Suppose G has a Petrie cycle partition $\mathscr{C}=\{C_1,\ldots,C_p\}$. For any vertex $v\in V$, two edges incident to v belong to a cycle $C_i\in\mathscr{C}$ and its third incident edge is not in any $C_j\in\mathscr{C}$. We call the third edge a *non-cycle edge* with respect to $\mathscr{C}$.

A 3-regular plane graph G is called a *multi-3-gon* if all of its faces have degrees divisible by 3. The following lemma was proved in Ivanço and Jendrol' (1999), Ivančo et al. (1994).

Lemma 1. If a 3-regular plane graph has a Petrie Hamiltonian cycle, then it is a multi-3-gon.

The following lemma generalizes Lemma 1. The proof is essentially the same as in Ivanço and Jendrol' (1999). We include it here for completeness.

Lemma 2. Any 3-regular plane graph G with a Petrie cycle partition must be a multi-3-gon.

Proof. Let $\mathscr{C}=\{C_1,\ldots C_p\}$ be a Petrie cycle partition of G. Consider any face F of G and a cycle $C_i\in\mathscr{C}$ that travels at least one edge of F. Suppose C_i travels two edges $e_1=(u,v)$ and $e_2=(v,w)$ from u to v to w where e_1 is not an edge of F, but e_2 is an edge of F. Let $e_3=(w,x)$ and $e_4=(x,y)$ be the two edges of F following e_2 in the direction from v to w. When C_i travels from e_1 to e_2, suppose it turns left at v (see Figure 3A). Since C_i is a Petrie cycle, it turns right at w and travels e_3. Then, C_i must make a left-turn at x. So e_4 is a non-cycle edge with respect to $\mathscr{C}$. If C_i turns right at v, it must turn left at w and travel e_3 (Figure 3B). Then, C_i must turn right at x. So e_4 must be a non-cycle edge with respect to $\mathscr{C}$.

Thus, the edges on the boundary of F can be partitioned into subpaths of length 3: The first two edges of a subpath belong to a Petrie cycle $C_i\in\mathscr{C}$, while the third edge of the subpath is a non-cycle edge. Hence, the length of F must be a multiple of 3. □

https://doi.org/10.1017/S0960129522000238 Published online by Cambridge University Press

If $G = (V, E)$ has a proper 3-edge-coloring $\lambda : E \to \{1, 2, 3\}$, the *Heawood valuation* (or simply valuation) associated with λ is a mapping $\lambda^* : V \to \{-1, 1\}$ defined as follows. For any vertex $v \in V$, if the three edges incident to v are colored 1,2,3 in cw order, then $\lambda^*(v) = 1$. Otherwise, define $\lambda^*(v) = -1$. The following lemma is well known (Ringel 1959, p. 18, Theorem 5):

Lemma 3. A 3-regular plane graph $G = (V, E)$ has a 3-edge-coloring if and only if there exists a mapping $\kappa : V \to \{-1, 1\}$ such that the sum of the values $\kappa(v)$ for all vertices on the boundary of any face F of G is divisible by 3. If κ is such a mapping, then there exists a 3-edge-coloring λ of G such that its associated valuation $\lambda^* = \kappa$.

Theorem 1. Every 3-regular multi-3-gon $G = (V, E)$ has exactly three Petrie cycle partitions.

Proof. Define a mapping $\kappa : V \to \{-1, 1\}$ by setting $\kappa(v) = 1$ for all $v \in V$. Since G is a multi-3-gon, the condition for κ in Lemma 3 is satisfied. Thus, G has a 3-edge-coloring λ such that its associated valuation $\lambda^*(v) = \kappa(v) = 1$ for all $v \in V$.

Let E_i $(i = 1, 2, 3)$ be the subset of the edges of color i in the coloring λ. Then, each E_i is a perfect matching of G. Let G_{12} be the subgraph of G induced by the edge set $E_1 \cup E_2$. Clearly each vertex in G_{12} has degree 2. Hence, G_{12} is a collection $\mathscr{C}_{12} = \{C_1, \ldots, C_p\}$ of disjoint cycles. For each $C_i \in \mathscr{C}$, the edges of C_i alternate between E_1 and E_2. Imagine we travel along C_i passing three consecutive edges $e_1 = (u, v)$, $e_2 = (v, w)$, $e_3 = (w, x)$, where $e_1, e_3 \in E_1$ and $e_2 \in E_2$. $\lambda^*(v) = 1$ implies C_i turns left at v. $\lambda^*(w) = 1$ implies C_i turns right at w. Repeating this argument, we see C_i alternately turns left and right. Thus,C_i is a Petrie cycle and $\mathscr{C}_{12}$ is a Petrie cycle partition of G.

Similarly, we can define the subgraph G_{13} induced by $E_1 \cup E_3$ and the subgraph G_{23} induced by $E_2 \cup E_3$. By the same argument, each of them defines a distinct Petrie cycle partition of G.

Next we show G has at most three distinct Petrie cycle partitions. Pick any vertex v of G with three incident edges e_1, e_2, e_3. Consider any Petrie cycle partition $\mathscr{C} = \{C_1, \ldots, C_p\}$ of G. Assume e_1 and e_2 belong to $C_i \in \mathscr{C}$ and e_3 is a non-cycle edge with respect to $\mathscr{C}$. Since C_i is a Petrie cycle, and two edges e_1 and e_2 are in C_i, we can uniquely trace all edges of C_i by alternately turning left and right. Any vertex w of C_i has a non-cycle edge (w, x) incident to it. The two other edges incident to x belong to a Petrie cycle $C_j \in \mathscr{C}$ (possibly $C_i = C_j$). Thus, we can uniquely trace all edges of C_j. Since G is connected, we can trace all Petrie cycles in $\mathscr{C}$ by repeating this process. (Note that there is no guarantee this process can successfully lead to a Petrie cycle partition of G). In summary, the fact that an edge e_3 incident to v is a non-cycle edge uniquely determines entire $\mathscr{C}$. If we pick e_1 or e_2 as a non-cycle edge, we can determine at most two other Petrie cycle partitions. So G has at most three distinct Petrie cycle partitions.

On the other hand, we have shown G has at least three distinct Petrie cycle partitions. Thus, G has exactly three Petrie cycle partitions. (This also shows the process described above always produces a valid Petrie cycle partition of G). □

Figure 4 shows a 3-regular multi-3-gon plane graph G and two Petrie cycle partitions $\mathscr{C}_{12}$ and $\mathscr{C}_{23}$. $\mathscr{C}_{12}$ contains two Petrie cycles. $\mathscr{C}_{23}$ contains a single Petrie Hamiltonian cycle. We end this section by proving the following::

Theorem 2. A 3-regular plane graph G has a Petrie cycle partition if and only if it is a multi-3-gon. Such G has exactly three Petrie cycle partitions, which can be found in linear time.

Proof. The proof immediately follows from Lemma 2 and Theorem 1. The implementation of the algorithm follows the proof of Theorem 1. First, we find a 3-edge-coloring of G by 3 colors $\{1, 2, 3\}$ so that, for every vertex v of G, the three edges incident to v are colored by 1, 2, 3 in cw order. To do this, pick any vertex v and color its three incident edges this way. Then, we can propagate the colors to other edges in G. Because G is a multi-3-gon, this process never causes

https://doi.org/10.1017/S0960129522000238 Published online by Cambridge University Press

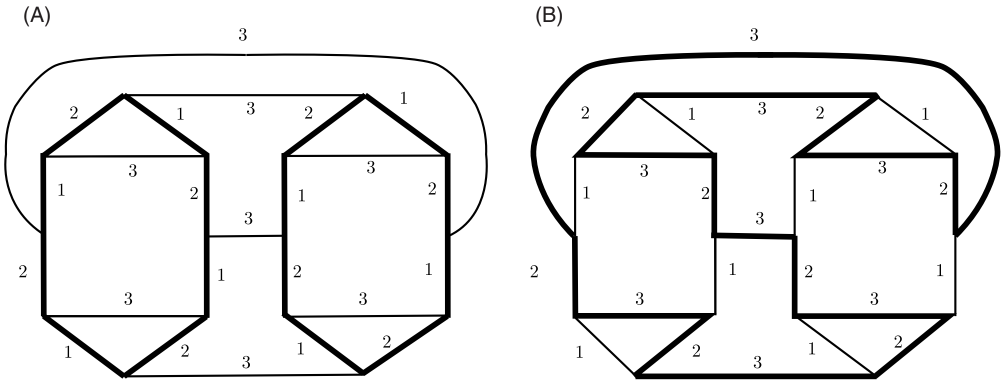

Figure 4. (A) Petrie cycle partition $\mathscr{C}_{12}$; (B) Petrie cycle partition $\mathscr{C}_{23}$. (The thick lines are the edges in Petrie cycles. The thin lines are non-cycle edges.)

color conflicts. Once the 3-edge-coloring is obtained, the Petrie cycle partitions $\mathscr{C}_{12}, \mathscr{C}_{13}, \mathscr{C}_{23}$ can be easily obtained. Entire algorithm clearly takes linear time. □

4. Characterization of 4-Regular Plane Graphs with Petrie Tour Partition

In this section, we study Petrie tour partitions of a 4-regular plane graph G. The special case of this problem where the Petrie tour partition contains only one tour (i.e. a Petrie Eulerian tour) was studied in Žitnik (2002). We generalize the results in Žitnik (2002). Throughout this section, $G = (V, E)$ denotes a 4-regular connected plane graph with no self-loops, but may have parallel edges.

4.1 Characterization

Let $P = e_1 \ldots e_k$ be a Petrie walk of G. Let $P^* = e_1^* \ldots e_k^*$ be the sequence of the dual edges e_i^* in the dual graph G^*. The following simple observation is crucial to our results.

Observation 1. (Žitnik 2002) If P is a Petrie tour of G, then the sequence P^* of the dual edges is a Petrie tour of the dual graph G^*.

This observation is illustrated in Figure 5A. The sequence 12345 is a Petrie walk in G. The sequence $F_1F_2F_3F_4F_5$ is the corresponding Petrie walk in the dual graph G^*. In this figure, the thick solid lines are the edges of the Petrie walk P of G. The thin solid lines are the edges of G but not in P. The dashed thick lines are the dual edges in the dual Petrie walk P^* of G^*. The doted thin lines are the edges of G^* but not in P^*.

Based on Observation 1, Žitnik (2002) showed that a 4-regular plane graph with a Petrie Eulerian tour must be bipartite. The following lemma generalizes this result.

Lemma 4. If $G = (V, E)$ has a Petrie tour partition, then G must be bipartite.

Proof. Let $\mathscr{P} = \{P_1, P_2, \ldots, P_q\}$ be a Petrie tour partition of G. By Observation 1, each P_i^* is a Petrie tour in the dual graph $G^* = (V^*, E^*)$. Because the edge set E of G one-to-one corresponds to the dual edge set E^* of G^*, the Petrie tours in $\mathscr{P}^* = \{P_1^*, \ldots, P_q^*\}$ partition the edge set E^*. Consider any node v_F in G^*. When a Petrie tour $P_i^* \in \mathscr{P}$ visits v_F, two edges in P_i^* are consumed:

https://doi.org/10.1017/S0960129522000238 Published online by Cambridge University Press

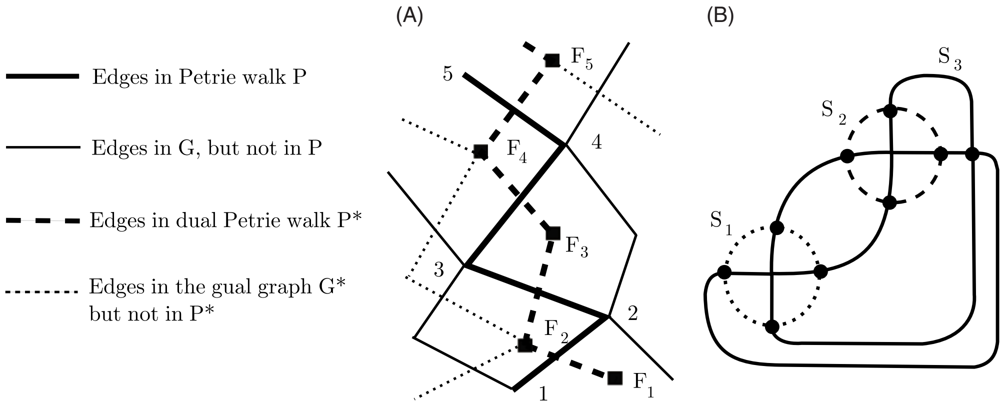

Figure 5. (A) An example of Observation 1. (B) A graph G and its S-tours $\mathscr{S}(G) = \{S_1, S_2, S_3\}$.

one enters v_F, one leaves v_F. Hence, every node in V^* has even degree in G^*. Namely every face F in G has even degree, as to be shown. □

Let n, m, f denote the number of vertices, edges, and faces of G, respectively. Since G is 4-regular, we have $2m = \sum_{v \in V} \deg(v) = 4n$. Let f_{2i} ($i \geq 1$) be the number of faces of G with degree $2i$. Then $f = \sum_{i \geq 1} f_{2i}$. By counting the sum of the degrees of the faces of G, we also have $2m = \sum_{F \text{ is a face of } G} \deg(F) = \sum_{i \geq 1} 2if_{2i}$. By Euler formula: $m = n + f - 2$. Putting these equations together, we have: (i) $m = 2n$; (ii) $f = n + 2$; and (iii) $f_2 = 4 + \sum_{i \geq 3} (i-2)f_{2i}$.

Note that $f_2 \geq 4$ is the number of degree-2 faces of G, and a degree-2 face is a pair of parallel edges. This explains why we have not restricted to simple graphs in this section.

Consider a tour P of G. Since G is 4-regular, at every vertex of P, there are three ways to continue the tour: turn left, go straight, or turn right. A tour S of G consisting of only going-straight steps is called a *straight tour* (or an *S-tour*). It is easy to see the edge set E of G can be uniquely partitioned into S-tours. Denote this partition by $\mathscr{S}(G) = \{S_1, \ldots, S_k\}$. An S-tour may visit a vertex of G twice. An S-tour is called *simple* if it is a cycle in G. Figure 5B shows a 4-regular plane graph G. $\mathscr{S}(G)$ contains three S-tours: S_1 and S_2 are simple. S_3 is not. Two S-tours are said *independent* if they do not intersect. The following theorem was proved in Jaeger and Shank (1981):

Theorem 3. Let $G = (V, E)$ be a 4-regular plane graph, and let $\mathscr{S}(G) = \{S_1, \ldots, S_k\}$ be the set of S-tours of G. Then, G is bipartite if and only if (i) all S-tours $S_i \in \mathscr{S}(G)$ are simple; and (ii) $\mathscr{S}$ can be partitioned into two subsets $\mathscr{S}_1$ and $\mathscr{S}_2$ such that each $\mathscr{S}_i$ ($i = 1, 2$) consists of mutually independent S-tours.

By Lemma 4 and Theorem 3, all graphs with a Petrie tour partition have a special structure: the set $\mathscr{S}(G)$ is partitioned into two subsets $\mathscr{S}_1$ and $\mathscr{S}_2$; $\mathscr{S}_1$ is a collection of simple cycles; $\mathscr{S}_2$ is also a collection of simple cycles; and the two sets of cycles are *overlaid* with each other. Such graphs can be complex: Even if $\mathscr{S}_1$ has only one cycle S_1 and $\mathscr{S}_2$ has only one cycle S_2, S_1 and S_2 can cross each other many times in complex ways.

In the following, we show that every 4-regular bipartite plane graph $G = (V, E)$ has exactly two distinct Petrie tour partitions.

Since G is bipartite, we can color the vertices of G by two colors red and green. Note that (1) the boundary of every face of G has the same number of red and green vertices, and (2) every vertex v is incident to exactly four faces. These two facts imply the number of red vertices equals the number of green vertices.

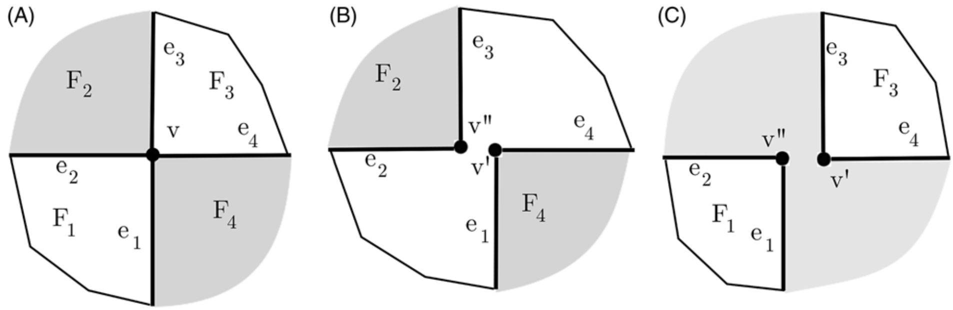

Figure 6. (A) A vertex v and its incident edges and faces; (B) After the white merge operation at v; (C) After the black merge operation at v.

Since G is 4-regular, we can color the faces of G using two colors white and black. (The number of the white faces and the number of the black faces of G may be different).

Definition 2. Let v be a vertex of G with four incident edges e_i $(1 \leq i \leq 4)$ in cw order and four incident faces F_i $(1 \leq i \leq 4)$ where F_1, F_3 are white and F_2, F_4 are black. Assume e_i, e_{i+1} $(1 \leq i \leq 4)$ are the edges of F_i (see Figure 6A.)

(1) The *white merge* operation at v is (Figure 6B):
- Replace v by two new vertices v' and v'';
- Make the edges e_1 and e_4 incident to v'; and make e_2 and e_3 incident to v''.

(2) The *black merge* operation at v is (Figure 6C):
- Replace v by two new vertices v' and v'';
- Make the edges e_1 and e_2 incident to v''; and make e_3 and e_4 incident to v'.

Note that after either merge operation at v, the two new vertices v' and v'' have degree 2. After the white merge operation at v, the two white faces F_1 and F_3 become one face. After the black merge operation at v, the two black faces F_2 and F_4 become one face.

Definition 3. (1) The *red-white-merge graph*, denoted by G_{rwm}, is the graph obtained from G by applying the white merge operation at every red vertex of G and the black merge operation at every green vertex of G. (See Figure 7B.)

(2) The *red-black-merge graph*, denoted by G_{rbm}, is the graph obtained from G by applying the black merge operation at every red vertex of G and the white merge operation at every green vertex in G. (See Figure 7C.)

By construction, every vertex v in G_{rwm} has degree 2 and the edge set of G_{rwm} one-to-one corresponds to the edge set of G. The same properties also hold for G_{rbm}. We can similarly define the *green-black-merge* graph G_{gbm} and the *green-white-merge* graph G_{gwm}. Obviously, $G_{rbm} = G_{gwm}$ and $G_{rwm} = G_{gbm}$.

Theorem 4. Every 4-regular plane bipartite graph G has exactly two Petrie tour partitions.

Proof. Consider the graph G_{rbm}. Since every vertex in G_{rbm} has degree 2, G_{rbm} is a disjoint union of simple cycles. Let $\mathscr{C} = \{C_1, \ldots, C_q\}$ be these cycles. Note that the edge set of G_{rbm} one-to-one corresponds to the edge set of G. For each cycle $C_i \in \mathscr{C}$, let P_i be the sequence of the edges of G corresponding to the edges of C_i. Then, P_i is a tour of G alternately traveling red and green vertices. Imagine we travel P_i so that the black faces are on right side. By the construction of G_{rbm},

https://doi.org/10.1017/S0960129522000238 Published online by Cambridge University Press

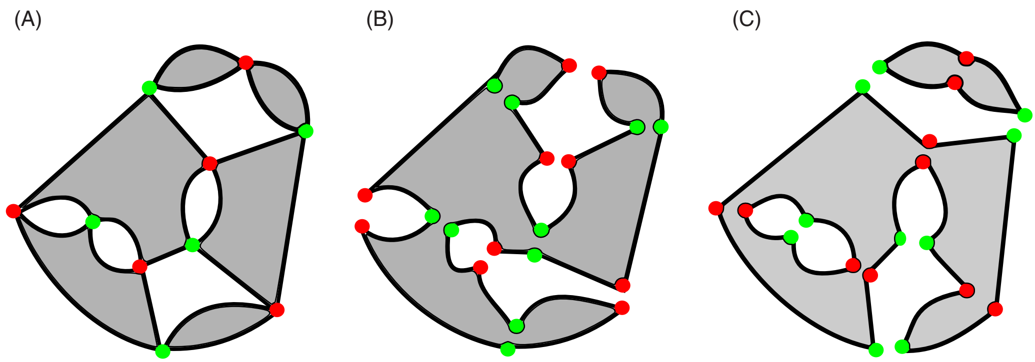

Figure 7. (A) G; (B) The red-white-merge graph G_{rwm}; (C) The red-black-merge graph G_{rbm}.

P_i always turns left at red vertices and right at green vertices (see Figure 7C). Hence, P_i is a Petrie tour of G. Let $\mathscr{P}_{rbm} = \{P_1, \ldots, P_q\}$. Since the edge set of G_{rbm} one-to-one corresponds to the edge set of G, every edge of G belongs to exactly one $P_i \in \mathscr{P}_{rbm}$. Thus, $\mathscr{P}_{rbm}$ is a Petrie tour partition of G. Similarly, we can show the red-white-merge graph G_{rwm} corresponds to another Petrie tour partition $\mathscr{P}_{rwm}$ of G.

Next we show $\mathscr{P}_{rbm}$ and $\mathscr{P}_{rwm}$ are the only Petrie tour partitions of G. Let $\mathscr{Q} = \{Q_1, \ldots, Q_t\}$ be any Petrie tour partition of G. Since G is bipartite, each $Q_i \in \mathscr{Q}$ alternately travels red and green vertices. Consider any tour $Q_i \in \mathscr{Q}$ and three consecutive edges $e_1 = (u, v)$, $e_2 = (v, w)$ and $e_3 = (w, x)$ of Q_i, where u, w are green; v, x are red. Let F_1 be the face with e_1 and e_2 on its common boundary. Let F_2 be the face with e_2 and e_3 on its common boundary.

Case 1: Q_i turns left at the red vertex v between e_1 and e_2 and F_1 is a white face. Since Q_i is a Petrie tour, it turns right at the green vertex w between e_2 and e_3. Since F_1 and F_2 share e_2 as common boundary and F_1 is white, F_2 must be black. This corresponds to performing the black merge operation at the red vertex v, and the white merge operation at the green vertex w. Repeating this argument, we see that all $Q_i \in \mathscr{Q}$ are obtained by performing the black merge operation at all red vertices and performing the white merge operation at all green vertices of G. Thus, $\mathscr{Q}$ is the same as the Petrie tour partition $\mathscr{P}_{rbm}$.

Case 2: Q_i turns left at the red vertex v between e_1 and e_2 and F_1 is a black face. By using same argument as in Case 1, we can show $\mathscr{Q}$ is the same as $\mathscr{P}_{rwm}$.

Case 3: Q_i turns right at the red vertex v between e_1 and e_2 and F_1 is a black face. By using same argument as in Case 1, we can show $\mathscr{Q}$ is the same as $\mathscr{P}_{rwm}$.

Case 4: Q_i turns right at the red vertex v between e_1 and e_2 and F_1 is a white face. By using same argument as in Case 1, we can show $\mathscr{Q}$ is the same as $\mathscr{P}_{rbm}$.

This completes the proof. □

Figure 7A shows a 4-regular plane bipartite graph G. Figure 7B shows the graph G_{rwm} corresponding to a Petrie tour partition of G with a single Petrie Eulerian tour. Figure 7C shows the graph G_{rbm} corresponding to a Petrie tour partition of G with three Petrie tours.

We end this subsection by proving the following:

Theorem 5. A 4-regular plane graph G has a Petrie tour partition if and only if it is bipartite. Such G has exactly two Petrie tour partitions, which can be found in linear time.

Proof. The proof immediately follows from Lemma 4 and Theorem 4. The linear time implementation of the algorithm can be done as follows.

First we construct the dual graph G^* of G. Color the vertices of G red and green. Color the faces of G^* white and black. Consider a vertex v of G. Let e_1, e_2, e_3, e_4 be the four edges incident to v in cw order. Split v into two vertices v' and v''. If v is red, make its two edges bounding a black face incident to v' and its other two edges bounding a black face incident to v''. If v is green, make its two edges bounding a white face incident to v' and its other two edges bounding a white face incident to v''. The resulting graph is the red-white-merge graph G_{rwm}. By Theorem 4, this is a Petrie tour partition of G. The other Petrie tour partition G_{rbm} of G can be obtained similarly. The whole process clearly takes linear time. □

4.2 Graph theoretic interpretation of the size of Petrie tour partition

In this subsection, we explain the meaning of the size of Petrie tour partitions of G.

Definition 4. The *white graph* $G^*_{white} = (V^*_{white}, E^*_{white})$ of G is defined as follows. (To avoid confusion, we call the members of V^*_{white} *nodes* and the members of E^*_{white} *lines*.)

- The node set V^*_{white} is the set of the white faces of G.
- Let v_{F_1} and v_{F_2} be two nodes in V^*_{white} corresponding to the two white faces F_1 and F_2 of G. v_{F_1} and v_{F_2} are connected by a line $e = (v_{F_1}, v_{F_2}) \in E^*_{white}$ if and only if F_1 and F_2 share a vertex v of G on their boundary. We denote this edge by $e(v)$. Moreover, if v is red, $e(v)$ is called a red line. If v is green, $e(v)$ is called a green line.
- The *white-red subgraph* of G^*_{white}, denoted by $G^*_{white,red}$, is the subgraph of G^*_{white} induced by its red lines. The *white-green subgraph* of G^*_{white}, denoted by $G^*_{white,green}$, is the subgraph of G^*_{white} induced by its green lines.

We can embed G^*_{white} as follows: Place the node v_F corresponding to a white face F in the center of F. Draw the line $e(v) = (v_{F_1}, v_{F_2})$ as a curve connecting two nodes v_{F_1} and v_{F_2}, passing through the vertex v that defines $e(v)$. Clearly G^*_{white} is a plane graph. Figures 9A and B show the graph $G^*_{white,red}$ and $G^*_{white,green}$, respectively, overlaid with G. Since the numbers of red and green vertices in G are the same, the number of red lines in $G^*_{white,red}$ and the number of green lines in $G^*_{white,green}$ are the same.

Definition 5. The *black graph* $G^*_{black} = (V^*_{black}, E^*_{black})$ of G is defined as follows:

- The node set V^*_{black} is the set of the black faces of G.
- Let v_{F_1} and v_{F_2} be two nodes in V^*_{black} corresponding to the two black faces F_1 and F_2 of G. v_{F_1} and v_{F_2} are connected by a line $e = (v_{F_1}, v_{F_2}) \in E^*_{black}$ if and only F_1 and F_2 share a common vertex v of G on their boundary. We denote this edge by $e(v)$. Moreover, if v is red, $e(v)$ is called a red line. If v is green, $e(v)$ is called a green line.
- The *black-red subgraph* of G^*_{black}, denoted by $G^*_{black,red}$, is the subgraph of G^*_{black} induced by its red lines. The *black-green subgraph* of G^*_{black}, denoted by $G^*_{black,green}$, is the subgraph of G^*_{black} induced by its green lines.

It is known the graphs G^*_{white} and G^*_{black} are dual graphs to each other (Berman and Shank 1979; Kidwell and Bruce Richter 1987).

Consider the white-red subgraph $G^*_{white,red}$ and a node $v_F \in V^*_{white,red}$ corresponding to a white face F of G. Let $e_1, \ldots, e_t$ be the lines in $G^*_{white,red}$ incident to v_F in cw order. Each line e_i passes a red vertex v_i on the boundary of F. A pair of consecutive lines defines an *angle* $\theta_i = (e_i, e_{i+1})$ of $G^*_{white,red}$ ($1 \leq i \leq t$ where $e_{t+1} = e_1$). The *subpath of F for θ_i*, denoted by $p(\theta_i)$, is the clockwise

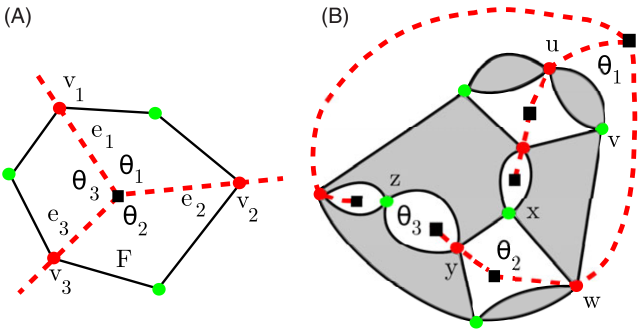

Figure 8. (A) Subpaths of *F* for angles; (B) A tour associated with an elementary circuit.

subpath on the boundary of F between the vertex v_i and v_{i+1}. Note that $\cup_{i=1}^{t} p(\theta_i)$ is the boundary of F (see Figure 8A.)

Embed $G^*_{white,red}$ in the plane (without the embedding of G). Let D be a connected component of $G^*_{white,red}$. Each face (including the exterior face) of D is called an *elementary circuit* of D. The *elementary circuit number* of D, denoted by $ecn(D)$, is the number of elementary circuits of D. If D has a nodes and b lines, then $ecn(D) = b - a + 2$. Let $D_1, \ldots, D_s$ be the connected components of $G^*_{white,red}$. All elementary circuits of each D_i are the elementary circuits of $G^*_{white,red}$. The *elementary circuit number* of $G^*_{white,red}$ is defined to be $ecn(G^*_{white,red}) = \sum_{i=1}^{s} ecn(D_i)$. For example, if $G^*_{white,red}$ is a tree, then $ecn(G^*_{white,red}) = 1$. As another example, if $G^*_{white,red}$ has two connected components D_1 and D_2 where D_1 is a tree and D_2 is the union of a spanning tree and two non-tree edges, then $ecn(D_1) = 1$, $ecn(D_2) = 3$ and $ecn(G^*_{white,red}) = 4$. In general, we have:

Fact 1. If $G^*_{white,red}$ has a nodes, b lines and k connected components, then $ecn(G^*_{white,red}) = b - a + 2k$.

Let c be an elementary circuit of $G^*_{white,red}$. Imagine we travel along c in cw order so that the interior of c is on the right side. (If c is the exterior face, we travel c in ccw order so that the exterior of c is on the right side). Let $\Theta(c) = \theta_1, \ldots, \theta_t$ be the sequence of angles of c we encounter on the right side of c. For each i $(1 \le i \le t)$, let $p(\theta_i)$ be the subpath for θ_i. Let $p(c)$ be the concatenation of $p(\theta_i)$ $(1 \le i \le t)$. Note that $p(c)$ is a tour of G and is called the Petrie tour associated with c.

Figure 8B illustrates these terms. The white-red subgraph $G^*_{white,red}$ is a single tree D_1. So $ecn(G^*_{white,red}) = 1$, and it has only one elementary circuit c which is just *walking around* D_1 in ccw order. Three angles $\theta_1, \theta_2, \theta_3$ of c are shown. $p(\theta_1) = \{(u, v), (v, w)\}$. $p(\theta_2) = \{(w, x), (x, y)\}$. θ_3 is defined by a leaf node v_{F_1} of D_1. So $p(\theta_1) = \{(y, z), (z, y)\}$ (the two parallel edges on the boundary of F_1).

Theorem 6. Let G be a 4-regular bipartite plane graph. Let $G^*_{white,red}$ and $G^*_{white,green}$ be the subgraphs defined above.

(1) Let $\Gamma_1 = \{c_1, \ldots, c_t\}$ be the set of elementary circuits of $G^*_{white,red}$. For each c_i $(1 \le i \le t)$, let $p(c_i)$ be the Petrie tour associated with c_i. The union $\cup_{i=1}^{t} p(c_i)$, which is called the graph associated with $G^*_{white,red}$, is the same as the red-white-merge graph G_{rwm}.

(2) Let $\Gamma_2 = \{c'_1, \ldots, c'_s\}$ be the set of elementary circuits of $G^*_{white,green}$. For each c'_i $(1 \le i \le s)$, let $p(c'_i)$ be the Petrie tour associated with c'_i. The union $\cup_{i=1}^{s} p(c'_i)$, which is called the graph associated with $G^*_{white,green}$, is the same as the green-white-merge graph G_{gwm}.

https://doi.org/10.1017/S0960129522000238 Published online by Cambridge University Press

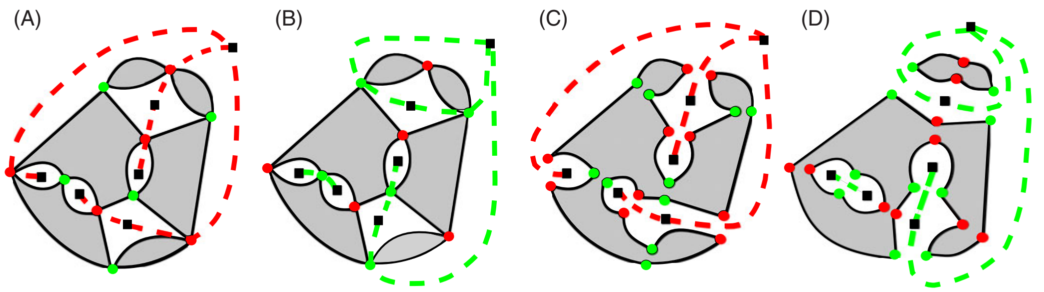

Figure 9. (A) $G^*_{white,red}$ overlaid with G; (B) $G^*_{white,green}$ overlaid with G; (C) the graph G_{rwm}; (D) the graph G_{gwm}. The solid small squares are the nodes of G^*_{white}. The thick red dashed lines are the lines of $G^*_{white,red}$. The thick green dashed lines are the lines of $G^*_{white,green}$.

Proof. We only prove the Statement 1. The proof of Statement 2 is similar.

The red-white-merge graph G_{rwm} is a collection $\mathscr{C} = \{C_1, \ldots, c_q\}$ of cycles. It is uniquely characterized by the following properties:

(1) Each edge of G belongs to exactly one $C_i \in \mathscr{C}$.
(2) For each red vertex v in C_i, the two edges in C_i incident to v belong to a back face of G.
(3) For each green vertex w in C_i, the two edges in C_i incident to w belong to a white face of G.

We show the graph $\cup_{i=1}^{t} p(c_i)$ also satisfies these properties.

(1) Every edge e of G is on the boundary of exactly one white face F of G. So it belongs to exactly one subpath $p(\theta)$ for an angle θ at node v_F. Every angle θ of $G^*_{white,red}$ belongs to exactly one elementary circuit c_i of $G^*_{white,red}$. So each edge e of G belongs to exactly one Petrie tour $p(c_i)$ associated with c_i.
(2) For each red vertex v in $p(c_i)$, by construction, the two edges in $p(c_i)$ incident to v belong to a black face of G.
(3) Similarly, for each green vertex w in $p(c_i)$, the two edges in $p(c_i)$ incident to w belong to a white face of G.

This proves Statement 1. □

Figure 9C shows the graph G_{rwm} which is also the graph associated with the white-red subgraph $G^*_{white,red}$. It is a Petrie tour partition of G consisting of a single Petrie Eulerian tour. Figure 9D shows the graph G_{gwm} which is also the graph associated with the white-green subgraph $G^*_{white,green}$. It is a Petrie tour partition of G consisting of three Petrie tours.

The following theorem immediately follows from Theorem 6.

Theorem 7. Given a 4-regular plane bipartite graph G, the size of the minimum Petrie tour partition of G is $\min\{ecn(G^*_{white,red}),\ ecn(G^*_{white,green})\}$.

Note: The following Theorem was proved in Žitnik (2002).

Theorem 8. Let $G = (V, E)$ be a 4-regular plane bipartite graph. Then, G has a Petrie Eulerian tour if and only if the following conditions hold.

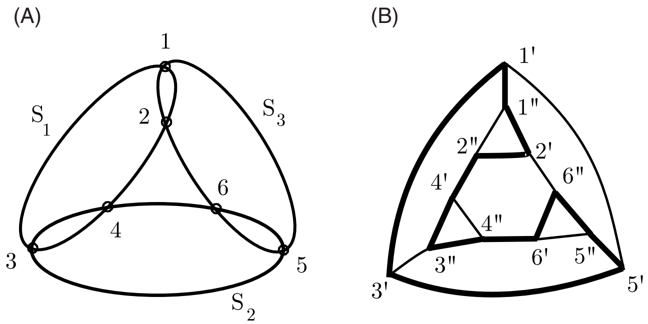

Figure 10. (A) A 4-regular plane graph G; (B) A 3-regularization G_3 of G.

- The number of black faces and the number of white faces in G are the same.
- Either the subgraph $G^*_{white,red}$ or the subgraph $G^*_{white,green}$ is a spanning tree of G^*_{white}.

Note that Theorem 7 generalizes Theorem 8: if the two conditions in Theorem 8 hold, then either $ecn(G^*_{white,red}) = 1$ or $ecn(G^*_{white,green}) = 1$.

The proof of the following theorem is similar to the proof of Theorem 6.

Theorem 9. Let G be a 4-regular bipartite plane graph. Let $G^*_{black,red}$ and $G^*_{black,green}$ be the subgraphs as defined above.

(1) Let $\Gamma_3 = \{c_1, \ldots, c_t\}$ be the set of elementary circuits of $G^*_{black,red}$. For each c_i $(1 \le i \le t)$, let $p(c_i)$ be the Petrie tour associated with c_i. Then, the union $\cup_{i=1}^{t} p(c_i)$ is the same as the red-black-merge graph G_{rbm}.

(2) Let $\Gamma_4 = \{c'_1, \ldots, c'_s\}$ be the set of elementary circuits of $G^*_{black,green}$. For each c'_i $(1 \le i \le s)$, let $p(c'_i)$ be the Petrie tour associated with c'_i. Then, the union $\cup_{i=1}^{s} p(c'_i)$ is the same as the green-black-merge graph G_{gbm}.

5. Characterization of Petrie Partitionable 4-Regular Plane Graphs

In this section, G always denotes a 4-regular plane graph, not necessarily bipartite. $\mathscr{S}(G) = \{S_1, \ldots, S_k\}$ denotes the set of S-tours of G.

Definition 6. *Let G be a 4-regular plane graph, and G_3 be a 3-regularization of G. A Petrie cycle partition $\mathscr{C}_3$ is called* full *if every edge of G belongs to a cycle $C_i \in \mathscr{C}_3$. (This implies all split edges of G_3 are non-cycle edges with respect to $\mathscr{C}_3$).*

A full Petrie cycle partition of G_3 corresponds to a Petrie tour partition of G. So the problem considered in Section 4, characterizing G with Petrie tour partitions, is to determine when G has a 3-regularization G_3 that has *full* Petrie cycle partitions. In computer graphics applications, the restriction to full Petrie cycle partitions of G_3 is not necessary. In this section, we study the general problem: characterize the Petrie partitionable graphs. In other words, determine when G has a 3-regularization G_3 that has Petrie cycle partition (full or not).

Figure 10A shows a 4-regular plane graph G with three S-tours S_1, S_2, S_3 where each pair of them intersect. By Lemma 4 and Theorem 3, G has no Petrie tour partitions. Figure 10B shows a 3-regularization G_3 of G with a Petrie cycle partition $\mathscr{C}_3$, consisting of a single Petrie Hamiltonian cycle C (denoted by thick lines in the figure).

We have the following simple characterization of Petrie partitionable graphs.

https://doi.org/10.1017/S0960129522000238 Published online by Cambridge University Press

Theorem 10. A 4-regular plane graph G with S-tour set $\mathscr{S}(G)$ is Petrie partitionable if and only if the following hold: (i) All S-tours in $\mathscr{S}(G)$ are simple, and (ii) $\mathscr{S}(G)$ can be partitioned into at most three subsets $\mathscr{S}_0, \mathscr{S}_1, \mathscr{S}_2$ of mutually independent S-tours.

Proof. By Theorem 2, G is Petrie partitionable if and only if G has a 3-regularization G_3 that is a multi-3-gon. By Theorems 5 and 3 and the remark after Definition 6, G has a 3-regularization G_3 with a full Petrie cycle partition if and only if all S-tours in $\mathscr{S}(G)$ are simple and $\mathscr{S}(G)$ can be partitioned into two subsets of mutually independent S-tours.

Suppose G has a 3-regularization G_3 that is a multi-3-gon. By Lemma 3, there exists a 3-edge-coloring λ of G_3 (with three colors 0, 1, 2) such that the valuation κ associated with λ satisfies $\kappa(u) = 1$ for all vertices u in G_3. In other words, the three edges in G_3 incident to u are colored by 0, 1, 2 in cw order. Consider any vertex v in G. Let e_1, e_2, e_3, e_4 be the edges in G incident to v in cw order. Let $e(v) = (v', v'')$ be the split edge in G_3 corresponding to v. Because the edge color pattern around v' and v'', it is clear that $\lambda(e_1) = \lambda(e_3)$, $\lambda(e_2) = \lambda(e_4)$ and $\lambda(e_1) \neq \lambda(e_2)$. Treat λ as an edge coloring of G, and let $\mathscr{S}_i$ ($i \in \{0, 1, 2\}$) be the set of edges in G with color i. Clearly, each $\mathscr{S}_i$ is a collection of mutually independent simple S-tours of G. Thus, $\mathscr{S}_0, \mathscr{S}_1, \mathscr{S}_2$ satisfy the condition of the theorem.

Now suppose the set of S-tours $\mathscr{S}(G) = \{\mathscr{S}_0, \mathscr{S}_1, \mathscr{S}_2\}$ of G satisfies the condition of the theorem. For each edge e of G, if e belongs to an S-tour in $\mathscr{S}_i$ ($i \in \{0, 1, 2\}$) color e by color i. Consider any vertex v in G and let e_1, e_2, e_3, e_4 be the edges in G incident to v in cw order. Let F_1, F_2, F_3, F_4 be the faces of G incident to v where e_i, e_{i+1} are on the boundary of F_i. For each $i = 1, 2, 3, 4$, call the triple $\alpha = (e_i, v, e_{i+1})$ an *angle of* F_i and denote it by $\alpha \in F_i$. Let $\mathscr{A}$ be the set of all angles of G. Define a mapping $\pi : \mathscr{A} \to \{-1, +1\}$ as follows. Consider any angle $\alpha = (e_i, v, e_{i+1}) \in \mathscr{A}$, let c_i and c_{i+1} be the color of e_i and $e_{i+1,}$ respectively. Define: $\pi(\alpha) = +1$ if $c_{i+1} - c_i \equiv +1 \pmod 3$; and $\pi(\alpha) = -1$ if $c_{i+1} - c_i \equiv -1 \pmod 3$.

For any face F of G, clearly $\sum_{\alpha \in F} \pi(\alpha) \equiv 0 \pmod 3$. Consider any vertex v in G, with four angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ incident to v in cw order. Clearly, $\pi(\alpha_1) = \pi(\alpha_3)$; $\pi(\alpha_2) = \pi(\alpha_4)$; and $\pi(\alpha_1) \neq \pi(\alpha_2)$. We now define a 3-regularization G_3 of G as follows:

- If $\pi(\alpha_1) = \pi(\alpha_3) = -1$, we split v in a way so that the faces F_1 and F_3 share the split edge $e(v)$ on their common boundary.
- If $\pi(\alpha_2) = \pi(\alpha_4) = -1$, we split v in a way so that the faces F_2 and F_4 share the split edge $e(v)$ on their common boundary.

For any face F' of G_3, let F be the face in G corresponding to F'. For each vertex v incident to F in G, let α be the angle of F incident to v. By the construction of G_3, if $\pi(\alpha) = -1$, v is split into two vertices in G_3 incident to F'; if $\pi(\alpha) = +1$, v corresponds to one vertex in G_3 incident to F'. Thus, the degree of F' in G_3 is: $\sum_{\alpha \in F \text{ and } \pi(\alpha) = +1} 1 + \sum_{\alpha \in F \text{ and } \pi(\alpha) = -1} 2 \equiv \sum_{\alpha \in F \text{ and } \pi(\alpha) = +1} 1 + \sum_{\alpha \in F \text{ and } \pi(\alpha) = -1} (-1) \equiv \sum_{\alpha \in F} \pi(\alpha) \equiv 0 \pmod 3$.

Since F' is any face in G_3, G_3 is a multi-3-gon, as to be shown. □

6. Determining if a 4-Regular Plane Graph is Petrie Partitionable is NP-Complete

Definition 7. *The* Petrie Partitionability of 4-regular plane graphs *(PP4R for short) problem is: given a 4-regular plane graph G, determine if G is Petrie partitionable or not.*

Definition 8. *The* Planar Graph 3-Colorability *(PG3C for short) problem is: given a plane graph H, determine if H has a vertex coloring using 3 colors.*

In this section, we show the following:

https://doi.org/10.1017/S0960129522000238 Published online by Cambridge University Press

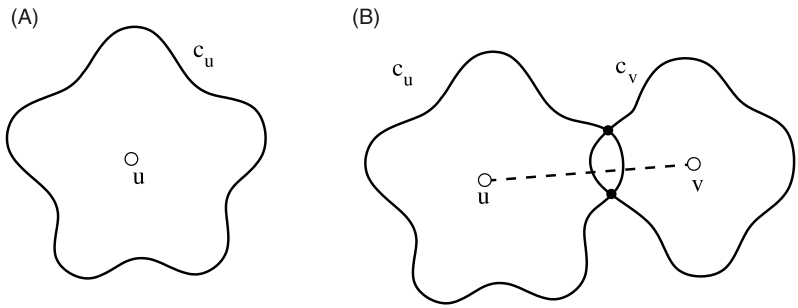

Figure 11. (A) A flower shaped cycle with 5 petals; (B) A degree-5 vertex u is adjacent to a degree-4 vertex v in H. The cycles c_u and c_v intersects at 2 points in G.

Theorem 11. The PP4R problem is NP-Complete.

Proof. It is known PG3C problem is NP-complete (Garey et al. 1976). We reduce the PG3C problem to the PP4R problem. This will establish the NP-completeness of the PP4R problem.

Let $H = (V_H, E_H)$ be a plane graph. We construct a 4-regular plane graph $G = (V_G, E_G)$ as follows. For each degree-k vertex u in V_H, we draw a *flower shaped cycle c_u with k petals* around it. (See Figure 11A). Each petals is associated with an edge incident to u in H. If (u, v) is an edge in E_H, then the petal of the cycle c_u associated with the edge (u, v) intersects with the petal of the cycle c_v associated with (u, v) at two points. (See Figure 11B). Do this for every vertex in V_H in such a way that these are the only intersection points of the cycles c_u's. So there are $2k$ points on the cycle c_u, and c_u is divided by these points into $2k$ *cycle segments*.

The vertices of G are these intersection points, and the edges of G are these cycle segments. From the construction, the following facts hold:

- G is a 4-regular plane graph, which can be constructed from H in polynomial time.
- Let $\mathscr{S}(G) = \{S_1, \ldots, S_k\}$ be the set of S-tours of G. Each S-tour in $\mathscr{S}(G)$ is just a cycle c_u for some vertex u in H. In other words, $\mathscr{S}(G) = \{c_u \mid u \in V_H\}$. Hence all S-tours of H are simple.

By Theorem 10, G is Petrie partitionable if and only if $\mathscr{S}(G)$ can be partitioned into three independent subsets $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3$, which is true if and only if the graph H is 3 vertex colorable. (If the cycle $c_u \in \mathscr{S}_i$ $(i = 1, 2, 3)$, then we color the vertex u by the color i in H. This defines a valid 3 vertex coloring of H.) Hence, G is Petrie partitionable if and only if H is 3 vertex colorable. This completes the polynomial time reduction from the PG3C problem to the PP4R problem and the proof of the theorem. □

7. Conclusion and Open Problems

We studied the problems of partitioning 3-regular plane graphs by Petrie cycles and partitioning 4-regular plane graphs by Petrie tours. We found simple characterizations for graphs having such partitions. This leads to simple linear time algorithms for finding minimum partitioning.

For 4-regular plane graphs, we discovered a nice characterization of Petrie partitionable graphs. We also showed that the problem of determining if a input 4-regular plane graph is Petrie partitionable is NP-complete.

The general version of these problems is motivated by applications in computer graphics, which require finding minimum partitioning of 3-regular plane graphs by Petrie paths and/or Petrie cycles, and finding minimum partitioning of 4-regular plane graphs by Petrie walks and/or Petrie

https://doi.org/10.1017/S0960129522000238 Published online by Cambridge University Press

tours. The general version of the problem is NP-complete. It is interesting to see if the insights discovered in this paper can lead to better heuristic algorithms and/or more efficient exact algorithms for solving the general version of these problems.

For the Petrie cycle partition problem, we considered only the 3-regular plane graphs. This is motivated by its connection to computer graphics applications. But it is also an interesting graph-theoretic problem for r-regular plane graphs with r=4 or 5. Is there a simple characterization for such graphs with Petrie cycle partitions?

Conflicts of interest. The authors declare none.

References

Berman, K. A. and Shank, H. (1979). Full 4-colorings of 4-regular maps. *Journal of Graph Theory* **3** 291–294.

Bommes, D., Campen, M., Ebke, H.-C., Alliez, P. and Kobbelt, L. (2013). Integer-grid maps for reliable quad meshing. *ACM Transactions on Graphics* **32** (4) 98:1–98:12, Article 98.

Bommes, D., L'vy, B., Pietroni, N., Puppo, E., Silva, C., Tarini, M. and Zorin, D. (2012). State of the art in quad meshing. In: *Eurographics STARS*, http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.363.6797.

Bondy, J. A. and Murty, U. S. R. (1979). *Graph Theory with Applications*, New York, North-Holland.

Dong, S., Bremer, P.-T., Garland, M., Pascucci, V. and Hart, J. C. (20061). Spectral surface quadrangulation. In: *ACM SIGGRAPH'06*, 1057–1066.

Estkowski, R., Mitchell, J. S. B. and Xiang, X. (2002). Optimal decomposition of polygonal models into triangle strips. In *Proceedings of the 18th ACM SoCG*, 254–263.

Fouquet, J. L. and Jolivet, J. L. (1982). Strong edge-coloring of cubic planar graphs. In: Adrian Bondy, J. and Murty, U. S. R. (eds.) *Progress in graph theory. Proceedings of the conference on combinatorics held at the University of Waterloo*, 247–264.

Garey, M. R., Johnson, D. S. and Stockmeyer, L. (1976). Some simplified NP-complete graph problems. *Theoretical Computer Science* **1** (3) 237–267.

Ivančo, J. and Jendroľ, S. (1999). On an eberhard-type problem in cubic polyhedral graphs having petrie and hamiltonian cycles. *Tatra Mountains Mathematical Publications* **18** 57–62.

Ivančo, J., Jendroľ, S. and Tkśč, M. (1994). Note on petrie and hamiltonian cycles in cubic polyhedral graphs. *Commentationes Mathematicae Universitatis Carolinae* **35** (2) 413–417.

Jaeger, F. and Shank, H. (1981). On the edge-coloring problem for a class of 4-regular maps. *Journal of Graph Theory* **5** 269–275.

Kidwell, M. E. and Bruce Richter, R. (1987). Trees and euler tours in a planar graph and its relatives. *American Mathematical Monthly* **94** 618–630.

Porcu, M. B. and Scateni, R. (2003). An interactive strpification algorithm based on dual graph operations. In: *Eurographics*.

Ringel, G. (1959). *Färbungsprobleme auf Flächen und Graphen*, Berlin.

Shank, H. (1975). The theory of left-right paths. *Combinatorial Mathematics III, Lecture Notes in Mathematics* **452** 42–54.

Šíma, J. (2005). Optimal triangle stripifications with minimum energy states in hopfields nets. In: *Internatioal Conference on Artificial Neural Networks: Biological Inspirations (ICANN 2005)*, LNCS, vol. 3965, 199–204.

Xiang, X., Held, M. and Mitchell, J. S. B. (1999). Fast and effective stripification of polygonal surface models. In: *1999 Symposium on Interactive 3D Graphics Layout*, 71–78.

Žitnik, A. (2002). Plane graphs with eulerian petrie walks. *Discrete Mathematics* **244** 539–549.

Cite this article: He X, Zhang H and Han Y (2022). On Petrie cycle and Petrie tour partitions of 3- and 4-regular plane graphs. *Mathematical Structures in Computer Science* **32**, 240–256. https://doi.org/10.1017/S0960129522000238