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AF-Skeletons and Real Rank Zero Algebras with the Corona Factorization Property

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Abstract. Let A be a stable, separable, real rank zero C^*-algebra, and suppose that A has an AF-skeleton with only finitely many extreme traces. Then the corona algebra $\mathcal{M}(A)/A$ is purely infinite in the sense of Kirchberg and Rørdam, which implies that A has the corona factorization property.

1 Introduction

Recall that in [7] Kirchberg and Rørdam called a C^*-algebra C purely infinite if it had no characters, and if furthermore for every pair of positive elements a, b in C such that b lies in the closed two-sided ideal generated by a, there is a sequence of elements $(r_n)_{n=1}^{\infty}$ such that $r_n a r_n^*$ converges in norm to b. This definition generalizes Cuntz's definition of purely infinite simple C^*-algebras.

Given a separable stable C^*-algebra B, it is of interest to determine when the corona algebra $\mathcal{M}(B)/B$ is purely infinite in the sense of Kirchberg and Rørdam. This property is connected with fundamental results about the structure of B, in particular, the corona factorization property, which will be defined later. One reason for the interest in the corona factorization property is that this condition is an algebraic characterization of a basically topological (or perhaps category-theoretical[1]) property: every full extension $\tau\colon A \to \mathcal{M}(B)/B$ of the C^*-algebra is absorbing in the nuclear sense.

We should mention that there is another notion of pure infiniteness for nonsimple algebras, due to Zhang. An algebra is purely infinite in the sense of Zhang if every nonzero hereditary subalgebra has an infinite projection. This definition is neither stronger nor weaker than the Kirchberg–Rørdam purely infinite property, except in the special case of simple algebras (then the two properties are equivalent). However, even though the corona of a stable separable and real rank zero C^*-algebra is purely infinite in Zhang's sense, this does not imply, unlike the Kirchberg-Rørdam property, that the original algebra has the corona factorization property.

In this paper, we give a sufficient condition for the corona algebra $\mathcal{M}(B)/B$ of a stable separable real rank zero C^*-algebra B to be purely infinite in the Kirchberg–Rørdam sense. Our condition is stated in terms of an AF-skeleton of B.

Definition 1.1 Let B be a C^*-algebra. Then, a pair (A, ι) is an AF-skeleton of B if A is an AF-algebra and $\iota\colon B \to A$ is a quasi-unital $*$-homomorphism, inducing an

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[1]It is interesting to note that Higson's construction [2] of a theory that is in many cases equivalent to KK-theory involved an explicitly category-theoretical construction.

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isomorphism on primitive ideal spaces, such that every projection in B is Murray–von Neumann equivalent to a projection in $\iota(A)$.

When the context is clear, we will sometimes refer to A as "the AF-skeleton," which is of course a misnomer, as A is not unique. A homomorphism is said to be *quasi-unital* (the term *proper* has also been used) if it maps some (hence all) approximate unit(s) of the domain to an approximate unit of the target algebra. Perera and Rørdam[10] actually used the equivalent form $\overline{\iota(B)A\iota(B)} = A$

The term quasi-unital is not entirely standard, and sometimes other terms or slightly different definitions of this term are used in the literature. Actually, AF-skeletons have an additional technical property (involving the order structure in the projection monoid) that we have omitted from the above list since we do not need it.

Lin [9] has shown that if B is a simple separable unital real rank zero C^*-algebra, with stable rank one and with $K_0(B)$ a dimension group, then B has an AF-skeleton. Perera and Rordam [10] have shown more generally that arbitrary separable real rank zero C^*-algebras have AF-skeletons.

In this paper, we prove the following result.

Theorem 1.2 *Let B be a stable separable real rank zero C^*-algebra having an AF-skeleton with finitely many extremal tracial rays. Then $\mathcal{M}(B)/B$ is purely infinite in the sense of Kirchberg and Rordam.*

We have been unsuccessful in removing the AF-skeleton from the hypothesis, but this may form a topic for future investigation. On the other hand, the use of AF-skeletons allows us to consider a fairly wide class of algebras A. In [11], Rørdam asks for an example of a real rank zero simple algebra which contains an infinite and a finite projection. If such an example exists, it would not be surprising if it had a corona that is purely infinite in the sense of Kirchberg and Rørdam, and one could hope that our theorem might be applicable to show this.

As a corollary of our main result, we get that algebras B satisfying the hypothesis above have the corona factorization property [8].

Definition 1.3 Let B be a separable, stable C^*-algebra. We say that B has the *corona factorization property* if full projections in $\mathcal{M}(B)/B$ are properly infinite.

Recall that a positive element is full if the norm-closed $*$-ideal generated by the element is the whole algebra. It can be shown that, at least in our setting of stable algebras, the corona factorization property is equivalent to having full projections of the multiplier C^*-algebra $\mathcal{M}(B)$ be properly infinite. From the above definition and Theorem 1.2, we have the following corollary.

Theorem 1.4 *Suppose that B is a stable separable real rank zero C^*-algebra and suppose that B has an AF-skeleton with only finitely many extreme tracial rays. Then B has the corona factorization property.*

This theorem implies that every full extension of B is absorbing in the nuclear sense. As an additional application of our main theorem, notice that it can readily be used

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to show, via the two of three property of purely infinite algebras, that the multiplier algebra of a real rank zero purely infinite algebra is purely infinite whenever the AF-skeleton has finitely many extremal tracial rays.

2 Main result

Our hypothesis is stated in terms of the tracial cone of a stable AF-algebra. Of course, the traces on the algebra are not continuous. Hence, it is most natural to view the traces as forming a convex cone of positive linear functionals on the ordered K_0-group. The natural topology on any collection of linear functionals is usually, as in this case, the weak topology, coming from evaluation at points of the underlying space. A *base* for such a cone T is a convex subset K such that any nonzero trace in T can be expressed uniquely as a scalar multiple of some element of K. (For example, one can construct a base by intersecting T with a hyperplane missing the origin.)

We denote the tracial cone of A by $T^{+}(A)$, and, fixing some choice of base, we denote the chosen base by $T(A)$. The base[2] is a Choquet simplex when it is compact, but of course it may or may not be compact in general. For example, $T(C_0(\mathbb{R}))$, where $C_0(\mathbb{R})$ is regarded as a C^*-algebra, is noncompact.

We use the following compactness criterion of Elliott and Handelman [3, Theorem 3.1, part (iii) $\Leftrightarrow$ (vii)] as a lemma.

Lemma 2.1 *Let A be a separable AF-algebra with no nonzero unital quotients. Then the space $T(A)$ is compact if and only if there exists a unital copy of O_∞ in the corona $\mathcal{M}(A)/A$.*

The proof of the above very interesting result is impacted by a counterexample due to Goodearl [4, p. 479], and hence we give an outline of the changes needed (with thanks to George Elliott). The counterexample affects [3, Theorem 2.10], and [3, Theorem 2.11] must be modified, with the conclusion weakened to only allow traces (extended) from B, and the hypothesis strengthened to include compactness of the base of the trace cone of B. It turns out that the equivalence of nine conditions in their Theorem 3.1 is still valid if conditions (v) and (vi) are omitted. In particular, the equivalence of their condition (iii) and condition (vii) is what we need for our proof.

We now give more detail on the modifications that need to be made to the proof in [3]. At the bottom of page 109 of [3], the stated inequality is now only for $\tau \in T(A)$. The first line of page 110 of [3] needs to be modified to read:

> Hence by the proof of Theorem 2.11 (with the conclusion weakened to the statement with $T(\mathcal{M}(A))$ replaced by $T(A)$, and the hypotheses strengthened to include compactness of $T(A)$), there exists a projection $e \in \mathcal{M}(A)$ such that...

In the next line $T(\mathcal{M}(A))$ should again be replaced by $T(A)$.

[2]The base is in many cases unique up to affine homeomorphism (see [5, 6.17, 10.2]). The case of non-compact base is not too well understood, but it is likely that different choices of base are still at least homotopy equivalent in a suitable sense.

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Lemma 2.2 *Let A be a stable AF-algebra such that the cone $T^+(A)$ of traces of A has only finitely many extremal rays. Projections in $\mathcal{M}(A)$ have an image that is either zero or properly infinite in the corona algebra.*

We thank the referee for suggesting the use of the Elliott–Handelman result to simplify the proof of this lemma, and for suggesting that we generalize to the case of non-simple AF-algebras.

Proof We may as well assume that P is not in A. If P is a projection in the multiplier algebra with $\overline{APA}$ equal to A, then by the result of Combes and Zettl [1], the tracial cone of the full hereditary subalgebra PAP is algebraically isomorphic to that of A, and hence is finite-dimensional. In particular it follows that $T(PAP)$ is finite-dimensional and hence compact. Since P is not in A/I for any ideal I, it follows that PAP has no unital quotients. By Lemma 2.1, there thus is a unital copy of O_∞ in the corona $\mathcal{M}(PAP)/PAP = P(\mathcal{M}(A)/A)P$. This implies that P has properly infinite image in the corona.

If P is a projection in the multiplier algebra with $\overline{APA}$ not equal to A, then we can reduce to the previous case, simply by perturbing P to $q := P + c$ where c is strictly positive in $(1-P)A(1-P)$. This element q is equal to P in the corona, which is all we need. By the Combes–Zettl result quoted above, the tracial cone of $\overline{qAq}$ is compact. We need only check that there are no unital quotients, and then we can proceed as before. To verify this, suppose that in some quotient $B := A/I$ it happens that $\overline{qBq}$ is unital. Denoting the unit of the hereditary subalgebra by $u \in B$, it follows that $(u-1)q$ multiplies B into zero, and hence is zero in $\mathcal{M}(B)$. But then $uq = q$ and this contradicts the fact that (the image of) q is not in B. ■

Looking at the preceding result, one might hope that the finite-dimensionality hypothesis could be replaced by the compactness of any base of the tracial cone. However, this form of compactness is rather delicate and in particular is not always preserved under passage to full hereditary subalgebras. In fact, as several people have pointed out to us, the converse of the above lemma also holds: for an AF-algebra, purely infinite corona implies finite-dimensional tracial cone.

Theorem 2.3 *Let B be a stable separable real rank zero C^*-algebra such that the AF-skeleton has only finitely many extremal tracial rays. Then every projection P in $\mathcal{M}(B)$ is either zero or properly infinite in the corona algebra $\mathcal{M}(B)/B$.*

Proof By hypothesis, let (A, ι) be an AF-skeleton for B that has only finitely many extreme traces. We can stabilize both A and B without changing the hypothesis (we just replace ι by $\iota \otimes 1$), thus, we may as well assume that A is stable. The monomorphism ι maps an approximate unit to an approximate unit, thus is strictly continuous, and extends to a strictly continuous map of multiplier algebras $\Psi\colon \mathcal{M}(A) \longrightarrow \mathcal{M}(B)$. The extended map, Ψ, is a unital monomorphism.[3]

Since Ψ maps A into B, it also induces a unital map of corona algebras.

[3]This is because the kernel of the extended map Ψ is the strict closure of the kernel of the given map ι.

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By the definition of an AF-skeleton, we have that every projection in B is Murray–von Neumann equivalent to a projection in $\Psi(A)$. It follows from this that the map Ψ maps the projection monoid $V(A)$ onto the projection monoid $V(B)$. We claim that the same is true for the projection monoids of the multiplier algebras. Certainly, Ψ induces a map, say Ψ_V, from the projection monoid $V(\mathcal{M}(A))$ into the projection monoid $V(\mathcal{M}(B))$, and we are to show that this map is onto.

Suppose then that Q is a projection in $\mathcal{M}(B)$. Since B has real rank zero, QBQ has real rank zero. Choose a countable approximate unit $\{q_n\}_{n=1}^{\infty}$ for QBQ, consisting of an increasing sequence of projections. Set $e_1 := q_1$ and $e_n := q_n - q_{n-1}$ for $n \geq 2$. Then $\sum_{n=1}^{\infty} e_n$ converges to Q in the strict topology of $\mathcal{M}(B)$. Now by the AF-skeleton property, for each $n \geq 1$, there is a projection f_n in A such that e_n is Murray–von Neumann equivalent to $\Psi(f_n)$. Considering $R := \sum v_n f_n v_n^*$ where the v_i are the usual sequence of orthogonal multiplier isometries of $\mathcal{M}(A)$ that come from the stability of A, we obtain a projection R in $\mathcal{M}(A)$.

Since Ψ is strictly continuous, we have $\Psi_V(R) = \sum_{n=1}^{\infty} \Psi(v_n f_n v_n^*)$, where the sum converges since $\sum_{n=1}^{\infty} v_n f_n v_n^*$ is known to converge in the strict topology in $\mathcal{M}(B)$. Since Ψ preserves Murray–von Neumann equivalence, $\Psi(v_n f_n v_n^*)$ is equivalent to $\Psi(f_n)$, which is in turn equivalent to e_n. Let w_i be a partial isometry in B with initial projection e_n and range projection $\Psi(v_n f_n v_n^*)$.

The sum $W = \sum_{n=1}^{\infty} w_n$ converges in the strict topology in $\mathcal{M}(B)$ and has initial projection Q, range projection $\Psi(R)$. Hence, $\Psi(R)$ is Murray–von Neumann equivalent to Q, and thus Ψ_V maps $V(\mathcal{M}(B))$ *onto* $V(\mathcal{M}(A))$.

Now let P be a projection in $\mathcal{M}(B)$ that is not in B. This projection has been shown to be equivalent to a projection, say $\Psi(U)$, in the image of Ψ. Clearly, the preimage U cannot be in the canonical ideal, so by Lemma 2.2 the projection U has properly infinite image in the corona $\mathcal{M}(A)/A$. (The AF-skeleton is stable, and hence has no unital quotients, so Lemma 2.2 can be applied.) As pointed out earlier, the map Ψ induces a homomorphism of corona algebras, and therefore the image $\Psi(U) + B$ in the corona is the image of the properly infinite projection $U + A$ under a homomorphism. A homomorphism maps a properly infinite projection to either zero or another properly infinite projection, so $\Psi(U) + B$ is properly infinite in the corona. Of course Murray–von Neumann equivalence also preserves this property, so that the original projection P has properly infinite image in the corona, as asserted. ∎

For the main result, we need two more results from the literature. First of all, from Zhang's work [12, Theorem 1.1], we have the following result.

Lemma 2.4 *Let B be a σ-unital real rank zero C^*- algebra. A hereditary subalgebra of the multiplier C^*-algebra $\mathcal{M}(B)$ is the linear span of its projections.*

We also need the following sufficient condition, due to Rørdam and Kirchberg [7, Proposition 4.7], for an algebra to be purely infinite in their sense. We note that the hypothesis is strictly stronger than the property of being purely infinite in the sense of Zhang.

Lemma 2.5 *Let C be a C^*-algebra with the property that every nonzero hereditary C^*-subalgebra in every quotient of C contains an infinite projection. Then C is purely infinite in the sense of Kirchberg and Rordam.*

We now prove the main result.

Proof of Theorem 1.2 By Lemma 2.5, it is enough to show that every hereditary subalgebra of every quotient of $\mathcal{M}(B)/B$ contains an infinite projection. By Theorem 2.3, nonzero projections in the corona of B that lift to projections of the multiplier C^*-algebra $\mathcal{M}(B)$ are properly infinite in the corona. It follows from Lemma 2.4 that every hereditary subalgebra of $\mathcal{M}(B)/B$ therefore is (linearly) spanned by properly infinite projections.

Let I be a two-sided norm-closed ideal in $\mathcal{M}(B)/B$. If H is a nonzero hereditary subalgebra of the quotient $(\mathcal{M}(B)/B)/I$, then the pre-image of H in the multiplier C^*-algebra $\mathcal{M}(B)$ is again a hereditary subalgebra and as before is spanned by projections. It follows that H is itself the span of projections coming from $\mathcal{M}(B)$. Since the map from $\mathcal{M}(B)$ to $(\mathcal{M}(B)/B)/I$ factors through $\mathcal{M}(B)/B$, these projections in H are, by Theorem 2.3, each the (nonzero) image of a properly infinite corona projection under a homomorphism, and since the properly infinite property is preserved by homomorphisms, it follows that they are properly infinite in H. Hence, by Lemma 2.5, the corona $\mathcal{M}(B)/B$ must be purely infinite in the sense of Kirchberg and Rordam. ■

Note From a forthcoming paper by Kucerovsky and Perera, it now appears that all stable real rank zero C^*-algebras have the corona factorization property.

References

[1] F. Combes and H. Zettl, *Order structures, traces and weights on Morita equivalent C^*-algebras.* Math. Ann. **265**(1983), no. 1, 67–81.

[2] A. Connes and N. Higson, *Déformations, morphismes asymptotiques et K-thèorie bivariante.* C. R. Acad. Sci. Paris Sèr. I Math. **311**(1990), no. 2, 101–106.

[3] G. A. Elliott and D. E. Handelman, *Addition of C^*-algebra extensions.* Pacific J. Math. **137**(1989), no. 1, 87–121.

[4] K. R. Goodearl, *K_0 of multiplier algebras of C^*-algebras with real rank zero K-theory.* **10**(1996), no. 5, 419–489.

[5] ———, *Partially ordered abelian groups with interpolation.* Mathematical Surveys and Monographs 20, American Mathematical Society, RI, 1986.

[6] G. G. Kasparov, *The operator K-functor and extensions of C^*-algebras.* Math. USSR Izvestia **16**(1981), 513–572.

[7] E. Kirchberg and M. Rørdam, *Nonsimple purely infinite C^*-algebras.* www.math.ku.dk/~rordam/manus/pi1.ps.gz.

[8] D. Kucerovsky, *Large Fredholm Triples.* J. Funct. Anal. **236**(2006), no. 2, 395–408.

[9] H. Lin, *Skeleton C^*-algebras.* Canad. J. Math. **44**(1992), no. 2, 324–341.

[10] F. Perera and M. Rørdam, *AF-embeddings into C^*-algebras of real rank zero.* J. Funct. Anal. **217**(2004), 142–170.

[11] M. Rørdam, *A simple C^*-algebra with a finite and an infinite projection.* Acta Math. **191**(2003), no. 2, 109–142.
[12] S. Zhang, *On the structure of projections and ideals of corona algebras.* Canad. J. Math. **41**(1989), 721–742.

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